Combinatorics of general balanced trees

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1 Introduction

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data F a = F0 a | F1 (F (G a))
data G a = G0 a | G1 a (G a)
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The generating function $f(x)$ of general balanced trees satisfies the equations

$$\begin{aligned} f(x) &= x + f(g(x)) \\ g(x) &= x + xg(x) \end{aligned}$$

$$f(x) = x + \epsilon f(g(x)) \tag{1}$$

$$f(x) = x + \epsilon g(x) + \epsilon^2 g(g(x)) + \epsilon^3 g(g(g(x))) + \ldots$$

$$\begin{aligned} g_0(x) &= x \\ g_1(x) &= g(x) \\ g_n(x) &= g_{n-1}(g(x)) \end{aligned}$$

$$f(x) = \sum_{n=0}^{\infty} g_n(x)\epsilon^n \tag{2}$$

Since

$$g(x) = \frac{x}{1-x} \tag{3}$$

one can easy show by induction that $\forall n \in \mathbb{N}$

$$g_n(x) = \frac{x}{1-nx}$$

Moreover $\forall n \in \mathbb{N}^+$

$$g_n(x) = \frac{g(nx)}{n}$$

and $\forall n, m \in \mathbb{N}^+$

$$g_{nm}(x) = \frac{g_n(mx)}{m} = \frac{g_m(nx)}{n}$$

$g_n(x)$ can be expanded in power series

$$g_n(x) = x + nx^2 + n^2x^3 + \dots + n^{k-1}x^k + \dots$$

$\forall k \in \mathbb{N}^+$

$$[x^k]g_n(x) = n^{k-1}$$

$$[\epsilon^n]f(x) = g_n(x)$$

$$[x^k\epsilon^n]f(x) = n^{k-1}$$

The functions $g_n(x)$ form a comutative monoid under function composition with identity element $g_0(x) = x$ since $\forall n, m, l \in \mathbb{N}$ we have

$$g_0 \circ g_n = g_n \circ g_0 = g_n$$

$$g_n \circ g_m = g_m \circ g_n = g_{n+m}$$

$$(g_n \circ g_m) \circ g_l = g_n \circ (g_m \circ g_l) = g_{n+m+l}$$

$$f(g(x)) = f(x) - x = \sum_{n=1}^{\infty} g_n(x)$$

$$f(g(x)) = \sum_{n=1}^{\infty} \frac{g(nx)}{n}$$

$$g(x) = \sum_{m=1}^{\infty} \mu(m) \frac{f(g(mx))}{m}$$

$$\frac{f(g(mx))}{m} = \sum_{n=1}^{\infty} \frac{g_n(mx)}{m} = \sum_{n=1}^{\infty} g_{nm}(x)$$

$$g(x) = \sum_{m=1}^{\infty} \mu(m) \sum_{n=1}^{\infty} g_{nm}(x)$$

Which can quickly be verified by looking at the coefficients in front x^k for $k \in \mathbb{N}^+$

$$1 = \sum_{m=1}^{\infty} \mu(m) \sum_{n=1}^{\infty} n^{k-1}m^{k-1} = \zeta(1-k) \sum_{m=1}^{\infty} \mu(m)m^{k-1} = \frac{\zeta(1-k)}{\zeta(1-k)}$$