# Rules of Differentiation

1. **Linearity** :

$$f(x) \xrightarrow{\frac{d}{dx}} f'(x) \qquad g(x) \xrightarrow{\frac{d}{dx}} g'(x)$$

$$\frac{d}{dx}\left[a f(x) + b\, g(x)\right] = a \frac{df}{dx} + b \frac{dg}{dx} = a f'(x) + b\, g'(x)$$

$$\frac{d}{dx}\left[2 f(x) + 5 g(x)\right] = 2 f'(x) + 5 g'(x)$$

## Differentiation of some functions

| $f(x)$ | $f'(x)$ | example |
|---|---|---|
| 1. $x^n$ | $n x^{n-1}$ | $x^5 \xrightarrow{\frac{d}{dx}} 5x^4$ , $x^{7/2} \xrightarrow{\frac{d}{dx}} \frac{7}{2} x^{5/2}$ |
| 2. $c$ (const) | $0$ | |
| 3. $e^x$ | $e^x$ | |

| | $f(x)$ | $f'(x)$ |
|---|---|---|
| 4. | $e^{\alpha x}$ | $\alpha e^{\alpha x}$ |
| 5. | $\log(x)$ | $\frac{1}{x}$ |
| 6. | $\underbrace{\log(\beta x)}_{\log \beta + \log x}$ | $\beta \cdot \frac{1}{\beta x} = \frac{1}{x}$ |
| 7. | $\underbrace{\log(x^n)}_{n \log x}$ | $\frac{n}{x}$ |
| 8. | $\sin(x)$ | $\cos(x)$ |
| | $\sin(\alpha x)$ | $\alpha \cos(\alpha x)$ |

examples

$$e^{5x} \xrightarrow{d/dx} 5e^{5x}$$

| | $f(x)$ | $f'(x)$ |
|---|---|---|
| 9. | $\cos(x)$ | $-\sin(x)$ |
| . | $\cos(\beta x)$ | $-\beta \sin(\beta x)$ |
| 10. | $\tan(x)$ | $\sec^2(x) = \frac{1}{\cos^2(x)}$ |

2. multiplication

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = f(x)\cdot\frac{dg}{dx} + g(x)\cdot\frac{df}{dx} = f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

$$\frac{d}{dx}(x\sin x) = x\frac{d}{dx}(\sin x) + \sin(x)\cdot\frac{d}{dx}(x)$$

$$= x\cos(x) + \sin(x)$$

$$\frac{d}{dx}\left(e^{-x}\cdot x^5\right) = e^{-x}\frac{d}{dx}(x^5) + x^5\frac{d}{dx}(e^{-x})$$

$$= 5e^{-x}x^4 - e^{-x}x^5$$

$$\frac{d}{dx}(\sin x\cdot\cos x) = \sin(x)\cdot\frac{d}{dx}(\cos x) + \cos(x)\cdot\frac{d}{dx}(\sin x)$$

$$= -\sin^2(x) + \cos^2(x) = \cos^2(x) - \sin^2(x)$$

3. <u>Division Rule</u>

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] \qquad (g(x) \neq 0)$$

$$\Rightarrow \frac{g(x) \cdot \frac{df}{dx} - f(x) \cdot \frac{dg}{dx}}{(g(x))^2} = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{(g(x))^2}$$

$$\frac{d}{dx}\left(\frac{1}{x+3}\right) = \frac{(x+3)\frac{d}{dx}(1) - 1 \cdot \frac{d}{dx} \cdot (x+3)}{(x+3)^2}$$

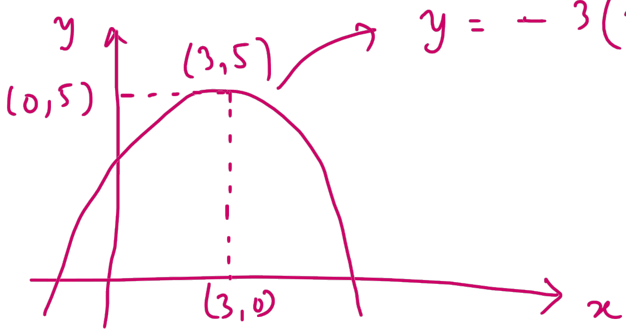
$$= \frac{0-1}{(x+3)^2}$$

$$= -\frac{1}{(x+3)^2}$$

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

# Maxima & Minima




$$y = -3(x-3)^2 + 5$$

$$y_{max} = 5 \text{ at } x = 3$$

$$y = -3(x-3)^2 + 5$$

$$\frac{dy}{dx} = -6 \cdot (x-3) = 0 \Rightarrow x - 3 = 0 \Rightarrow x = 3$$

$$= -6x + 18$$

$x = 3$ is a point of maximum.

$$\frac{d^2y}{dx^2} = -6 < 0$$

$$\frac{d^2y}{dx^2} = 0 \Rightarrow \frac{dy}{dx} = c \Rightarrow y = cx + d$$

## Composite functions

$$x \rightarrow \boxed{f(\ )} \rightarrow f(x) \rightarrow \boxed{g(\ )} \rightarrow \overbrace{\underbrace{g(f(x))}_{g \circ f(x)}}^{\text{composite function}} \rightarrow \boxed{h(\ )} \rightarrow \underbrace{h(g(f(x)))}_{h \circ g \circ f(x)}$$

$$f(x) = x^2$$

$$g(x) = \sin(x)$$

$$g \circ f(x) = \sin(x^2) \quad \Big| \quad f \circ g(x) = \sin^2(x)$$

$$\boxed{\begin{aligned} f(x) &= \alpha x + \beta \\ g(x) &= e^{-x} \\ h(x) &= \frac{1}{1+x} \end{aligned}}$$

$$h \circ g \circ f(x) = ?$$

$$g \circ f(x) = e^{-(\alpha x + \beta)}$$

$$h \circ g \circ f(x) = \frac{1}{1 + e^{-(\alpha x + \beta)}}$$

## Chain Rule of differentiation :-

$$\frac{d}{dx}\left[g(f(x))\right] = ?$$

$$f(x) = u$$

$$\frac{d}{dx}\left[g(u)\right] = \frac{d}{du} g(u) \cdot \frac{du}{dx}$$

$$\sin(\cos(x)) \qquad u = \cos(x)$$

$$\frac{d}{dx}[\sin(u)] = \frac{d}{du}(\sin(u)) \cdot \frac{du}{dx}$$

$$= \cos(u) \cdot (-\sin(x))$$

$$= -\sin(x) \cdot \cos(\cos(x))$$

$$h \circ g \circ f(x) \Big\} \quad u = f(x) \Rightarrow h \circ g(u) \Big\} \; v = g(u) \Rightarrow h(v)$$

$$\frac{d}{dx}(h \circ g \circ f(x)) = \frac{d}{dx}[h(v)] = \frac{d}{dv}[h(v)] \cdot \frac{dv}{dx} = \frac{dh}{dv} \times \frac{dv}{du} \times \frac{du}{dx}$$

$$\left.\begin{aligned} f(x) &= \alpha x + \beta \\ g(x) &= e^{-x} \\ h(x) &= \frac{1}{1+x} \end{aligned}\right\} \quad h \circ g \circ f(x) = \frac{1}{1 + e^{-(\alpha x + \beta)}}$$

$$\frac{du}{dx} = \alpha$$

$$u = f(x) = \alpha x + \beta \quad , \quad v = g(u) = e^{-u}$$

$$\frac{dv}{du} = -e^{-u}$$

$$h(v) = \frac{1}{1+v}$$

$$\frac{dh}{dv} = -\frac{1}{(1+v)^2}$$

$$\frac{d}{dx}\left[h \circ g \circ f(x)\right] = \frac{d}{dx}\left[h(v)\right] = \frac{d}{dv} h(v) \cdot \frac{dv}{du} \cdot \frac{du}{dx}$$

$$= -\frac{1}{(1+v)^2} \cdot \left(-e^{-u}\right) \cdot \alpha$$

$$= +\frac{\alpha}{(1+e^{-u})^2} e^{-u} = \frac{\alpha e^{-u}}{(1+e^{-u})^2} = \frac{\alpha \cdot e^{-(\alpha x + \beta)}}{\left[1 + e^{-(\alpha x + \beta)}\right]^2}$$

$$\frac{d}{dx}\Bigg[\underbrace{\frac{1}{1+e^{-(\alpha x+\beta)}}}_{\sigma(x)}\Bigg] = \frac{\alpha \cdot e^{-(\alpha x+\beta)}}{\left(1+e^{-(\alpha x+\beta)}\right)^2} = \frac{\alpha \cdot \left[1+e^{-(\alpha x+\beta)}-1\right]}{\left(1+e^{-(\alpha x+\beta)}\right)^2}$$

$$= \frac{\alpha}{1+e^{-(\alpha x+\beta)}} \cdot \left[\frac{1+e^{-(\alpha x+\beta)}-1}{1+e^{-(\alpha x+\beta)}}\right]$$

$$= \frac{\alpha}{1+e^{-(\alpha x+\beta)}} \cdot \left[1 - \frac{1}{1+e^{-(\alpha x+\beta)}}\right]$$

$$\frac{d}{dx}[\sigma(x)] = \alpha \cdot \sigma(x) \cdot [1-\sigma(x)]$$

$$S(x) = \frac{1}{1+e^{-x}} \Rightarrow S'(x) = S(x)[1-S(x)]$$

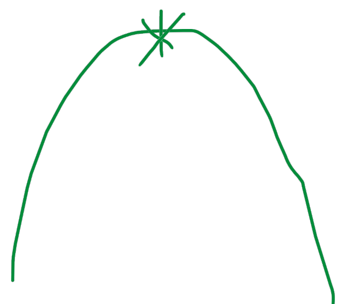
$$f(x) = 3x^4 - 2x + 1$$

$$\frac{df}{dx} = 12x^3 - 2 = 0 \Rightarrow x^3 = \frac{2}{12} = \frac{1}{6}$$

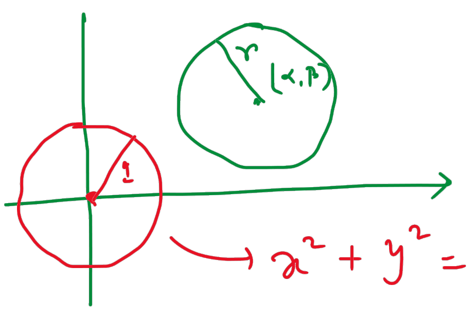
$$\Rightarrow x = \sqrt[3]{\frac{1}{6}} = 0.5503$$

$$-3x^2 + 5$$

$$f_{min}(x) = f(x)\Big|_{x = \sqrt[3]{\frac{1}{6}}}$$



$$f(x) = 4x^2 - 3x + 2$$




$$(x - \alpha)^2 + (y - \beta)^2 = r^2$$

$$x^2 + y^2 = 1 \Rightarrow y = \pm\sqrt{1 - x^2}$$

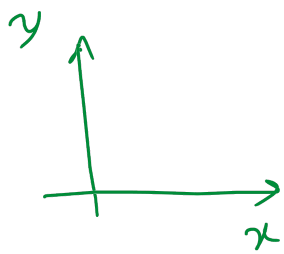
# Multi-variable function

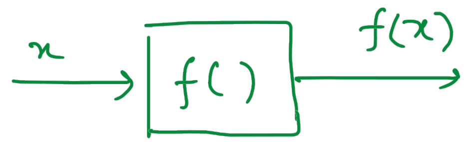
$f(x, y, z)$

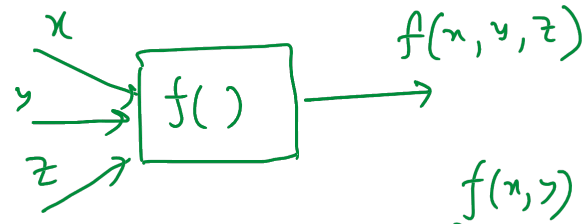
ex.1: $3x^2 + 2e^{-y} + \sin(\log(z))$

ex.2: $5e^{-xyz}$

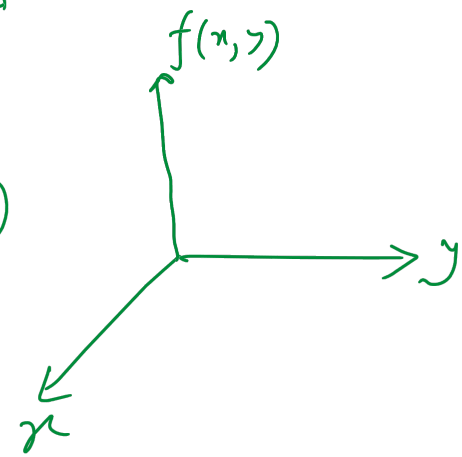
$y = f(x)$



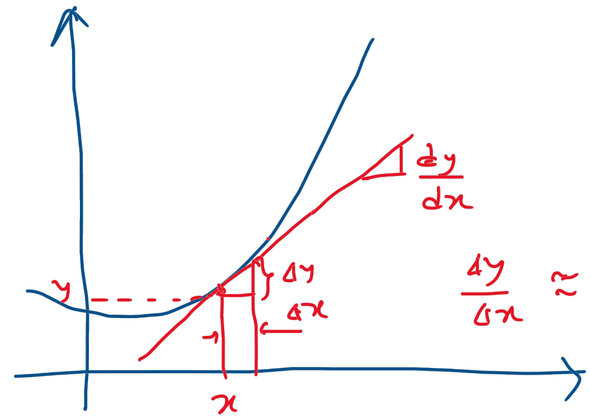

$z = f(x, y)$



For function of single variable

$$y = f(x)$$

$$\Delta y = \frac{dy}{dx} \cdot \Delta x$$




$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx} \Rightarrow \Delta y \approx \frac{dy}{dx} \cdot \Delta x$$

$f(x, y, z)$

$df = ?$

$$df = \boxed{\frac{\partial f}{\partial x}} dx + \boxed{\frac{\partial f}{\partial y}} dy + \boxed{\frac{\partial f}{\partial z}} dz$$

Partial differentiation of $f$ wrt $x$.

diff of $f$ wrt $x$ considering $y$ & $z$ constant

rate_male_student = 500

rate_female_ .. = 400

rate_teacher = 1000

$\Delta m = 2$

$\Delta f = -3$

$\Delta t = 1$

$$\Delta B = 500 \times 2 + 400 \times (-3) + 1000 \times (1)$$

$$f(x,y) = \boxed{xy}$$

$$\frac{d}{dx}\left[f(x) \cdot g(x)\right]$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$= y\, dx + x\, dy$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xy) = y \frac{\partial x}{\partial x} = y$$

$$\frac{\partial f}{\partial y} = x$$

$$f(x,y,z) = 3x^2 e^{-y} \sin(\log z)$$

$$\frac{\partial f}{\partial x} = 6x\, e^{-y} \sin(\log(z)), \quad \frac{\partial f}{\partial y} = -3x^2 e^{-y} \sin(\log z), \quad \frac{\partial f}{\partial z} = \frac{3x^2 e^{-y} \cdot \cos(\log(z))}{z}$$

## Optimize multi-variable function

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial z} = 0$$

$\Bigg\}$ optimum $(x, y, z)$