Continuous Probability

- continuous r.v. : a r.v. whose # can be any number on an entire interval on a # line

PMF are hard, so we use Prob. Density Fxn.

Ex. Train problem, leaves every 5 min., what is the prob. we will wait more than 2 min?

$X \equiv$ amt. of time we wait for train (mins) continuous

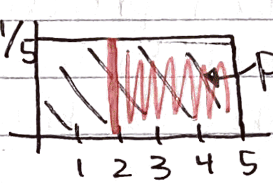
$Y \equiv$ the # of mins (rounded up) we wait for next train discrete

support for $Y = \{1,2,3,4,5\}$

support for $X = [0,5]$

pmf for Y

y	1	2	3	4	5
$p(y)$	$1/5$	$1/5$	$1/5$	$1/5$	$1/5$

pdf 

$$f(x) = \begin{cases} 1/5 & 0 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

Density b/c ∞ # of #s in our support - gets impossible to 'spread' a finite probability

Density Fxn - think of it the same way that you do a density of metals - we need interval to calculate a prob. mass.

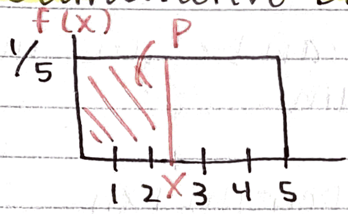
Uniform Dist. - has same prob. b/t any 2 intervals of the same length

$X \sim U(0,5)$

$P(X \ge 2) = \underbrace{1 - P(X \le 2)}_{\text{uses the CDF}}$

$$P(X \ge 2) = \underbrace{\int_2^5 \underbrace{1/5}_{\text{density}}\, \underbrace{dx}_{\text{width}}}_{\text{mass}} = \frac{1}{5}x\Big|_2^5 = 1 - \frac{2}{5} = 3/5$$

Cumulative Dist. Fxn (cdf)



$P(X \le x) = \int_{-\infty}^{x} f(y)\,dy = \int_{-\infty}^{0} 0\,dy + \int_{0}^{x} \frac{1}{5}\,dy$

$= \frac{1}{5}x = F(x)$

$1 - F(2) = 1 - \frac{1}{5}(2) = 1 - \frac{2}{5} = \frac{3}{5}$ ✓

Expected Value

$E(X) = \int_{-\infty}^{\infty} \overbrace{x \cdot f(x)\,dx}^{p(x)} = \int_0^5 x \cdot \frac{1}{5}\,dx = \frac{1}{5}\frac{x^2}{2}\Big|_0^5 = \frac{5}{2}$

$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \int_0^5 x^2 \cdot \frac{1}{5}\,dx = \frac{1}{5}\frac{x^3}{3}\Big|_0^5 = \frac{25}{3}$

Variance

$V(X) = E(X^2) - [E(X)]^2 = \frac{25}{3} - \left(\frac{5}{2}\right)^2 = \frac{25}{12}$

Poisson Process

a stochastic process $\{N(t), t \ge 0\}$ is said to be a counting process if $N(t)$ represents the total # of events that occur up to time t

→ most important stochastic process having rate α, $\alpha > 0$ if:

can be verified by our knowledge of the problem

1) $N(0) = 0$ - the counting process begins $t=0$
2) the process has independent increments
3) total # of events in any interval of length t is Poisson Dist. with $\mu = \alpha t$

$P(N(t+s) - N(t) = n) = \frac{e^{-\alpha t}(\alpha t)^n}{n!}$

x - event occurs

|xx xx|
s s+t

$X \equiv$ the # of events that occur in the time length t interval t

N(t) number of events

$X \sim \text{Poisson}(\alpha t)$

↳ only parameter, rate·time = mean

$P(X = x) = \frac{e^{-\alpha t}(\alpha t)^x}{x!}$

X	0	1	2	...
$p(x) = P(X=x)$	$\frac{e^{-\alpha t}(\alpha t)^0}{0!}$	$\frac{e^{-\alpha t}(\alpha t)^1}{1!}$	$\frac{e^{-\alpha t}(\alpha t)^2}{2!}$	

EX. customers arrive at ice cream parlor, rate $\alpha = 2/hr$

a) prob. no one shows b/t 3 and 4 PM? give pmf

$X \equiv$ # of customers that visit parlor bt 2PM + 3PM

$X \sim$ Poisson$(2 \cdot 1)$

$P(X=0) = \frac{e^{-2}(2)^0}{0!} = e^{-2} = 0.135335$

2 — 3, $\underbrace{\quad\quad}_{t = 1\,hr}$

dpois(0,2·1)

N(t) expect 2/hr

- pmf for many possibilities SEE R

b) starting at noon what is the expected time of the 6th customer?

noon ——— ?, $\underbrace{\quad\quad}_{t = 3\,hrs}$

$2 \cdot t = 6 \quad t = 3$

c) prob 2 or more visit b/t 6PM and 8PM? pmf and cdf

6 ——— 8

$X \equiv$ # of customers b/t 6 and 8 ($t = 2hrs$)

$X \sim$ Pois$(2 \cdot 2)$

Note: parameter $\mu = 2 \cdot 2$ changed from last problem

$P(X \geq 2) = 1 - P(X \leq 1) = 1 - [P(X=0) + P(X=1)]$

$1 - \left[\frac{4^0 e^{-4}}{0!} + \frac{4^1 e^{-4}}{1!}\right] = 1 - 5e^{-4} = 0.908422$

SEE R

df of px and Fx

what is the time between occurrences?

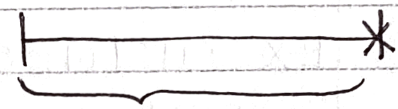
- Assume Poisson process, rate $= \lambda$, for counting process

- $W \equiv$ time until next event, $F(w) = P(W \leq w) = 1 - P(W \geq w)$ — when no one comes in

$1 - P(W \geq w)$ if we are assuming the next event, then no event has occurred up to that time w

Define another r.v.

$X \equiv$ the # of events in time w



t = length $w = w$

$P(X = 0) \qquad X \sim Pois(\lambda w)$

$\hookrightarrow \dfrac{(\lambda w)^0 e^{-\lambda w}}{0!} = e^{-\lambda w} = \boxed{P(W \geq w)}$

$F(w) = 1 - e^{-\lambda w}$ This is the cdf for an exponential with a rate of λ Exponential (λ)

Exponential Distn. (λ)

$X \sim exp(\lambda)$

cdf $\quad F(x) = 1 - e^{-\lambda x}$ $\quad \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

pdf $\quad f(x) = F'(x) = \lambda e^{-\lambda x}$ $\quad \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{ow} \end{cases}$

$E(X) = \int_0^\infty x \cdot \lambda e^{-\lambda x} = \text{I.B.P} = 1/\lambda$

$V(X) = 1/\lambda^2 \qquad \sigma = 1/\lambda$

- $E(X)$ is the same as S.D. for Exponential

EX: Burger King, Poisson, rate 400/day

- prob. no arrivals next 5 min?

Exponential Dist.

$X \equiv$ the time of the next arrival

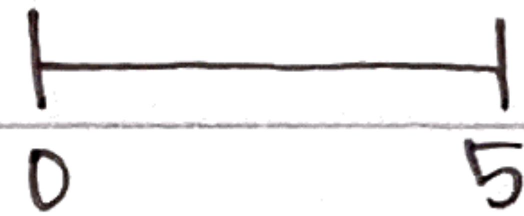
$X \sim exp\left(\dfrac{400 \text{ cust}}{\text{day}} \cdot \dfrac{1}{24\text{hr}} \cdot \dfrac{1\text{hr}}{60\text{min}}\right) \sim exp(0.2778)$ cust/min

$P(X \geq 5) = 1 - P(X \leq 5) = 1 - \left[1 - e^{-.2778 \cdot 5}\right] = 0.249$

5 min = 0.003472 days

Poisson Process

$X \equiv$ # of arrivals in next 5 mins



$X \sim \text{Poisson}(400 \cdot t)$

$X \sim \text{Poisson}(400 \cdot 0.003472)$

$X \sim \text{Poisson}(1.389)$

$$P(X=0) = \frac{(1.389)^0 e^{-1.389}}{0!} = e^{-1.389} = 0.249$$

- Normal Dist. $X \sim N(\mu, \sigma)$

$\mu = E(X)$

$\sigma = \sqrt{V(X)}$