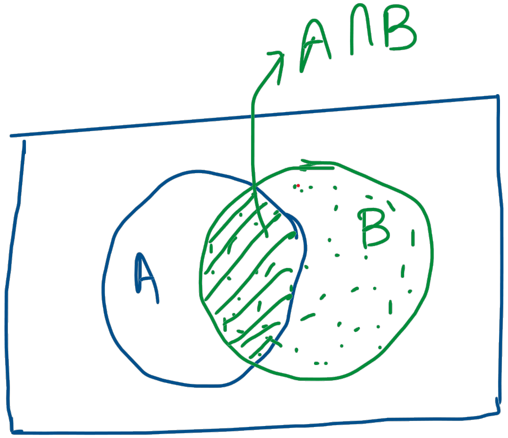
Conditional Probability

Event - A, Event - B

Suppose event - B has happened. Given than event B has happened what is the probability that A will happen.

$$P(A \text{ given that } B \text{ has happened}) = \underbrace{P(A|B)}_{\text{Prob. of A given B.}}$$



$$\boxed{P(A|B) = \frac{P(A\cap B)}{P(B)}}$$

$$\boxed{P(A\cap B) = P(B) \cdot P(A|B)}$$

Multiplication rule of Prob.

Ex :-

		Smoker		
		S	NS	Total
Gender	M	187	53	240
	F	57	203	260
	Total	244	256	500

What is the probability that a randomly chosen female student is smoker ?

A = being a smoker

B = being a female.

$$P(A|B) = \frac{\text{no. of female smoker}}{\text{no. of females}} = \frac{57}{260} \checkmark$$

$$P(A \cap B) = \frac{57}{500}$$

$$P(B) = \frac{260}{500}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{57/500}{260/500} = \frac{57}{260} \checkmark$$

Ex: 3 times coin toss. Find the prob. of getting more no. of heads than tails given that the outcome of the first toss is head.

→ A = more no. of heads than tails

B = outcome of the first toss is head

$\Sigma' = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$A = \{HHH, HHT, HTH, THH\}$, $B = \{HHH, HTH, HHT, HTT\}$

$A \cap B = \{HHH, HTH, HHT\}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{3/8}{4/8} = \frac{3}{4}$$

Ex: 2 times 6-sided fair dice roll. What is the prob that the sum of the outcomes = 9 given that the first dice outcome is even.

$A = \{\text{sum of outcome} = 9\}$, $B = \{\text{first dice outcome is even}\}$

$= \{(3,6), (6,3), (4,5), (5,4)\}$

$B = \{(x,y) : x \in \{2,4,6\}, y \in \{1,2,3,4,5,6\}\}$

$P(A \cap B) = \frac{2}{36} = \frac{1}{18}$

$P(B) = \frac{18}{36} = \frac{1}{2}$

$\therefore P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/18}{1/2} = \frac{1}{9}$

Multiplication Rule of Probability

$$P(A|B) = \frac{P(A\cap B)}{P(B)} \Rightarrow \boxed{P(A\cap B) = P(B)\cdot P(A|B)}$$

$$\boxed{P(A\cap B\cap C) = \frac{P(A\cap B\cap C)}{P(A\cap B)}\cdot\frac{P(A\cap B)}{P(A)}\cdot P(A)}$$

$$= P(C|A\cap B)\cdot P(B|A)\cdot P(A)$$

$$P(A_1\cap A_2\cap A_3) = P(A_3|A_1\cap A_2)\cdot P(A_2|A_1)\cdot P(A_1)$$

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P\left(A_n \middle| \bigcap_{i=1}^{n-1} A_i\right)\cdot P\left(A_{n-1} \middle| \bigcap_{i=1}^{n-2} A_i\right)\cdots\cdots P(A_2|A_1)\cdot P(A_1)$$

Ex: Three cards are drawn from a deck of 52 ordinary randomly shuffled cards. What is the prob that none of the three cards is a 'heart'. Note that the cards are drawn without replacement.

$$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2 | A_1)$$
$$= 39/52 \cdot 38/51$$

$A_1 = \{1^{st}$ card is <u>not</u> heart$\}$

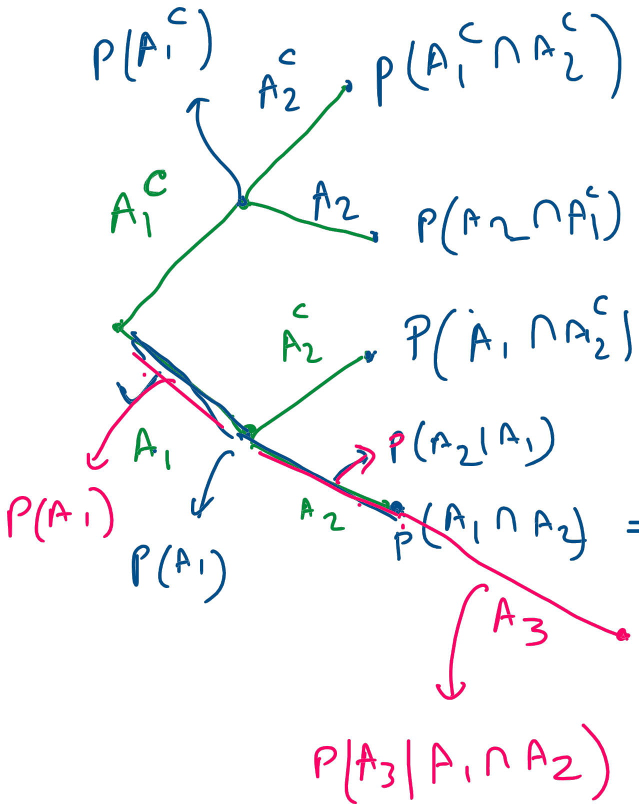
$A_2 = \{2^{nd}$ card is not heart$\}$

$A_3 = \{3^{rd}$ card is not heart$\}$

$$P(A_1 \cap A_2 \cap A_3)$$

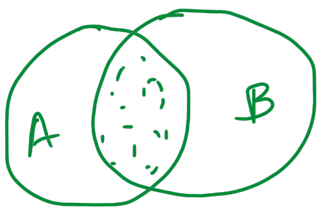
$$P(A_1 \cap A_2 \cap A_3) = \underbrace{P(A_1)}_{39/52} \cdot \underbrace{P(A_2 | A_1)}_{38/51} \cdot \underbrace{P(A_3 | A_1 \cap A_2)}_{37/50}$$

$$P(A_1 \cap A_2 \cap A_3) = 39/52 \times 38/51 \times 37/50 =$$



$$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2 | A_1)$$

$$\underline{P(A_1 \cap A_2 \cap A_3)} = P(A_1) \cdot P(A_2 | A_1) \cdot P(A_3 | A_1 \cap A_2)$$



$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

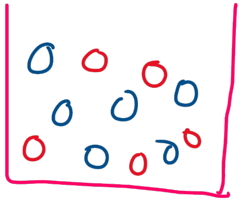
Now, if one event is independent of another event. then

$$P(A|B) = P(A)$$

$$\therefore P(A \cap B) / P(B) = P(A) \quad \therefore \boxed{P(A \cap B) = P(A) \cdot P(B)}$$

A & B are independent

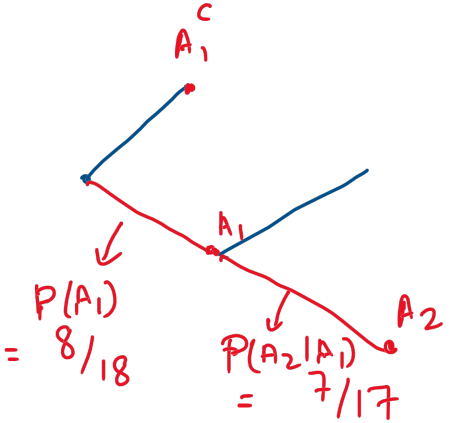
Exp-1



$\#$ Blue markel $= 10$

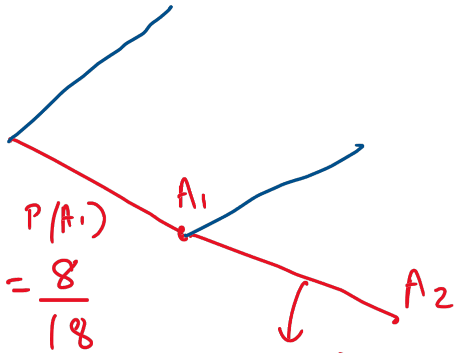
$\#$ Red markel $= 8$

If I draw two marbles one after another without replacement then what is the prob. that both are red.



$$P(A_1 \cap A_2) = \frac{8}{18} \times \frac{7}{17}$$

If I draw two marbles with replacement then what is the prob. that both are red?



$$P(A_2|A_1) = P(A_2) = \frac{8}{18}$$

$$\therefore P(A_1 \cap A_2) = P(A_1) \cdot P(A_2)$$

$$= \left(\frac{8}{18}\right)^2$$

Examples of conditional Prob :-

(SP Prediction) 1) Given the price of stock for last 7 days, what is the price tomorrow?

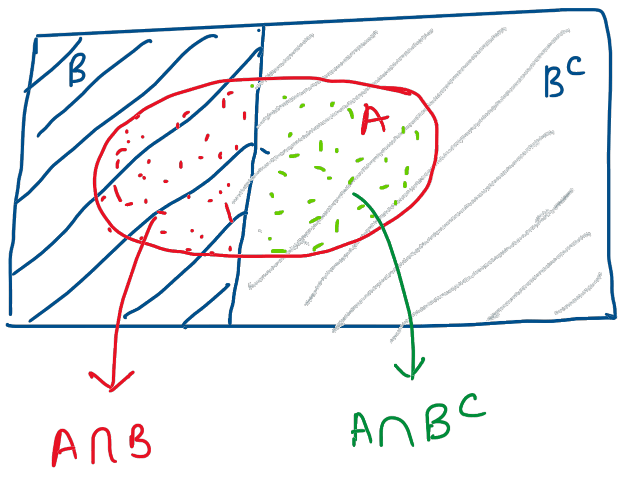
(image classi) 2) Given an image, what is the prob of it is a image of cat?

(classification) 3) Given the age, income, credit score what is the prob that a person will be eligible for a loan?

(weather forecasting) 4) Given past 7 days weather what will be the weather today?

(NLP) 5) Given last 5 words are w_1, w_2, w_3, w_4 & w_5, what is the probability of w_6 being the next word?

Theorem of Total Probability :-



$$P(A) = P(A \cap B) + P(A \cap B^c)$$

$$\Rightarrow \boxed{P(A) = P(B) \cdot P(A|B) + P(B^c) \cdot P(A|B^c)}$$

Theorem of total probability.

Ex: In a country 40% people have covid-19 symptoms. Now out of these people 80% are covid +ve. Studies suggest that there are 50% chances of asymptomatic people to have covid +ve.

If I select a random person from the country what is the prob that he/She is Covid +ve?

$A = \{$ a randomly chosen person is covid +ve $\}$

$P(A) = ?$

$B = \{$ a person has covid symptoms $\}$

$$P(A) = \underbrace{P(A \cap B)}_{\text{Prob of a person having covid +ve and covid symptoms.}} + \underbrace{P(A \cap B^c)}_{\text{Prob of a person having covid +ve but no symptoms.}}$$

$$P(A \cap B) = \underbrace{P(B)}_{P(\text{covid symptom})} \cdot \underbrace{P(A \mid B)}_{P(\text{covid +} \mid \text{covid sym})}$$

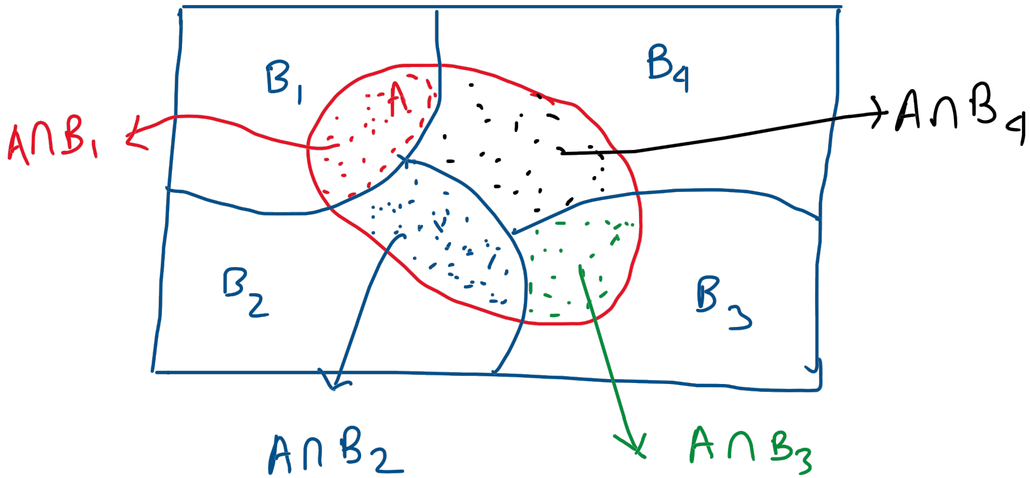
$$P(A \cap B) = \underbrace{P(B)}_{0.4} \cdot \underbrace{P(A \mid B)}_{0.8} = 0.32$$

$$P(A \cap B^c) = \underbrace{P(B^c)}_{0.6} \times \underbrace{P(A \mid B^c)}_{0.5}$$

$P(A \cap B^c) = 0.3$

$\therefore P(A) = 0.32 + 0.3 = 0.62$

Law of Total Probability for multiple disjoint events :-



B_1, B_2, B_3, B_4 are MECE, mutually exclusive and collectively exhaustive

$B_i \cap B_j = \phi$ and $\bigcup_{i=1}^{n} B_i = \Sigma$ (sample space)

$$A = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3) \cdots \cup (A \cap B_n)$$

$$\therefore P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3) + \cdots + P(A \cap B_n)$$

$$\therefore P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(B_i) \times P(A | B_i)$$

Ex: GATE exam : An exam in india conducted for engineering streams for Post-Graduation entrance.

Suppose it happens for 5 stream : 1) Mechanical 2) Electrical 3) Civil , 4) chemical , 5) Comp. Sc.

appearance : 25% Mech, 20% Elec, 28% civil, 17% chem, 10% CS

Passing : 30% Mech, 40% Elec, 32% civil, 50% chem, 60% CS

Q. Suppose I choose a gate aspirant randomly. What is the Prob that the student will pass?

→ $B_1 \to$ mech, $B_2 \to$ Elect, $B_3 \to$ Civil, $B_4 \to$ chem, $B_5 \to$ CS

→ $A \to$ Person has passed $P(A) = ?$

$$P(A) = P(B_1) \cdot P(A|B_1) + P(B_2) \cdot P(A|B_2) + P(B_3) \cdot P(A|B_3) + P(B_4) \cdot P(A|B_4) + P(B_5) \cdot P(A|B_5)$$

$$= 0.25 \times 0.3 + 0.2 \times 0.4 + 0.28 \times 0.32 + 0.17 \times 0.5 + 0.1 \times 0.6 = \underline{0.39}$$

Baye's Theorem :

$P(A|B) \rightarrow$ Known , $P(A)$, $P(B)$

$\therefore P(B|A) = ?$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)}$$

$$\therefore \boxed{P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)}}$$

Baye's theorem.

Likelihood

Support

Prior Probability

Posterior Probability

$$\boxed{P(B|A)} = \boxed{\frac{P(A|B)}{P(A)}} \cdot P(B)$$

total Probability

Ex: 40% people with Covid-19 symptoms. Out of these 80% have covid +,
50% asymptomatic people has covid +ve.
I choose a random person who is covid +ve. What is the prob that the person is asymptomatic?

$B \to$ Asymptomatic, $\quad A \to$ covid +ve

$P(B) = 0.6$, $P(B^c) = 0.4$

$P(A|B) = 0.5$

$P(A|B^c) = 0.8$

$$P(A) = P(B) \cdot P(A|B) + P(B^c) \cdot P(A|B^c) = 0.62$$

$$\therefore P(B|A) = \frac{P(A|B)}{P(A)} \cdot P(B)$$

$$= \frac{0.5}{0.62} \times 0.6$$

$$= 0.4838$$

Ex: In the prev example of GATE exam.

Q: Given a person has passed the GATE, what is the prob that the person is from electrical Engg?

$$P(B_2 | A) = \frac{P(A | B_2)}{P(A)} \cdot P(B_2)$$

$$= \frac{0.4}{0.39} \times 0.2 = 0.205$$

$$P(B_5 | A) = \frac{P(A | B_5)}{P(A)} \cdot P(B_5) = \frac{0.6}{0.39} \times 0.1 = 0.158$$

Ex: Suppose there is a test for a rare disease which is 96% accurate. Chances of the rare disease happen to the population is 1%. A person is tested +ve for the disease, then what is the probability that the person actually has the disease?

$A \rightarrow$ has disease. | $A^c \rightarrow$ doesn't have disease

$B \rightarrow$ test +ve | $B^c \rightarrow$ test -ve

(+ve) B
- $A|B \rightarrow$ True +ve
- $A^c|B \rightarrow$ False +ve

B^c (-ve)
- $A|B^c \rightarrow$ False -ve
- $A^c|B^c \rightarrow$ True -ve

$$P(B) = P(B|A)\cdot P(A) + \underbrace{P(B|A^c)}_{5\%}\cdot P(A^c)$$

$$= 0.96 \times 0.01 + 0.05 \times 0.99$$

$$= 0.0591$$

$$P(A|B) = ?$$

$$P(A|B) = \frac{P(B|A)}{P(B)} P(A)$$

$$= \frac{0.96}{0.0591} \times 0.01 = \underline{\underline{0.162}}$$