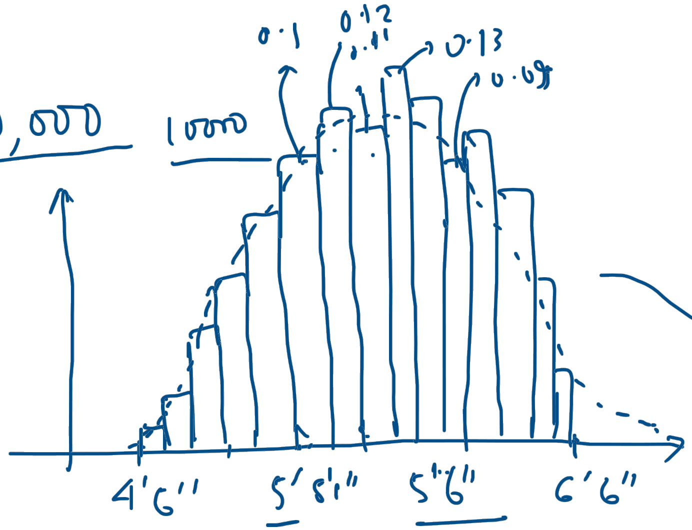
Continuous Random Variable :-

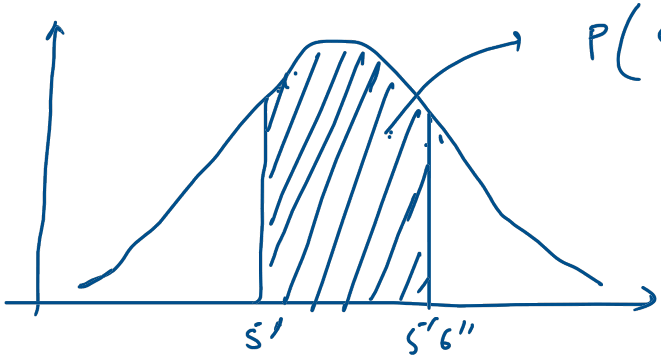
Height of a person :

10 people

100,000 1000



frequency histogram plot



P(5 ≤ h ≤ 5'6")

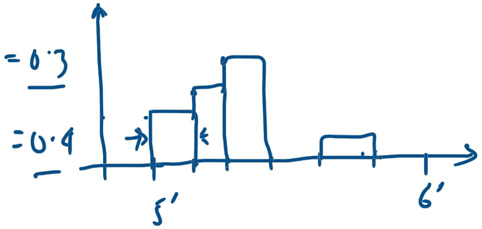
5' – 5'2" : 2/10 = 0.2 (2")

5'2" – 5'4" : 3/10 = 0.3

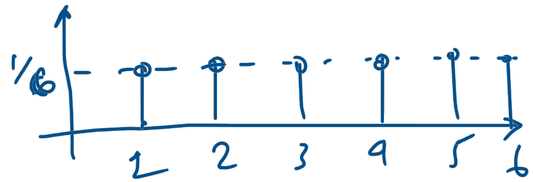
5'4" – 5'6" : 4/10 = 0.4

5'6' – 5'8" : 0

5'8" – 5'10" : 1/10 = 0.1

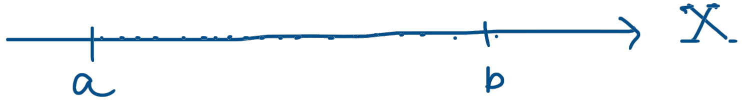


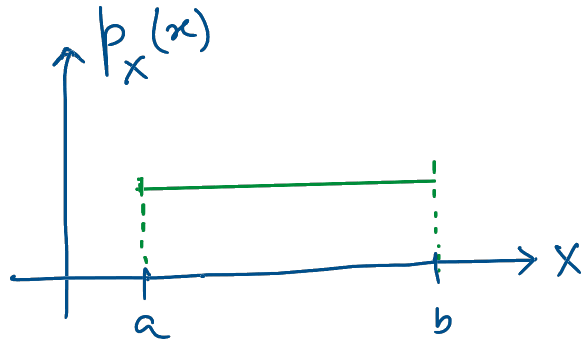
Rolling of 6 face dice



Uniform discrete RV

Continuous R.V. are defined within a range and can take any real value within that range.





Discrete RV

1) Discrete values within a range
2) Probability mass function (PMF)

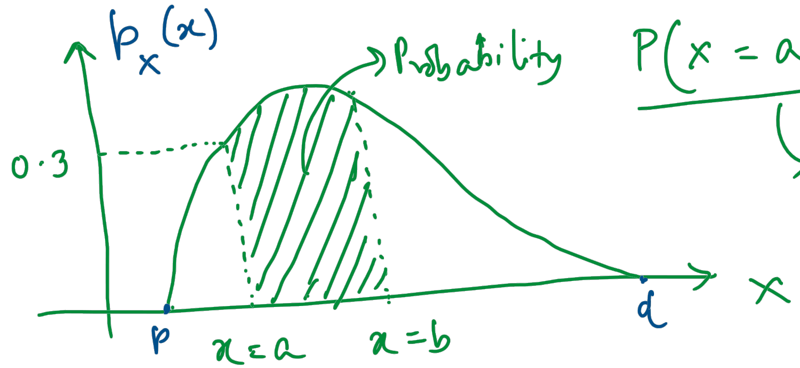
Continuous RV

1) Any real value within the range
2) Probability density function (PDF)

Probability Density Function :-

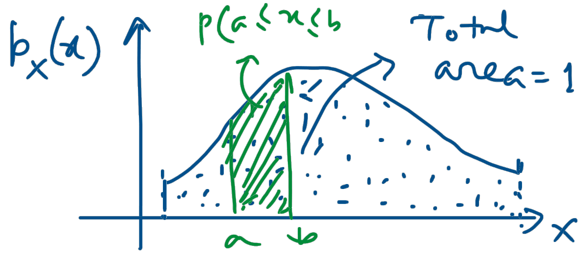
$\neq$ Probability

$p_X(x)$ $\therefore$ $\underbrace{\int_x p_X(x)\, dx = 1}_{\text{total area}}$



$P(X = a) = 0$ → Probability of a certain exact value $= 0$

$p_X(x) \longrightarrow$ Probability Density Function

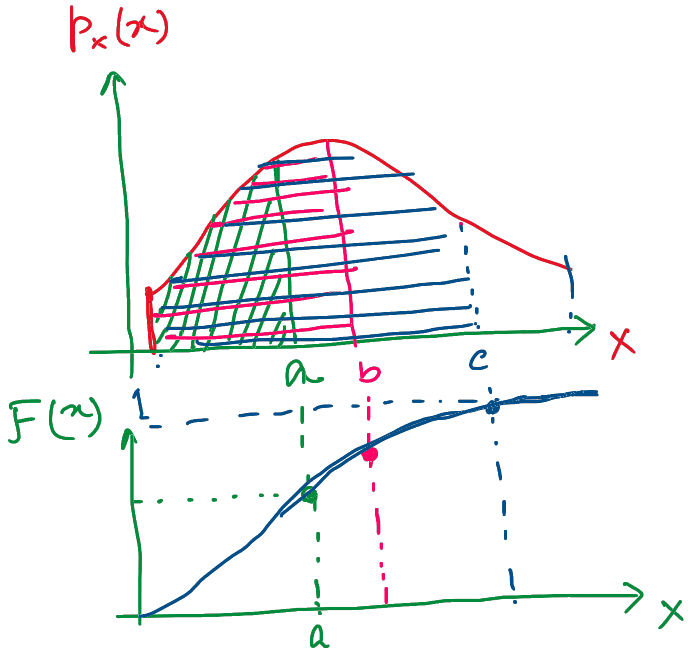


$$\int_X p_X(x)\, dx = 1 \longrightarrow \text{The area under the PDF} = 1$$

$$\int_a^b p_X(x)\, dx = P(a \leq X \leq b) \quad : \text{Probability of the R.V. between a to b}$$

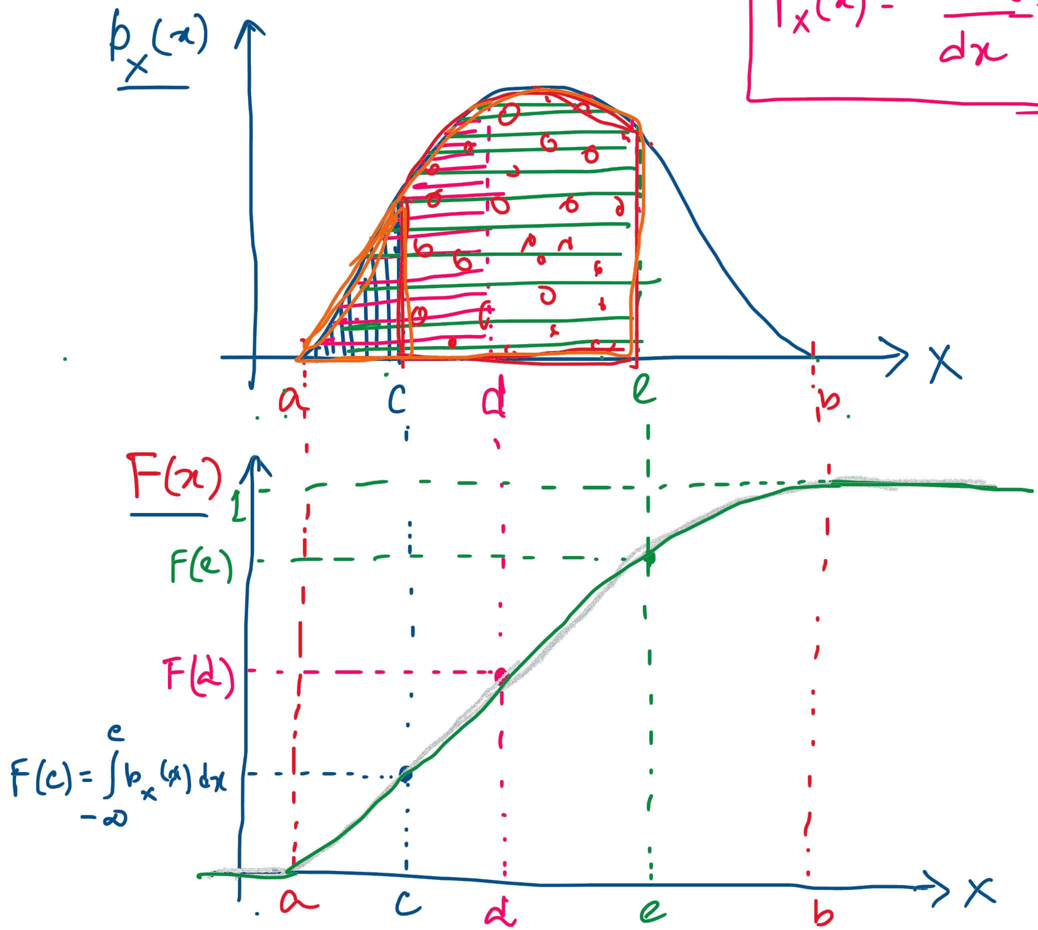
Cumulative distribution function (CDF)

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} p_X(y)\, dy$$



$\max(F(x)) = 1$ $\quad$ F(x) monotonically increasing function

PDF & CDF



for any Cnt. R.V.

$$\boxed{p_X(x) = \frac{dF(x)}{dx}}$$

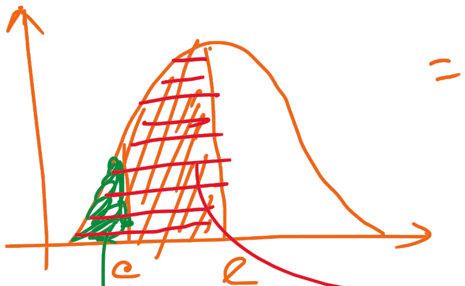
$f(x)$ is a monotonically increasing function.

$x_1 < x_2$ then $f(x_1) \leq f(x_2)$

$$F(x) = \int_{-\infty}^{x} p(y)\,dy$$

$$P(c \leq X \leq e) = \int_{-\infty}^{e} p_X(x)\,dx - \int_{-\infty}^{c} p_X(x)\,dx$$

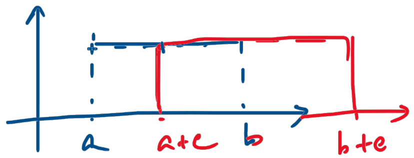
$$= F(e) - F(c)$$



$$\int_{0}^{c} p_X(x)\,dx \qquad \int_{0}^{e} p_X(x)\,dx$$

Expectation of a Continuous R.V.

$$\mathbb{E}[X] = \int_X x\, p_X(x)\, dx$$



In case of Discrete R.V.

x	x_1	x_2	x_3	$\cdots$	x_n
$p_X(x)$	p_1	p_2	p_3		p_n

$$\mathbb{E}[X] = \sum_{i=1}^{n} x_i p_i$$

$$Y = aX + c \quad \therefore \mathbb{E}[Y] = a\,\mathbb{E}[X] + c$$

$$\mathbb{E}[Y] = \int_Y y\, p_Y(y)\, dy = \int_X (ax + c)\, p_X(x)\, dx = a \overbrace{\int_X x\, p_X(x)\, dx}^{\mathbb{E}[X]} + c \overbrace{\int_X p_X(x)\, dx}^{1}$$

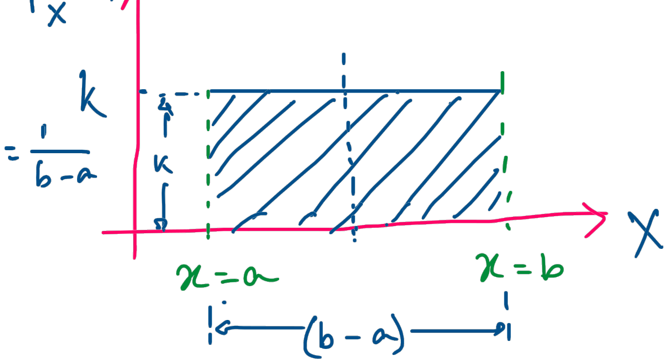
$$= a\,\mathbb{E}[X] + c$$

$$Z = aX + bY \quad \therefore \mathbb{E}[Z] = a\,\mathbb{E}[X] + b\,\mathbb{E}[Y]$$: Linearity of Expectation.

Variance: $\mathbb{E}[X] = \mu$

$$\therefore \mathrm{Var}(X) = \int_X (x - \mu)^2\, p_X(x)\, dx$$

Uniform distribution

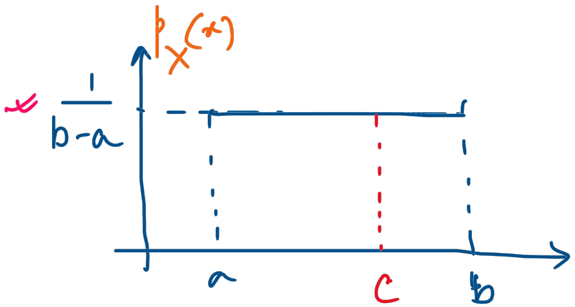


$$\int_a^b p_X(x)\,dx = 1$$

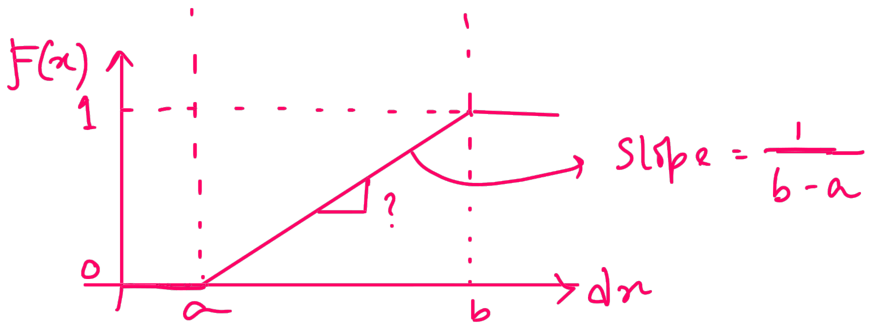
$$p_X(x) = k \quad ; \; a \le x \le b$$

$$\Rightarrow \int_a^b k\,dx = 1 \qquad \Rightarrow k(b-a) = 1$$

$$\Rightarrow k = \frac{1}{b-a}$$



$$P(x=c) = 0$$



Expectation (mean):

$$E[X] = \int_a^b x \underbrace{p_X(x)}_{=\frac{1}{b-a}}\,dx = \int_a^b \frac{x}{b-a}\,dx$$

$$= \frac{1}{b-a}\int_a^b x\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{1}{b-a}\,\frac{1}{2}(b^2-a^2)$$

$$= \frac{1}{2}\,\frac{1}{\cancel{(b-a)}}\cdot(b+a)\cancel{(b-a)} = \frac{a+b}{2}$$

Variance:

$$\mathbb{E}[x] = \frac{a+b}{2} = \mu \Rightarrow 2\mu = a+b$$

$$x^3 - y^3 = (x-y)(x^2+xy+y^2)$$

$$Var(x) = \int_X (x-\mu)^2 \, p_x(x) \, dx = \frac{1}{b-a}\int_a^b (x-\mu)^2 \, dx$$

$$= \frac{1}{b-a} \cdot \left[\frac{(x-\mu)^3}{3}\right]_a^b$$

$$b-\mu = b - \frac{a+b}{2} = \frac{b-a}{2}$$

$$\mu - a = \frac{a+b}{2} - a = \frac{b-a}{2}$$

$$= \frac{1}{3} \cdot \frac{1}{b-a} \cdot \left[(b-\mu)^3 - (a-\mu)^3\right]$$

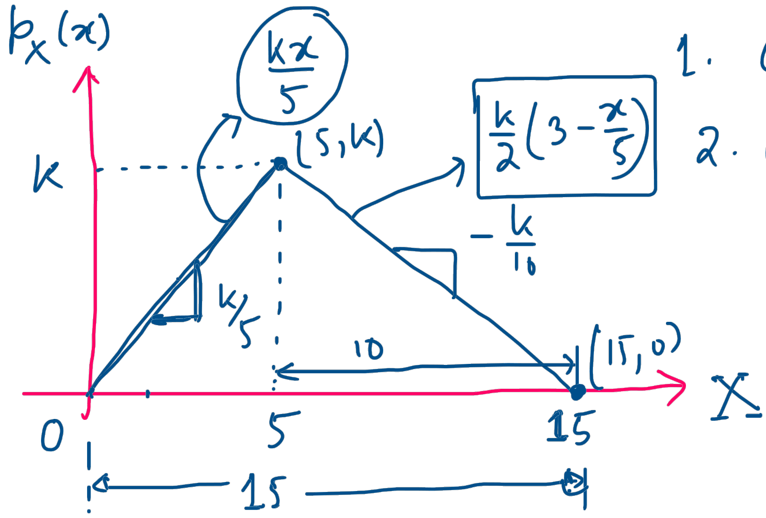
$$= \frac{1}{3} \cdot \frac{1}{\cancel{b-a}} \cdot (\cancel{b-\mu-a+\mu}) \cdot \left[(b-\mu)^2 + (b-\mu)(a-\mu) + (a-\mu)^2\right]$$

$$= \frac{1}{3} \cdot \left[(b-\mu)^2 + 2(b-\mu)(a-\mu) + (a-\mu)^2 - (b-\mu)(a-\mu)\right]$$

$$= \frac{1}{3} \cdot \left[(\underbrace{b+a-2\mu}_{0})^2 - (b-\mu)(a-\mu)\right]$$

$$= \frac{1}{3} \cdot (b-\mu)(\mu-a) = \frac{(b-a)^2}{12}$$

Ex:



1. Calculate k.
2. Calculate the Expectation of R.V.

$y = mx + c$

$\Rightarrow 0 = 15\cdot\left(-\frac{k}{10}\right) + c$

$\Rightarrow c = \frac{15k}{10} = \frac{3k}{2}$

$y = -\frac{kx}{10} + \frac{3k}{2}$

$= \frac{k}{2}\left(3 - \frac{x}{5}\right)$

$-\frac{5k}{10} + \frac{3k}{2}$

$= \frac{-5k+15k}{10}$

$= \frac{10k}{10} = k$

1. Finding k:

Total Area under PDF $= 1$

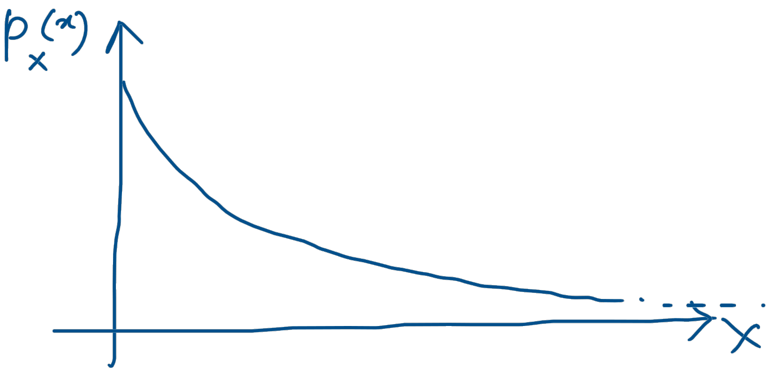
$$\frac{1}{2} \times k \times 15 = 1 \Rightarrow k = \frac{2}{15}$$

(2) $$\mathbb{E}[X] = \int_x x\, p_X(x)\, dx = \int_0^5 x\, p_X(x)\, dx + \int_5^{15} x\, p_X(x)\, dx$$

$$= \frac{k}{5}\int_0^5 x^2\, dx + \frac{k}{2}\int_5^{15} x\left(3 - \frac{x}{5}\right) dx$$

$= - - - - - - -$

Exponential distribution :-



$$p_X(x) = \lambda e^{-\lambda x} \quad 0 \leq x < \infty \qquad (\lambda > 0)$$

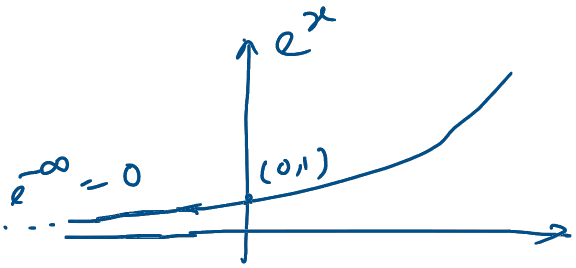
$$\int_0^\infty p_X(x)\,dx = \int_0^\infty \lambda e^{-\lambda x}\,dx \qquad e^{-\infty} = 0$$

$$= \lambda \cdot \int_0^\infty e^{-\lambda x}\,dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^\infty$$

$$= \lambda \left[\frac{e^{-\infty} - e^{0}}{-\lambda}\right] = (-1)[0 - 1]$$

$$= 1$$



$$\mathbb{E}[X] = \lambda \int_0^\infty x e^{-\lambda x}\,dx$$

$$\therefore \mathbb{E}[X] = \frac{1}{\lambda}$$

$$\mathrm{Var}(X) = \frac{1}{\lambda^2}$$

($\lambda \rightarrow$ parameter of exponential distribution)

Normal Distribution :- [Gaussian Distribution] $X \sim \mathcal{N}(\mu, \sigma^2)$

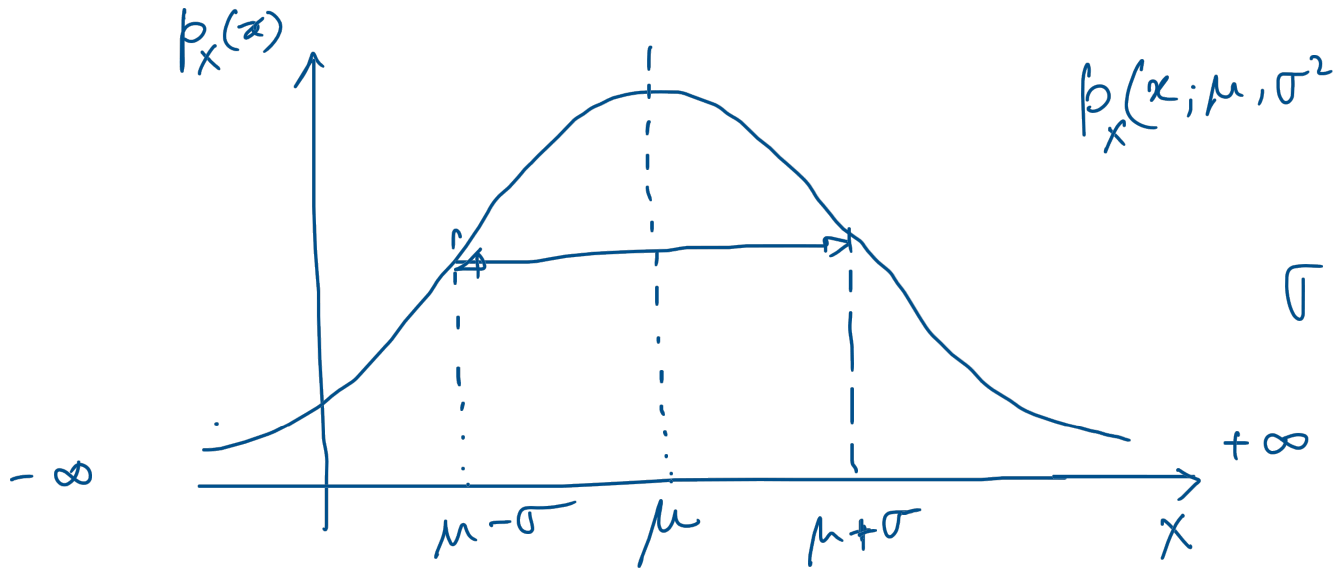
Two parameters : μ, σ^2

μ : mean / expectation

σ^2 : variance

$$p_X(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

σ : std. deviation



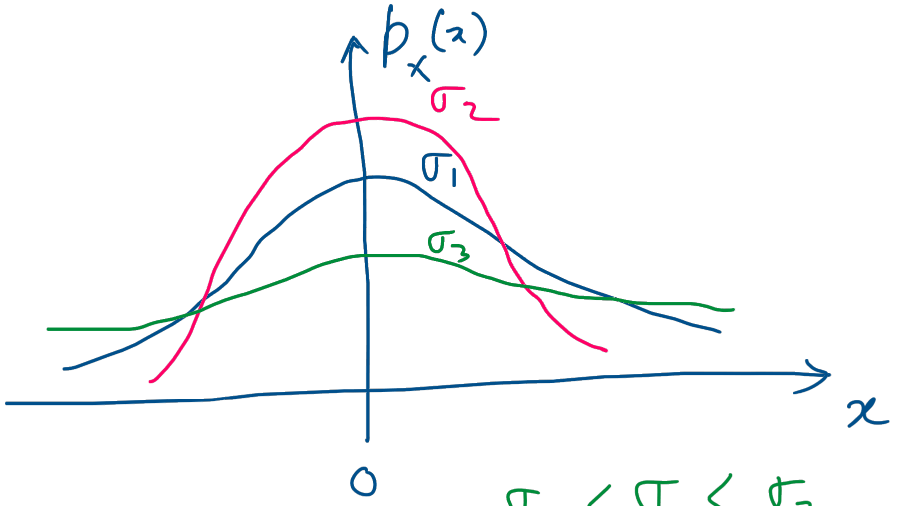
$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1$$

$\mathbb{E}[X] = \mu$

$\mathrm{Var}[X] = \sigma^2$

Standard Normal distribution :

$\mu = 0$ and $\sigma^2 = 1$



$\sigma_2 < \sigma_1 < \sigma_3$

$$p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$\sigma \uparrow \rightarrow$ flattens

$$X \sim N(0,1)$$

X, $(Y = aX + b)$

$\therefore \underline{Var(Y) = a^2 Var(X)}$

Suppose $X \sim \underline{\underline{N(\mu, \sigma^2)}}$ then $Z \sim \underline{\underline{N(0, 1)}}$

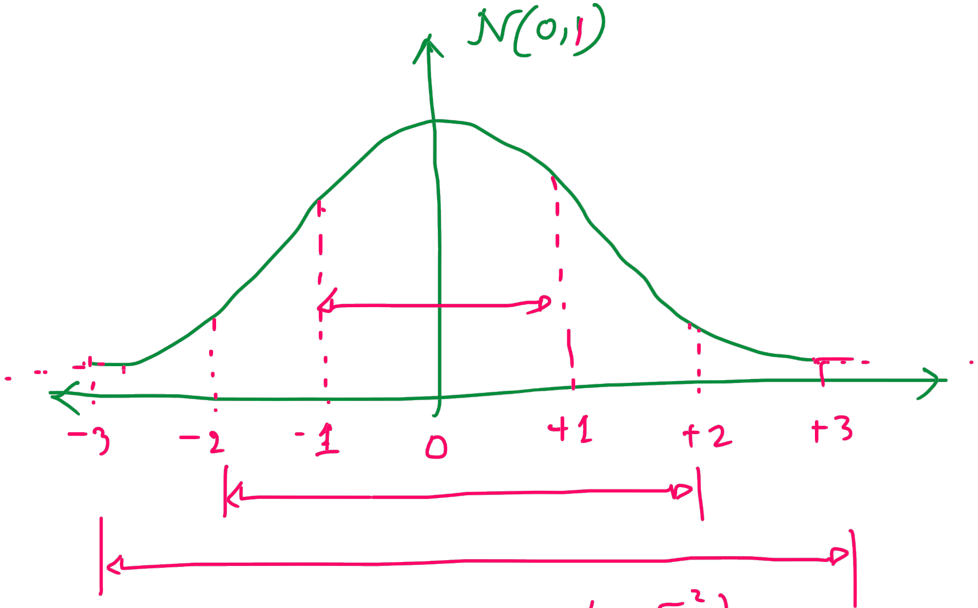
$$\left(\underline{Z} = \frac{X - \mu}{\sigma} \right) \longrightarrow \text{Standardization.}$$

$$Z = \frac{x - \mu}{\sigma}$$

$$\Rightarrow Z = \frac{1}{\sigma} X - \frac{\mu}{\sigma}$$

$\underline{\mathbb{E}[\underline{Z}]} = \frac{1}{\sigma}(\mathbb{E}[X] - \mu) = \underline{0}$ $\quad \therefore Var(\underline{Z}) = \frac{1}{\sigma^2} Var(X) = \frac{\sigma^2}{\sigma^2} = \underline{1}$

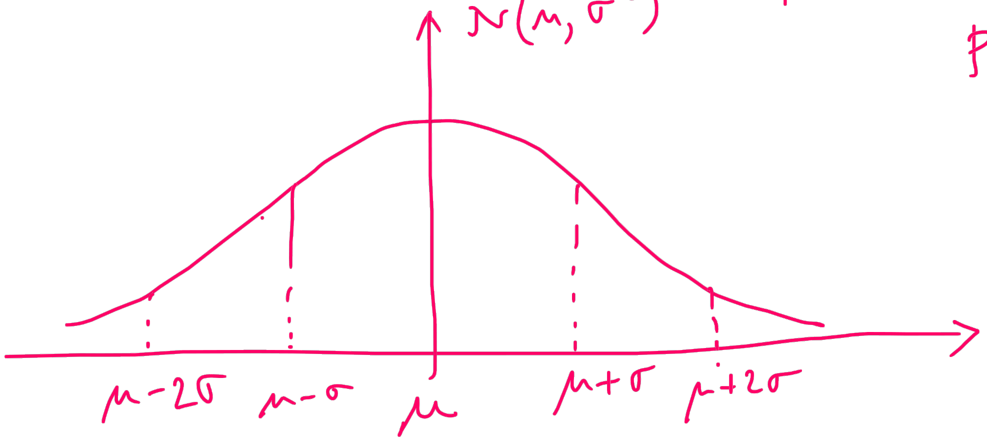
Some Properties of Normal Distribution



$P(-1 < X \leq +1) = 0.6826 \sim 68.26\%$

$P(-2 < X \leq +2) = 0.9544 \sim 95.44\%$

$P(-3 < X \leq +3) = 0.997 \sim 99.7\%$



$P(\mu-\sigma < X \leq \mu+\sigma) \simeq 68.26\%$

$P(\mu-2\sigma < X \leq \mu+2\sigma) \simeq 95.4\%$

$P(\mu-3\sigma < X \leq \mu+3\sigma) \simeq 99.7\%$

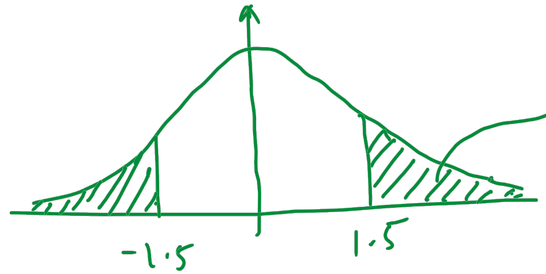
Ex 1: $P(-1.5 < x < 1.5) = ?$ given $X \sim N(0, 1)$

$$= F(1.5) - F(-1.5)$$

$$= 0.9332 - (1 - 0.9332)$$

$$= 2 \times 0.9332 - 1$$

$$= 1.8664 - 1 = 0.8664$$



$F(\infty) - F(1.5)$
$= 1 - 0.9332$

$$P(-k < x < +k) = 2F(k) - 1 =$$

for $k = 1$: $P(-1 < x \leq +1) = 2 \times 0.8413 - 1 = 1.6826 - 1$
$= 0.6826$

Suppose there is a factory which makes T-shirts from size 25 to 60. The mean size is 35 and the std. deviation is 5. Calculate the probability of a t-shirt size to be within 30 to 45?

$\Rightarrow$ $z = \dfrac{x - \mu}{\sigma}$

x	z
30	$-5/5 = -1$
45	$+10/5 = +2$

$P(-1 \leq z \leq +2) = ?$

$= F(2) - F(-1)$

$= F(2) - [1 - F(1)]$

$= F(1) + F(2) - 1$

$= 0.8413 + 0.9772 - 1$

$= 1.8185 - 1 = 0.8185$