Double integral in Fourier method is defined as follows

$$G(p,q) \equiv \int_a^b \int_0^{2\pi} f(r,\theta)\phi_p(r)e^{iq\theta}d\theta dr. \tag{1}$$

Let V be the discrete sampling of above function $f(r,\theta)$,then V is 2x2 array of components

$$v_{\sigma,\tau} = f(r_\sigma, \theta_\tau) \tag{2}$$

where $\sigma = 0, \cdots, N_r - 1$ and $\tau = 0, \cdots, N_\theta - 1$ with N_r is number of quadrature points for all elements and N_θ is number of modes in θ direction element.

For numerical evaluation, we use Gauss-Lobatto Quadrature formula for r direction integral and Fourier transform for θ direction integral.

Note that there're weight factors $\{w_\sigma\}_{\sigma=0}^{N_r}$ which are used for weighted sum of integrand on each $\{r_\sigma\}$.

For fixed r_0, the discrete Fourier transform for $f(r_0,\theta)$is defined by

$$f(r_0,\theta_\tau) = \sum_{k=-N_\theta/2+1}^{N_\theta/2} \widehat{f(r_0)}_k e^{ik\theta_\tau} \tag{3}$$

where

$$\widehat{f(r_0)}_k = \frac{1}{2\pi}\int_0^{2\pi} f(r_0,\theta)e^{-ik\theta}d\theta \tag{4}$$

with k $\in \{-\frac{N_\theta}{2}+1, \cdots, \frac{N_\theta}{2}\}$.

The flow I understood in last meeting is as follows:

$$\begin{aligned}
G(p,q) &= \int_a^b r^2\phi_p(r)\int_0^{2\pi} f(r,\theta)e^{iq\theta}d\theta dr && (5)\\
&= \int_a^b r^2\phi_p(r)\int_0^{2\pi} \sum_{k=-N_\theta/2+1}^{N_\theta/2} \widehat{f(r)}e^{ik\theta}e^{iq\theta}d\theta dr && (6)\\
&= \int_a^b r^2\phi_p(r)\widehat{f(r)} \sum_{k=-N_\theta/2+1}^{N_\theta/2} \int_0^{2\pi} e^{ik\theta}e^{iq\theta}d\theta dr && (7)\\
&= \int_a^b r^2\phi_p(r)\widehat{f(r)}dr \sum_{k=-N_\theta/2+1}^{N_\theta/2} \delta_{kq} && (8)\\
&= \int_a^b r^2\phi_p(r)\widehat{f(r)}dr && (9)\\
&= \sum_\sigma w_\sigma r_\sigma^2\phi_p(r_\sigma)\widehat{f(r_\sigma)} && (10)
\end{aligned}$$

This derivation **assumes** that

$$f(r_0,\theta) = \sum_{k=-N_\theta/2+1}^{\frac{N_\theta}{2}} \widehat{f(r_0)}e^{ik\theta} \tag{11}$$

where $\theta \in [0, 2\pi]$ and

$$\widehat{f(r_0)}_k = \frac{1}{2\pi} \int_0^{2\pi} f(r_0, \theta) e^{-ik\theta} d\theta \tag{12}$$

The expression about each $\widehat{f(r_0)}_k$ is based on Trapezoidal Rule.