design used is 25% over-designed compared with that obtained by method I (areas are intermingled using a pseudorandom selector). Linear search termination criterion is 10^{-6}, band tightness is 10^{-4}, convergence criteron is 10^{-4} and forward finite difference parameter is 10^{-4}.

(3) If the optimization iterations are initiated from the design obtained by method I, the OPT requires 561 reanalyses and 208 function evaluations. In brief, the reduction in numerical computations is *ca* 66% compared with the earlier initial design ($= 1.25\times$ design by method I).

Which of the two procedures (method I or method II) should be preferred? The recommendation is to prefer method I, even though method II typically provides a solution closer to the optimum by a few percentage points. Method I is preferred not only because LP requires more computation but also because method II has a tendency to alter the configuration of the given structure (see Fig. 1). If any sizing variable has zero value, the initial design can become infeasible. Initiation of optimization from an infeasible design is not recommended as too many computations may be required to generate a feasible design.

6. CONCLUSION

The integrated force method as an analytical tool can enhance optimum structural design because of the following four factors: (1) it provides a good initial design, (2) reanalysis is inexpensive, but accurate, (3) gradients of behaviour constraints can be calculated in closed form, and (4) last but not least, IFM ensures robustness in structural optimization by avoiding the singularity condition.

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ON THE IMPLEMENTATION OF THE GALERKIN APPROACH IN THE BOUNDARY ELEMENT METHOD

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Abstract—In the last few years there has been an increasing interest in the Galerkin approach to the direct boundary element method (BEM). However, unlike standard point collocation, the Galerkin procedure requires double integrals on $\Gamma \times \Gamma$. This paper deals with the numerical evaluation of double singular integrals arising in the Galerkin BEM for plane problems. Some differences in the type of singularities encountered in the implementation of the collocation and Galerkin schemes are pointed out. They result in some integration procedures that are peculiar to the Galerkin approach. Numerical results on plane elastostatics problems are presented.

1. INTRODUCTION

Until recently, the direct boundary element method (BEM) was usually implemented by using nodal collocation procedures. However, in the last few years increasing attention has been given to the Galerkin formulation of BEM whose asymptotic convergence to the exact solution has been proved by Wendland and his co-workers [1].

The Galerkin approach seems to be promising for the development of self-adaptative codes based on *a posteriori* error indicators and estimators [2–4], and more recently, for devising symmetric alternative formulation of the direct BEM [5–7], which should result in great advantages in elasto-plastic problems [8, 9].

A Galerkin formulation of the indirect BEM for electrical engineering problems was presented by Lean *et al.* [10]. Some details on the numerical integration of singular functions were published in a subsequent paper [11]. The analysis of two-dimensional elastostatic problems by the Galerkin direct BEM was considered in [3], where double integrations were performed using a semi-analytical method.

The aim of the present paper is to present a general comprehensive scheme for the double numerical integration of the singular functions which arise in the Galerkin formulations of the direct BEM for plane problems. Unlike the collocation procedure, the Galerkin approach requires the evaluation of double integrals, and the integrand functions can be singular along a line of their domain of definition. As a typical example for a broad class of engineering problems, the elastostatic case will be considered. However, what is relevant is only the order of singularity of the kernel functions involved [12].

In Sec. 2, a brief description is given of the direct BEM together with the discretization procedure. In Sec. 3, the system of algebraic equations is obtained by the application of a Galerkin approach. Section 4 deals with the problem of the evaluation of strongly singular integrals which have to be defined in the Cauchy principal value (CPV) sense. Weakly singular integrals are considered in Sec. 5. The two cases of 'coincident' and 'adjacent' elements are analysed separately. Section 6 reports on numerical results obtained to illustrate the application of the proposed procedures. Concluding remarks are presented in the last section.

2. DISCRETIZED BOUNDARY INTEGRAL EQUATION

The direct version of the BEM for elastostatic problems is based on the following boundary integral equation [12, 13]:

$$c_{ij}(z)u_j(z) + \int_\Gamma t^*_{ij}(z,x)u_j(x)\,\mathrm{d}\Gamma(x) = \int_\Gamma u^*_{ij}(z,x)t_j(x)\,\mathrm{d}\Gamma(x), \quad (1)$$

where, for the sake of brevity, no body force terms are considered. In this equation, $t^*_{ij}(z,x)$ and $u^*_{ij}(z,x)$ represent the well-known fundamental solutions of Kelvin's problem, z being the load (or source) point and x the field point, both lying on the boundary Γ of an elastic domain of dimensionality N_d ($N_d = 2$ or 3); $u_j(x)$ and $t_j(x)$ ($j = 1, \ldots, N_d$) denote the displacement and traction boundary fields, respectively. The so-called free-term coefficients $c_{ij}(z)$ depend only on the local geometry of Γ in z [14]. If Γ is smooth in z, $c_{ij}(z) = 1/2\delta_{ij}$. As is well known, the kernel functions for plane problems ($N_d = 2$) have the following order of singularity when x approaches z: $u^*_{ij}(z,x) = O(\log r)$ and $t^*_{ij}(z,x) = O(1/r)$, where $r = |x - z|$. Thus, the first integral in (1) is denoted by $\fint$ because it must be understood in the Cauchy principal value (CPV) sense.

The numerical solution of the problem can be obtained by dividing the boundary into N_e elements, with

$$\Gamma = \bigcup_{n=1}^{N_e} \Gamma_n \quad \text{and} \quad \Gamma_n \cap \Gamma_m = \varnothing \quad (m \neq n). \tag{2}$$

On each boundary element Γ_n, the displacement and the traction components in the i coordinate direction are approximated by N^n_i degrees of freedom:

$$u_i(x) = \sum_{k=0}^{N^n_i - 1} \varphi^{kn}(x) u^{kn}_i \quad (x \in \Gamma_n)$$

$$t_i(x) = \sum_{k=0}^{N^n_i - 1} \varphi^{kn}(x) t^{kn}_i \quad (x \in \Gamma_n). \tag{3}$$

In these equations, $\varphi^{kn}(x)$ is the interpolation function associated with the kth d.o.f. on Γ_n. If constant, linear, parabolic, etc., approximations for the unknowns are to be considered, then $N^n_i = 1, 2, 3, \ldots$ etc. Thus, the total number of d.o.f. of the discrete model is

$$N = \sum_{n=1}^{N_e} \sum_{i=1}^{N_d} N^n_i. \tag{4}$$

Substituting the discretization defined in (3) in the boundary integral equation (1), we obtain

$$r_i(z) = \sum_{l=0}^{N^m_j - 1} \sum_{j=1}^{N_d} \left[c_{ij}(z) \varphi^{ln}(z) + \fint_{\Gamma_n} t^*_{ij}(z,x) \varphi^{ln}(x)\, d\Gamma(x) \right] u^{ln}_j$$
$$+ \sum_{\substack{m=1\\(m\neq n)}}^{N_e} \sum_{l=0}^{N^m_j - 1} \sum_{j=1}^{N_d} \left[\int_{\Gamma_m} t^*_{ij}(z,x) \varphi^{lm}(x)\, d\Gamma(x) \right] u^{lm}_j$$
$$- \sum_{m=1}^{N_e} \sum_{l=0}^{N^m_j - 1} \sum_{j=1}^{N_d} \left[\int_{\Gamma_m} u^*_{ij}(z,x) \varphi^{lm}(x)\, d\Gamma(x) \right] t^{lm}_j \quad z \in \Gamma_n, \tag{5}$$

where $r_i(z)$ denotes the residual function, which can be proved to be related with the interpolation error of the boundary unknowns [3, 4].

3. GALERKIN APPROACH

To determine the values of the N d.o.f. of the discrete model, a system of N simultaneous equations is necessary, which can be obtained by a Galerkin approach. In this case the residual function defined in (5) is weighted on each Γ_n using the same interpolation functions $\varphi^{kn}(x)$ as in (3) by

$$\int_{\Gamma_n} \varphi^{kn}(z) r_i(z)\, d\Gamma(z) = 0, \tag{6}$$

with $i = 1, \ldots, N_d$; $k = 0, \ldots, (N^n_i - 1)$ and $n = 1, \ldots, N_e$.

Bearing in mind that when using the Galerkin approach the source point z always lies strictly within the element Γ_n (i.e. smooth boundary), the substitution of eqn (5) in (6) yields

$$\sum_{l=0}^{N^m_j - 1} \sum_{j=1}^{N_d} \left[\int_{\Gamma_n} \varphi^{kn}(z) \tfrac{1}{2}\delta_{ij} \varphi^{ln}(z)\, d\Gamma(z) \right.$$
$$\left. + \int_{\Gamma_n} \varphi^{kn}(z) \fint_{\Gamma_n} t^*_{ij}(z,x) \varphi^{ln}(x)\, d\Gamma(x)\, d\Gamma(z) \right] u^{ln}_j$$
$$+ \sum_{\substack{m=1\\(m\neq n)}}^{N_e} \sum_{l=0}^{N^m_j - 1} \sum_{j=1}^{N_d} \left[\int_{\Gamma_n} \varphi^{kn}(z) \int_{\Gamma_m} t^*_{ij}(z,x) \varphi^{lm}(x)\, d\Gamma(x)\, d\Gamma(z) \right] u^{lm}_j$$
$$= \sum_{m=1}^{N_e} \sum_{l=0}^{N^m_j - 1} \sum_{j=1}^{N_d} \left[\int_{\Gamma_n} \varphi^{kn}(z) \int_{\Gamma_m} u^*_{ij}(z,x) \varphi^{lm}(x)\, d\Gamma(x)\, d\Gamma(z) \right] t^{lm}_j, \tag{7}$$

which can be rewritten in a more compact form by

$$H^{knlm}_{ij} u^{lm}_j = G^{knlm}_{ij} t^{lm}_j, \tag{8}$$

where

$$G^{knlm}_{ij} = \int_{\Gamma_n} \varphi^{kn}(z) \int_{\Gamma_m} u^*_{ij}(z,x) \varphi^{lm}(x)\, d\Gamma(x)\, d\Gamma(z), \tag{9}$$

while

$$H^{knlm}_{ij} = \int_{\Gamma_n} \varphi^{kn}(z) \int_{\Gamma_m} t^*_{ij}(z,x) \varphi^{lm}(x)\, d\Gamma(x)\, d\Gamma(z), \tag{10}$$

if $n \neq m$, and

$$H^{knln}_{ij} = \tfrac{1}{2}\delta_{ij} \int_{\Gamma_n} \varphi^{kn}(z) \varphi^{ln}(z)\, d\Gamma(z)$$
$$+ \int_{\Gamma_n} \varphi^{kn}(z) \fint_{\Gamma_n} t^*_{ij}(z,x) \varphi^{ln}(x)\, d\Gamma(x)\, d\Gamma(z) \tag{11}$$

if $n = m$.

After enforcing the prescribed boundary conditions, eqns (8) can be rearranged to obtain the final system of equations [12] in its standard form

$$Ay = b. \tag{12}$$

The aim of this paper is to present a comprehensive numerical scheme for the evaluation of double integrals defined in (9)–(11) when referred to plane problems. Emphasis will be placed only on the integration of singular functions because the regular case does not present any particular problems.

4. STRONGLY SINGULAR INTEGRALS

In this section the evaluation of the integrals defined by expression (11) will be considered. The first, due to the integration of the free term, does not present any special problem because it is a one-dimensional integral of a smooth function. Note that the free-term in (11) is always $1/2\delta_{ij}$ even if Γ_n has a corner at one (or both) end point(s). The second term is a double integral in which the inner term must be defined in the CPV sense.

As has been shown in [15] and [16] the CPV integrals arising in the collocation BEM can be conveniently evaluated in a direct manner by means of Gauss–Legendre quadrature formulae [17].

Owing to the definition of CPV, the integral considered can be conveniently rewritten in the following form (Fig. 1)

$$\int_{\Gamma_n} \varphi^{kn}(z) \left[\fint_{\Gamma_n} t^*_{ij}(z,x) \varphi^{ln}(x)\, d\Gamma(x) \right] d\Gamma(z)$$
$$= \int_{s_i}^{s_f} \varphi^{kn}(z) \left\{ \lim_{\epsilon \to 0^+} \left[\int_{s_i}^{s_z - \epsilon} t^*_{ij}(z,x) \varphi^{ln}(x)\, ds_x \right.\right.$$
$$\left.\left. + \int_{s_z + \epsilon}^{s_f} t^*_{ij}(z,x) \varphi^{ln}(x)\, ds_x \right] \right\} ds_z, \tag{13}$$

where s is the curvilinear abscissa defined on the contour Γ.

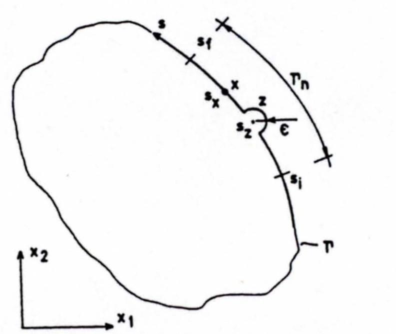

Fig. 1. Curvilinear abscissa on the boundary and definition of CPV integral.

To approximate the geometry of the boundary on Γ_n, we can introduce, as usual in the BEM, a parametric representation of an intrinsic variable for both x and z, thus obtaining $x = x(\xi)$, $z = z(\eta)$, with $\xi, \eta \in [-1, +1]$, and $ds_x = J(\xi)\, d\xi$, $ds_z = J(\eta)\, d\eta$. Hence, expression (13) becomes

$$\int_{-1}^{1} \varphi^{kn}[z(\eta)] \left(\lim_{\delta\xi \to 0^+} \left\{ \int_{-1}^{\eta - \delta\xi} t^*_{ij}[z(\eta), x(\xi)] \right.\right.$$
$$\times \varphi^{ln}[x(\xi)] J(\xi)\, d\xi + \int_{\eta + \delta\xi}^{1} t^*_{ij}[z(\eta), x(\xi)]$$
$$\left.\left. \times \varphi^{ln}[x(\xi)] J(\xi)\, d\xi \right\} \right) J(\eta)\, d\eta, \tag{14}$$

where, by expanding ϵ in its Taylor series,

$$\epsilon = J(\eta)\delta\xi + O(\delta\xi^2) \tag{15}$$

and therefore

$$\delta\xi = O(\epsilon). \tag{16}$$

Let

$$\frac{h(\eta, \xi)}{\xi - \eta} = \varphi^{kn}[z(\eta)] t^*_{ij}[z(\eta), x(\xi)] \varphi^{ln}[x(\xi)] J(\xi) J(\eta), \tag{17}$$

where it can be shown that $h(\eta, \xi)$ is a continuous and regular function. For brevity, in the definition of this function all indices have been dropped. Expression (14) can now be written in a more compact and useful form:

$$\int_{-1}^{1} \left\{ \lim_{\delta\xi \to 0^+} \left[\int_{-1}^{\eta - \delta\xi} \frac{h(\eta, \xi)}{\xi - \eta}\, d\xi \right.\right.$$
$$\left.\left. + \int_{\eta + \delta\xi}^{1} \frac{h(\eta, \xi)}{\xi - \eta}\, d\xi \right] \right\} d\eta. \tag{18}$$

Adding and subtracting $h(\eta, \eta)/(\xi - \eta)$ we obtain

$$\int_{-1}^{1} \left\{ \lim_{\delta\xi \to 0^+} \left[\int_{-1}^{\eta - \delta\xi} \frac{h(\eta, \xi) - h(\eta, \eta)}{\xi - \eta}\, d\xi \right.\right.$$
$$+ \int_{\eta + \delta\xi}^{1} \frac{h(\eta, \xi) - h(\eta, \eta)}{\xi - \eta}\, d\xi$$
$$\left.\left. + h(\eta, \eta) \left(\int_{-1}^{\eta - \delta\xi} \frac{1}{\xi - \eta}\, d\xi + \int_{\eta + \delta\xi}^{1} \frac{1}{\xi - \eta}\, d\xi \right) \right] \right\}$$
$$= \int_{-1}^{1} \int_{-1}^{1} \frac{h(\eta, \xi) - h(\eta, \eta)}{\xi - \eta}\, d\xi\, d\eta$$
$$+ \int_{-1}^{1} h(\eta, \eta) \left[\lim_{\delta\xi \to 0^+} (\log|\delta\xi| - \log|-1 - \eta| \right.$$
$$\left. + \log|1 - \eta| - \log|\delta\xi|) \right] d\eta, \tag{19}$$

and finally

$$\int_{\Gamma_n} \varphi^{kn}(z)\left[\int_{\Gamma_n} t^*_{ij}(z,x)\varphi^{ln}(x)\,\mathrm{d}\Gamma(x)\right]\mathrm{d}\Gamma(z)$$

$$= \int_{-1}^{1}\int_{-1}^{1} \frac{h(\eta,\xi)-h(\eta,\eta)}{\xi-\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta$$

$$+ \int_{-1}^{1} h(\eta,\eta)[\log(1-\eta)-\log(1+\eta)]\,\mathrm{d}\eta. \tag{20}$$

$$h(\eta,\eta) = \begin{cases} 0 & \text{if } i=j \\ -\dfrac{1-2\nu}{4\pi(1-\nu)}\varphi^{kn}[z(\eta)]\varphi^{ln}[x(\eta)]J(\eta) & \text{if } i=1\wedge j=2 \\ +\dfrac{1-2\nu}{4\pi(1-\eta)}\varphi^{kn}[z(\eta)]\varphi^{ln}[x(\eta)]J(\eta) & \text{if } i=2\wedge j=1. \end{cases} \tag{24}$$

Thus, the evaluation of the original double integral with a CPV has been reduced to the computation of a double integral of a regular function, plus a one-dimensional integral of a weak singular function.

The numerical evaluation of the regular integral can be obtained by a standard application of product one-dimensional Gauss–Legendre formulae:

$$\int_{-1}^{1}\int_{-1}^{1} \frac{h(\eta,\xi)-h(\eta,\eta)}{\xi-\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta$$

$$\simeq \sum_{i=1}^{m_1} w_i \sum_{j=1}^{m_2} \frac{h(\eta_i,\xi_j)}{\xi_j-\eta_i}w_j - \sum_{i=1}^{m_1} w_i h(\eta_i,\eta_i)$$

$$\times \sum_{j=1}^{m_2} \frac{1}{\xi_j-\eta_i}w_j, \quad (\xi_j \neq \eta_i), \tag{21}$$

in which w_i, w_j and ξ_j, η_i are referred to the Gaussian weights and points, respectively.

Letting

$$\eta_1 = \frac{1-\eta}{2} \quad \text{and} \quad \eta_2 = \frac{1+\eta}{2}, \tag{22}$$

the second integral can be computed as follows:

$$\int_{-1}^{1} h(\eta,\eta)[\log(1-\eta)-\log(1+\eta)]\,\mathrm{d}\eta$$

$$= \int_{-1}^{1} h(\eta,\eta)\left[\log\left(\frac{1-\eta}{2}\right)-\log\left(\frac{1+\eta}{2}\right)\right]\mathrm{d}\eta$$

$$= 2\left[\int_0^1 h(1-2\eta_1, 1-2\eta_1)\log\eta_1\,\mathrm{d}\eta_1\right.$$

$$\left.-\int_0^1 h(2\eta_2-1, 2\eta_2-1)\log\eta_2\,\mathrm{d}\eta_2\right]$$

$$\simeq 2\sum_{i=1}^{m_1}[h(2\bar\eta_i-1, 2\bar\eta_i-1)$$

$$-h(1-2\bar\eta_i, 1-2\bar\eta_i)]\bar w_i, \tag{23}$$

where $\bar w_i$ and $\bar\eta_i$ are referred to as the weights and points of logarithmic modified Gauss formulae, respectively.

We note [15, 16] that $h(\eta,\eta)$ has the following simple expressions, depending on the indices i and j of $t^*_{ij}(z,x)$:

An alternative scheme for the evaluation of these integrals was recently proposed in [18].

5. WEAKLY SINGULAR INTEGRALS

In addition to the strongly singular integrals of type (11), which have been considered in the previous section, other singular cases have to be dealt with. When $\Gamma_n \equiv \Gamma_m$ [Fig. 2, case (I)], integrals of type (9) have integrand functions with singularity along the diagonal $\xi=\eta$ in the $(-1,+1)\times(-1,+1)(\xi,\eta)$ square. Moreover, when the elements Γ_n and Γ_m are adjacent, i.e. have one common point [Fig. 2, cases (II) and (III)], both integrals of types (9) and (10) have integrand functions with a singularity at the points either $(\xi,\eta)=(+1,-1)$ or $(\xi,\eta)=(-1,+1)$.

5.1. *Singularity 'along the diagonal $\xi=\eta$' (coincident elements)*

As has been mentioned above, integrals of type (9) show a singularity along the line $z=x$ when $n=m$ and also $i=j$:

$$G^{knln}_{ii} = \int_{\Gamma_n} \varphi^{kn}(z)\int_{\Gamma_n} u^*_{ii}(z,x)\varphi^{ln}(x)\,\mathrm{d}\Gamma(x)\,\mathrm{d}\Gamma(z) \tag{25}$$

where

$$u^*_{ii}(z,x) = C[-\log r(z,x) + g_{ii}(z,x)] \tag{26}$$

with C a constant and $g_{ii}(z,x)$ a continuous and regular function [12].

After the introduction of the usual element-wise parametric representation of the geometry, let

$$f(\xi,\eta) = C\varphi^{kn}[z(\eta)]\varphi^{ln}[x(\xi)]J(\eta)J(\xi), \tag{27}$$

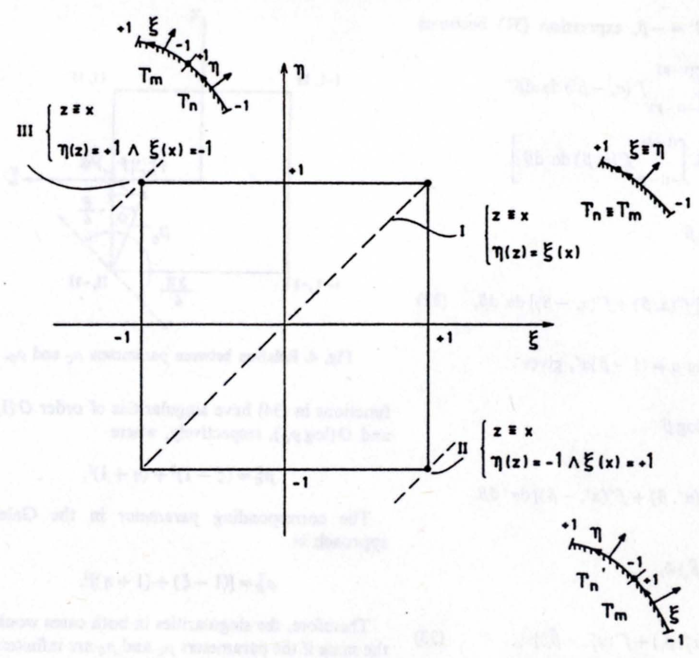

Fig. 2. Types of singularity: 'along a diagonal' (I) and 'at a vertex' (II and III).

then the singular part of (25) can be written as

$$-\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\log r[z(\eta),x(\xi)]\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{28}$$

which can be transformed into the sum of a regular and a singular part,

$$-\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\log\left\{\frac{2r[z(\eta),x(\xi)]}{|\xi-\eta|}\frac{|\xi-\eta|}{2}\right\}\mathrm{d}\xi\,\mathrm{d}\eta$$

$$= -\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\log\left\{\frac{2r[z(\eta),x(\xi)]}{|\xi-\eta|}\right\}\mathrm{d}\xi\,\mathrm{d}\eta$$

$$-\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\log\left(\frac{|\xi-\eta|}{2}\right)\mathrm{d}\xi\,\mathrm{d}\eta. \tag{29}$$

For the evaluation of the (weak) singular integral, it is convenient to introduce the following coordinate transformation [2] (Fig. 3)

$$\begin{cases} \alpha = \dfrac{\eta+\xi}{2} \\ \beta = \dfrac{\eta-\xi}{2} \end{cases} \quad \text{or} \quad \begin{cases} \xi = \alpha-\beta \\ \eta = \alpha+\beta \end{cases} \Rightarrow \mathrm{d}\xi\,\mathrm{d}\eta = 2\,\mathrm{d}\alpha\,\mathrm{d}\beta, \tag{30}$$

to obtain

$$-2\left[\int_{-1}^{0} \log|\beta|\int_{-(1+\beta)}^{(1+\beta)} f'(\alpha,\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta\right.$$

$$\left.+\int_0^1 \log\beta\int_{-(1-\beta)}^{(1-\beta)} f'(\alpha,\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta\right]. \tag{31}$$

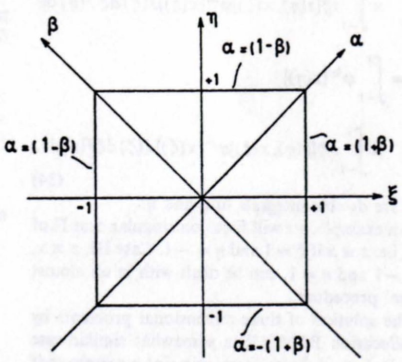

Fig. 3. (ξ,η) in (α,β) coordinate transformation.

Now, letting $\beta' = -\beta$, expression (31) becomes

$$-2\left[\int_0^1 \log\beta' \int_{-(1-\beta')}^{(1-\beta')} f'(\alpha, -\beta')\,d\alpha\,d\beta' + \int_0^1 \log\beta \int_{-(1-\beta)}^{(1-\beta)} f'(\alpha,\beta)\,d\alpha\,d\beta\right]$$

$$= -2\int_0^1 \log\beta \times \int_{-(1-\beta)}^{(1-\beta)} [f'(\alpha,\beta)+f'(\alpha,-\beta)]\,d\alpha\,d\beta, \quad (32)$$

which, considering $\alpha = (1-\beta)\alpha'$, gives

$$-2\int_0^1 (1-\beta)\log\beta \times \int_{-1}^{1} [f''(\alpha',\beta)+f''(\alpha',-\beta)]\,d\alpha'\,d\beta$$

$$\simeq 2\sum_{i=1}^{m_1}(1-\bar{\beta}_i)\bar{w}_i \times \sum_{j=1}^{m_2}[f''(\alpha_j',\bar{\beta}_i)+f''(\alpha_j',-\bar{\beta}_i)]w_j, \quad (33)$$

in which the numerical integration with respect to α' has been performed by Gauss–Legendre formulae, and the integration with respect to β has been done by Gauss type formulae with logarithmic weight.

5.2. *Singularity 'at a vertex' (adjacent elements)*

After the introduction of the element-wise parametric representation, integrals (9) and (10) become

$$H_{ij}^{knlm} = \int_{-1}^{1} \varphi^{kn}[z(\eta)] \times \int_{-1}^{1} t_{ij}^*[z(\eta), x(\xi)]\varphi^{lm}[x(\xi)]J(\xi)\,d\xi\,J(\eta)\,d\eta$$

$$G_{ij}^{knlm} = \int_{-1}^{1} \varphi^{kn}[z(\eta)] \times \int_{-1}^{1} u_{ij}^*[z(\eta), x(\xi)]\varphi^{lm}[x(\xi)]J(\xi)\,d\xi J(\eta)\,d\eta, \quad (34)$$

which are double integrals in ξ and η.

As an example, we will focus on singular case II of Fig. 2, i.e. $z \equiv x$ if $\xi = 1$ and $\eta = -1$. Case III, $z \equiv x$, if $\xi = -1$ and $\eta = 1$, can be dealt with in an almost identical procedure.

In the solution of three-dimensional problems by the collocation BEM [13], a somewhat similar case arises. In fact, if the collocation point z corresponds to point $(+1, -1)$ in the (ξ, η) square, the integrand

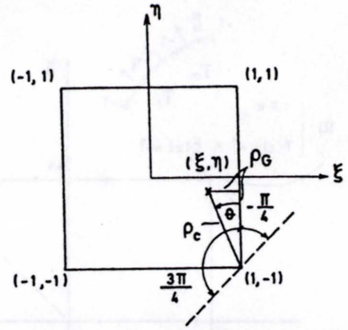

Fig. 4. Relation between parameters ρ_C and ρ_G.

functions in (34) have singularities of order $O(1/\rho_C)$ and $O(\log\rho_C)$, respectively, where

$$\rho_C^2 = (\xi-1)^2 + (\eta+1)^2. \quad (35)$$

The corresponding parameter in the Galerkin approach is

$$\rho_G^2 = [(1-\xi)+(1+\eta)]^2. \quad (36)$$

Therefore, the singularities in both cases would be the same if the parameters ρ_C and ρ_G are infinitesimal quantities of the same order as (ξ, η) approaches $(1, -1)$ (Fig. 4).

$$\lim_{(\xi,\eta)\to(1,-1)} \frac{\rho_C^2}{\rho_G^2} = \lim_{\rho_C\to 0} \frac{\rho_C^2}{(\rho_C\sin\theta + \rho_C\cos\theta)^2}$$

$$= \frac{1}{1+\sin(2\theta)}, \quad \theta\in[0,\pi/2]. \quad (37)$$

Note that ratio $\rho_C/\rho_G \to \infty$ only when $\theta = -\pi/4 + k\pi$ $(k = 0, 1, 2, \ldots)$, whereas, in the range of interest $0 \leq \theta \leq \pi/2$, $(\lim_{(\xi,\eta)\to(+1,-1)}\rho_C/\rho_G) \in [1/2, 1]$.

Therefore, we can basically use the methods previously developed for the collocation 3D BEM, such as the triangle-to-square transformation [13]. More precisely, the following double coordinate transformation can be considered (Fig. 5):

$$\begin{cases}\alpha = \dfrac{\eta+\xi}{2}\\ \beta = \dfrac{\eta-\xi}{2}\end{cases} \text{ or } \begin{cases}\xi = \alpha-\beta\\ \eta = \alpha+\beta\end{cases} \Rightarrow d\xi\,d\eta = 2\,d\alpha\,d\beta \quad (38)$$

and

$$\begin{cases}\delta = \dfrac{\alpha}{\beta+1}\\ \gamma = 2\beta+1\end{cases} \text{ or } \begin{cases}\alpha = \dfrac{\gamma+1}{2}\delta\\ \beta = \dfrac{\gamma-1}{2}\end{cases} \Rightarrow d\alpha\,d\beta = \frac{\gamma+1}{4}\,d\delta\,d\gamma. \quad (39)$$

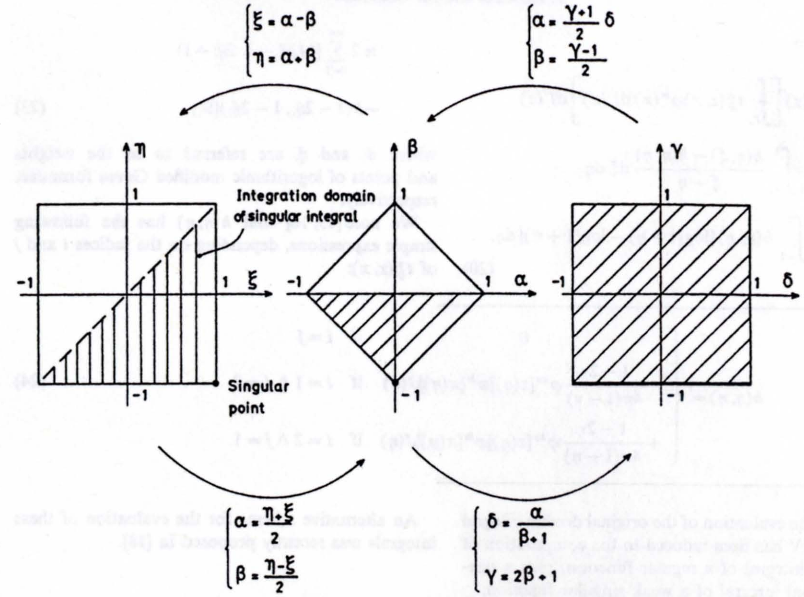

Fig. 5. Triangle-to-square coordinate transformation.

Let, for brevity,

$$T(\xi,\eta) = \varphi^{kn}[z(\eta)]t_{ij}^*[z(\eta), x(\xi)]\varphi^{lm}[x(\xi)]J(\eta)J(\xi) = O(1/\rho_G)$$

$$U(\xi,\eta) = \varphi^{kn}[z(\eta)]u_{ij}^*[z(\eta), x(\xi)]\varphi^{lm}[x(\xi)]J(\eta)J(\xi) = O(\log\rho_G), \quad (40)$$

then, integrals (34) become

$$\int_{-1}^{1}\int_{-1}^{1} T(\xi,\eta)\,d\xi\,d\eta$$

$$= 2\left[\int_{-1}^{0}\int_{-(1+\beta)}^{(1+\beta)} T'(\alpha,\beta)\,d\alpha\,d\beta + \int_0^1\int_{-(1-\beta)}^{(1-\beta)} T'(\alpha,\beta)\,d\alpha\,d\beta\right]. \quad (41)$$

On the right-hand side we have now a singular and a regular integral. The singular integral can be transformed into a regular one as well.

$$2\int_{-1}^{0}\int_{-(1+\beta)}^{(1+\beta)} T'(\alpha,\beta)\,d\alpha\,d\beta$$

$$= \frac{1}{2}\int_{-1}^{1}(\gamma+1)\int_{-1}^{1} T''(\delta,\gamma)\,d\delta\,d\gamma$$

$$\simeq \frac{1}{2}\sum_{i=1}^{m_1}(\gamma_i+1)w_i\sum_{j=1}^{m_2} T''(\delta_j,\gamma_i)w_j. \quad (42)$$

To obtain more accurate results for the quasi-singular integrals defined in eqn (42), the new coordinate transformation [19]

$$\gamma = \tfrac{1}{2}(\zeta+1)^2 - 1 \Rightarrow d\gamma = (\zeta+1)\,d\zeta \quad (43)$$

can also be used. Hence,

$$\frac{1}{2}\int_{-1}^{1}(\gamma+1)\int_{-1}^{1} T''(\delta,\gamma)\,d\delta\,d\gamma$$

$$= \frac{1}{4}\int_{-1}^{1}(\zeta+1)^3\int_{-1}^{1} T'''(\delta,\zeta)\,d\delta\,d\zeta$$

$$\simeq \frac{1}{4}\sum_{i=1}^{m_1}(\zeta_i+1)^3 w_i\sum_{j=1}^{m_2} T'''(\delta_j,\zeta_i)w_j. \quad (44)$$

Exactly the same procedure can be used for the integration of $U(\xi,\eta)$.

6. NUMERICAL EXAMPLES

To test the numerical integration procedure presented in this paper, the following plane stress problems with known closed form solutions have been considered.

6.1. *Square plate uniformly tractioned or distorted*

First, two plane problems of a square plate either uniformly tractioned or distorted are studied (Fig. 6). Young's modulus $E = 1$ and Poisson's ratio $\nu = 0$ are assumed.

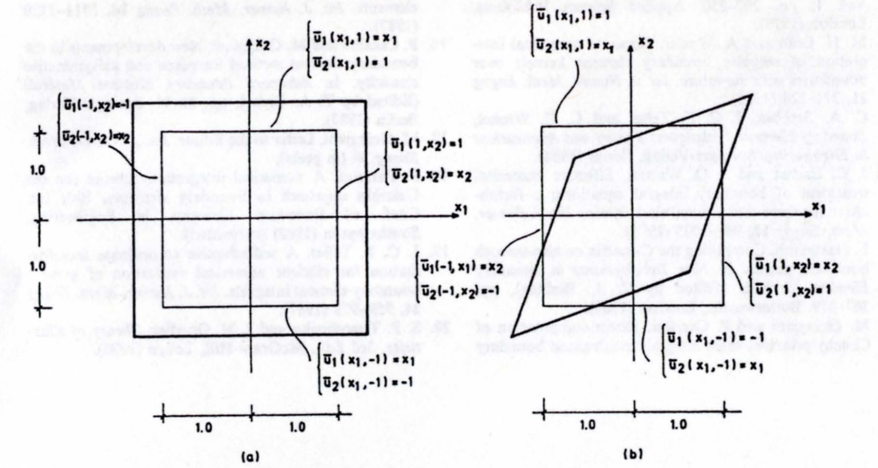

Fig. 6. Square plate uniformly tractioned (a) and distorted (b).

The displacement domain fields of the two problems are, respectively,

$$\begin{cases} u_1(x_1,x_2)=x_1 \\ u_2(x_1,x_2)=x_2 \end{cases} \quad \text{and} \quad \begin{cases} u_1(x_1,x_2)=x_2 \\ u_2(x_1,x_2)=x_1. \end{cases} \tag{45}$$

The exact solution for the traction's boundary fields of the two problems are a unit uniform positive normal stress and a unit shear stress, respectively.

Here, the two problems are analysed using meshes of four linear elements. Prescribed boundary displacements are given as data, obtaining the boundary tractions as results. Considering the linearity of the exact solutions, it is clear that the error of the solutions is due to the numerical evaluation of the integrals.

The following orders of numerical integration were used: 2 × 2 [or 3 × 2 to avoid $\xi_j \neq \eta_i$ in eqn (21)] for the two-dimensional integrals and 2 for the one-dimensional ones when the elements are coincident and 6 × 6 (to compute G_{ij}^{kl}) or 7 × 7 (to compute H_{ij}^{kl}) when the elements are neither coincident nor adjacent.

To study the efficiency of the coordinate transformations presented for adjacent elements, three cases were analysed: (a) when the procedure presented in [18] is used; (b) when the procedure presented here is used together with the formula (42) and (c) when the same procedure is used together with the formula (44). Each one of the three cases was resolved with 4 × 4, 6 × 6, 8 × 8, 10 × 10, 12 × 12 and 14 × 14 points of integration.

The error of the solutions and the CPU times (VAX 8300) required to set up the systems of equations are presented in Tables 1 and 2. This

Table 1. Error and CPU time for the square plate uniformly tractioned

Order of integration	Cases (a) and (b)				Case (c)		
	$\Vert E \Vert_t$	e_t (%)	CPU (a)	CPU (b)	$\Vert E \Vert_t$	e_t (%)	CPU
4 × 4	5.63×10^{-3}	5.51×10^{-1}	4.80	4.62	1.09×10^{-3}	9.49×10^{-2}	6.84
6 × 6	1.12×10^{-3}	1.11×10^{-1}	8.35	8.02	1.10×10^{-4}	1.07×10^{-2}	11.03
8 × 8	3.46×10^{-4}	3.43×10^{-2}	14.15	12.64	6.14×10^{-6}	6.02×10^{-4}	19.38
10 × 10	1.45×10^{-4}	1.43×10^{-2}	21.03	19.21	3.97×10^{-7}	3.71×10^{-5}	30.01
12 × 12	7.16×10^{-5}	7.10×10^{-3}	28.75	28.46	5.73×10^{-8}	9.04×10^{-6}	41.09
14 × 14	3.94×10^{-5}	3.90×10^{-3}	39.26	34.93	1.96×10^{-8}	2.97×10^{-6}	52.80

Table 2. Error and CPU time for the square plate distorted

Order of integration	Cases (a) and (b)				Case (c)		
	$\Vert E \Vert_t$	e_t (%)	CPU (a)	CPU (b)	$\Vert E \Vert_t$	e_t (%)	CPU
4 × 4	3.58×10^{-3}	3.57×10^{-1}	4.62	4.59	4.61×10^{-3}	3.77×10^{-1}	6.55
6 × 6	1.10×10^{-3}	1.07×10^{-1}	8.44	8.06	1.53×10^{-4}	1.02×10^{-2}	11.24
8 × 8	3.94×10^{-4}	3.91×10^{-2}	13.25	12.84	4.94×10^{-6}	2.89×10^{-4}	18.32
10 × 10	1.68×10^{-4}	1.67×10^{-2}	19.89	18.52	2.20×10^{-7}	2.21×10^{-5}	28.55
12 × 12	8.33×10^{-5}	8.28×10^{-3}	32.03	27.69	5.88×10^{-8}	9.71×10^{-6}	41.67
14 × 14	4.58×10^{-5}	4.55×10^{-3}	42.28	42.60	2.55×10^{-8}	4.09×10^{-6}	50.50

error is characterized by its norm $\Vert E \Vert_t$ and by the average e_t (in percent) of the error at the nodes. The first value is given by

$$\Vert E \Vert_t = \left\{ \sum_{i=1}^{2} \sum_{e=1}^{4} \int_{-1}^{1} [\bar{t}_i^e(\xi) - t_i^e(\xi)]^2 \, d\xi/2 \right\}^{1/2}, \tag{46}$$

where $\bar{t}_i^e(\xi)$ and $t_i^e(\xi)$ denote the exact and the approximate solutions on Γ_e, respectively. The second value is obtained by

$$e_t = \frac{\sum_{i=1}^{2} \sum_{n=1}^{8} (\bar{t}_i^n - t_i^n)/\bar{t}_i^n}{16} \times 100, \tag{47}$$

where $\bar{t}_i^n$ and t_i^n are, respectively, the exact and the calculated values of the ith component of the traction at node n: a double node is considered at each corner.

The results in Tables 1 and 2 show that the use of the presented scheme leads to the same results as that presented in [18] with a slight reduction of computer time. The use of coordinate transformation (43) gives more accurate results to the evaluation of the quasi-singular integrals arising when adjacent elements are considered.

6.2. *Hollow cylinder submitted to internal uniform pressure*

Finally, the plane problem of a cylinder submitted to internal unit uniform pressure (Fig. 7), also with $E = 1$ and $\nu = 0$, is considered. Because of the symmetry only a quarter part of it was studied.

The exact solution in polar coordinates [20] is

$$\begin{cases} u_r = \dfrac{r^2+4}{3r} \\ u_\theta = 0 \end{cases} \quad \text{and} \quad \begin{cases} \sigma_r = \dfrac{r^2-4}{3r^2} \\ \sigma_\theta = \dfrac{r^2+4}{3r^2}. \end{cases} \tag{48}$$

Two meshes of boundary elements were used (Fig. 7): one with four elements and the other with eight elements, which was obtained from the former by dividing each element into two. The geometry of the curved elements was approximated using parabolic Lagrange shape functions. In each mesh the unknown boundary fields were approximated by linear and parabolic discontinuous interpolating functions. In Figs 8 and 9 the displacement and the traction fields on the straight side are plotted and compared with the exact solution.

7. CONCLUSIONS

A totally numerical general scheme for the evaluation of double singular integrals arising in the Galerkin direct BEM has been presented. The main results can be summarized as follows:

(1) double integrals with a Cauchy principal value were reduced to the computation of a double regular integral plus the evaluation of a one-dimensional integral with logarithmic singularity [eqn (20)];

(2) double integrals of weakly singular function 'along a diagonal' were transformed into double integrals with weak singularity in one variable only [eqn (33)]; and

(3) double integrals of weakly singular functions at 'a vertex' were reduced to the evaluation of regular

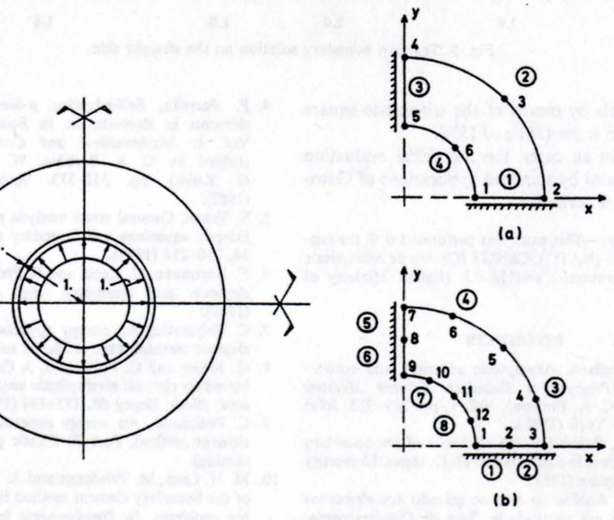

Fig. 7. Hollow cylinder with uniform internal pressure.

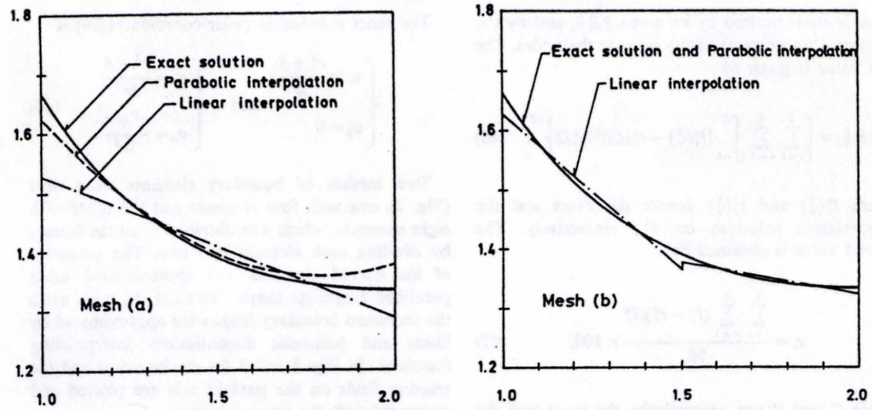

Fig. 8. Displacement boundary solution on the straight side.

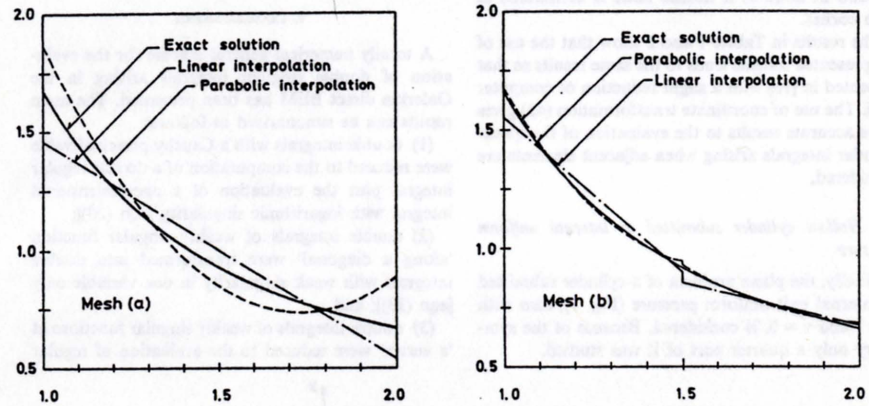

Fig. 9. Traction boundary solution on the straight side.

double integrals by means of the triangle-to-square transformation [eqns (38) and (39)].

Therefore, in all cases, the numerical evaluation can be performed by standard applications of Gaussian quadrature formulae.

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