(v) $\int \frac{dx}{\sqrt[3]{x}\sqrt[3]{(1+x)^5}}$

$\rightarrow$ Let $I = \int \frac{dx}{\sqrt[3]{x}\sqrt[3]{(1+x)^5}}$ And, $1+x = ux$

or, $ux - x = 1$;

$x = \frac{1}{u-1}$

Again, $u = \frac{1}{x} + 1$

$\therefore du = -\frac{1}{x^2} + 1$

Or, $dx = -x^2 du$

$\therefore I = \int \frac{-x^2 du}{x^{1/3} u^{\frac{5}{3}} x^{5/3}}$

$= -\int \frac{x^2 du}{x^2 u^{\frac{5}{3}}}$

$= -\int \frac{du}{u^{5/3}}$

$= -\int u^{-\frac{5}{3}} d$

$= 3/2\, u^{-2/3} + c$

$= \frac{3}{2} \left(\frac{x+1}{x}\right)^{-2/3} + c$

$= \frac{3}{2} \left(\frac{x}{x+1}\right)^{2/3} + c$ (Ans)

4(ii) $\int \cos 2\cot^{-1}\sqrt{\frac{1-x}{x+1}}\,dx$

$\rightarrow$ Let $I = \int \cos 2\cot^{-1}\sqrt{\frac{1-x}{x+1}}\,dx$

And $x = \cos\theta$

So, $dx = (-\sin\theta)d\theta$

$I = \int \cos 2\cot^{-1}\sqrt{\frac{1-x}{x+1}}\,dx$

$= \int \cos 2\cot^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\,(-\sin\theta)d\theta$

$= \int \cos 2\cot^{-1}\sqrt{\frac{2\sin^2\theta/2}{2\cos^2\frac{\theta}{2}}}\,(-\sin\theta)d\theta$

$= \int \cos 2\cot^{-1}\tan\theta/2\,(-\sin\theta)d\theta$

$= \int \cos 2\cot^{-1}\cot\left(\frac{\pi}{2}-\frac{\theta}{2}\right)(-\sin\theta)d\theta$

$= \int \cos 2\left(\frac{\pi}{2}-\frac{\theta}{2}\right)(-\sin\theta)d\theta$

$= \int \cos 2(\pi-\theta)\,(-\sin\theta)d\theta$

$= \int(-\cos\theta)(-\sin\theta)d\theta$

$= \frac{1}{2}\int 2\cos\theta\sin\theta\,d\theta$

$= \frac{1}{2}\int \sin 2\theta\,d\theta$

$= -\frac{1}{4}\cos 2\theta + c$

$= -\frac{1}{4}(2\cos^2\theta - 1) + c$

$= -\frac{1}{4}(2x^2-1)+c$

(Ans)

(iii) $\int \frac{\sin^3 x}{\cos^9 x} dx$

Ans: $\int \frac{\sin^3}{\cos^9 x} dx$

$= \int \frac{\sin^2 x \sin x}{\cos^9 x} dx$

$= \int \frac{(1-\cos^2 x)\sin x}{\cos^9 x} dx$

$= -\int \frac{(1-u^2)}{u^9} du$

$= -\int \left(\frac{1}{u^9} - \frac{u^2}{u^9}\right) du$

$= -\int \left(\frac{1}{u^9} - \frac{1}{u^7}\right) du$

$= -\int u^{-9} du + \int u^{-7} du$

$= \frac{u^{-8}}{(-8)} + \frac{u^{-6}}{(-6)} + c$

$= \frac{1}{8u^8} - \frac{1}{6u^6} + c$

$= \frac{1}{8\cos^8 x} - \frac{1}{6\cos^6 x} + c$ (Ans)

(vi) $\int \frac{7x-9}{x^2-2x+35} dx$

$= \frac{7\ln(x^2-2x+35}{2} - \frac{2\tan^{-1}\left(\frac{x-1}{\sqrt{34}}\right)}{\sqrt{34}} + c$

$= \frac{7\ln(x^2-2x+35}{2} - \frac{2\tan^{-1}\left(\frac{2x-2}{2\sqrt{34}}\right)}{\sqrt{34}} + c$ (Ans)

(i) $\int e^{2x}\left(\frac{1+\sin 2x}{1+\cos 2x}\right) dx$

$= \int e^x \frac{(2+\sin 2x)}{1+\cos 2x} dx$

$= e^x \tan(x) + c$ (Answer)

(iv) $\int \frac{1}{\sqrt[3]{\cos x}\sqrt[3]{\sin^5 x}}\,dx$

$= \int \frac{dx}{(\cos x)^{1/3}(\sin x)^{5/3}}$

$= \int \frac{dx}{\frac{(\cos x)^{1/3}}{(\sin x)^{1/3}} \cdot \sin^2 x}$

$= \int \frac{\operatorname{cosec}^2 x\,dx}{(\cot x)^{1/3}}$

$= \int \frac{-dz}{z^{\frac{1}{3}}}$

$= -\int z^{-1/3}\,dz$

$= -\frac{z^{-1/3+1}}{-1/3+1} + c$

$= -\frac{z^{2/3}}{2/3} + c$

$= -\frac{3}{2}\, z^{2/3} + c$

$= -3/2\, (\cot x)^{2/3} + c$ (Ans)

let,

$\cot x = z$

$\Rightarrow -\operatorname{cosec}^2 x\,dx = dz$

$\Rightarrow \operatorname{cosec}^2 x\,dx = -dz$

3. (a) State and prove Lagrange's mean value theorem.

Statement: Lagrange's mean value theorem (MVT) states that if a function $f(x)$ is continuous on a closed interval $[a,b]$ and differentiable on the open interval (a,b) then there exists at least one point say $(x = c)$ within (a,b) where

$$f(b) - f(a) = f'(c)\,(b-a)$$

$$\text{or, } f'(c) = \frac{f(b) - f(a)}{b-a}$$

This theorem (also known as First Mean Value Theorem) allows to express the increment of a function on an interval through the value of the derivative at an intermediate point of the segment.

Proof:

Consider the auxiliary function

$$F(x) = f(x) + \lambda x$$

We choose a number λ such that the condition $F(a) = F(b)$ is satisfied. Then

$f(a) + \lambda a = f(b) + \lambda b$

$\Rightarrow f(b) - f(a) = \lambda(a-b)$

$\Rightarrow \lambda = \frac{f(b)-f(a)}{b-a)}$

As a result, we have, $F(x) = f(x) \frac{f(b)-f(a)}{b-a} x$

The function $F(x)$ is continuous on the closed interval $[a,b]$, differentiable on the open interval (a,b) and takes equal values at the endpoints of the interval. Therefore, it satisfies all the conditions of Rolle's theorem. Then there is a point c in the interval (a,b) such that, $F(c)=0$

It follows that,

$f'(c) \frac{f(b)-f(a)}{b-a} = 0$

or, $f(b)-f(a) = f(c)(b-a)$. [Proved]

(b) Prove that, x^x has a minimum value for $x = \frac{1}{e}$

$\rightarrow$ Let $y = x^x$

Taking ln both side

$\ln y = x \ln x$ —(1)

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Now, differentiating (1) both sides with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + \ln x \cdot 1$$

$$\Rightarrow \frac{dy}{dx} = y(1+\ln x) \text{ ---- (2)}$$

Now, differentiate (2) with respect to x

$$\frac{d^2y}{dx^2} = (1+\ln x)\frac{dy}{dx} + y\frac{1}{x}$$

$$= (1+\ln x)\frac{dy}{dx} + x^x \frac{1}{x}$$

For find the minimum value $\frac{dy}{dx} = 0$

$\rightarrow y(1+\ln x) = 0$ But here $y \neq 0$

So, $(1+\ln x) = 0$

$\Rightarrow \ln x = -1 \Rightarrow \ln x = -\ln e$

$\Rightarrow \ln x = \ln\left(\frac{1}{e}\right)$ or

$\therefore x = \frac{1}{e}$

When, $x = \frac{1}{e}$ then,

$$\Rightarrow \frac{d^2y}{dx^2} = (1+\ln {}^1/e)\frac{dy}{dx} + \left(\frac{1}{e}\right)^{1/e} \cdot e$$

$$\Rightarrow \frac{d^2x}{dx^2} = 0 + \left(\frac{1}{e}\right)^{1/e} \cdot e > 0$$

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So, x^x has a minimum value for $x = 1/e$

(Proved)

(Q) Evaluate of the following

(i) $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$ (ii) $\lim\limits_{x \to 0} (\sin x)^x$

(i) →

$$\lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x} \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{e^x - e^{-x}}{0 + \sin x} \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{e^x + e^{-x}}{\cos x} = \frac{e^0 + e^0}{\cos 0} = \frac{1+1}{1}$$

$= 2$ (Ans)

(ii) Given, $\lim\limits_{x \to 0} (\sin x)^x$

Let, $y = \lim\limits_{x \to 0} (\sin x)^x$ [0^0 form]

$\Rightarrow \ln y = \lim\limits_{x \to 0} x \ln \sin x$ [$0 \times \infty$ form]

$$= \lim_{x \to 0} \frac{\ln \sin x}{1/x} \quad \left[\frac{\alpha}{\alpha} \text{ form}\right.$$

$$= \lim_{x \to 0} \frac{1/\sin x \cdot \cos x}{-\frac{1}{x^2}}$$

$$= \lim_{x \to 0} \left(-\frac{\cot x}{x^2}\right) = \lim_{x \to 0} \left(-\frac{x^2}{\tan x}\right)$$

$$= -\lim_{x \to 0} \frac{2x}{\sec^2 x}$$

$$= -\frac{2 \times 0}{\sec^2 0} = \frac{0}{1} = 0$$

$$\Rightarrow \ln y = 0 \Rightarrow \ln y = \ln(1) \therefore y = 1$$

$$\Rightarrow \lim_{x \to 0} (\sin x)^x = 1 \text{ (Ans)}$$

V.V.I

Given that, $y = f(x) = |x| + |x-1| + |x-2|$

$$= \begin{cases} x+x-1+x-2 \;;\; \text{when } x \in [0,\infty) \cap [1,\infty) \cap [2,\infty) \\ x+x-1-(x-2) \;;\; \text{when } x \in [0,\infty) \cap [1,\infty) \cap (-\infty,2) \\ x-(x-1)-(x-2) \;;\; \text{when } x \in [0,\infty) \cap (-\infty,1) \cap (-\infty,2) \\ -x-(x-1)-(x-2) \;;\; \text{when } x \in (-\infty,0) \cap (-\infty,1) \cap (-\infty,2) \end{cases}$$

$$\begin{cases} 3x-3 \;;\; \text{when } x \in [2,\infty) \\ x+1 \;;\; \text{when } x \in [1,2) \\ -x+3 \;;\; \text{when } x \in [0,1) \\ -3x+3 \;;\; \text{when } x \in (-\infty,0) \end{cases}$$

$+ + +$
$+ + -$
$+ - -$
$- - -$

$x \geq 0 \Rightarrow [0,\infty)$

$x-1 \geq 0$
$\Rightarrow x \geq 1 \Rightarrow [1,\infty)$
$x \geq 2 \Rightarrow [2,\infty)$

$x-2 < 0$
$x < 2$
$(-\infty,2)$

$x-1 < 0 \Rightarrow x < 1$
$\Rightarrow (-\infty,1)$

Domain of $f(x)$, $= [2,\infty) \cup [1,2) \cup [0,1) \cup (-\infty,0)$
$= (-\infty,\infty)$

Again, for Range,

$y = 3x-3$ in the domain $[2,\infty)$ has Range $[3,\infty)$

$y = x+1$ " " " $[1,2)$ " " $[2,3)$

$y = -x+3$ in the domain $[0,1)$ has Range $(2,3]$

$y = -3x+3$ in the domain $(-\infty,0)$ has Range $(3,\infty)$

Hence the Range of $f(x)$, $R_f = [3,\infty) \cup [2,3) \cup (2,3] \cup (3,\infty)$
$= [2,\infty)$

Ans.