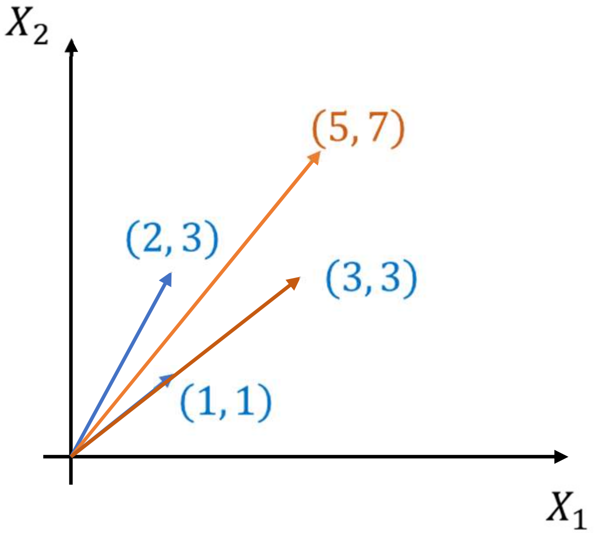
Eigenvalues & Eigenvectors

Matrix Vector Multiplication



- Consider the vector $\vec{x}_1 = [2\,,3]^T$
- If the matrix $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$ is post multiplied with $\vec{x}_1$, then we obtain another vector $\vec{x}_1'$.

$$\vec{x}_1' = A\vec{x}_1 \;\Rightarrow \vec{x}_1' = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

- So we can say that multiplying a vector with a matrix changes the direction and magnitude of the vector.

- However for a given square matrix there are some vectors whose direction remain unchanged after multiplying with the matrix.
- Consider the vector $\vec{x}_2 = [1\,,1]^T$. If we multiply A with $\vec{x}_2$ then, $\vec{x}_2' = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- Notice that only the magnitude changed but direction remain unchanged. i.e. the transformed vector $\vec{x}_2'$ is a scalar multiplication of the original vector $\vec{x}_2$.

Eigenvalues And Eigenvectors

- For a given square matrix A if we find a vector $\vec{v}$. Such that multiplying A with $\vec{v}$ doesn't change the direction of $\vec{v}$ but scales it, then that vector $\vec{v}$ is known as the **eigenvector** of A and the scalar value by which it will be scaled after being multiplied with A is known as the corresponding **eigenvalue**.

- Mathematically, $if \;\; A\vec{v} = \lambda\vec{v}\,, then\; \lambda\; is\; called\; the\; eigenvalue\; and\; \vec{v}\; the\; eigenvector\; of\; matrix\; A$

- $A\vec{v} = \lambda\vec{v} \;\Rightarrow (A - \lambda I)\vec{v} = 0$, here I is the identity matrix of the same order of matrix A.

- For non-trivial solution of the eigenvectors, $|A \;- \lambda I| = 0$

- The above equation is a polynomial of λ of degree same as the order of the matrix A. This equation is also known as the characteristic equation of matrix A.

- We can calculate the eigenvalues by solving the characteristic equation.

- Putting the value of the eigenvalue one can calculate the corresponding eigenvectors.

Lagrange’s Multiplier

Constrained Optimization

The unconstrained optimization problem can be stated as: **Find the extreme value of** $y = f(x)$

And to solve the problem we find the first derivative of *y* with respect to *x* and set that to zero. $$\frac{dy}{dx} = 0$$

In general a dependent variable could be functions of several independent variable. Then the unconstrained optimization problem is stated as: **Find the extreme value of** $y = f(x_1, x_2, x_3, \ldots, x_n)$

And to solve the problem we find the first derivative of *y* with respect to *all independent variables* and set those to zero.

$$\frac{\partial y}{\partial x_1} = 0 \quad \frac{\partial y}{\partial x_2} = 0 \quad \frac{\partial y}{\partial x_3} = 0 \quad \bullet\bullet\bullet \quad \frac{\partial y}{\partial x_n} = 0$$

However in real life we often come across the situations, where we have to optimize the function subject to certain conditions (called the constraints).

Find the extreme value of $y = f(x)$ **,**
subjected to the condition $g(x) = 0$

This type of problems are known as constrained optimization problem.

Lagrange's Multiplier

Find the extreme value of $y = f(x)$ **,**
subjected to the condition $g(x) = 0$

To solve the constrained optimization problem we formulate the **Lagrangian** as following:

$$L(x,\lambda) = f(x) - \lambda g(x)$$

Where, λ is a dummy variable known as **Lagrange's Multiplier.** Then we take derivative of the Lagrangian and set that to zero to find the optimal value.

$$\frac{\partial L}{\partial x} = 0 \rightarrow \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 0 \rightarrow \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$$

In general, there could be ***n*** many independent variables and ***m*** many constraints.

Find the extreme value of $y = f(x_1, x_2, x_3, \ldots, x_n)$ **,**
subjected to: $g_1(x_1, x_2, x_3, \ldots, x_n) = 0\ ;\ g_2(x_1, x_2, x_3, \ldots, x_n) = 0\ ;\ \ldots\ ;\ g_m(x_1, x_2, x_3, \ldots, x_n) = 0$

Then the **Lagrangian** is formulated as following.

$$L(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_m) = f - \lambda_1\, g_1 - \lambda_2\, g_2 - \ldots - \lambda_m\, g_m$$

Lagrange's Multiplier

Example: Maximize $y = 5x_1x_2$, subjected to $2x_1 + x_2 = 100$

Step-1: Formulate the Lagrangian. $L(x_1, x_2, \lambda) = 5x_1x_2 - \lambda\,(2x_1+x_2-100)$

Step-2: Take the partial derivatives of Lagrangian wrt x_1, x_2 and set them to zero.

$$\frac{\partial L}{\partial x_1} = 0 \rightarrow 5x_2 - 2\lambda = 0 \quad ---(1) \qquad \frac{\partial L}{\partial x_2} = 0 \rightarrow 5x_1 - \lambda = 0 \quad ---(2)$$

Step-3: Along with these we make use of the constrained equation. $2x_1 + x_2 - 100 = 0 \quad ---(3)$

Step-4: Solving these three equations we get:

$$\lambda = 125$$

$$x_1 = 25$$

$$x_2 = 50$$

Step-5: Hence the maximum value of y ; $y_{max} = 5 \cdot 25 \cdot 50 = 6250$

Thank You