K-Nearest Neighbour (K-NN) Classifier

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OUTLINE

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- Other Distance Metrics
- K-NN Classifier
- Decision Boundary of K-NN Classifier
- Choosing the value of K
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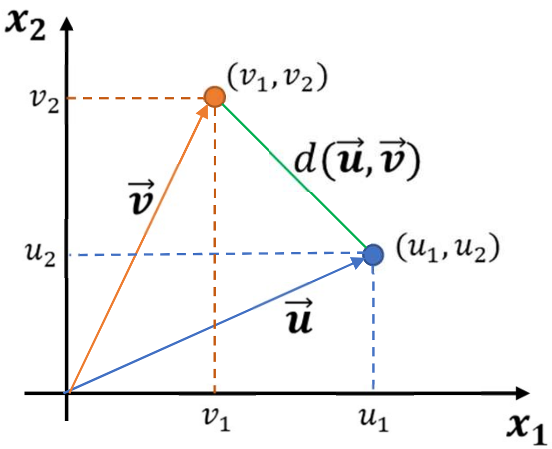
EUCLIDEAN DISTANCE



- Consider the points in two-dimension. Each point in two-dimension can be represented by a vector of dimension two.
- The point (u_1, u_2) can be represented by the vector $\vec{\boldsymbol{u}} = [u_1, u_2]^T$
- And the point (v_1, v_2) can be represented by the vector $\vec{\boldsymbol{v}} = [v_1, v_2]^T$
- The Euclidean distance between the points is:

$$d(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$$

- In general a point in n-dimensional space is represented as $\vec{\boldsymbol{u}} = [u_1, u_2, u_3, \dots, u_n]^T$, a n-D vector
- Hence, the Euclidean distance between two points in n-dimensional space is represented as:

$$d(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2 + \dots + (u_n - v_n)^2} = \sqrt{\sum_{i=1}^{n} (u_i - v_i)^2}$$

- In general in vector notation, the Euclidean distance is written as: $d(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) = \sqrt{(\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}})^T (\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}})}$

OTHER DISTANCE METRICS

- **Manhattan Distance:** For two data points denoted by $\boldsymbol{x}$ and $\boldsymbol{y}$ the Manhattan distance is defined as:

$$dist_{manhattan}\,(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{n} |x_i - y_i|$$

- **Minkowski Distance:** For two data points denoted by $\boldsymbol{x}$ and $\boldsymbol{y}$ the Minkowski distance is defined as:

$$dist_{minkowski}\,(\boldsymbol{x}, \boldsymbol{y}, h) = [\sum_{i=1}^{n} (x_i - y_i)^h]^{\left(\frac{1}{h}\right)}$$

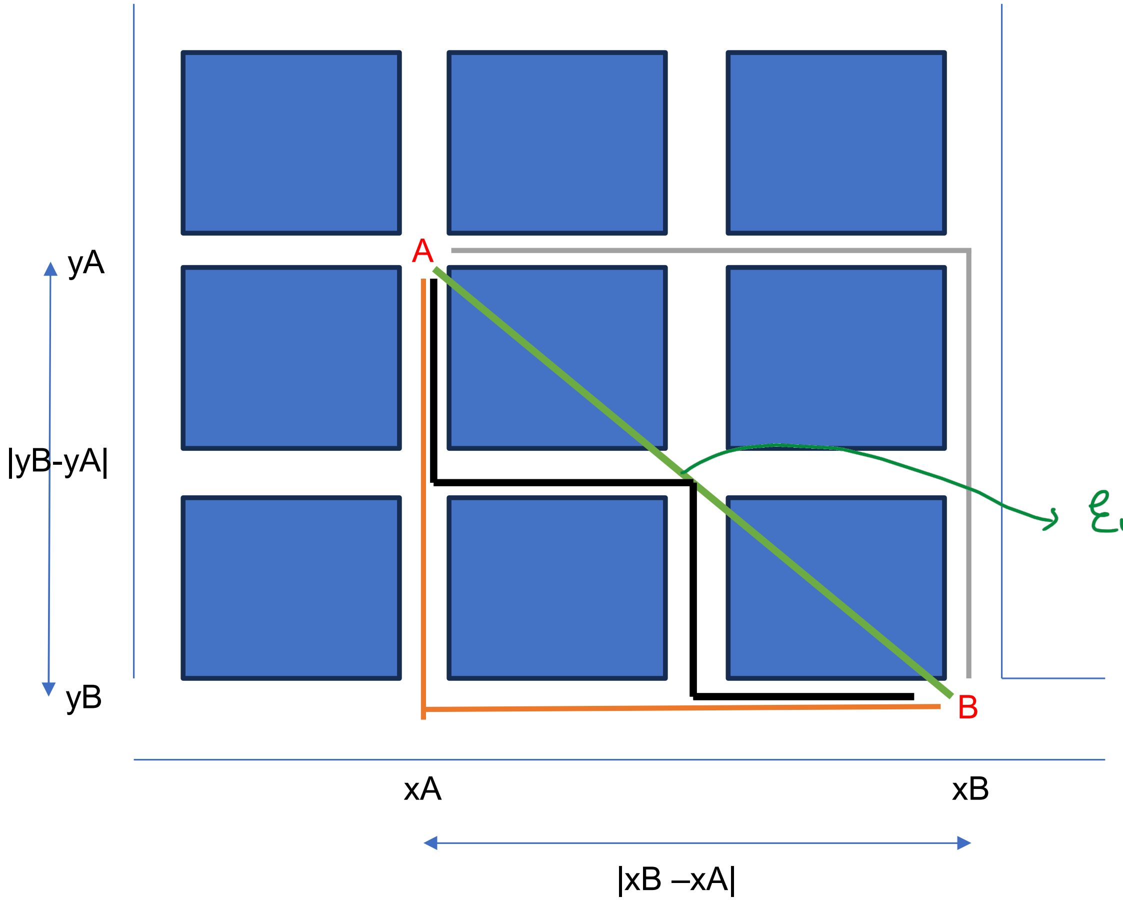
Note: for $h = 2$, Minkowski Distance is same as Euclidean Distance and for $h = 1$, it is Manhattan Distance

- **Chebyshev Distance:** For two data points in n-dimensional space it is defined as:

$$dist_{chebyshev}\,(\boldsymbol{x}, \boldsymbol{y}) = \max(|x_1 - y_1|,\ |x_2 - y_2|, |x_3 - y_3|, \ldots, |x_n - y_n|)$$

There are other distance metric like: Mahalanobis Distance, Bhattacharya Distance etc. which are used for advanced statistical pattern recognition tasks.

Manhattan Distance or city-block distance



Euclidean distance (✗)

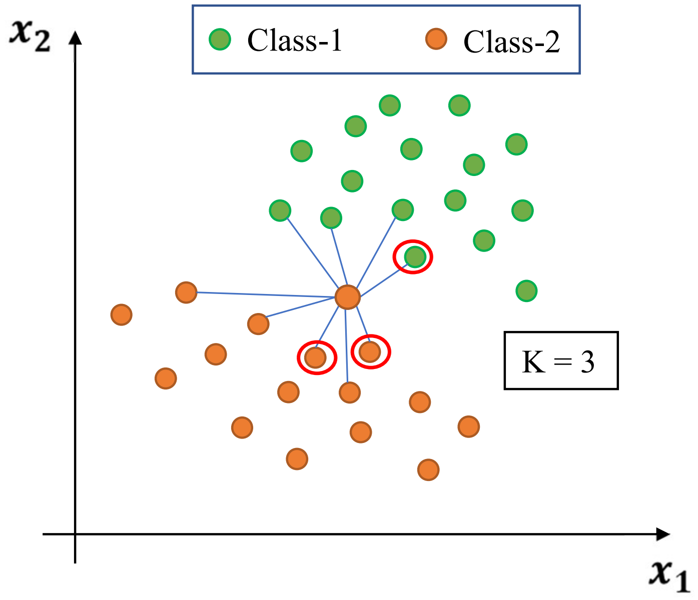
The shortest distance between A to B around the city ring road

$$\underbrace{|x_B - x_A| + |y_B - y_A|}_{\text{City block distance}}$$

Manhattan "

K-NN CLASSIFIER

- **Intuition:** One is known by the company one keeps.



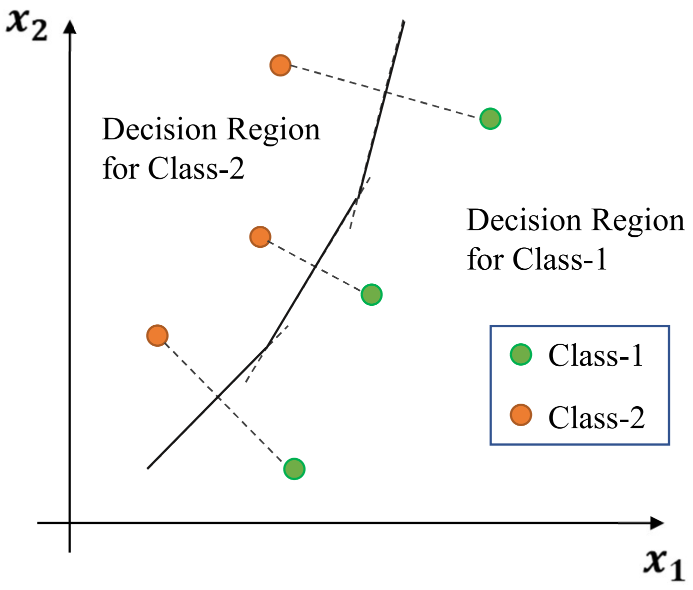
- **K-NN Algorithm:**

1. All the training samples/ points are available beforehand.
2. When a new test sample arrives calculate its **distance** from **all training points.**
3. Choose K-nearest neighbours based on the distance calculated. Usually the K is a positive odd integer and supplied by user.
4. Assign the class label of the test sample **based on majority**. i.e. for a test sample if most number of neighbours among those K-Nearest Neighbours belong to one particular class-c, then assign the class label of test sample as c.

- **Characteristics of K-NN Classifier:**

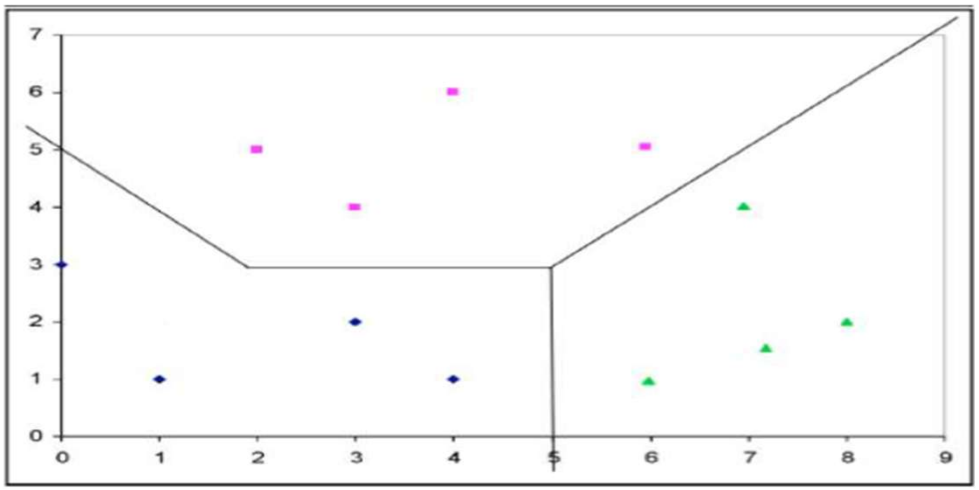
It doesn't create model based on the training patterns in advance. Rather, when a test instance comes for testing, runs the algorithm to get the class prediction of that particular testing instance. Hence, there is no learning in advance.
Hence, k-NN classifier is also known as **Lazy Learner**.

K-NN CLASSIFIER: DECISION BOUNDARY



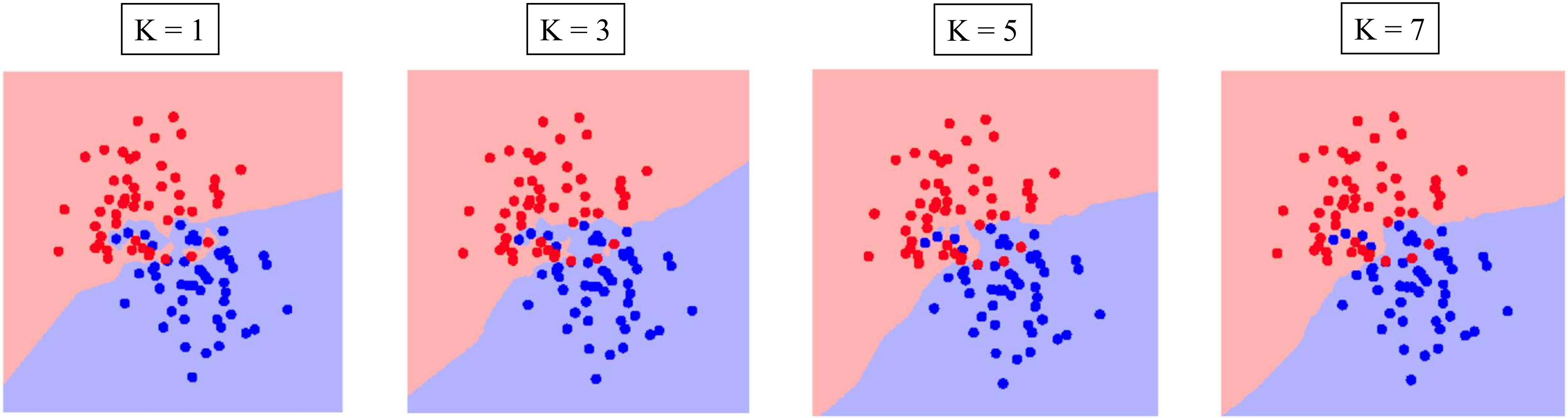
- Boundary are the points those are equidistant between the points of Class-1 and Class-2
- Construct lines between closest pairs of points in different classes.
- Draw perpendicular bisectors. End bisectors at intersections.
- Note that locally the boundary is linear.
- Hence the decision boundary is piecewise linear curve.

For multiclass classification also the same thing is done to find the decision boundary.



K-NN: CHOOSING THE VALUE OF K

- Increasing the 'K' simplifies the decision boundary. Because majority voting implies less emphasis on individual points



- However increasing the K also increases computational cost.
- Hence, choosing K is an optimization between how much simplified decision boundary we want vs. how much computational cost we can afford.
- Usually K = 5, 7, 9, 11 works fine for most practical problems.

K-NN CLASSIFIER: MERITS AND DEMERITS

Merits:

- K-NN Classifier often works very well for practical problems.
- It is very easy to implement, as there is no complex learning algorithm involved.
- Robust to Noisy Data.

Demerits:

- Choosing the value of K may not be straightforward. Often the same training samples are used for different values of K, and we choose the most suitable value of K based on minimum misclassification errors on test samples.
- Doesn’t work well for categorical attributes.
- Can encounter problem with sparse training data. (i.e. data points are located far away from each other)
- Can encounter problems in very high-dimensional spaces.
  - Most points are at corners.
  - Most points are at the edge of the space.

  This problem is known as ***Curse of Dimensionality*** and affect many other Machine Learning algorithms.

1-D case:

define edge as: $[0,\epsilon] \cup [1-\epsilon, 1]$

say $\epsilon = 0.05$

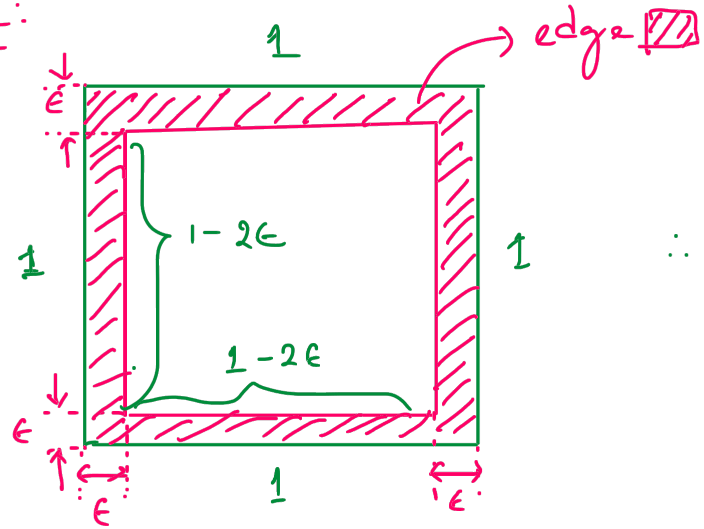
edge is $[0, 0.05] \cup [0.95, 1]$

x is uniformly distributed $[0,1]$, what is the probability that x is at the edge of the line?

$$P(\text{edge}) = \frac{2\epsilon}{L} = 2 \times 0.05 = 0.1$$

$$p = 1 - (1 - 2\epsilon)$$

2-D case:

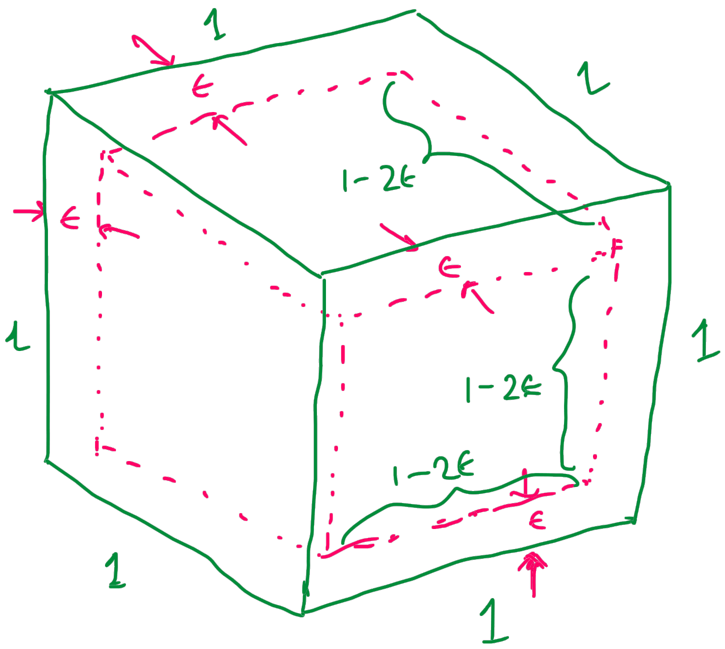


x is a point uniformly distributed in the unit square

$\therefore$ P(edge) = area of unit square − area of the blank region

$$= 1 - (1 - 2\epsilon)^2 = 1 - 0.9^2 = 1 - 0.81 = 0.19$$

3D-case:



x is uniformly distributed in the cube

$\therefore P(\text{edge}) = $ volume of unit cube − volume of the inside cube

$= 1 - (1-2\epsilon)^3$

$\epsilon = 0.05$, $\therefore P(\text{edge}) = 1 - 0.9^3 = 1 - 0.729$

$= 0.271$

For a n-dimensional hyper-cube $\boxed{P(\text{edge}) = 1 - (1-2\epsilon)^n}$

for $n = 20$, $\epsilon = 0.05$ $P(\text{edge}) = 1 - (0.9)^{20} = 1 - 0.12 = 0.88$

for $n = 50$, $\epsilon = 0.01$ $P(\text{edge}) = 1 - (0.98)^{50} = 1 - 0.36 = 0.64$

Thank You