Monte Carlo in Movie Production

Ray Tracing and Sampling

Slavomir Kaslev
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December 17, 2016

About me

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- I work at Worldwide FX as Head of R&D

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Studio demo reel

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- "Physically Based Rendering: From Theory to Implementation" by Matt Pharr and Greg Humphreys

Bussiness card ray tracer

Slavomir

fabiensanglard.net/rayTracing_back_of_business_card/

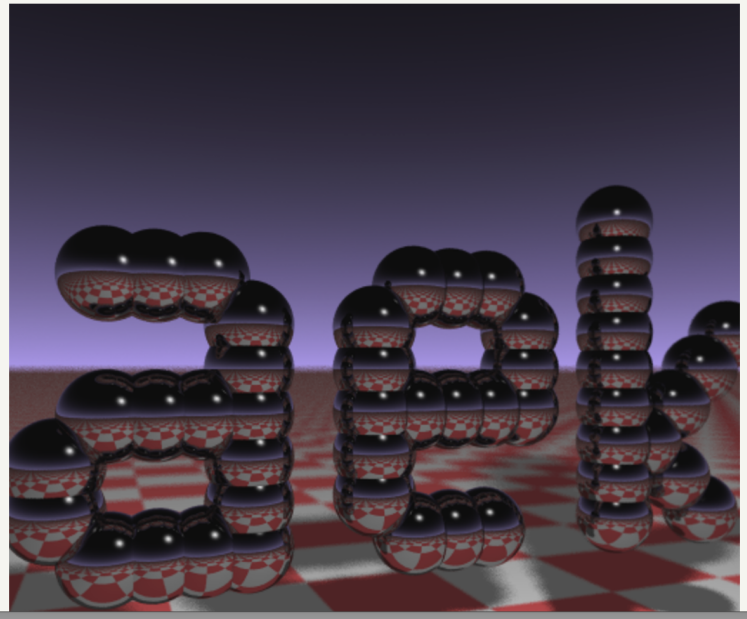
The Business Card Code

```
#include <stdlib.h>   // card > aek.ppm
#include <stdio.h>
#include <math.h>
typedef int i;typedef float f;struct v{
f x,y,z;v operator+(v r){return v(x+r.x
,y+r.y,z+r.z);}v operator*(f r){return
v(x*r,y*r,z*r);}f operator%(v r){return
x*r.x+y*r.y+z*r.z;}v(){}v operator^(v r
){return v(y*r.z-z*r.y,z*r.x-x*r.z,x*r.
y-y*r.x);}v(f a,f b,f c){x=a;y=b;z=c;}v
operator!(){return*this*(1/sqrt(*this%*
this));}};i G[]={247570,280596,280600,
249748,18578,18577,231184,16,16};f R(){
return(f)rand()/RAND_MAX;}i T(v o,v d,f
&t,v&n){t=1e9;i m=0;f p=-o.z/d.z;if(.01
<p)t=p,n=v(0,0,1),m=1;for(i k=19;k--;)
for(i j=9;j--;)if(G[j]&1<<k){v p=o+v(-k
,0,-j-4);f b=p%d,c=p%p-1,q=b*b-c;if(q>0
){f s=-b-sqrt(q);if(s<t&&s>.01)t=s,n=!(
p+d*t),m=2;}}return m;}v S(v o,v d){f t
;v n;i m=T(o,d,t,n);if(!m)return v(.7,
.6,1)*pow(1-d.z,4);v h=o+d*t,l=!(v(9+R(
),9+R(),16)+h*-1),r=d+n*(n%d*-2);f b=l%
n;if(b<0||T(h,l,t,n))b=0;f p=pow(l%r*(b
>0),99);if(m&1){h=h*.2;return((i)(ceil(
h.x)+ceil(h.y))&1?v(3,1,1):v(3,3,3))*(b
*.2+.1);}return v(p,p,p)+S(h,r)*.5;}i
main(){printf("P6 512 512 255 ");v g=!v
(-6,-16,0),a=!(v(0,0,1)^g)*.002,b=!(g^a
)*.002,c=(a+b)*-256+g;for(i y=512;y--;)
for(i x=512;x--;){v p(13,13,13);for(i r
=64;r--;){v t=a*(R()-.5)*99+b*(R()-.5)*
99;p=S(v(17,16,8)+t,!(t*-1+(a*(R()+x)+b
*(y+R())+c)*16))*3.5+p;}printf("%c%c%c"
,(i)p.x,(i)p.y,(i)p.z);}}
```

The code looks confusing but it compiles and run flawlessly !! On my MacBook Air, I can save this file on my Desktop as `card.cpp`, open Terminal and type :

```
c++ -O3 -o card card.cpp
./card > card.ppm
```

Within 27 seconds, the following image is generated:



The Rendering Equation

$$L(p, \omega_o) = L_e(p, \omega_o) + \int_{S^2} f(p, \omega_o, \omega_i) L(t(p, \omega_i), -\omega_i) \left|\cos \theta_i\right| d\omega_i$$

Analytical solution: Operator formulation

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- The light transport operator

$$L = L_e + \boldsymbol{T} L$$

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- Power series representation: $L = L_e + \boldsymbol{T} L_e + \boldsymbol{T}^2 L_e + ...$

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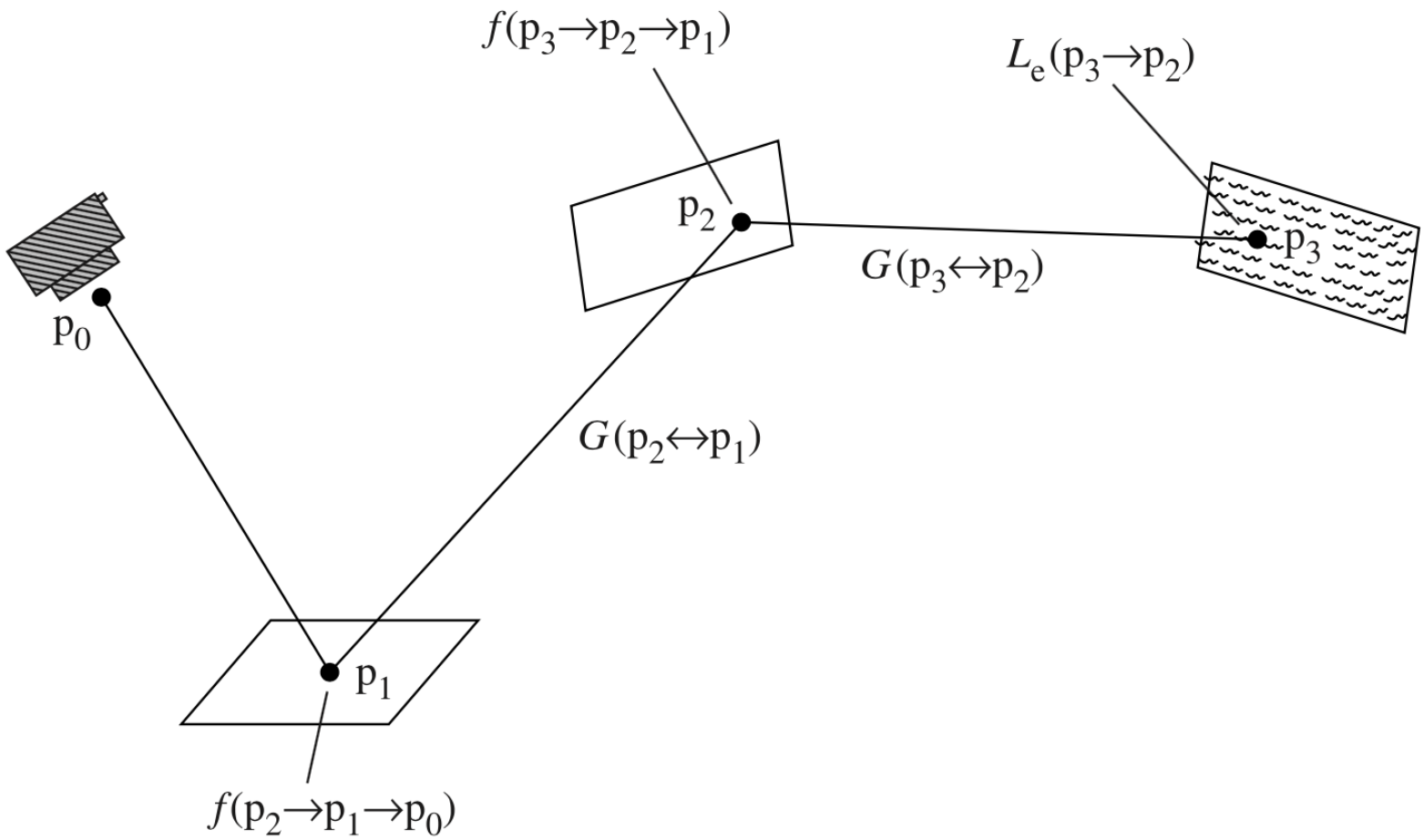
$$\boldsymbol{S} = (\boldsymbol{I} - \boldsymbol{T})^{-1}$$
$$L = \boldsymbol{S}L_e$$

Numerical solution: Integral over paths

$$L(p' \to p) = L_e(p' \to p) + \int_A f(p'' \to p' \to p) L(p'' \to p') G(p'' \leftrightarrow p') dA(p'')$$

Generating paths



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- The modern version of the Monte Carlo method was invented in the late 1940s by Stanislaw Ulam
- Monte Carlo methods were central to the simulations required for the Manhattan Project

Monte Carlo method I

Suppose we want to find the value of I where

$$I = \int_{x \in [0,1]^s} f(x) dx$$

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Well by definition of expected value we know that

$$\mathbf{E}[f(X)] = \int_{x \in [0,1]^s} f(x) p(x) dx$$

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Therefore

$$I = \mathbf{E}[\frac{f(X)}{p(X)}]$$

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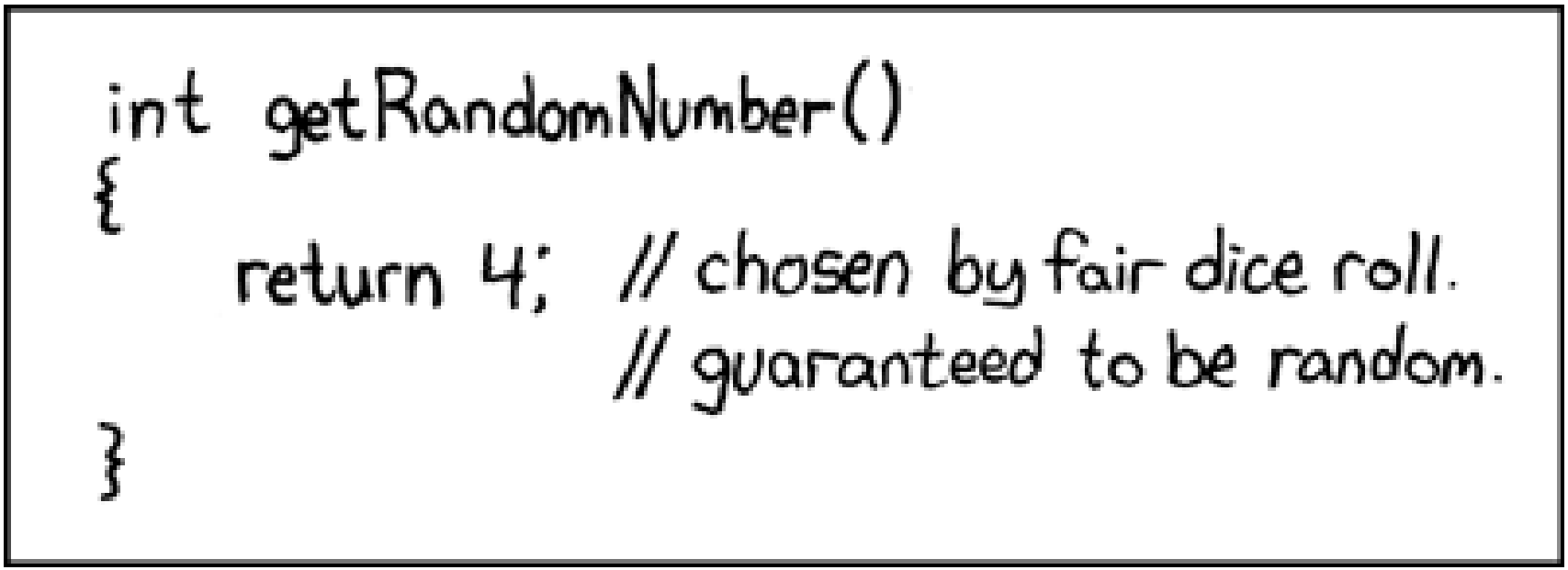
$$\mathbf{E}[\widetilde{I_N}] = I$$

$$\mathbf{V}[\widetilde{I_N}] = \frac{1}{N} \mathbf{V}[\frac{f(X)}{p(X)}]$$

$$\sigma[\widetilde{I_N}] = \frac{1}{\sqrt{N}} \sigma[\frac{f(X)}{p(X)}]$$

Random Numbers: xkcd #221



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- Low discrepancy sequences: van der Corput, Halton, Hammersley, Sobol and others

$$D_N^*(x_1, ..., x_N) \leq C \frac{(\ln N)^s}{N}$$

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- https://github.com/skaslev/pyman

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"All problems in computer graphics can be solved with a matrix inversion." Jim Blinn

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- Plug that back in the equation and solve for a_n
- Sum the series and substitue $\epsilon = 1$

Further reading

- "Physically Based Rendering: From Theory to Implementation" by Matt Pharr and Greg Humphreys
- "Robust Monte Carlo Methods for Light Transport Simulation", Eric Veach, Ph.D. dissertation
- "Light Transport Simulation with Vertex Connection and Merging" by Iliyan Georgiev, Jaroslav Kivnek, Tom Davidovi, Philipp Slusallek
- "Random Number Generation and Quasi-Monte Carlo Methods" by Harald Niederreiter
- "Quantum Mechanics and Path Integrals" by Richard Feynman
- "An Introduction to the Analysis of Algorithms" by Robert Sedgewick and Phillipe Flajolet
- "Mathematical Physics", Carl Bender, https://www.youtube.com/watch?v=LYNOGk3ZjFM
- "fract", source code from this presentation, https://github.com/skaslev/fract

Questions?

Thank you