Random Variable :

(Random variable) is a function that maps each of the element of the sample space to some real value.

A random variable (R.V.) is a real-valued function of the outcome of a random experiment.

Example: Toss 5 coins in succession (R.E.)

R.V. : Number of heads in the 5 coin toss (X)

$X = 0$ (T T T T T) $\quad {}^5C_0$

$X = 1 \quad {}^5C_1$

$X = 2 \quad {}^5C_2$

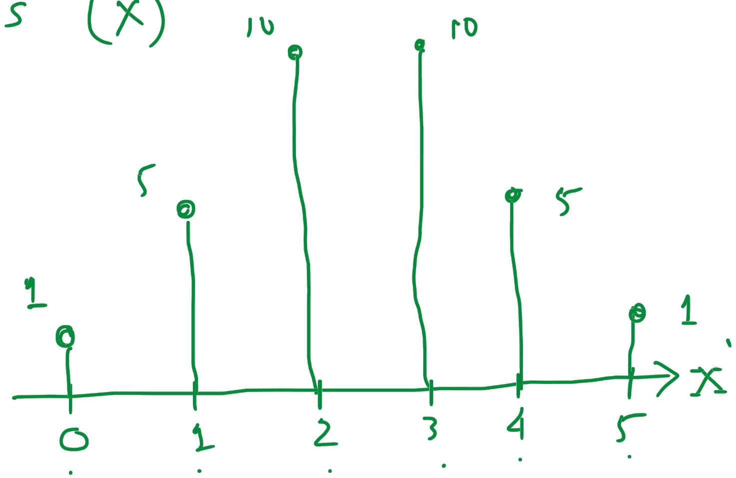
$X = 3 \quad {}^5C_3$

$X = 4$

$X = 5$

Sample Space

HHHHH
HHHHT
HHHTH
⋮
TTTTT



$X = 0, 1, 2, 3, 4, 5$

$Y = (\text{no. of head})^2, \quad Y = X^2$

$(Y = 0, 1, 4, 9, 16, 25)$

Main concepts related to R.V.

→ A R.V. is real valued function of the outcome of a random experiment

→ A function of a R.V. is also R.V.

→ We can associate with each R.V. certain 'aggregation' such as: mean, standard deviation, variance.

Random Variable

→ Discrete Random Variable (outcome of the R.E. are discrete)

Ex. Tossing coins.
Rolling a dice

→ Continuous Random Variable (outcome are continuous)

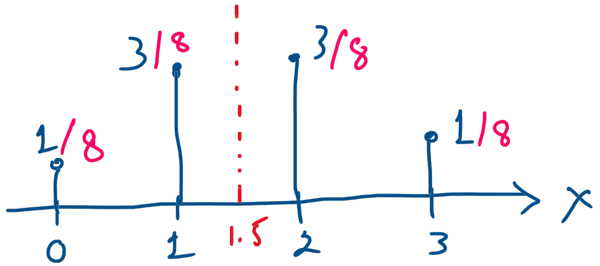
Ex. Rain fall
Stock price

Discrete R.V.

Ex. 1 A coin is tossed three times. X = no. of tails in the outcome.

$X = \{0, 1, 2, 3\}$

$$\Sigma = \{\underset{0}{HHH}, \underset{1}{HHT}, \underset{1}{HTH}, \underset{2}{HTT}, \underset{1}{THH}, \underset{2}{THT}, \underset{2}{TTH}, \underset{3}{TTT}\} \rightarrow 8$$



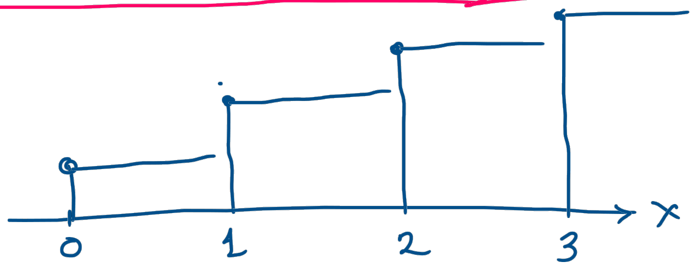
Probability mass function (P.M.F.)

$P(X=0) = 1/8$, $P(X=1) = \frac{3}{8}$, $P(X=2) = \frac{3}{8}$, $P(X=3) = \frac{1}{8}$

$$P(X=x) = p_X(x) \quad \} \text{ PMF}$$

Cumulative Distribution function:

$$P(X \leq x) = F_X(x)$$



$P(X \leq 0) = \frac{1}{8}$

$P(X \leq 1) = \frac{4}{8} = \frac{1}{2}$

$P(X \leq 2) = 7/8$

$P(X \leq 3) = 1$

$P(X \leq 1.5)$
$= P(X \leq 1)$

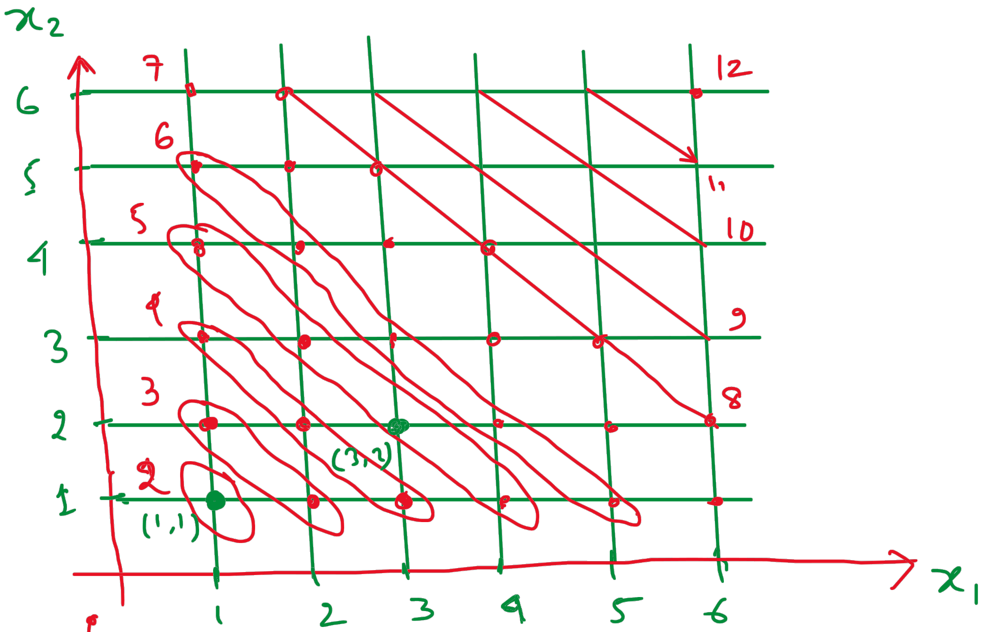
Ex: Suppose I roll two dice in succession.

X = Sum of the outcomes of two dice.

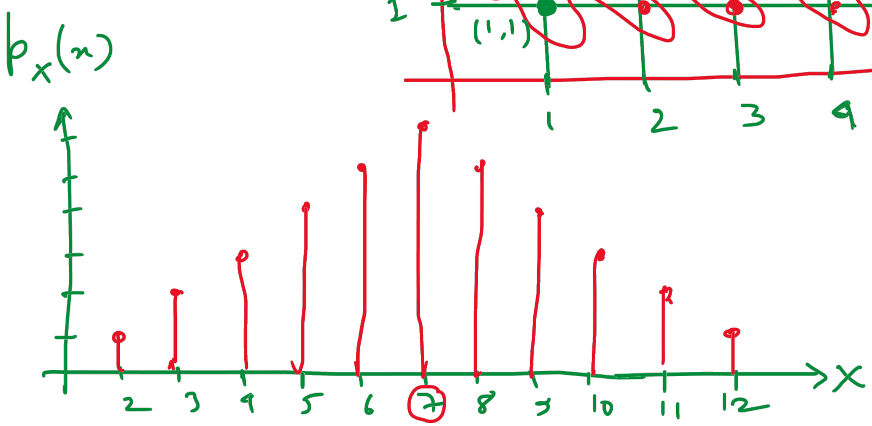
$X = \{2, 3, 4, \ldots\ldots, 12\}$ $\quad\quad X = x_1 + x_2$

x_1: o/p of 1^{st} dice

x_2: o/p of 2^{nd} dice



X	$p_X(x)$
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36



Expectation of a R.V.

Expectation $\longrightarrow$ weighted average.

Simple average: $x_1, x_2, x_3, \ldots, x_n$

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

if $\omega_1 = \omega_2 = \omega_3 \cdots = \omega_n = 1$

Weighted average:

values: $x_1, x_2, x_3, \ldots, x_n$

weights: $\omega_1, \omega_2, \omega_3, \ldots, \omega_n$

$\left(\sum_i \omega_i = 1\right)$

$$\bar{x}_w = \frac{\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_n x_n}{\underbrace{\omega_1 + \omega_2 + \cdots + \omega_n}_{=1}} = \underbrace{\sum_{i=1}^{n} \omega_i x_i}$$

Expectation of a R.V. Let the R.V. is X with PMF $p_X(x)$

$X \in \{x_1, x_2, \ldots, x_n\}$ corresponding PMF $p_X(x_1), p_X(x_2), \ldots, p_X(x_n)$

$$\sum_x p_X(x) = 1$$

$$\therefore \boxed{\mathbb{E}(X) = \sum_x p_X(x) \cdot x}$$

Ex: Three coin toss X = no. of tail

X	$p_X(x)$
0	1/8
1	3/8
2	3/8
3	1/8

$$\therefore \mathbb{E}[X] = \sum_x p_X(x) \cdot x = \frac{1}{8} \times 0 + \frac{3}{8} \times 1 + \frac{3}{8} \times 2 + \frac{1}{8} \times 3$$

$p_X(x)$, x

$$= \frac{12}{8} = \underline{1.5}$$

Ex: Two dice throwing. $\underline{X = x_1 + x_2}$, x_1 : outcome of first dice
x_2 : " " second dice

$$\underline{\mathbb{E}[X]} = \frac{252}{36} = \underline{7}$$

Few properties of Expectations :-

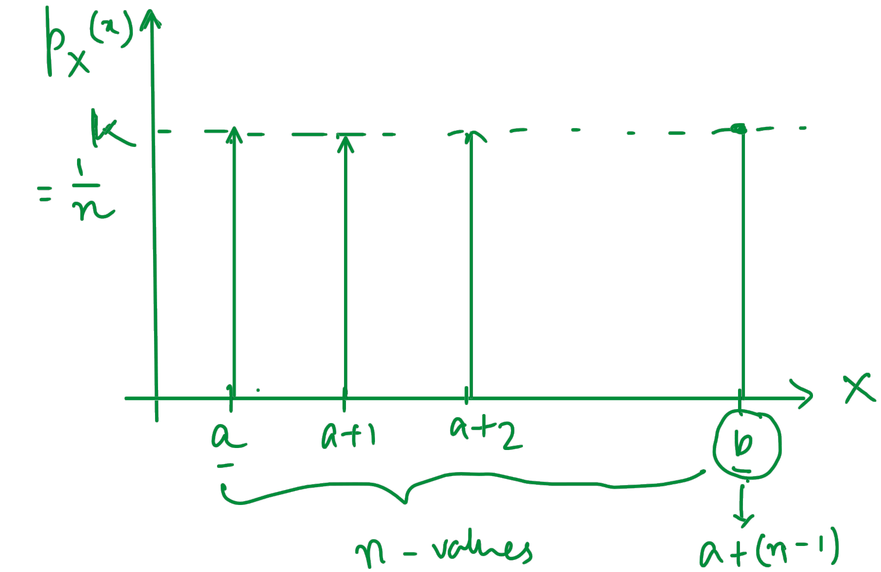
1. $X = \text{constant}(k)$: $\mathbb{E}(X) = k$

2. $Y = aX$, $\mathbb{E}[Y] = a \cdot \mathbb{E}[X]$

3. Suppose X & Y are two R.V.
   Z is another R.V. such that $Z = aX + bY$

   then $\underbrace{\mathbb{E}[Z] = a\,\mathbb{E}[X] + b\,\mathbb{E}[Y]}_{\text{Linearity of Expectation.}}$

Uniform Discrete R.V. :-



$$p_X(x) = k \qquad k = ?$$

$$\sum_x p_X(x) = 1$$

$$\Rightarrow \sum_x k = 1 \Rightarrow nk = 1 \Rightarrow k = \frac{1}{n}$$

$$E[X] = k \cdot \left[a + (a+1) + \dots + \{a + (n-1)\}\right]$$

$$= k\left[n \cdot a + (1 + 2 + 3 + \dots + (n-1))\right]$$

$$= k\left[na + \frac{n(n-1)}{2}\right]$$

$$= nk\left[a + \frac{(n-1)}{2}\right]$$

$$= \frac{2a + (n-1)}{2} = \frac{a + \{a + (n-1)\}}{2} = \frac{a+b}{2}$$

Variance of RV:-

X is a Random variable. $\mathbb{E}(X) \longrightarrow$ Expectation of RV.

Then variance of R.V. is defined as: $\mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$

$$(X - \mathbb{E}[X])^2 = X^2 - 2X\mathbb{E}[X] + \mathbb{E}[X]^2$$

$$\mathrm{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}\left[X^2 - 2X\,\mathbb{E}[X] + \mathbb{E}[X]^2\right]$$

$$= \mathbb{E}(X^2) - \mathbb{E}(2X\,\mathbb{E}[X]) + \mathbb{E}\left[\mathbb{E}[X]^2\right]$$

$$= \mathbb{E}[X^2] - 2\mathbb{E}[X]\cdot\mathbb{E}[X] + \mathbb{E}[X]^2$$

$$= \mathbb{E}[X^2] - 2\,\mathbb{E}[X]^2 + \mathbb{E}[X]^2$$

$$\boxed{\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2}$$

Standard deviation

$$\sigma(X) = \sqrt{\mathrm{Var}(X)}$$

$$= \sqrt{\mathbb{E}\left[(X - \mathbb{E}(X))^2\right]}$$

Ex:- 3 coin toss. X = # tails. $\mathbb{E}[X] = 3/2$

X	$p_X(x)$	$X - \mathbb{E}[X]$	$(X - \mathbb{E}[X])^2$
0	$1/8$.	$0 - 3/2 = -3/2$	$9/4$
1	$3/8$.	$1 - 3/2 = -1/2$	$1/4$
2	$3/8$.	$2 - 3/2 = \frac{1}{2}$	$1/4$
3	$1/8$.	$3 - \frac{3}{2} = \frac{3}{2}$	$9/4$

$$\mathrm{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}(X))^2\right]$$

$$= \frac{1}{8} \times \frac{9}{4} + \frac{3}{8} \times \frac{1}{4} + \frac{3}{8} \times \frac{1}{4} + \frac{1}{8} \times \frac{9}{4}$$

$$= \frac{9 + 3 + 3 + 9}{32} = \frac{24}{32}$$

$$= \frac{3}{4} = \underline{0.75}$$

X	$p_X(x)$	X^2
0	$1/8$	0
1	$3/8$	1
2	$3/8$	4
3	$1/8$	9

$\therefore \mathbb{E}[X^2]$

$$= \frac{1}{8} \times 0 + \frac{3}{8} \times 1 + \frac{3}{8} \times 4 + \frac{1}{8} \times 9$$

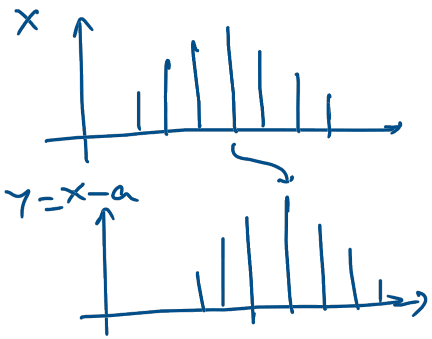
$$= \frac{3 + 12 + 9}{8} = \frac{24}{8} = 3$$

$$\mathbb{E}[X]^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

$$\therefore \mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 3 - \frac{9}{4} = \frac{12 - 9}{4} = \frac{3}{4} = \underline{0.75}$$

Properties of variance :-

X is R.V. and $\mathbb{E}[X]$ is expectation, $\mathrm{Var}(X)$ the variance.



$$Y = X - a \quad , \quad \mathbb{E}[Y] = \mathbb{E}[X] - a$$

$$Y^2 = X^2 - 2aX + a^2 \quad , \quad \mathbb{E}[Y^2] = \mathbb{E}[X^2] - 2a\,\mathbb{E}[X] + a^2$$

$$\mathrm{Var}(Y) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2$$

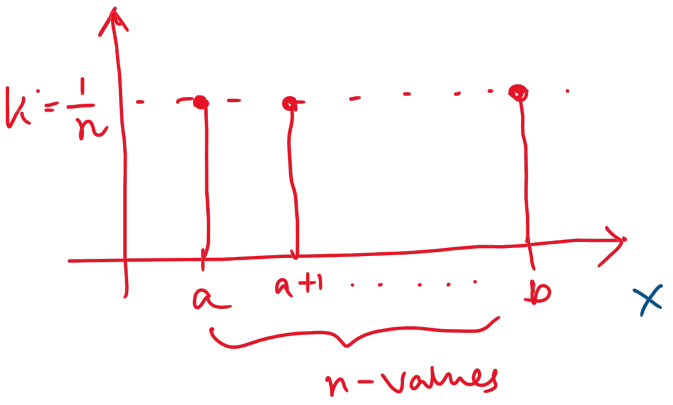
$$= \left(\mathbb{E}[X^2] - 2a\,\mathbb{E}[X] + a^2\right) - \left(\mathbb{E}[X] - a\right)^2$$

$$= \left(\mathbb{E}[X^2] - 2a\,\mathbb{E}[X] + a^2\right) - \left(\mathbb{E}[X]^2 - 2a\,\mathbb{E}[X] + a^2\right)$$

$$= \mathbb{E}[X^2] - \cancel{2a\,\mathbb{E}[X]} + \cancel{a^2} - \mathbb{E}[X]^2 + \cancel{2a\,\mathbb{E}[X]} - \cancel{a^2}$$

$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \qquad \therefore\ \mathrm{Var}(Y) = \mathrm{Var}(X)$$

Variance of Uniform discrete R.V.



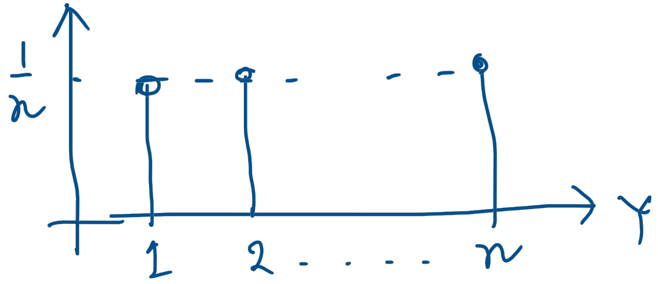
$Var(X) = ?$

$\mathbb{E}[X] = \frac{a+b}{2}$

X	$p_X(x)$	$Y = X - a + 1$
a	$1/n$	1
$a+1$	$1/n$	2
$a+2$	$\vdots$	3
$\vdots$		$\vdots$
$a+(n-1)$	$1/n$	n

$$\mathbb{E}[Y] = \mathbb{E}[X] - a + 1$$
$$= \frac{a+b}{2} - a + 1$$
$$= \frac{a+b-2a+2}{2}$$
$$= \frac{b-a+2}{2}$$
$$= \frac{\cancel{a}+(n-1)-\cancel{a}+2}{2}$$

$$\mathbb{E}[Y] = \frac{n+1}{2}$$



Y	Y^2
1	1
2	4
$\vdots$	$\vdots$
n	n^2

$$\therefore \mathbb{E}[Y^2] = \sum_{i=1}^{n} \frac{1}{n} \cdot i^2 = \frac{1}{n} \sum_{i=1}^{n} i^2$$
$$= \frac{1}{\cancel{n}} \cdot \frac{\cancel{n}(n+1)\cdot(2n+1)}{6}$$

$$\mathbb{E}[Y^2] = \frac{(n+1)(2n+1)}{6}$$

$$\mathrm{Var}(Y) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$= \frac{n+1}{2}\left(\frac{2n+1}{3} - \frac{n+1}{2}\right)$$

$$= \frac{n+1}{2}\left(\frac{4n+2-3n-3}{6}\right) = \frac{n+1}{2}\cdot\frac{n-1}{6}$$

$$= \frac{n^2-1}{12}$$

$$\therefore \mathrm{Var}(Y) = \frac{n^2-1}{12} = \mathrm{Var}(X)$$

Property of variance

$$X \to \mathbb{E}[X] \longrightarrow \mathrm{Var}(X)$$

$Y = aX$. then $\mathrm{Var}(Y) = ?$ $\quad \mathrm{Var}(Y) = a^2\,\mathrm{Var}(X)$

$$\mathrm{Var}(Y) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = \mathbb{E}[a^2X^2] - (a\cdot\mathbb{E}[X])^2 = a^2\{\mathbb{E}[X^2] - \mathbb{E}[X]^2\}$$

$$= a^2\,\mathrm{Var}(X)$$

Bernouli R.V. : It models a single trial of an experiment that has two possible outcomes : 'success' , 'failure'.

Ex:- Biased coin toss with probability of head $= p$ $\quad 0 \le p \le 1$

Ex:- Exam attempt with success prob $= p$ and failure prob $= 1 - p$

Ex:- Throwing a ball to the basketball net.

Parameter : Probability of success $= p$ $\quad (0 \le p \le 1)$

" " failure $= 1 - p$

$$X = \begin{cases} 1, & \text{success} \\ 0, & \text{failure} \end{cases} \qquad p_X(0) = 1 - p, \quad p_X(1) = p$$

$$\therefore \mathbb{E}[X] = 0 \cdot (1-p) + 1 \cdot p = \underline{p}$$

$$\underline{X^2} = \begin{cases} 1 \\ 0 \end{cases} \qquad \therefore \mathbb{E}[X^2] = p$$

$$\therefore \mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = p - p^2 = p(1-p)$$

Binomial R.V. The Binomial R.V. models the number of successes in a fixed number of <u>independent</u> Bernoulli trials.

<u>parameters</u>: p : probability of success in a trial ; $1-p$: probability of failure

n : number of trials.

<u>Random variable</u> : X = number of success (can be $0, 1, 2, \dots, n$)

<u>PMF</u> : $P(X = k) = {}^nC_k \cdot p^k \cdot (1-p)^{n-k}$

Special case : $p = \frac{1}{2}$. $(1-p) = \frac{1}{2}$. $p^k \cdot (1-p)^{n-k} = p^n = \frac{1}{2^n}$

$P(X = k) = {}^nC_k \left(\frac{1}{2^n}\right)$

<u>Expectation</u> : $\mathbb{E}[X] = \underline{np}$ and <u>Variance</u> : $\underline{np(1-p)}$

$$\mathbb{E}[X] = \sum_{k=0}^{n} P(X=k)\cdot k = \sum_{k=0}^{n} k \cdot {}^{n}C_{k} \cdot p^{k} \cdot (1-p)^{n-k}$$

For $k=0$, $k \cdot P(X=k) = 0 \cdot P(X=0) = 0$ & ${}^{n}C_{k} = \dfrac{n!}{k!\,(n-k)!}$

$$\therefore \mathbb{E}[X] = \sum_{k=1}^{n} k \frac{n!}{k!\,(n-k)!} \cdot p^{k} \cdot (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} k \cdot \frac{(n-1)!}{k!\,(n-k)!} \cdot p^{k-1}(1-p)^{n-k}$$

$$= np \cdot \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,(n-k)!} \cdot p^{k-1} \cdot (1-p)^{n-k}$$

if $j = k-1$

then $n-k$

$= (n-1) - (k-1)$

$= n-1-j$

$$= np \sum_{j=0}^{n-1} \frac{(n-1)!}{j!\,(n-1-j)!} p^{j} (1-p)^{(n-1)-j}$$

$$= np \cdot [p + (1-p)]^{n-1} = np \quad \therefore \boxed{\mathbb{E}[X] = np}$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

X is sum of n - independent bernouli trials.

$\therefore X = Y_1 + Y_2 + \cdots + Y_n$ where $Y_j \sim \text{Bernouli}(p)$

$$\therefore \mathrm{Var}(X) = \mathrm{Var}(Y_1 + Y_2 + \cdots + Y_n)$$

$$= \mathrm{Var}(Y_1) + \mathrm{Var}(Y_2) + \cdots + \mathrm{Var}(Y_n)$$

$$= n\,\mathrm{Var}(Y_j) = n \cdot p(1-p)$$

$$\therefore \mathrm{Var}(X) = np(1-p)$$