# Playing with function iteration

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## 1 General problem

The equations $f(x) = x + f(x^2 + x^3)$ and $f(x) = x + f(\frac{1}{1-x})$ are instances of the more general equation

$$f(x) = x + f(g(x)) \tag{1}$$

We want to find a function $f(x)$ such that (1) holds for all $x$, where the function $g(x)$ is known. Telescoping the equation gives

$$f(x) = x + g(x) + g(g(x)) + g(g(g(x))) + \ldots$$

Let's use the following notation for iterating a function

$$\begin{aligned} g^{[0]}(x) &= x \\ g^{[1]}(x) &= g(x) \\ g^{[n]}(x) &= g(g^{[n-1]}(x)) \end{aligned}$$

to rewrite equation (1) as

$$f(x) = \sum_{n=0}^{\infty} g^{[n]}(x) \tag{2}$$

We've reduced the problem from solving equation (1) to calculating the infinite sum (2).

## 2 Power series sleight of hand

The next step is instead of solving equation (1) we'll change the problem by inserting a small parameter $\epsilon$

$$f(x) = x + \epsilon f(g(x)) \tag{3}$$

Repeating the steps above, we telescope (3)

$$f(x) = x + \epsilon g(x) + \epsilon^2 g(g(x)) + \epsilon^3 g(g(g(x))) + \ldots$$

and finally we get $f(x)$ as power series of $\epsilon$

$$f(x) = \sum_{n=0}^{\infty} g^{[n]}(x)\epsilon^n \tag{4}$$

Now we have two procedures for computing $f(x)$ defined by equations (2) and (4). Of course, the two procedures will agree only if we finally set $\epsilon$ to 1.

## 3 Special case $g(x) = \frac{1}{1-x}$

Equation (3) takes the form

$$f(x) = x + \epsilon f(\frac{1}{1-x}) \tag{5}$$

Note that the function $g^{[n]}(x)$ is periodic in $n$ with period 3. That is

$$\begin{aligned} g^{[0]}(x) &= x \\ g^{[1]}(x) &= \frac{1}{1-x} \\ g^{[2]}(x) &= 1 - \frac{1}{x} \\ g^{[3]}(x) &= x \end{aligned}$$

So the series in (4) have the form

$$f(x) = x + \epsilon\frac{1}{1-x} + \epsilon^2(1 - \frac{1}{x}) + \epsilon^3 x + \epsilon^4\frac{1}{1-x} + \dots \tag{6}$$

We'll sum this series using a generic summation procedure $S$ with the properties

$$S(a_0 + a_1 + a_2 + \dots) = a_0 + S(a_1 + a_2 + a_3 + \dots) \tag{7a}$$

$$S(\sum \alpha a_n + \sum \beta b_n) = \alpha S(\sum a_n) + \beta S(\sum b_n) \tag{7b}$$

To simplify the calculation we'll substitute

$$s = f(x) \qquad a = x \qquad b = \frac{1}{1-x} \qquad c = 1 - \frac{1}{x}$$

in (6) to obtain

$$s = S(a + b\epsilon + c\epsilon^2 + a\epsilon^3 + b\epsilon^4 + c\epsilon^5 + \dots)$$

Next we'll pull the first term out of the sum two times by (7a) and bring $\epsilon$ outside the sum by (7b)

$$\begin{aligned} s &= S(a + b\epsilon + c\epsilon^2 + a\epsilon^3 + b\epsilon^4 + c\epsilon^5 + \dots) \\ s &= a + \epsilon S(b + c\epsilon + a\epsilon^2 + b\epsilon^3 + c\epsilon^4 + a\epsilon^5 + \dots) \\ s &= a + b\epsilon + \epsilon^2 S(c + a\epsilon + b\epsilon^2 + c\epsilon^3 + a\epsilon^4 + b\epsilon^5 + \dots) \end{aligned}$$

If we multiply the first equation by $\epsilon^2$, the second by $\epsilon$ then add them all up, the result is

$$(1 + \epsilon + \epsilon^2)s = (1 + \epsilon)a + \epsilon b + \epsilon^2(a + b + c)S(1 + \epsilon + \epsilon^2 + \epsilon^3 + \dots)$$

Notice that $S(1 + \epsilon + \epsilon^2 + \epsilon^3 + \dots) = \frac{1}{1-\epsilon}$ and now we have an explicit formula for $s$

$$s = \frac{(1 + \epsilon)a + \epsilon b}{1 + \epsilon + \epsilon^2} + \frac{\epsilon^2}{1 - \epsilon^3}(a + b + c)$$

Therefore the solution of (5) is

$$f(x) = \frac{(1+\epsilon)x + \epsilon\frac{1}{1-x}}{1+\epsilon+\epsilon^2} + \frac{\epsilon^2}{1-\epsilon^3}\frac{x^3-3x+1}{x(x-1)} \tag{8}$$

We're interested in the solution of this equation

$$f(x) = x + f(\frac{1}{1-x}) \tag{9}$$

which can be derived from (8) by seting $\epsilon$ to 1. Notice that when $\epsilon = 1$, the second term in the equation becomes infinite unless $x^3 - 3x + 1 = 0$.

We conclude that the solution $f(x)$ of equation (9) is infinite everywhere except for a finite set of points given by the roots of the equation

$$x^3 - 3x + 1 = 0$$

namely

$$x_1 \approx -1.87938524 \qquad x_2 \approx 0.34729636 \qquad x_3 \approx 1.53208889$$

where $f(x)$ has an explicit formula

$$f(x) = \frac{1}{3}\,\frac{1+2x-2x^2}{1-x}$$

and can be evalutated directly

$$f(x_1) \approx -1.13715804 \qquad f(x_2) \approx 0.7422272 \qquad f(x_3) \approx 0.39493084$$

Notice that iterating $g(x)$ over the numbers $x_1, x_2, x_3$ forms a cycle

$$g(x_1) = x_2 \qquad g(x_2) = x_3 \qquad g(x_3) = x_1$$

and

$$\sum_{k=1}^{3} x_k = 0 \qquad \sum_{k=1}^{3} f(x_k) = 0 \qquad \sum_{k=1}^{3} g(x_k) = 0$$