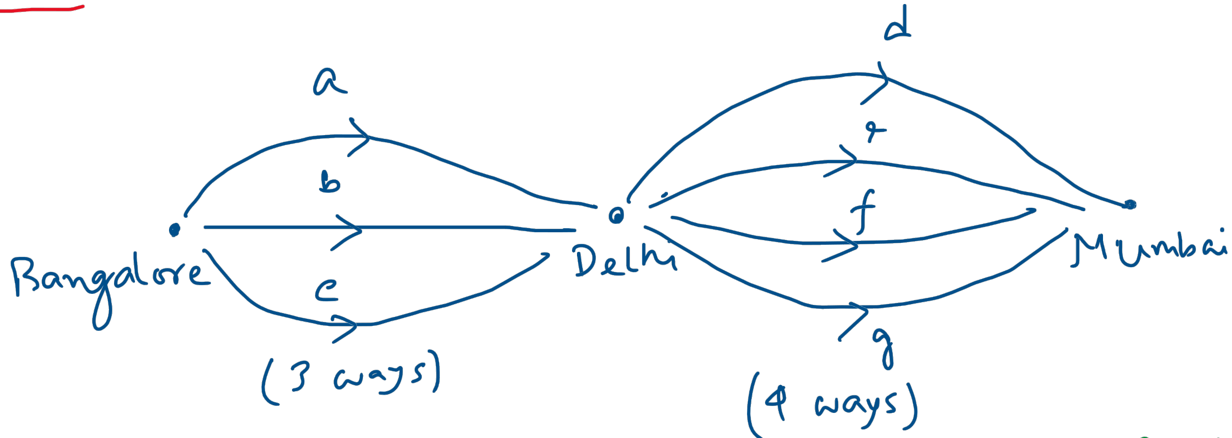
# Counting Principles

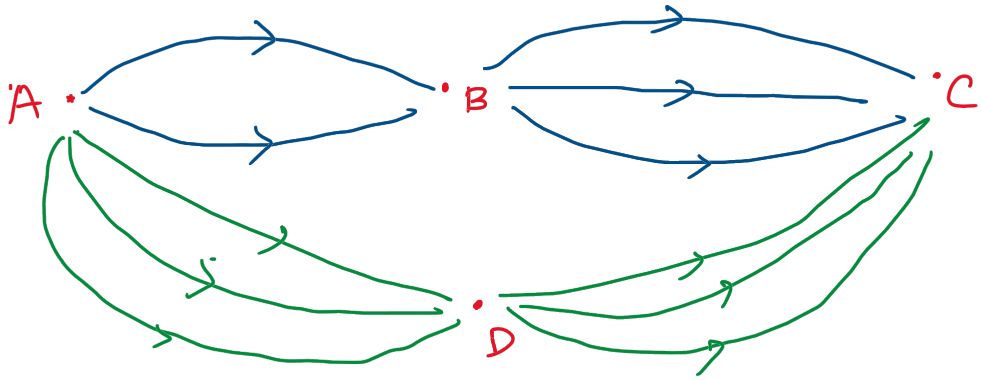



| | | |
|---|---|---|
| a-d | b-d | c-d |
| a-e | b-e | c-e |
| a-f | b-f | c-f |
| a-g | b-g | c-g |

$$\underbrace{\text{(the 12 pairs above)}}_{3 \times 4 = 12}$$

If $k$ events occurs one after another. If there are $n_1$ ways the first event can occur, $n_2$ ways the second event can occur, ...., $n_k$ ways the $k^{th}$ event can occur. Then the total number of ways all the events can occur is:

$$\underbrace{n_1 \times n_2 \times n_3 \times \cdots \times n_k}_{\text{multiplication}}$$




$A \to C$ ?

$A \to B \to C$ $\quad$ $A \to D \to C$

$2 \times 3 = 6$ $\quad$ $3 \times 3 = 9$

$6 + 9 = 15$

k-events (none of them occurs simultaneously)

$n_1, n_2, \dots, n_k$

$n_1 + n_2 + \dots + n_k$ (addition).

**Permutation (arrangements) of objects :-**

Arranging 'r' objects from n-objects order does matter. (r-permutation)
(Repetition not allowed) $(r \le n)$

$\overset{1}{\bigcirc} \quad \overset{2}{\bigcirc} \quad \overset{3}{\bigcirc} \; \dots \; \overset{n}{\bigcirc}$

$\underbrace{\underline{n} \quad \underline{(n-1)} \quad \underline{(n-2)}}_{r=3}$

$n \times (n-1) \times (n-2) \times \dots (n-r+1)$

$${}^{n}P_{r} = r\text{-permutation of } n\text{-objects} = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)$$

$${}^{n}P_{r} = \frac{n \times (n-1) \times (n-2) \times \cdots \times (n-r+1) \times [(n-r) \times (n-r-1) \times \cdots \times 3 \times 2 \times 1]}{(n-r) \times (n-r-1) \times \cdots \times 3 \times 2 \times 1}$$

$$\boxed{{}^{n}P_{r} = \frac{n!}{(n-r)!}}$$

Ex 1: Suppose I need to form a committee of 3 people (president, chairman, vice-chairman) from a group of 10 peoples. How many ways we can form the committee?

$\frac{a}{P}$ $\frac{b}{C}$ $\frac{c}{VC}$  $\frac{b}{P}$ $\frac{c}{C}$ $\frac{a}{VC}$

$${}^{10}P_{3} = \frac{10!}{7!}$$

$$= \frac{10 \times 9 \times 8 \times \cancel{7!}}{\cancel{7!}}$$

$$= 720$$

**Ex 2 :**

$$\boxed{\underbrace{\_\ \_}_{\text{Letter}}\ \ \overbrace{\_\ \_\ \_}^{\text{numbers}}}$$

1) Letters should be distinct
2) The number can't start from 0.

AE 234 (✓)

PP 312 (×)

AE 057 (×)

BX 555 (✓)

$$^{26}P_2 = \frac{26!}{24!} = \frac{26\times 25\times \cancel{24!}}{\cancel{24!}}$$

$$\underline{\ \ \ }\ \underline{\ \ \ } \qquad {}^{26}P_2 = 26\times 25 = 650$$

$$\underline{9}\ \ \underline{10}\ \ \underline{10} \qquad 9\times 10\times 10 = 900$$

# distinct number plates

$$650 \times 900 = 585\,000$$

## Combination :

r - combination of n-objects :

Order doesn't matter

Choosing r-objects from n-objects . $^nC_r$ $\binom{n}{r}$

$a_1$ $a_2$ $2!$ $\begin{pmatrix} a_1, a_2 \\ a_2, a_1 \end{pmatrix}$

$\boxed{a_1, a_2, a_3}$ $\begin{pmatrix} a_1\ a_2\ a_3 \\ a_1\ a_3\ a_2 \\ a_2\ a_1\ a_3 \\ a_2\ a_3\ a_1 \\ a_3\ a_1\ a_2 \\ a_3\ a_2\ a_1 \end{pmatrix}$ $3!$

$a_1, a_2, a_3, a_4$

$4!$ $(24)$

$\underbrace{\_\ \_\ \_\ \cdots\ \_}_{r\text{-object}}$ $(r!)$

$\overbrace{O\ \ O\ \ O\ \cdots\ O}^{n\text{-objects}}$

$\underbrace{\underline{n}\ \ \underline{(n-1)}\ \ \underline{(n-2)}}_{r\text{-place}}$

$$^nP_r = n \times (n-1) \times \cdots \times (n-r+1)$$
$$= r! \times {}^nC_r$$
$$\therefore {}^nC_r = \frac{^nP_r}{r!} = \frac{n!}{r! \times (n-r)!}$$

$\underline{Ex:}$ 10 students. I want to select 3 students from 10 students. How many different ways we can do it?

$$^{10}C_3 = \frac{10!}{7! \times 3!} = \frac{10 \times \cancel{9}^{3} \times \cancel{8}^{4} \times \cancel{7!}}{\cancel{7!} \times \cancel{3} \times \cancel{2}} = 120$$

$$^{n}C_r = \frac{n!}{r! \times (n-r)!}$$

$$^{n}C_r = \frac{n!}{(n-r)!\ r!} = {}^{n}C_{n-r}$$

$$^{10}C_3 = {}^{10}C_7$$

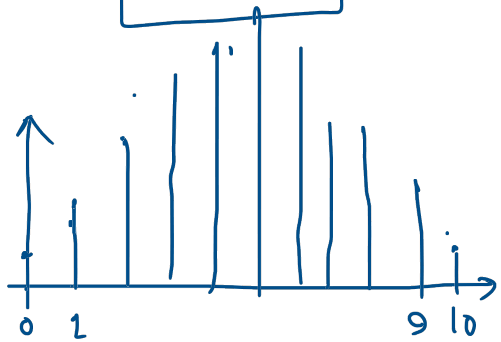
$$^{10}C_2 = \frac{10 \times 9}{2} = 45$$

$$^{n}C_0 = \frac{n!}{0! \times n!} = 1$$

$$\boxed{0! = 1}$$

$$^{n}C_n = {}^{n}C_0 = 1$$

$n = 10$. $\quad {}^{n}C_r \quad r = 0, \ldots, 10$



$$1 + x = 1 \cdot x^0 + 1 \cdot x^1$$

$$(1+x)^2 = 1 \cdot x^0 + 2x^1 + 1 \cdot x^2$$

$$(1+x)^3 = 1 \cdot x^0 + 3 \cdot x^1 + 3 \cdot x^2 + x^3$$

$$(1+x)^4 = 1x^0 + 4x^1 + 6x^2 + 4x^3 + x^4$$

$$(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$

<u>Pascal's triangle</u>

$$\begin{array}{ccccccccccc}
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

$2^1$

$2^2$

$2^3$

$2^4$

$$\boxed{(1+x)^n = {}^nC_0 + {}^nC_1 \cdot x + {}^nC_2\, x^2 + \cdots + {}^nC_r\, x^r + \cdots + {}^nC_n\, x^n}$$

Binomial Theorem

$(n+1)$ terms

$$(a+b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1} b + {}^nC_2\, a^{n-2} b^2 + \cdots + {}^nC_r\, a^{n-r} \cdot b^r + \cdots + {}^nC_{n-1} \cdot a b^{n-1} + {}^nC_n\, b^n$$

${}^nC_r \rightarrow$ binomial coefficients.

$$\begin{tikzcd} {}^{n-1}C_{r-1} \arrow[dr] & & {}^{n-1}C_{r} \arrow[dl] \\ & {}^{n}C_{r} & \end{tikzcd}$$

$$\boxed{{}^{n}C_{r} = {}^{n-1}C_{r-1} + {}^{n-1}C_{r}}$$

$${}^{5}C_{3} = {}^{4}C_{2} + {}^{4}C_{3} = 10$$

$${}^{4}C_{2} \to \frac{6}{{}^{3}C_{1} + {}^{3}C_{2}}$$

$${}^{3}C_{1} \to 3$$

$${}^{3}C_{2} \to {}^{2}C_{1} + {}^{2}C_{2} \to \frac{2 \qquad 1}{3}$$

$${}^{4}C_{3} \to \frac{4}{{}^{3}C_{2} + {}^{3}C_{3}}$$

$${}^{3}C_{2} \to {}^{2}C_{1} + {}^{2}C_{2} \to \frac{2 \qquad 1}{3}$$

$${}^{3}C_{3} \to 1$$

$$(1+x)^{n} = {}^{n}C_{0} + {}^{n}C_{1}\,x + {}^{n}C_{2}\,x^{2} + \cdots + {}^{n}C_{n}\,x^{n}$$

$$\Rightarrow \text{for } x=1: \quad 2^{n} = {}^{n}C_{0} + {}^{n}C_{1} + \cdots + {}^{n}C_{n}$$

$$\boxed{\sum_{r=0}^{n} {}^{n}C_{r} = 2^{n}}$$

**Probability** : A branch of mathematics that deals with uncertain events.

**Random Experiment (experiment)** : A random experiment is a process which will produce one of the several possible outcomes and the outcome can't be predicted beforehand.

Ex: Coin toss $\rightarrow$ H, T $\}$ 2 - possible outcomes

Rolling a dice (6 face) $\longrightarrow$ 6 possible outcomes

For an experiment the set of all possible outcomes is known as sample space ($\Sigma$) of the experiment.

Coin-toss :- $\Sigma = \{H, T\}$ , Rolling of 6 face dice: $\Sigma = \{1, 2, 3, 4, 5, 6\}$

Subset of the sample space (collection of possible outcomes) is called 'Event'.

**Ex 1:** Rolling a 6-face dice, getting an even number

$E_1$ = getting even number

$E_1 = \{2, 4, 6\}$

$\Sigma = \{1, 2, 3, 4, 5, 6\}$

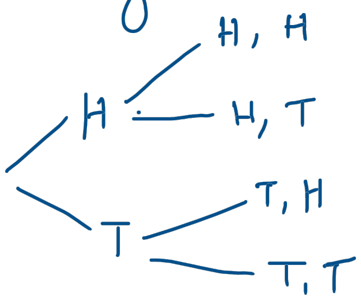
**How to find sample space :-**

$\Sigma = \{\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_n\}$

$\sigma_1, \sigma_2, \dots, \sigma_n$ should follow MECE

Mutually Exclusive and Collectively Exhaustive

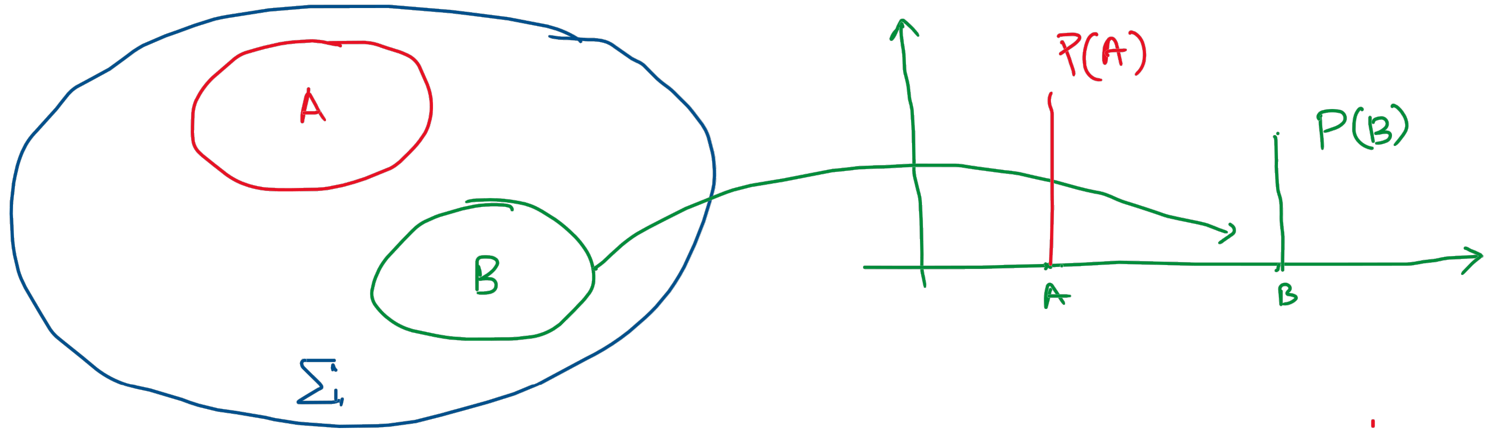
**Ex:-** Tossing two coins simultaneously




$\Sigma = \{(H, H), (H, T), (T, H), (T, T)\}$

$\Sigma$ → Sample space

**Probability:** The probability law assigns a non-negative number to an event.




$P(A)$: Probability of occuring event A. Which is a non-negative number which encodes our idea of the chances of that event to happen.

$$\boxed{0 \leq P(A) \leq 1}$$

Always between 0 to 1.

$P(A) = 0$ } impossible event

$P(A) = 1$ } sure event

$P(\Sigma) = 1$

$P(A) > P(B) \Rightarrow$ A is more likely to happen than B.

$P(\text{Rain}) > P(\text{Sunny})$

## Count Based Approach:

$\Sigma$ = sample space

$n = |\Sigma| \longrightarrow$ number of distinct possible outcome of the experiment.

$A \to$ Event. $\quad A \subseteq \Sigma \qquad |A| = n_A \to$ number of outcomes in A.

then $\boxed{P(A) = \dfrac{n_A}{n}}$ (if all the outcomes of the event are equally likely)

**Ex:** rolling a fair 6 sided dice. Event = getting an even number.

$\Sigma' = \{1, 2, 3, 4, 5, 6\}$ , $E = \{2, 4, 6\}$

$n = 6$ $\qquad n_E = 3$

$\therefore P(E) = \frac{3}{6} = \frac{1}{2}$

Ex: What is the probability of choosing one 'Ace' from a randomly shuffled deck of cards.

(2, 4, 6, 8, 10)

$$\left.\begin{array}{l} \diamondsuit \longrightarrow A, K, Q, J, 2, \ldots, 10 \\ \clubsuit \longrightarrow A, K, Q, J, 2, \ldots, 10 \\ \spadesuit \longrightarrow A, K, Q, J, 2, \ldots, 10 \\ \heartsuit \longrightarrow A, K, Q, J, 2, \ldots, 10 \end{array}\right\} \text{52 cards}$$

Event $(A) = \{A_d, A_c, A_s, A_H\}$

$n_A = 4$

$|\Sigma'| = 52$

$\therefore P(A) = \frac{4}{52} = \frac{1}{13}$

Ex:- Even numbered card  $n_E = 20$  $\therefore P(E) = \frac{20}{52} = \frac{5}{13}$

Ex:- A person throws two 6 face fair dices. what is the prob. that sum of outcomes of the dice = 7.

$\Sigma' = \{(1,1), (1,2), (1,3) \ldots, (6,4), (6,5), (6,6)\}$  $n = |\Sigma'| = 36$

$$B = \{(x,y) : x+y=7 \ \& \ x,y \in \{1,2,3,4,5,6\}\}$$

$$B = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\} \quad \therefore n_B = 6$$

$$\therefore P(B) = \frac{6}{36} = \frac{1}{6}$$

$$\frac{n_x}{n} \longrightarrow \frac{1}{6} \quad \text{as} \quad n \to \infty$$

Ex: Suppose I toss a coin three times. What is the prob that two-heads will occur?

$$\Sigma = \{(H,H,H), (H,H,T), (H,T,H), (H,T,T), (T,H,H), (T,H,T), (T,T,H), (T,T,T)\}$$

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$$n = |\Sigma| = 8 \qquad \therefore P(A) = \frac{3}{8}$$

## Complement Event:

$A$ is an event. then $\Sigma - A = A^C$

$$P(A^C) = 1 - P(A) \quad \therefore P(A) + P(A^C) = 1$$

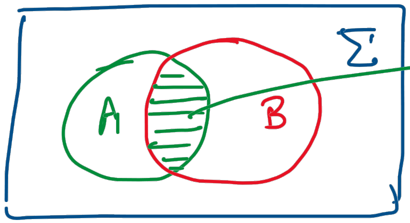
**Ex:** Toss a coin three times. What is the prob of getting atleast one head.

$A$ = atleast one head in three toss

$A^C$ = no-head in three toss $\longrightarrow (T, T, T)$

$$\therefore P(A^C) = \frac{1}{8} \qquad \therefore P(A) = 1 - P(A^C) = 1 - \frac{1}{8} = \frac{7}{8}$$

## Probability of two Events happening simultaneously:-




$\longrightarrow A \cap B$

$P(A \cap B)$

Ex: Suppose I am rolling two dice. What is the prob of getting sum of two dice = 7 and first dice value < second dice value.

$$A = \{(x, y) : x + y = 7 \ \& \ x, y \in \{1, 2, 3, 4, 5, 6\}\}$$

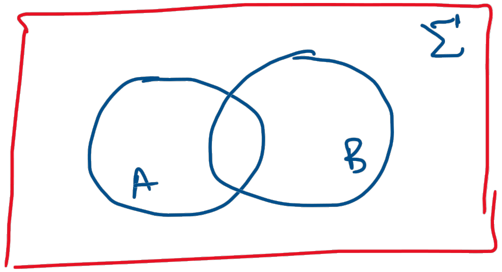
$$B = \{(x, y) : x < y \ \& \ x, y \in \{1, 2, 3, 4, 5, 6\}\}$$

$$A \cap B = \{(x, y) : x + y = 7 \ \& \ x < y \ \& \ x, y \in \{1, 2, 3, 4, 5, 6\}\}$$

$$= \{(1, 6), (2, 5), (3, 4)\}$$

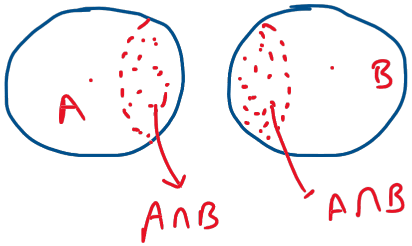
$$n = |\Sigma| = 36 \quad \therefore \ |A \cap B| = 3 \quad \therefore P(A \cap B) = \frac{3}{36} = \frac{1}{12}$$

## Probability of A or B :-




$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

Ex:- Two dice rolling. What is the probability that outcome of the first dice is exactly divisible by second. or the second dice value is divisible by 3.




$A = \{(x,y) : x \,\%\, y = 0\}$   $x, y \in \{1,2,3,4,5,6\}$

$B = \{(x,y) : y \,\%\, 3 = 0\}$

$A + B - (A\cap B)$

$A = \{(1,1), (2,1), (3,1), (4,1), (5,1), (6,1), (2,2), (4,2), (6,2), (3,3), (6,3), (4,4), (5,5), (6,6)\}$

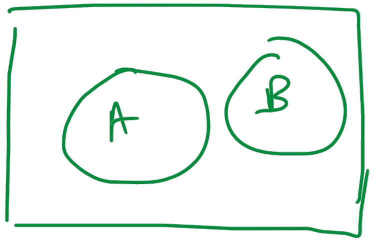
$(A\cup B)$.

$P(A) = ? = \frac{14}{36}$

$P(B) = \frac{12}{36}$ $\therefore$ $A\cap B = \{(3,3), (6,3), (6,6)\}$ $\therefore$ $P(A\cap B) = \frac{3}{36}$

$\therefore P(A\cup B) = \frac{14+12-3}{36} = \frac{23}{36}$

Probability of a <u>null set</u> (empty set) $P(\phi) = 0$




$A \cap B = \phi$ } A & B are disjoint sets.

$\therefore P(A \cup B) = P(A) + P(B)$

<u>Ex:</u> what is the probability I can draw a king or an ace.

A → draw a king  B → draw an ace

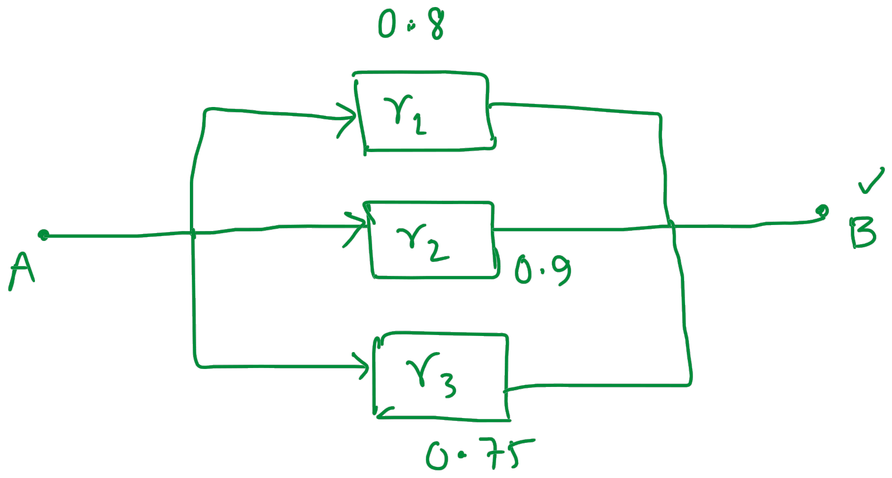
$$P(A \cup B) = P(A) + P(B) = \frac{4}{52} + \frac{4}{52} = \frac{2}{13}$$

<u>Independent Event:</u> A, B

$$P(A \cap B) = P(A) \times P(B)$$

<u>Ex:-</u> $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Ex:-




$P(r_1) = 0.8$ , $P(r_1^c) = 0.2$

$P(r_2) = 0.9$ , $P(r_2^c) = 0.1$

$P(r_3) = 0.75$ , $P(r_3^c) = 0.25$

$\therefore P(A \to B) = ?$

$$P(A \to B) = 1 - \underbrace{P(\text{B doesn't receive the message})}_{\text{if all the routers fail}}$$

$$\therefore P(r_1^c \cap r_2^c \cap r_3^c) = P(r_1^c) \times P(r_2^c) \times P(r_3^c)$$

$$= 0.2 \times 0.1 \times 0.25$$

$$= 0.02 \times 0.25 = 0.005$$

$\therefore P(A \to B) = 0.995 = 99.5\%$