The right hand side of weak form is

$$G(p,q) \equiv \int_a^b \int_0^{2\pi} f(r,\theta)\phi_p(r)e^{iq\theta}d\theta dr. \tag{1}$$

Let V be the discrete sampling of above function $f(r,\theta)$,then V is 2x2 array of components

$$v_{\sigma,\tau} = f(r_\sigma, \theta_\tau) \tag{2}$$

where $\sigma = 0,\cdots,N_r - 1$ and $\tau = 0,\cdots,N_\theta - 1$ with N_r is number of quadrature points for all elements and N_θ is number of modes in θ direction element.

For numerical evaluation, we use Gauss-Lobatto Quadrature formula for r direction integral and Fourier transform for θ direction integral.

Note that there're weight factors $\{w_\sigma\}_{\sigma=0}^{N_r}$ which are used for weighted sum of integrand on each $\{r_\sigma\}$.

For fixed r_0, the discrete Fourier transform for $f(r_0,\theta)$is defined by

$$f(r_0,\theta_\tau) = \sum_{k=-N_\theta/2+1}^{N_\theta/2} \widehat{f(r_0)}_k e^{ik\theta_\tau} \tag{3}$$

where $k \in \{-\frac{N_\theta}{2}+1,\cdots,\frac{N_\theta}{2}\}$.

The flow I understood in last meeting is as follows:

$$\begin{aligned} G(p,q) &= \int_a^b r^2\phi_p(r)\widehat{f(r)}dr \sum_{k=-N_\theta/2+1}^{N_\theta/2} \delta_{kq} && (4)\\ &= \int_a^b r^2\phi_p(r)\widehat{f(r)}dr && (5)\\ &= \sum_\sigma w_\sigma r_\sigma^2 \phi_p(r_\sigma)\widehat{f(r_\sigma)} && (6) \end{aligned}$$

The **first question** is how to deal with array form of $\widehat{f(r_\sigma)}$. Since given σ, $\widehat{f(r_\sigma)}$ would be defined from the vector $[f(r_\sigma,\theta_\tau]_{\tau=0}^{N_\theta-1}$ $G(p,q)$ will be vector form. This phenomena happens the same in using ifft() as follows:

In our solution form,

$$u(r,\theta) = \sum_{j=0}^{N_r} \sum_{k=-N_\theta/2+1}^{N_\theta/2} \hat{u}_{jk}\phi_j(r)e^{ik\theta} \tag{7}$$

, we can apply ifft to compute terms having $e^{ik\theta}$ like this:

$$\begin{aligned} u(r,\theta) &= \sum_{j=0}^{N_r} \phi_j(r) \sum_{k=-N_\theta/2+1}^{N_\theta/2} \hat{u}_{jk}e^{ik\theta} && (8)\\ &= \sum_{j=0}^{N_r} \phi_j(r) N_\theta ifft(\hat{u}_{jk}) && (9)\\ & && (10) \end{aligned}$$

here, since we can compute $ifft(\hat{u}_{jk})$ from the vector form $[\hat{u}_{jk}]_{k=-N_\theta/2+1}^{N_\theta/2}$ But this also produces vector form.

The **second question** is to verify Trefthan's definition of discrete Fourier transform (dft/inverse dft pair). In my code that implements dft/inverse dft, when I change scale term in front of summation differently from his book, I could obtain similar result to the result of matlab fft(), ifft(). I attach the code here.

In his book I omit h in (3.2) and change $\frac{1}{2\pi}$ to $\frac{1}{N_\theta}$ in (3.3). The result of this change is different from that of matlab fft(), ifft() in that

- The counting order
- sign of $Re(\hat{u})$ when $\theta \neq 0$.

which were talked in last meeting.

I think in last meeting I couldn't get the point how to implement what I learned. For example, the how to connect k in

$$M_3 + M_2 - k^2 M_1 \tag{11}$$

with fft() in RHS.

For this I decide to finish the write-up and verify all things. This would be good to prevent unexpected things from coming out during programming.