Simple Linear Regression

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OUTLINE

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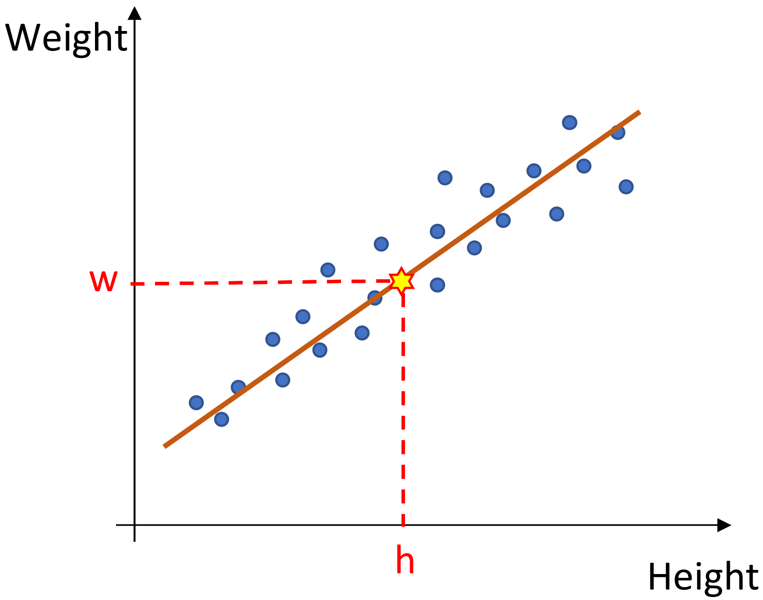
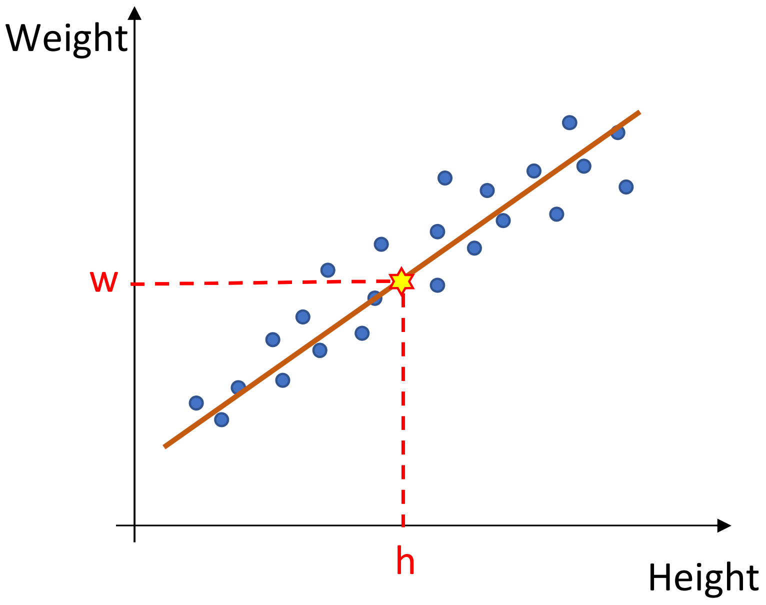
- Hypothesis function for Simple Linear Regression
- Mean Square Error Loss / Cost function
- Intuition of Cost Function
- Ordinary Least Square Regression

Simple Linear Regression : Intuition



- Consider the scatter plot of the Weight vs. Height of adults as shown beside.
- The trend or the form of the relationship is strongly positive.
- Now suppose we wish to estimate the **weight** of a person just by knowing his/ her **height**.
- In order to do so we first fit a straight line through our data points.
- Then from the graph, knowing the height we can find the weight of the corresponding person.
- Hence, we are intending to find out the **equation of the straight line** that ***best*** describes the relationship between Weight and Height.

Simple Linear Regression : Intuition



- There is only one predictor/input variable (Height) and one target variable (Weight) and we are intending to find out a relationship of the form:

$$\boldsymbol{y} = \boldsymbol{\theta_0} + \boldsymbol{\theta_1}\,\boldsymbol{x}$$

$$\textit{Here, } y \textit{ is the target variable and}$$
$$x \textit{ is the predictor variable}$$

- We have to find out $\boldsymbol{\theta_0}$ and $\boldsymbol{\theta_1}$, such that the straight line $\boldsymbol{y} = \boldsymbol{\theta_0} + \boldsymbol{\theta_1}\,\boldsymbol{x}$ fits into our dataset **best**.

- This is called Simple Linear Regression, because it has only one predictor variable and the relationship among target and predictor variable is linear.

Simple Linear Regression : Hypothesis

Simple Linear Regression Model with Single Predictor

$$\underbrace{\boldsymbol{y}}_{\text{Target Variable}} = \underbrace{\boldsymbol{\theta_0}}_{\text{Intercept}} + \underbrace{\boldsymbol{\theta_1}}_{\text{Slope}}\underbrace{\boldsymbol{x}}_{\text{Predictor Variable}} + \underbrace{\boldsymbol{\varepsilon}}_{\text{Random Residual Error}}$$

$\boldsymbol{\theta_0}$ and $\boldsymbol{\theta_1}$ are called model parameters

- We use our sample data to find estimates for the coefficients/ model parameters $\boldsymbol{\theta_0}$ and $\boldsymbol{\theta_1}$ i.e.: $\widehat{\boldsymbol{\theta_0}}$ and $\widehat{\boldsymbol{\theta_1}}$.
- We can then **predict** what the value of $\boldsymbol{y}$ should be corresponding to a particular value for $\boldsymbol{x}$ by using the Least Squares Prediction Equation (also known as our **hypothesis function**):

$$\widehat{\boldsymbol{y}} = \widehat{\boldsymbol{\theta_0}} + \widehat{\boldsymbol{\theta_1}}\,\boldsymbol{x}$$

Where $\widehat{\boldsymbol{y}}$ is our prediction for $\boldsymbol{y}$

Simple Linear Regression : Cost Function

Residuals and Residual Sum of Squares:

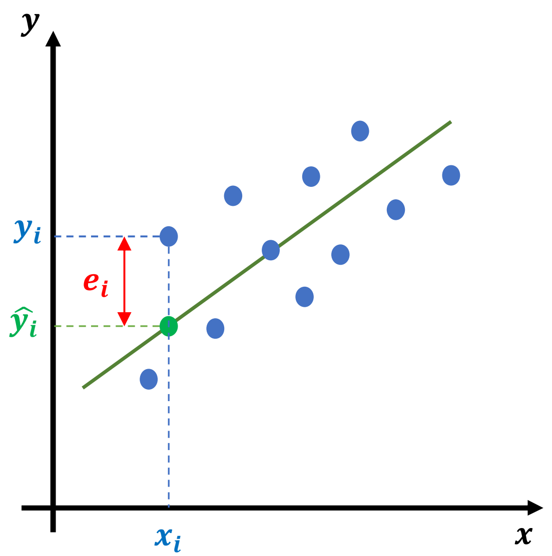
- For i^{th} sample $\langle \boldsymbol{x_i}\,,\ \boldsymbol{y_i} \rangle$ the predicted value of $\boldsymbol{y_i}$ is $\boldsymbol{\hat{y}_i}$, Which we obtain from the equation $\boldsymbol{\hat{y}_i} = \boldsymbol{\widehat{\theta_0}} + \boldsymbol{\widehat{\theta_1}}\,\boldsymbol{x_i}$

- Then, $\boldsymbol{e_i} = \boldsymbol{y_i} - \boldsymbol{\hat{y}_i}$ (actual – predicted) represents the i^{th} residual.

- We define **Residual Sum of Squares** (RSS) as:

$$RSS = \sum_{i=1}^{m} \boldsymbol{e_i}^2 = \sum_{i=1}^{m} (\boldsymbol{y_i} - \boldsymbol{\hat{y}_i})^2 = \sum_{i=1}^{m} (\boldsymbol{y_i} - (\boldsymbol{\widehat{\theta_0}} + \boldsymbol{\widehat{\theta_1}}\,\boldsymbol{x_i}))^2$$

There are total m no. of samples



Simple Linear Regression : Cost Function

Mean Square Error Cost Function:

- We can define the cost function as:

$$\boldsymbol{J}(\widehat{\boldsymbol{\theta}_0}, \widehat{\boldsymbol{\theta}_1}) = \frac{\mathbf{1}}{\mathbf{2}} \frac{\boldsymbol{RSS}}{\boldsymbol{Number\ of\ training\ samples}} = \frac{\mathbf{1}}{\mathbf{2m}} \sum_{i=1}^{m} (\boldsymbol{y}_i - (\widehat{\boldsymbol{\theta}_0} + \widehat{\boldsymbol{\theta}_1}\, \boldsymbol{x}_i))^2$$

Here a factor $\frac{1}{2}$ is multiplied just for computational simplicity. Otherwise, the cost function $\boldsymbol{J}(\widehat{\boldsymbol{\theta}_0}, \widehat{\boldsymbol{\theta}_1})$ is nothing but **mean or average of the Residual sum of squares.** (also known as **Mean Square Error (MSE)).**

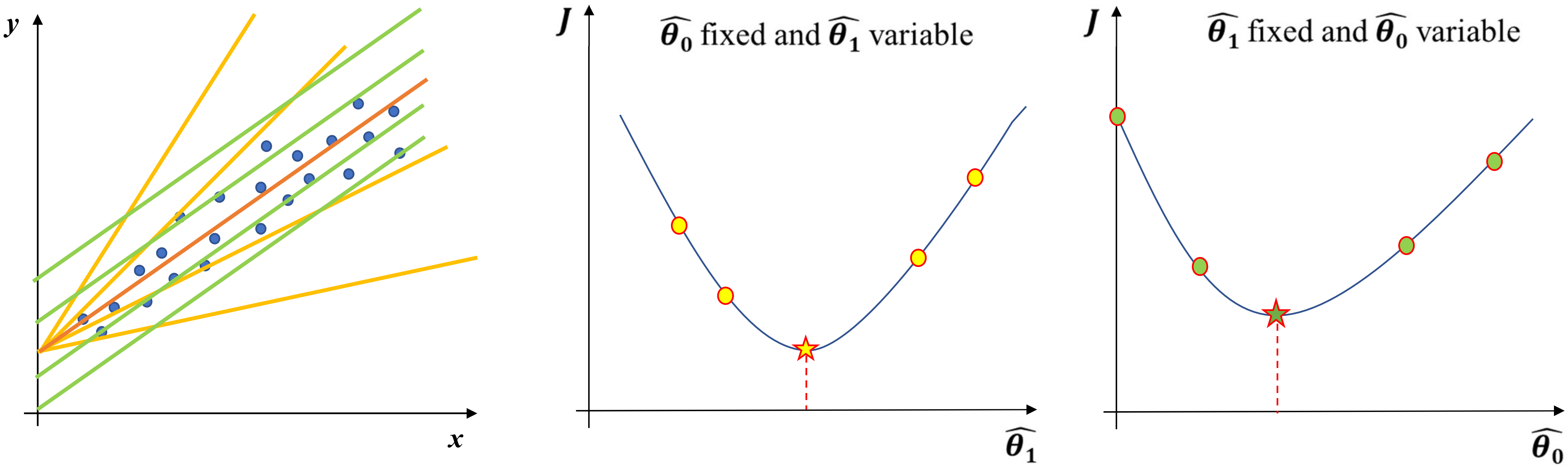
Our Objective:

- To find the suitable values of $\widehat{\boldsymbol{\theta}_0}$ and $\widehat{\boldsymbol{\theta}_1}$ such that the cost function $\boldsymbol{J}(\widehat{\boldsymbol{\theta}_0}, \widehat{\boldsymbol{\theta}_1})$ is minimized, in other words the Residual Sum of Square **(RSS)** is minimized. Then the straight line $\widehat{\boldsymbol{y}} = \widehat{\boldsymbol{\theta}_0} + \widehat{\boldsymbol{\theta}_1}\boldsymbol{x}$ will fit our data **best**. This is called **least squares fit**.

Simple Linear Regression : Cost Function

Intuition of Cost Function:

Consider the example of single predictor variable where the hypothesis function is $\hat{\boldsymbol{y}} = \widehat{\boldsymbol{\theta}_0} + \widehat{\boldsymbol{\theta}_1}\,\boldsymbol{x}$ and the cost function is $\boldsymbol{J}\left(\widehat{\boldsymbol{\theta}_0}, \widehat{\boldsymbol{\theta}_1}\right) = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{m}} \sum_{i=1}^{m} (\boldsymbol{y}_i - (\widehat{\boldsymbol{\theta}_0} + \widehat{\boldsymbol{\theta}_1}\,\boldsymbol{x}_i))^2$. Now we keep one parameter fixed and vary other. Let's see how $\boldsymbol{J}\left(\widehat{\boldsymbol{\theta}_0}, \widehat{\boldsymbol{\theta}_1}\right)$ varies.



- Our objective is to find the values of the parameters for which the cost function is minimized.

Simple Linear Regression : OLS fit

Solving for the best fit: Ordinary Least Squares (OLS) Regression:

- We have to Minimize $\boldsymbol{RSS}$ or $\boldsymbol{J}(\widehat{\boldsymbol{\theta}_0}, \widehat{\boldsymbol{\theta}_1})$ with respect to $\widehat{\boldsymbol{\theta}_0}$ and $\widehat{\boldsymbol{\theta}_1}$
- Hence we have to do, $\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\widehat{\boldsymbol{\theta}_0}}(\boldsymbol{RSS}) = \mathbf{0}$ and $\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\widehat{\boldsymbol{\theta}_1}}(\boldsymbol{RSS}) = \mathbf{0}$
- By solving the above two equations we get the following value of $\widehat{\boldsymbol{\theta}_1}$ and $\widehat{\boldsymbol{\theta}_0}$:

$$\widehat{\boldsymbol{\theta}_1} = \frac{\sum_{i=1}^{m}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{m}(x_i - \bar{x})^2} = r_{xy}\frac{\sigma_y}{\sigma_x} \quad and \quad \widehat{\boldsymbol{\theta}_0} = \bar{y} - \widehat{\boldsymbol{\theta}_1}\bar{x}$$

$where, \bar{x}\ is\ the\ mean\ of\ predictor\ variable\ x\ and\ \bar{y}\ is\ the\ mean\ of\ target\ variable\ y$

$\sigma_x\ is\ the\ standard\ deviation\ of\ x\ and\ \sigma_y\ is\ the\ standard\ deviation\ of\ y$

$and\ r_{xy}\ is\ the\ \boldsymbol{correlation\ coefficient}\ between\ x\ and\ y.$

OLS Regression :-

$$J(\hat{\theta}_0, \hat{\theta}_1) = \frac{1}{2m} \underbrace{\sum_{i=1}^{m} \left[y_i - (\hat{\theta}_0 + \hat{\theta}_1 x_i) \right]^2}_{RSS}$$

$$\frac{\partial J}{\partial \hat{\theta}_0} = 0 \; ; \quad \frac{\partial J}{\partial \hat{\theta}_1} = 0$$

$$RSS = \sum_{i=1}^{m} \left[y_i - (\hat{\theta}_0 + \hat{\theta}_1 x_i) \right]^2$$

$$= \sum_{i=1}^{m} \left[(\hat{\theta}_0 + \hat{\theta}_1 x_i) - y_i \right]^2$$

$$\frac{\partial}{\partial \hat{\theta}_0}(RSS) = 0 \; ; \quad \frac{\partial}{\partial \hat{\theta}_1}(RSS) = 0$$

$$\frac{1}{2m} \frac{\partial}{\partial \hat{\theta}_0} \left[\sum_{i=1}^{m} \{ (\hat{\theta}_0 + \hat{\theta}_1 x_i) - y_i \}^2 \right]$$

$$\Rightarrow \frac{1}{2m} \sum_{i=1}^{m} \frac{\partial}{\partial \hat{\theta}_0} \left[(\hat{\theta}_0 + \hat{\theta}_1 x_i) - y_i \right]^2 = 0$$

$$\Rightarrow \frac{1}{\cancel{2}m} \cancel{2} \sum_{i=1}^{m} \left(\hat{\theta}_0 + \hat{\theta}_1 x_i - y_i \right) = 0$$

$$\Rightarrow \frac{1}{m} \sum_{i=1}^{m} \hat{\theta}_0 + \hat{\theta}_1 \overbrace{\left(\frac{1}{m} \sum_{i=1}^{m} x_i \right)}^{\bar{x}} - \overbrace{\left(\frac{1}{m} \sum_{i=1}^{m} y_i \right)}^{\bar{y}} = 0$$

$$\hat{\theta}_0 + \hat{\theta}_1 \bar{x} - \bar{y} = 0$$

$$\Rightarrow \boxed{\hat{\theta}_0 = \bar{y} - \hat{\theta}_1 \bar{x}}$$

$$\frac{\partial}{\partial \hat{\theta}_1}\left[\frac{1}{2m}\sum_{i=1}^{m}\left(\hat{\theta}_0+\hat{\theta}_1 x_i - y_i\right)^2\right] = 0$$

$$\frac{d}{dx}(ax-b)^2$$

$$= 2\cdot(ax-b)\cdot a$$

$$\Rightarrow \frac{1}{2m}\sum_{i=1}^{m}\frac{\partial}{\partial \hat{\theta}_1}\left(\hat{\theta}_0+\hat{\theta}_1 x_i - y_i\right)^2 = 0$$

$$\Rightarrow \frac{1}{2m}\sum_{i=1}^{m}\frac{\partial}{\partial \hat{\theta}_1}\left(\bar{y}-\hat{\theta}_1\bar{x}+\hat{\theta}_1 x_i - y_i\right)^2 = 0$$

$$\Rightarrow \frac{1}{2m}\sum_{i=1}^{m}\frac{\partial}{\partial \hat{\theta}_1}\left[\hat{\theta}_1(x_i-\bar{x})-(y_i-\bar{y})\right]^2 = 0$$

$$\Rightarrow \frac{1}{\cancel{2}m}\cancel{2}\sum_{i=1}^{m}\left[\hat{\theta}_1(x_i-\bar{x})-(y_i-\bar{y})\right](x_i-\bar{x}) = 0$$

$$\Rightarrow \left[\hat{\theta}_1\sum_{i=1}^{m}(x_i-\bar{x})^2-\sum_{i=1}^{m}(y_i-\bar{y})\cdot(x_i-\bar{x})\right] = 0$$

$$\boxed{\hat{\theta}_1 = \frac{\sum_{i=1}^{m} (y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^{m} (x_i - \bar{x})^2}}$$

$$\hat{\theta}_1 = \frac{\frac{1}{m-1} \sum_{i=1}^{m} (y_i - \bar{y})(x_i - \bar{x})}{\frac{1}{m-1} \sum_{i=1}^{m} (x_i - \bar{x})^2}$$

$$\hat{\theta}_1 = \frac{\mathrm{Cov}(x, y)}{\mathrm{Var}(x)} = \frac{\sigma_x \cdot \sigma_y \cdot r_{xy}}{\sigma_x^2}$$

$$r_{xy} = \frac{\mathrm{cov}(x,y)}{\sigma_x \cdot \sigma_y}$$

$$\mathrm{Var}(x) = \sigma_x^2$$

$$\boxed{\hat{\theta}_1 = r_{xy} \cdot \frac{\sigma_y}{\sigma_x}}$$

Thank You