Matrix (Matrices in plural)

$$\vec{v} \ (\text{vector}) = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$M = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \cdots & \vec{v}_m \end{bmatrix}$$

$$= \left.\underbrace{\begin{bmatrix} v_{11} & v_{21} & v_{31} & \cdots & v_{m1} \\ v_{12} & v_{22} & v_{32} & & v_{m2} \\ v_{13} & v_{23} & v_{33} & & v_{m3} \\ \vdots & \vdots & \vdots & & \vdots \\ v_{1n} & v_{2n} & v_{3n} & & v_{mn} \end{bmatrix}_{n \times m}}_{m\text{-columns}}\right\} n\text{-Rows}$$

Row

Column

$$M = [v_{ij}]_{\substack{i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, n}}$$

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 5 \\ -3 & 0.5 & 7.5 \\ 4 & 2.8 & 9.7 \end{bmatrix}_{4 \times 3}$$

Different types of matrix

1. **Square matrix** : If A is a square matrix $row(A) = col(A)$

$$B = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 9 \\ 7 & 0 & 8 \end{bmatrix}_{3 \times 3} \longrightarrow B \text{ is a square matrix}$$

Principal diagonal

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nn} \end{bmatrix}_{n \times n}$$

$\{x_{11}, x_{22}, x_{33}, \ldots, x_{nn}\}$ Diagonal elements.

2. **Diagonal matrix** :

$$\begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ 0 & & \ddots & \\ & & & \sigma_n \end{bmatrix}_{n \times n} \rightarrow$$

A diagonal matrix has at-least one non-zero diagonal element but all-zero non-diagonal elements.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \text{Diag}(2, 3, 4)$$

The diagonal matrix are often denoted by $\boxed{\text{diag(diagonal elements)}}$

3. <u>Identity Matrix</u> :- Identity matrix has all the <u>diagonal</u> elements = 1 (same) and other values = 0.

$$\mathbb{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \text{identity matrix of order} = 2$$

$$\mathbb{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \text{identity matrix of order} = 3$$

Operations on matrix

1. Transpose :-

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}_{3\times 2} \qquad B = A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}_{2\times 3}$$

$A_{n\times m}$ $\quad B = A^T$ then $row(B) = m$ & $col(B) = n$

$\hookrightarrow m\times n$

2. Norm (Frobenius norm, F-norm)

$$\vec{v}\ (\text{vector}) = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad \|\vec{v}\|_2 = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

$$A = [a_{ij}]_{n\times m} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ a_{31} & a_{32} & \cdots & a_{3m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

$$\|A\|_F = \sqrt{(a_{11}^2 + a_{12}^2 + \cdots + a_{1m}^2) + (a_{21}^2 + a_{22}^2 + \cdots + a_{2m}^2) + \cdots + (a_{n1}^2 + a_{n2}^2 + \cdots + a_{nm}^2)}$$

$$= \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}^2}$$

$$B = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & -3 \end{bmatrix} \qquad \|B\|_F = ? \quad = \sqrt{2^2 + 3^2 + 1^2 + 0^2 + 5^2 + (-3)^2}$$

$$= \sqrt{4 + 9 + 1 + 0 + 25 + 9}$$

$$= \sqrt{48}$$

3. <u>Addition</u>

$A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$ } when two matrices have same number of rows and columns they can be added.

$$C = A + B = [c_{ij}]_{m \times n}$$

$$c_{ij} = a_{ij} + b_{ij}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 0.5 & 9 & 11 \\ 6.5 & 2.2 & 0.8 \end{bmatrix} = \begin{bmatrix} 1.5 & 11 & 14 \\ 10.5 & 7.2 & 6.8 \end{bmatrix}$$

$$\quad A \qquad\qquad\qquad B \qquad\qquad\qquad C$$

4. <u>Multiplying a matrix with a scalar</u>

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

$$\underbrace{\lambda}_{\text{Scalar value}} A = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \vdots & & & \\ \lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{mn} \end{bmatrix}$$

$$\| \lambda A \|_F = \lambda \| A \|_F$$

$$3 \times \begin{bmatrix} 2 & 1 & 7 \\ 5 & 0.5 & 9 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 21 \\ 15 & 1.5 & 27 \end{bmatrix}$$

5. Element - wise multiplication

$$A = [a_{ij}]_{n \times m}, \quad B = [b_{ij}]_{n \times m}$$

For element wise multiplication the matrices need to be of same dimension.

$$C = A * B = A \odot B$$

[Hadamard Product]

$$c_{ij} = a_{ij} \times b_{ij}$$

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 9 & 7 \end{bmatrix} \odot \begin{bmatrix} -1 & 2 & 9 \\ 0 & 5 & -6 \end{bmatrix} = \begin{bmatrix} -2 & 6 & 36 \\ 0 & 45 & -42 \end{bmatrix}$$

6. Multiplying a vector with matrix

Post-multiplication

Pre-multiplication.

Post-multiplication

$\vec{v}^T M \rightarrow$ Post multiplication of vector $\vec{v}$ with matrix M

$$\vec{v}_{m\times 1} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} \qquad M_{m\times n}$$

then $\vec{v}^T M = \vec{w}^T_{1\times n}$

$$\vec{v}^T_{1\times m} = [v_1 \quad v_2 \quad \dots \, v_m]_{1\times m} \quad M_{m\times n}$$

$\vec{v}^T M$

then $\dim(\vec{v}) = \text{row}(M)$

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \vec{v}^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \qquad M = \begin{bmatrix} 5 & -1 \\ 6 & 2 \\ 7 & 3 \end{bmatrix}_{3 \times 2}$$

$$\vec{v}^T M = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{array}{cc} \vec{a} & \vec{b} \\ \left[\begin{matrix} 5 \\ 6 \\ 7 \end{matrix}\right. & \left.\begin{matrix} -1 \\ 2 \\ 3 \end{matrix}\right] \end{array} = \begin{bmatrix} \vec{v}^T \vec{a} \,, & \vec{v}^T \vec{b} \end{bmatrix}$$

$$= \begin{bmatrix} \vec{v} \cdot \vec{a} \,, & \vec{v} \cdot \vec{b} \end{bmatrix}$$

$$= \begin{bmatrix} 38 \,, & 12 \end{bmatrix}_{1 \times 2} = \vec{w}^T$$

$$\vec{w} = \begin{bmatrix} 38 \\ 12 \end{bmatrix}$$

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

$$\vec{x}^T \vec{y}$$

Pre - Multiplication

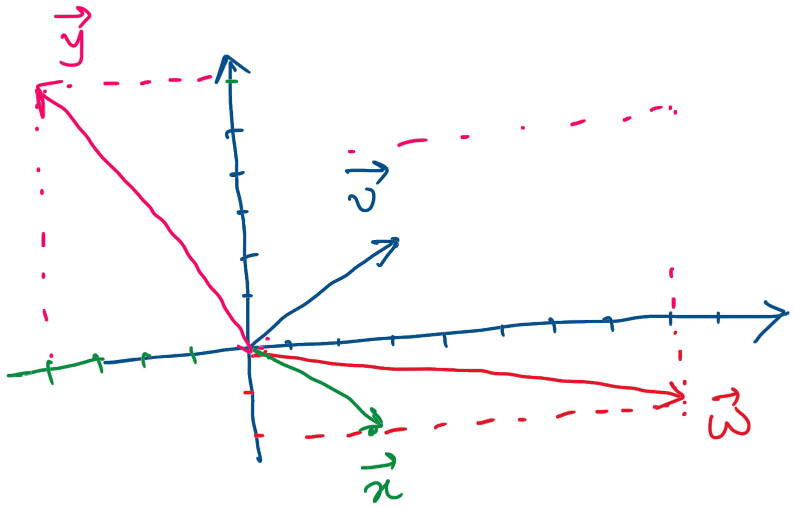
$M\vec{v} \longrightarrow$ Pre-multiplication of vector $\vec{v}$ with matrix M

$M_{m\times n} \; \vec{v}_{n\times 1} = (M\vec{v})_{m\times 1}$ $\qquad col(M) = dim(\vec{v})$

$$\begin{array}{cc} & \\ \vec{a}^T & \\ \vec{b}^T & \end{array}\begin{bmatrix} 2 & 3 & 0 & -1 \\ 6 & 7 & -2 & 1 \end{bmatrix}_{\underline{2\times 4}} \begin{array}{c} \vec{v} \\ \begin{bmatrix} 0.5 \\ 1 \\ -1 \\ 0 \end{bmatrix}_{\underline{4\times 1}} \end{array} = \begin{bmatrix} \vec{a}^T\vec{v} \\ \vec{b}^T\vec{v} \end{bmatrix} = \begin{bmatrix} \vec{a}\cdot\vec{v} \\ \vec{b}\cdot\vec{v} \end{bmatrix}$$

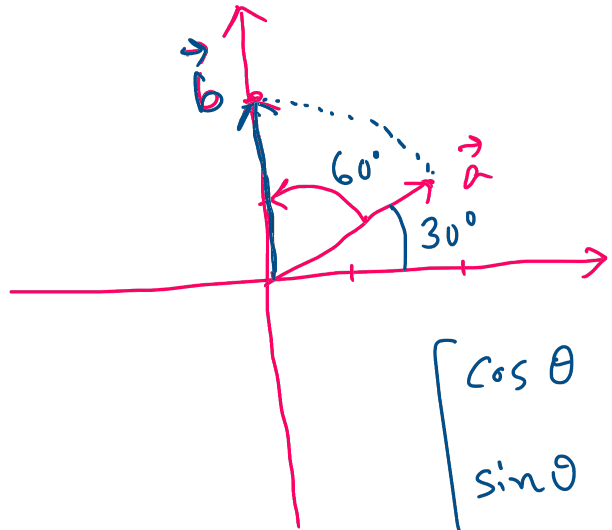
$$= \begin{bmatrix} 2\times 0.5 + 3\times 1 + 0\times(-1) + (-1)\times 0 \\ 6\times 0.5 + 7\times 1 + (-2)\times(-1) + 1\times 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \end{bmatrix}_{\underline{2\times 1}}$$

$$\begin{bmatrix} 2 & 1 \\ -1 & 0.5 \end{bmatrix} \underset{\vec{v}}{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} = \underset{\vec{w}}{\begin{bmatrix} 8 \\ -2 \end{bmatrix}}$$



$$\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \underset{\vec{x}}{\begin{bmatrix} 2 \\ -2 \end{bmatrix}} = \underset{\vec{y}}{\begin{bmatrix} -4 \\ 6 \end{bmatrix}}$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

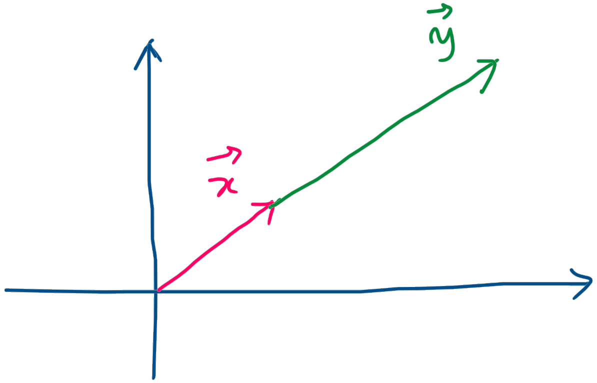


$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \underset{\vec{a}}{\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

2D rotation matrix

$$A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{y} = A\vec{x} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\vec{x}$$



$$\boxed{A\vec{x} = \lambda\vec{x}}$$

where λ is a scaler
A is a square matrix

$\vec{x}$ is called eigen-vector of A
and λ is called eigen-value of A

Eigen-value Problem

$A_{n \times n} \longrightarrow n$ - eigenvalues $(\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n)$

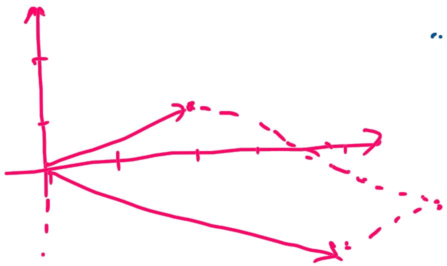
Corresponding to each distinct eigen values there exist a unit vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$, $\vec{v}$ is called the eigen vector

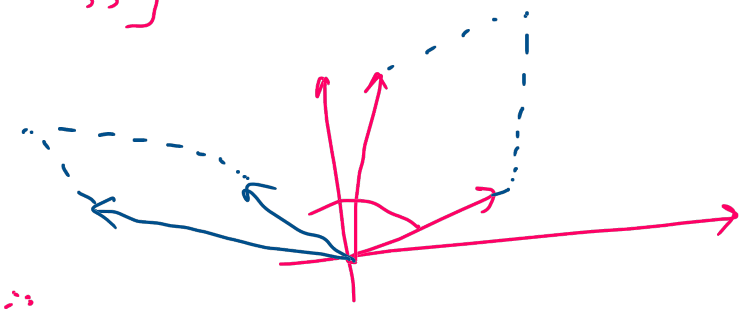
Determinant value of a square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \det(A) = |A| = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -2 \end{bmatrix} \qquad |A| = (2)(-2) - (1).(4) = -8$$

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \qquad |B| = b_{11} \cdot \begin{vmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{vmatrix} - b_{12} \begin{vmatrix} b_{21} & b_{23} \\ b_{31} & b_{33} \end{vmatrix} + b_{13} \begin{vmatrix} b_{21} & b_{22} \\ b_{31} & b_{32} \end{vmatrix}$$





Calculate Eigen-value of a square matrix

$$A\vec{x} = \lambda\vec{x}$$

$$\Rightarrow A\vec{x} - \lambda\vec{x} = 0$$

$$\Rightarrow (A - \lambda I)\vec{x} = 0$$

$$\Rightarrow |A - \lambda I| = 0$$

2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11} - \lambda) & a_{12} \\ a_{21} & (a_{22} - \lambda) \end{bmatrix}$$

$$|A - \lambda I|$$

$$= (a_{11} - \lambda) \cdot (a_{22} - \lambda) - a_{21} \cdot a_{12}$$

$$\boxed{(a_{11} - \lambda)(a_{22} - \lambda) - a_{21} a_{12} = 0} \rightarrow \lambda_1, \lambda_2$$

Characteristic equation

$$A_{n\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$$

$$\left.|A - \lambda I| = \underbrace{(a_{11}-\lambda)\cdot(a_{22}-\lambda)\cdots(a_{nn}-\lambda)}_{O(\lambda^n)} + \cdots + \cdots = 0 \right\}$$

$$\lambda_1, \lambda_2, \ldots, \lambda_n$$

If A is $n\times n$ square matrix then the characteristic equation is also a polynomial equation of degree-n, which has n-solutions in general.

$$A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}_{2\times 2}$$

$$A - \lambda I = 0$$

$$\Rightarrow \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)(1-\lambda) + 2 = 0$$

$$\Rightarrow 4 - 4\lambda - \lambda + \lambda^2 + 2 = 0$$

$$\Rightarrow \boxed{\lambda^2 - 5\lambda + 6 = 0} \longrightarrow \text{characteristic equation.}$$

Every square matrix satisfies its own characteristic equation

$$\boxed{A^2 - 5A + 6I = 0}$$

Cayley - Hamilton's theorem

$$\lambda^2 - 3\lambda - 2\lambda + 6 = 0$$

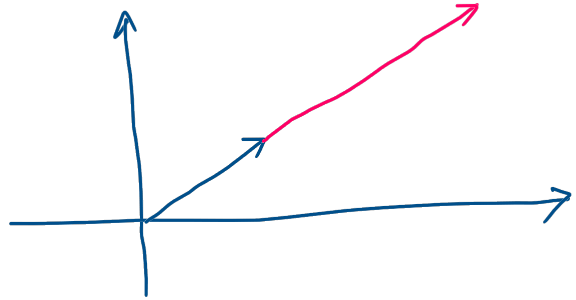
$$\Rightarrow \lambda(\lambda-3) - 2(\lambda-3) = 0$$

$$\Rightarrow (\lambda-2)(\lambda-3) = 0 \quad \Rightarrow \lambda = \underline{2},\ \lambda = \underline{3}$$

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} = \underline{3} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -3 \end{bmatrix} = \begin{bmatrix} -9 \\ -9 \end{bmatrix} = 3 \begin{bmatrix} -3 \\ -3 \end{bmatrix} \checkmark$$



$$\vec{u}_x = \frac{\vec{x}}{\|\vec{x}\|} = \frac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\sqrt{2}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

direction of eigen-vector

$$(A - \lambda I)\vec{x} = 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$2x_1 - x_2 = 0 \qquad x_1 = 1, \quad x_2 = 2$$

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Matrix - Matrix multiplication

$$A_{3\times 2} \times B_{2\times 4} = C_{3\times 4}$$

$$A_{m\times \underline{n}} \times B_{\underline{n}\times p} = C_{m\times p}$$

$$[a_{ij}]_{m\times n} \times [b_{ij}]_{n\times p} = [c_{ij}]_{m\times p}$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

$$A = \begin{array}{c} \vec{a}_1^T \\ \vec{a}_2^T \\ \vec{a}_3^T \end{array} \begin{bmatrix} 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 \\ 1 & -1 & 2 & 3 \end{bmatrix}_{3\times 4}$$

$$B = \begin{array}{cc} \vec{b}_1 & \vec{b}_2 \\ \left[\begin{array}{c} 1 \\ -1 \\ 0 \\ 2 \end{array}\right. & \left.\begin{array}{c} 3 \\ 0 \\ -2 \\ 1 \end{array}\right] \end{array}_{4\times 2}$$

$$C = A \times B = \begin{array}{cc} \vec{a}_1^T \vec{b}_1 & \vec{a}_1^T \vec{b}_2 \\ \left[\begin{array}{c} c_{11} \\ c_{21} \\ c_{31} \end{array}\right. & \left.\begin{array}{c} c_{12} \\ c_{22} \\ c_{32} \end{array}\right] \end{array}_{3\times 2} = \begin{bmatrix} \quad \end{bmatrix}$$

A is a square matrix, then $A^2 = A \times A$

$A^3 = A \cdot A \cdot A$. $\quad A^n = \underbrace{A \times A \times \cdots \cdots \times A}_{n\text{-times}}$

Inverse of a square matrix :

A is a square matrix and A is non-singular $(|A| \neq 0)$

Singular square matrix, $|A| = 0$ are not invertible

there is matrix B such that ; $\boxed{AB = I}$, then B is called the inverse of A

$$B = A^{-1}$$

$$AA^{-1} = I \quad \& \quad A^{-1}A = I$$

$(AB)^{-1} = B^{-1} A^{-1}$, $(A \cdot B)^T = B^T \cdot A^T$

if $\det|A| = a$ then $\det(A^{-1}) = \frac{1}{a}$