Fall 2003, Semester research;

Development of Numerical Solver using Spectral and Fourier Method

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Aug 28th, 2003

Abstract

This document is for specifying details of Spectral methods and Fourier method.

Contents

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1 A Steady-State Diffusion Problem

According to a report[1]

1.1 Diffusion Problem

1.2 Galerkin Method with Weak Solution

1.3 Boundary Conditions

2 Spectral Polynomial Methods in 1 Dimensional Space

Our problem is the Poisson equation

$$L(u) \equiv \nabla^2 u - f = 0, \tag{1}$$

for all $x \in \Omega$. In one dimensional case, 1 is written as

$$L(u)(x) \equiv \frac{d^2}{dx^2} u(x) - f(x) = 0, \tag{2}$$

for all x in $[a, b]$.

We solve this equation in weak sense. That is to say, we define a functional $(\cdot) : C^0 \to \Re$ such that

$$\left(\nabla^2 u, \nu\right) = \int_\Omega \frac{d^2}{dx^2} u(x) \cdot \nu(x) d\mu(x), \tag{3}$$

for each ν in C^0, where C^0 is a set of continuous functions. Then we find the solution u by solving the equation as follows:

$$\left(\nabla^2 u, \nu\right) = (f, \nu), \tag{4}$$

for each ν in C^0.

2.1 Basis Functions

The spectral approximation of solution u is generally represented as

$$u(x) = \sum_{i=0}^{N_{dof-1}} \hat{u}_i(x) \Psi_i(x) \tag{5}$$

on $[a, b]$. To construct this the global basis functions $\{\Psi_i(x)\}_{i=0}^{N_{dof-1}}$, each Ψ is represented by the linear combination of local basis functions ψ_i on each element in $[a, b]$, say Ω^e.

We define a basis functions ψ_i on Ω^{st} to be a real valued function with the Legendre polynomial $\{P_i^{1,1}\}$ as follows:

$$\psi_i(\xi) = \{ \begin{array}{ll} \frac{1-\xi}{2}, & i = 0 \\ \frac{1+\xi}{2}, & i = 1 \\ \left(\frac{1-\xi}{2}\right)\left(\frac{1+\xi}{2}\right) P_{i-2}^{1,1}(\xi), & i \geq 2 \end{array} \tag{6}$$

for all ξ in $[-1, 1]$.

Then on a single standard element Ω^{st}, the approximation $u(\xi)$ is represented as

$$u(\xi) = \sum_{i=0}^{N^e} \hat{u}_i^e \psi_i(\xi), \tag{7}$$

for ξ in Ω^{st}.

2.2 Spectral Polynomial Method in A Element

We apply the basis representation 7 to weak formulation 4 with the same test function $\{\psi_q\}$, then we obtain the following:

$$\sum_{p=0}^{N^e} \hat{u}_p^e \left(\nabla^2\psi_p, \psi_q\right) = \left(\nabla^2 \sum_{p=0}^{N^e} u_p^e \psi_p, \psi_q\right) = (f, \psi_q) \tag{8}$$

for $q = 0, \cdots, N^e$.

With this relation we can setup a system of linear equations for the coefficient $\{\hat{u}_p^e\}_{p=0}^{N^e}$ with $N^e + 1 \times N^e + 1$ matrix $\mathbf{L}_{N^e}$ defined as follows:

$$\mathbf{L}_{N^e} \cdot \hat{\mathbf{u}} = \mathbf{f}, \tag{9}$$

where

$$\mathbf{L}_{N^e}(p, q) = \int_{\Omega^e} \frac{d^2}{d\xi^2} \psi_p(\xi) \psi_q(\xi) d\xi, \tag{10}$$

$$\hat{\mathbf{u}} = \left[\hat{u}_p\right]_{p=0}^{N^e}, \tag{11}$$

$$\mathbf{f} = \left[\int_{\Omega^e} f(\xi) \psi_q(\xi) d\xi\right]_{q=0}^{N^e}. \tag{12}$$

2.3 H/P Refinement using Global Assembly

3 Results

In section 1, 2, we present the result of convergence in both h refinement and p refinement with the following steady-state Poisson differential equation:

$$\frac{d^2}{dx^2}u(x) = \sin(\pi x),$$

for all x in $[0, 1]$.

3.1 H-Convergence of 1-D Spectral Method

This test is to validate the relation between size of element and the accuracy of approximation. We apply equidistance element and investigate the movement of error scale. As shown in Figure 1 and 2, the smaller are the elements, the more exact is the solution. Moreover by testing with different order of basis, we also could see the fact that the higher are orders, the faster do they converge.

- Dirichlet-Dirichlet Case

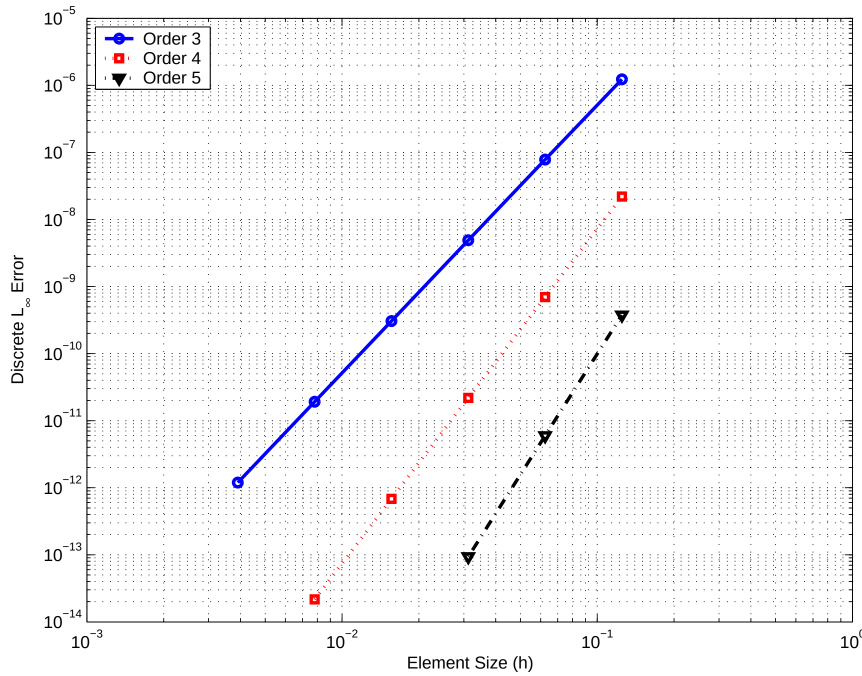

Figure 1: Graph showing change of errors by the increase of the number of elements: Dirichlet-Dirichlet

Table 1: Specification of Figure1 and their errors

Polynomial order	Error	Slope
3	$1.1940e-012$	3.9946
4	$2.2204e-015$	4.9875
5	$2.4147e-015$	5.9839

- Dirichlet-Neumann Case

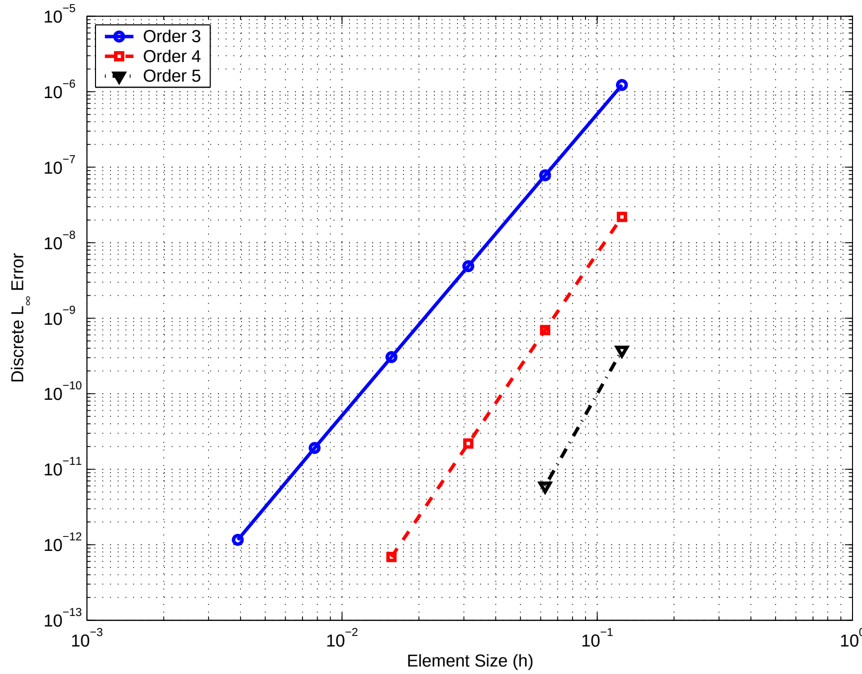

Figure 2: Graph showing change of errors by the increase of the number of elements: Dirichlet-Dirichlet

Table 2: Specification of Figure2 and their errors

Polynomial order	Error	Slope
3	$1.1620e-012$	4.0024
4	$4.6629e-014$	4.9877
5	$9.7367e-014$	5.9775

3.2 P-Convergence of 1-D Spectral Method

- Dirichlet-Dirichlet Case

Table 3: Specification of Figure3 and their errors

Element Size	Error
0.2	$7.7716e-016$
0.1	$1.1796e-016$

- Dirichlet-Neumann Case

3.3 Approximation of High order Polynomial solving 1D Poisson Equation

In this section we construct a polynomial P_n of order n defined on $[0,1]$, which satisfies the following.

$$P_n(0) = 0, \quad P_n(1) = 1$$
$$\frac{d^k}{dx^k}P_n(0) = 0, \quad \frac{d^k}{dx^k}P_n(1) = 0$$

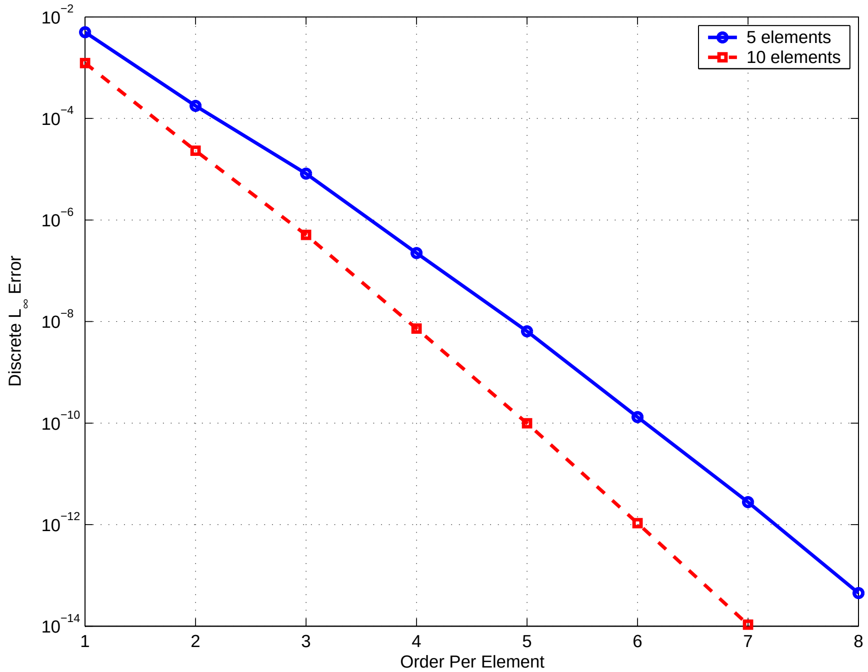

Figure 3: Graph showing change of errors by the increase of the number of elements: Dirichlet-Dirichlet

Table 4: Specification of Figure4 and their errors

Element Size	Error
0.2	$8.3267e-016$
0.1	$6.6613e-016$

for all $k = 1, \cdots, n-2$.
Then for each n, we obtain a polynomial P_n by solving a system of linear equations having unique solution which determines the set of coefficients of P_n. We will apply the spectral polynomial solver to approximate the second derivative Q_{n-2} of P_n.

3.3.1 Polynomial Representation of Approximation using Legendre Basis

Figure **??** is showing some samples of solution of order(n) 9.

Problem 3.1 *Consider the following differential equation for $u(x)$ such that*

$$\frac{d^2}{dx^2}u(x) = Q_{n-2},$$

for all x in $[0,1]$. Then the problem is to find approximation $p(x)$ of $u(x)$ using spectral polynomial method.

3.3.2 Existence of Approximate Solution

The Figure **??** and **??** are showing the result of spectral polynomial method approximating the solution of Problem 3.1. The maximum error value in Table **??** is showing the approximation is within numerically exact solution tolerance.

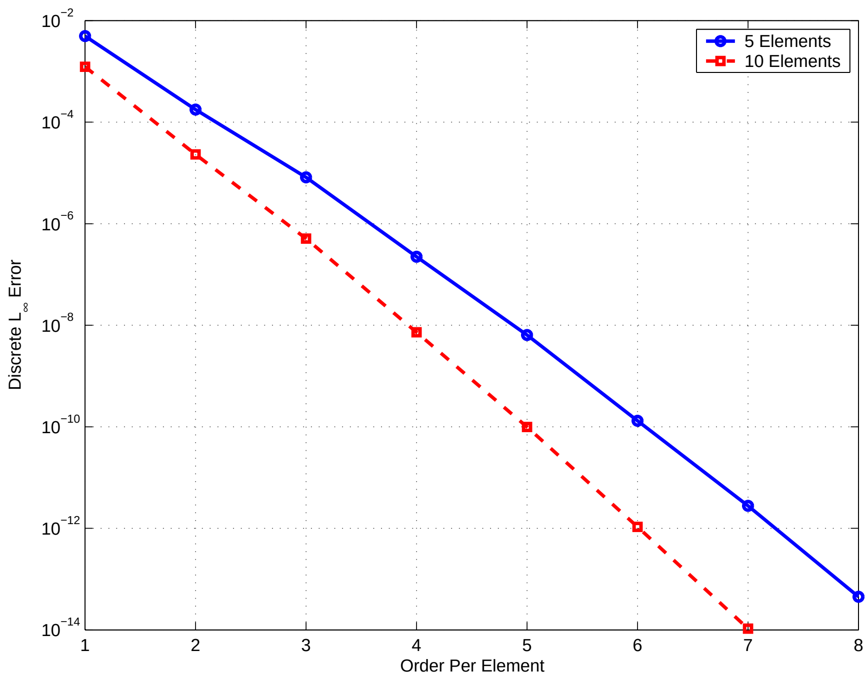

Figure 4: Graph showing change of errors by the increase of the number of elements: Dirichlet-Neumann

3.3.3 Convergence of Solution in H-test

- Dirichlet-Dirichlet Case

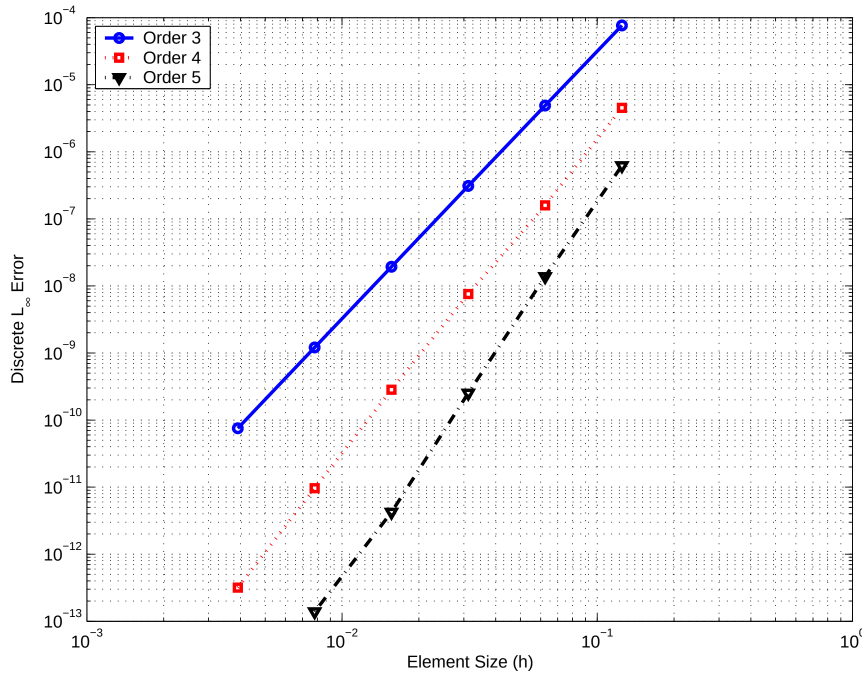

Figure 5: Graph showing change of errors by the increase of the number of elements: Dirichlet-Dirichlet

Table 5: Specification of Figure6 and their errors

Polynomial order	Error	Slope
3	$7.5505e-011$	3.9909
4	$3.1679e-013$	4.7538
5	$9.4591e-014$	5.5248

- Dirichlet-Neumann Case

Table 6: Specification of Figure6 and their errors

Polynomial order	Error	Slope
3	$7.5530e-011$	3.9908
4	$4.5619e-013$	4.6486
5	$4.1855e-013$	5.7218

3.3.4 Test of Convergence of Solution in Variable Ordered Elements

- Dirichlet-Dirichlet Case

- Dirichlet-Neumann Case

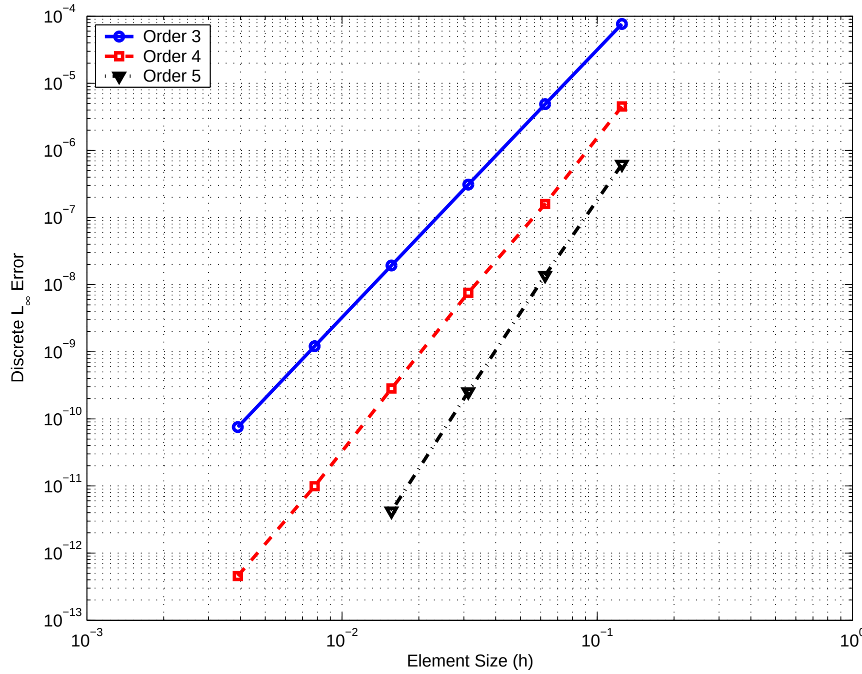

Figure 6: Graph showing change of errors by the increase of the number of elements: Dirichlet-Neumann

Table 7: Specification of Figure8 and their errors

Number of Elements	Error
5	$5.5067e-014$
10	$7.9936e-014$

References

[1] **Spectral/hp Element Method** Karniadarkis, Wat. Res., 24, 97-101. S.A. 1990.

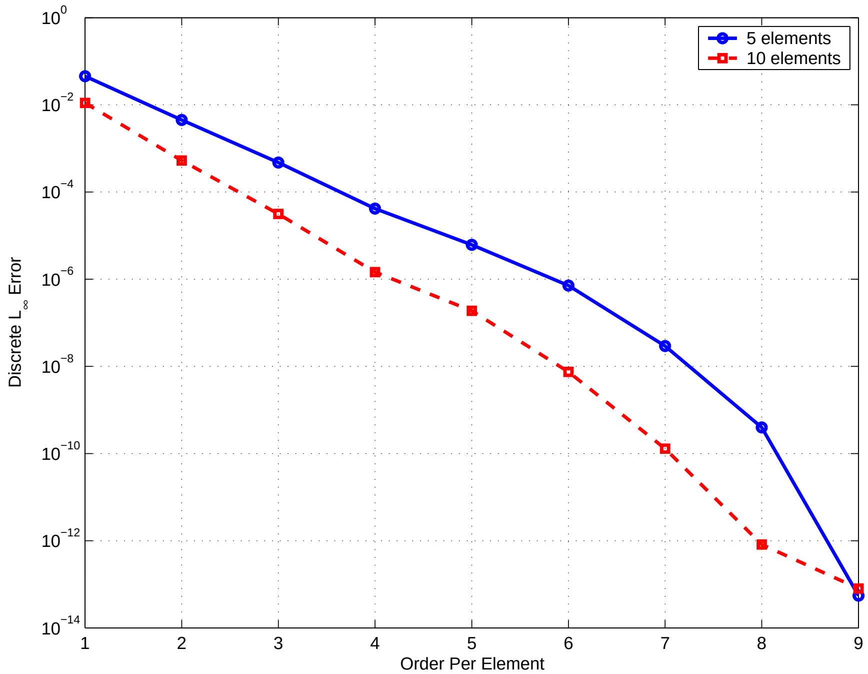

Figure 7: Graph showing change of errors by the increase of the number of elements: Dirichlet-Dirichlet

Table 8: Specification of Figure8 and their errors

Number of Elements	Error
5	$3.0431e-013$
10	$3.1186e-013$

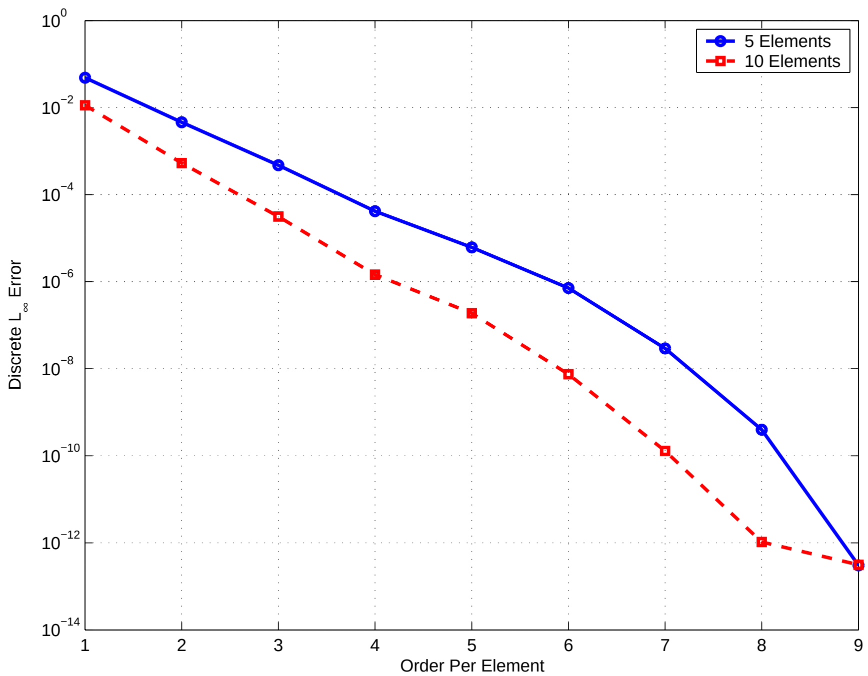

Figure 8: Graph showing change of errors by the increase of the number of elements: Dirichlet-Neumann