Solving
the Poisson Partial Differential Equatio
using Spectral Polynomial Methods

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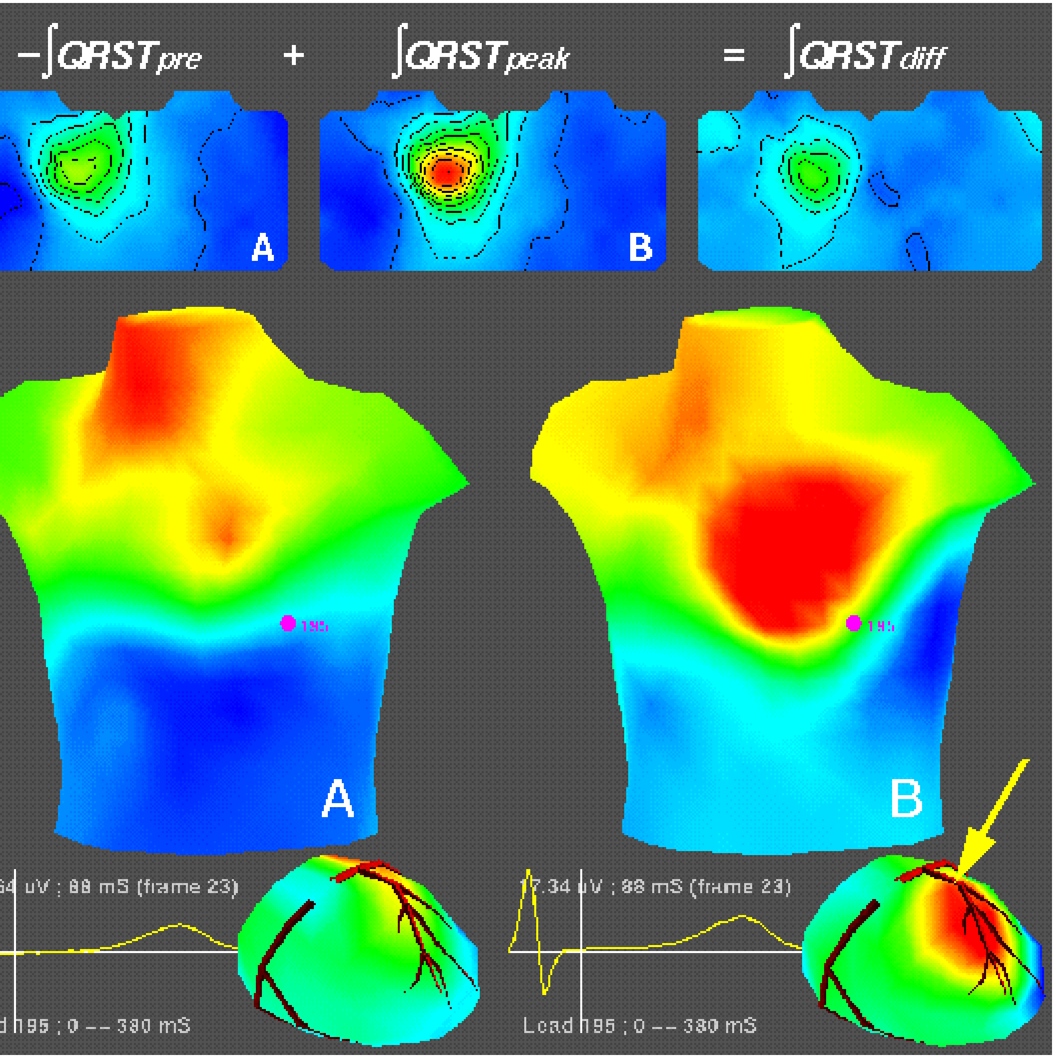
CONTENTS

Heart Arrhythmias

- Increasing the number of patient suffering heart disorder: Heart attack, Myocardinal infarction
- Limit of Electrocardiogram currently used in the field
- Infarction of myocardinal affects regular heart contraction and dilation
- Coronary angioplasty can recover mycardinal
- Figuring out the source of ischemic part of heart is objectives for medical treatement

Ischemic Change Localization



//www.cvrti.utah.edu/~macleod/research/images/ptca.gif

- Clue for solution:
  - Body is a conductor with various organs of unique conductivity
  - Vessel, heart chamber has t highest conductivity
  - Ischemic heart tissue results higher potential

Conductivities:

0	air	0.0
1	skin	0.00005
2	fat	0.0000375
3	muscle	0.000125
4	lung	0.000054
5	heartmuscle	0.000238
6	heartchamber	0.00068
7	fatpad	0.00005
8	vessel	0.00068

Laplace Equation with Current Density

- By Ohm's law the current density is represented by negative gradient of scalar potential with conductivity: $$\mathbf{J} = -\sigma \nabla \Phi$$

- As a special case of Poisson equation, we obtain

$$\nabla \cdot \sigma \nabla \Phi = 0 \qquad \text{in } \Omega$$

since the source density in the domain is zero: that is, sources lie outside or at the boundary of domain.

- Actually, the source comes from ventricle of heart.

- Since the air is non-conductive, we have zero Neumann condition on outer boundary.

One-dimensional Spectral Methods

Spectral method

- Is high-order method

$$u(x) = \sum_{i=0}^{N_{dof}-1} \hat{u}_i \Phi_i(x)$$

- Has system of equations in weak form

$$\textstyle\sum_{p=0}^{P_e} \hat{u}_p^e \langle \frac{d}{dx}\phi_p, \frac{d}{dx}\phi_q \rangle = \langle f, \phi_q \rangle + \left[\frac{d}{dx} u(x) \phi_q(x) \right]_{x_1}^{x_2}$$

- Is an extension of finite element method with basis on each element

$$\psi_p(\xi) = \begin{cases} \left(\frac{1-\xi}{2}\right) & p = 0 \\ \left(\frac{1-\xi}{2}\right)\left(\frac{1+\xi}{2}\right) P_{p-1}^{1,1}(\xi) & 0 < p < P \\ \left(\frac{1+\xi}{2}\right) & p = P. \end{cases}$$

- Each element has local boundary basis and interior basis

One-dimensional Spectral Methods

- Interior basis (bubble mode) works as and high-order basis

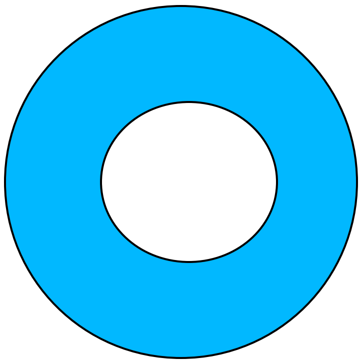
$$
\begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots 0 & \phi^0_{P_0-1,P_0-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0\cdots \\
\cdots -.5 \cdots 0 & 0 & 1 & 0 & \cdots & 0 & -.5 & 0 & 0\cdots \\
\cdots 0 & 0 & 0 & \phi^1_{1,1} & \cdots & 0 & 0 & 0 & 0\cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\cdots 0 & 0 & 0 & 0 & \cdots & \phi^1_{P_1-1,P_1-1} & 0 & 0 & 0\cdots \\
\cdots 0 & 0 & -.5 & 0 & \cdots & 0 & 1 & 0 & 0\cdots -5\cdots \\
\cdots 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \phi^2_{P_0,P_0} & 0\cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\begin{bmatrix}
\vdots \\ \hat{u}^0_{P_0-1} \\ \hat{u}^1_0 \\ \hat{u}^1_1 \\ \vdots \\ \hat{u}^1_{P_1-1} \\ \hat{u}^2_0 \\ \hat{u}^2_1 \\ \vdots
\end{bmatrix}
=
\begin{bmatrix}
\vdots \\ f^0_{P_0-1} \\ f^0_{P_0} + f^1_0 \\ f^1_1 \\ \vdots \\ f^1_{P_1-1} \\ f^1_{P_1} + f^2_0 \\ f^2_1 \\ \vdots
\end{bmatrix}
$$

- Two boundary conditions can be employed to the system

$$
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & A_{1,0} & \cdots & A_{1,N_{dof}-1} \\
\vdots & \vdots & \vdots & \vdots \\
0 & A_{N_{dof}-1,1} & \cdots & A_{N_{dof}-1,N_{dof}-1}
\end{bmatrix}
\begin{bmatrix}
\hat{u}_0 \\ \hat{u}_1 \\ \vdots \\ \hat{u}_{N_{dof}-1}
\end{bmatrix}
=
\begin{bmatrix}
0 \\ f_1 \\ \vdots \\ f_{N_{dof}-1}
\end{bmatrix}
+
\begin{bmatrix}
0 \\ \vdots \\ 0 \\ 1
\end{bmatrix}
\mathcal{G}_N -
\begin{bmatrix}
-1 \\ A_{1,0} \\ \vdots \\ A_{N_{dof}-1,0}
\end{bmatrix}
\mathcal{G}_D .
$$

Spectral Methods on 2D Annulus

2D Annulus Model



- Inner circle: Dirichlet Boundary Condition
- Outer circle: Zero Neumann Condition
- Solve Poisson equation for scalar potenti on the interior of the annulus

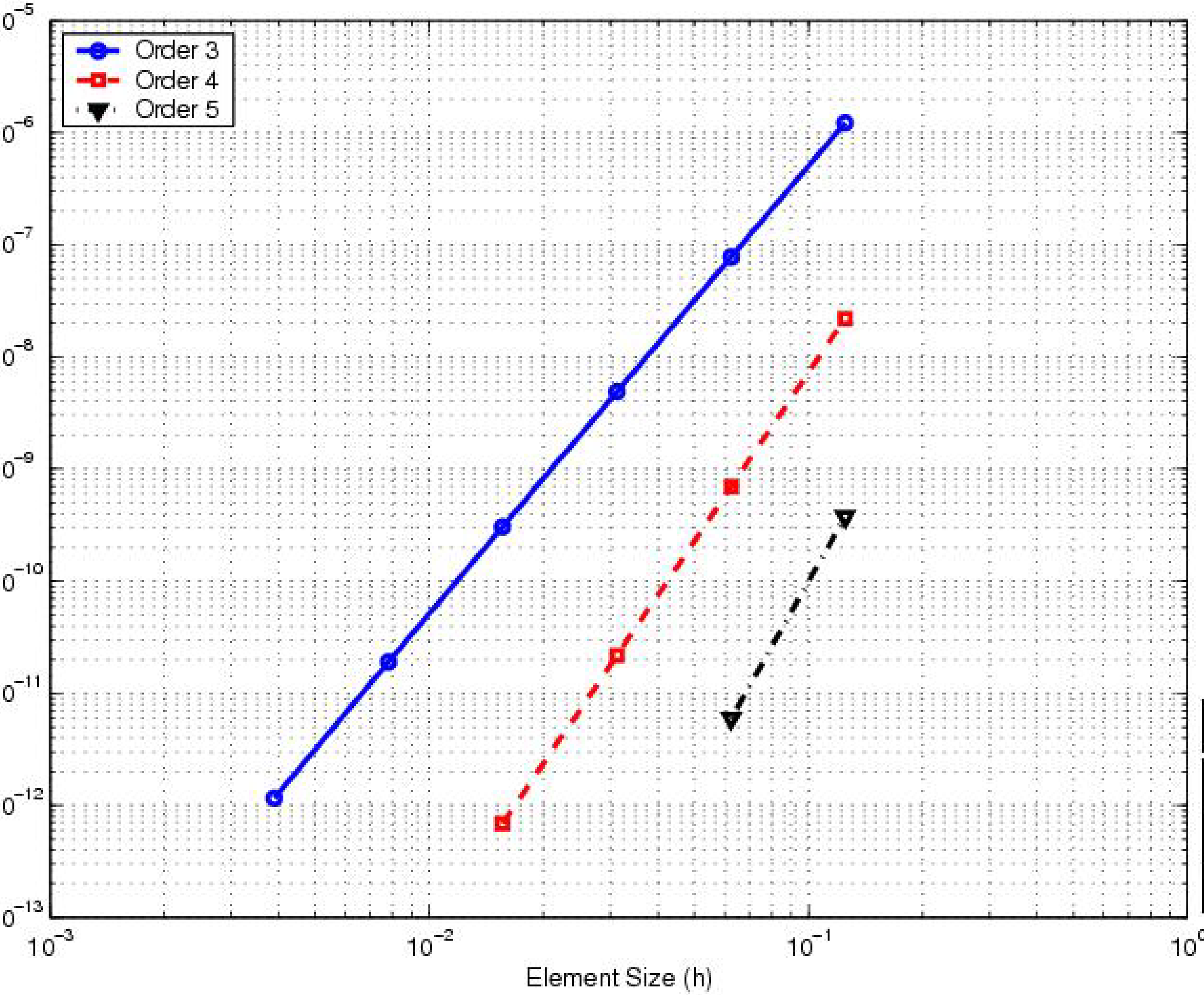
$$-\left[\frac{\partial}{\partial r}(\sigma(r,\theta)\frac{\partial}{\partial r}) + \frac{1}{r}(\sigma(r,\theta)\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial}{\partial \theta}(\sigma(r,\theta)\frac{\partial}{\partial \theta})\right] u(r,\theta) = f(r,\theta),$$

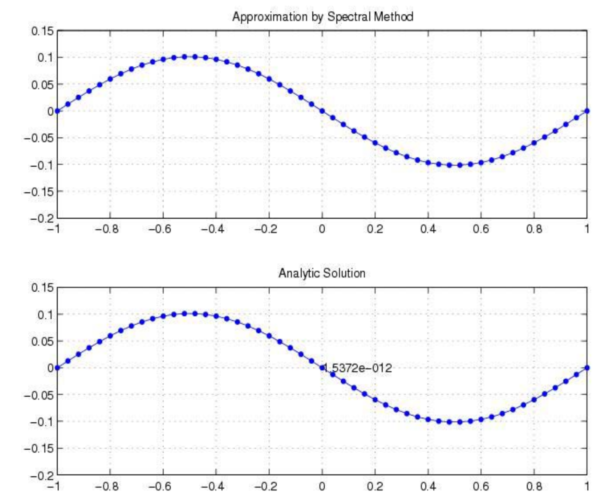
$$\text{with periodicity of } u, \quad u(r,0) = u(r,2\pi)$$

- We use an expansion set using tensor product of 1D basis and Fourier basis

$$u(r,\theta) = \sum_{j=0}^{N_r} \sum_{k=-N_\theta/2+1}^{N_\theta/2} \hat{u}_{jk}\phi_j(r)e^{ik\theta}$$

H-test for 1D Spectral Methods I

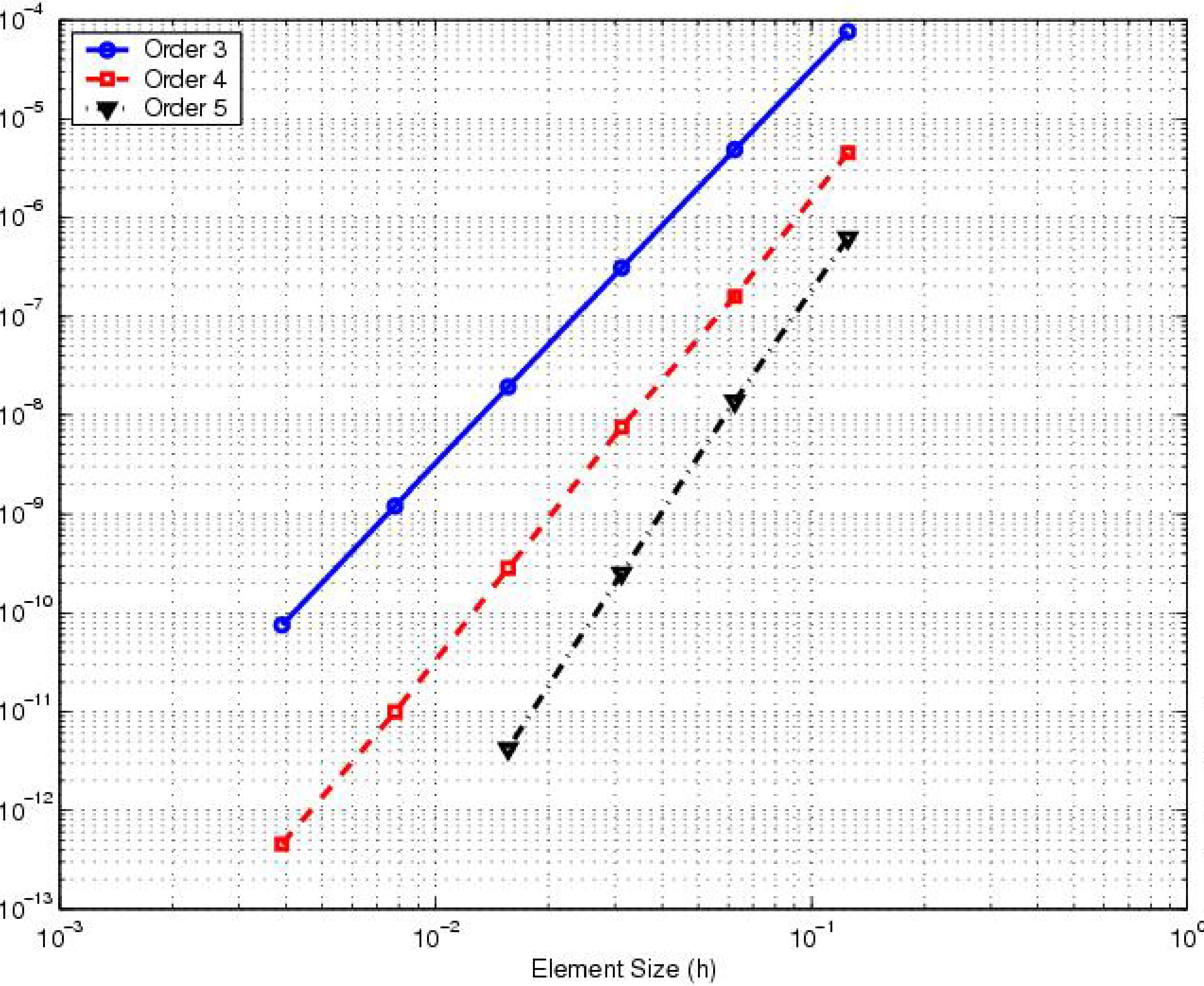


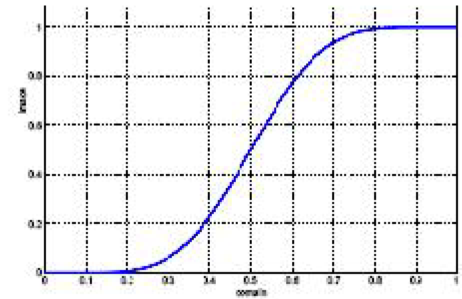


$$\frac{d^2}{dx^2}u(x) = \sin(\pi x)$$

Polynomial order	Error(L^∞)	Slop
3	$1.1620e-012$	4.00
4	$4.6629e-014$	4.98
5	$9.7367e-014$	5.97

H-test for 1D Spectral Methods II





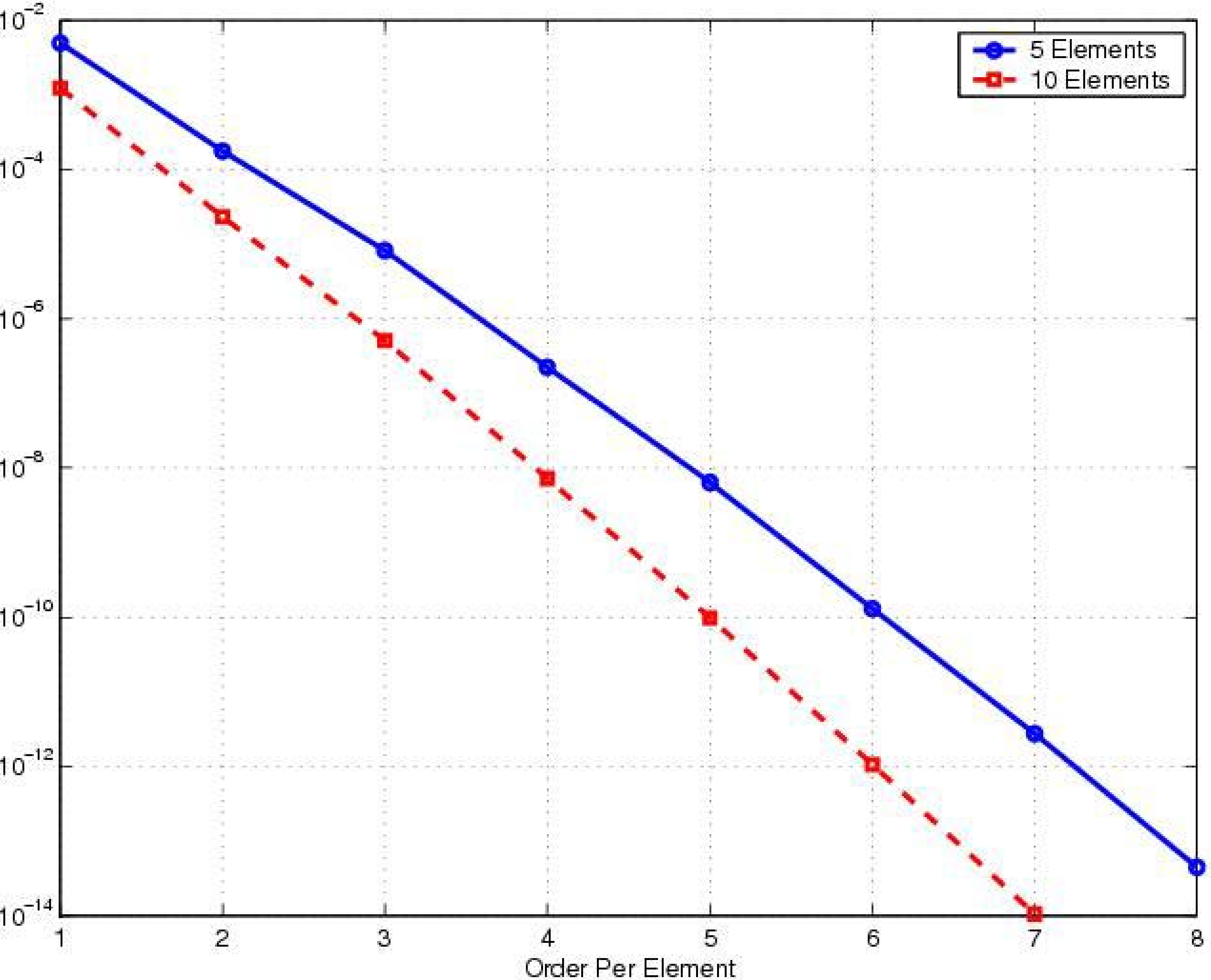
$$P_n(0) = 0, \qquad P_n(1) =$$

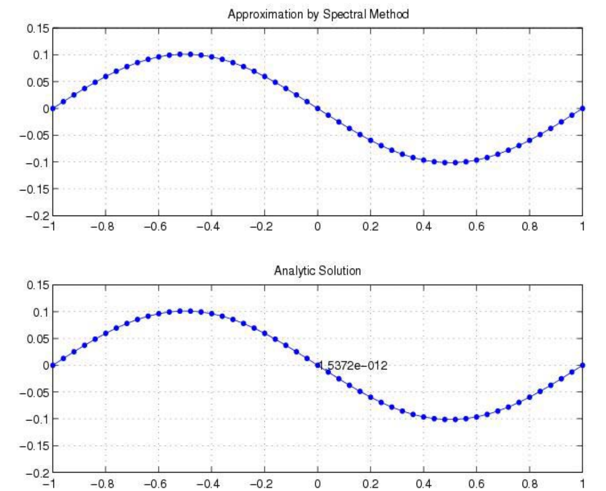
$$\frac{d^k}{dx^k} P_n(0) = 0, \quad \frac{d^k}{dx^k} P_n(1) =$$

$$\frac{d^2}{dx^2} u(x) = Q_{n-2}$$

Polynomial order	Error(L^∞)	Slo
3	$7.5530e-011$	3.9
4	$4.5619e-013$	4.6
5	$4.1855e-013$	5.7

P-test for 1D Spectral Methods I

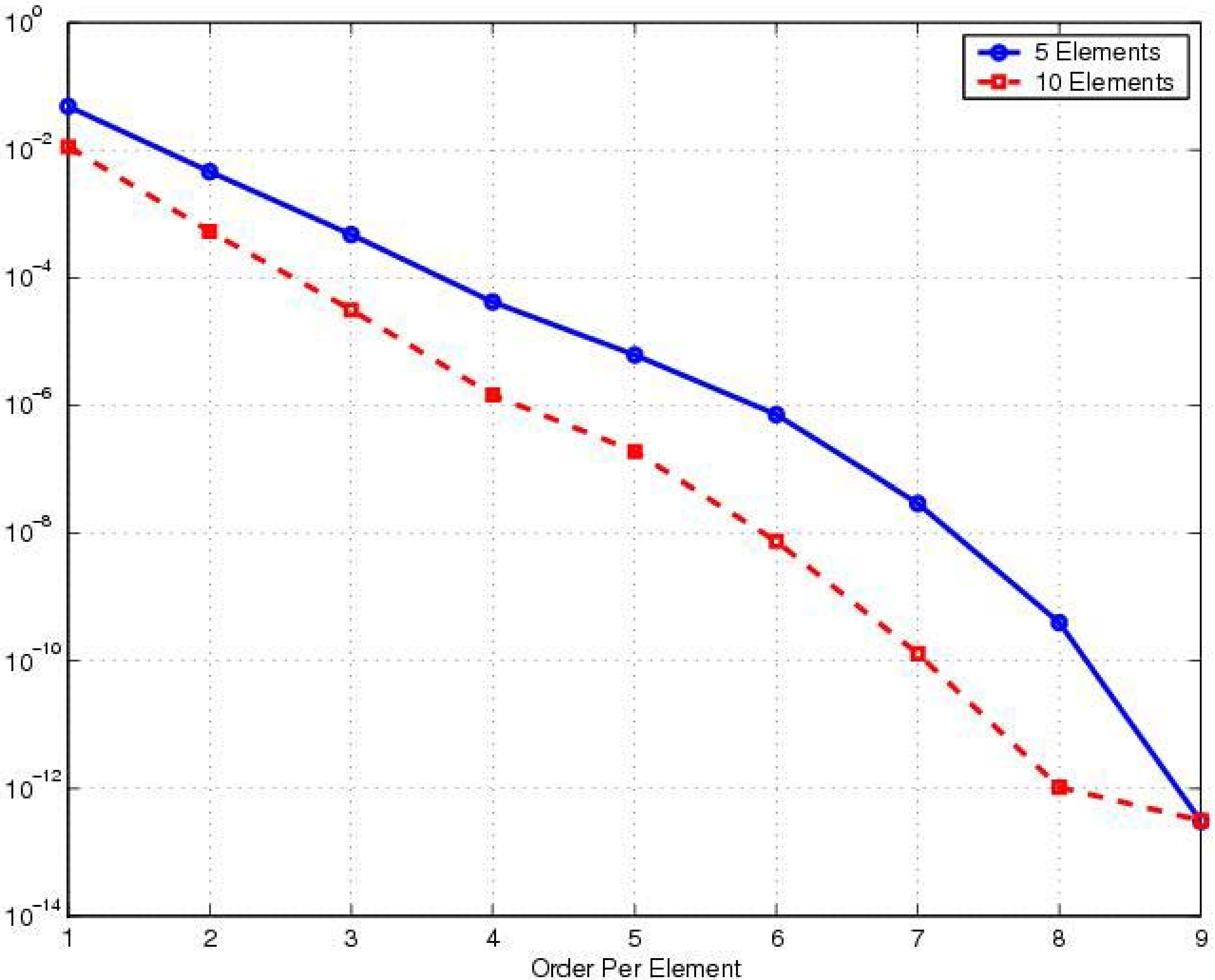


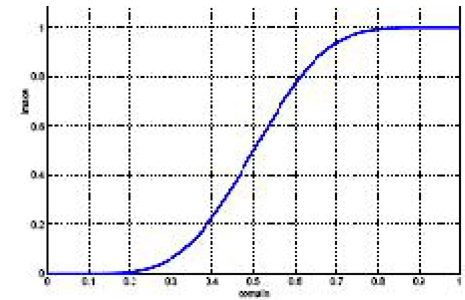


$$\frac{d^2}{dx^2}u(x) = \sin(\pi x)$$

Element Size	Error (L^∞)
0.2	$8.3267e-016$
0.1	$6.6613e-016$

P-test for 1D Spectral Methods II





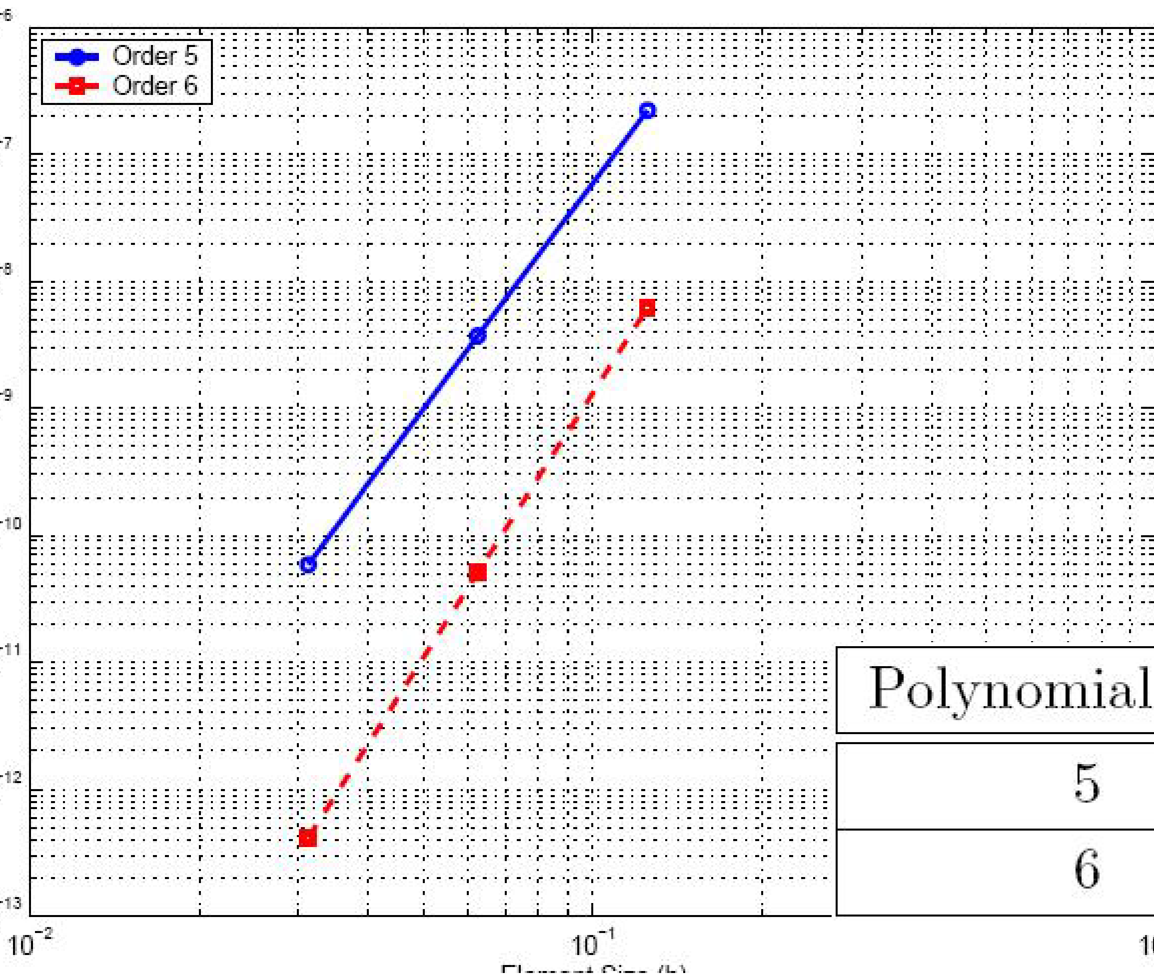
$$P_n(0) = 0, \qquad P_n(1) = 1$$

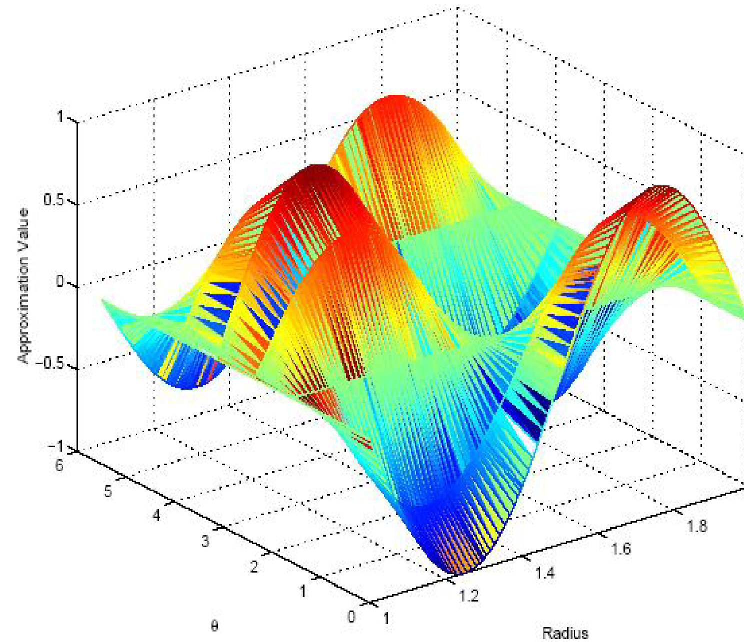
$$\frac{d^k}{dx^k} P_n(0) = 0, \quad \frac{d^k}{dx^k} P_n(1) =$$

$$\frac{d^2}{dx^2} u(x) = Q_{n-2}$$

Element Size	Error (L^∞)
0.2	$3.0431e-013$
0.1	$3.1186e-013$

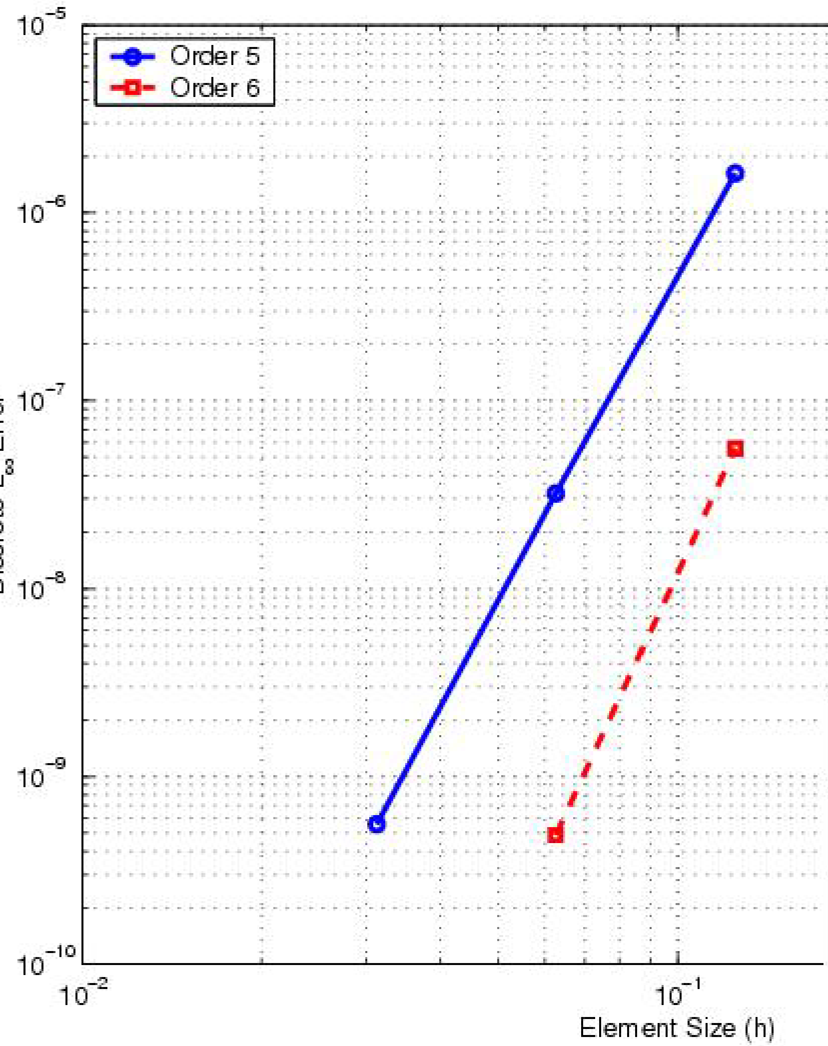
H-test for 2D Spectral Methods I

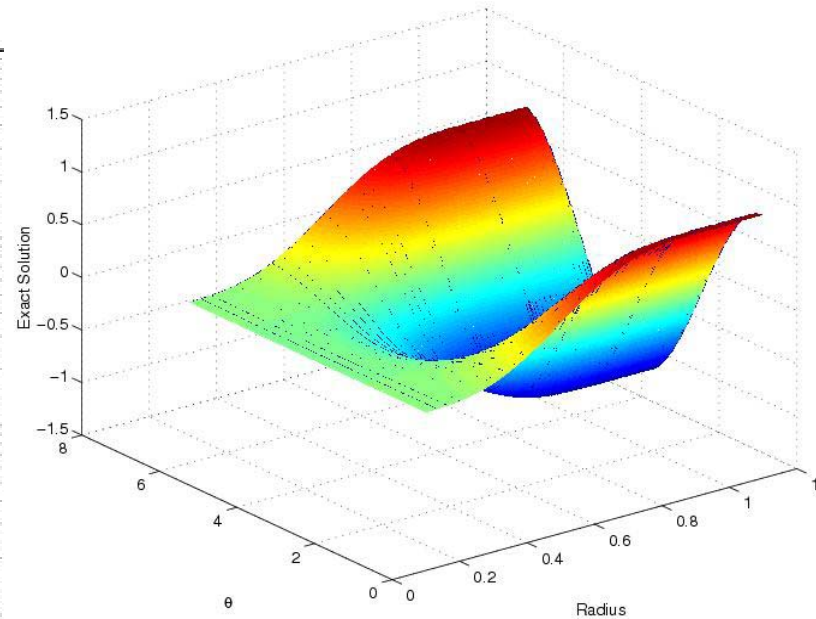




Polynomial order	Error(L^∞)	Slope
5	$9.3603e-013$	5.940
6	$8.6542e-014$	6.940

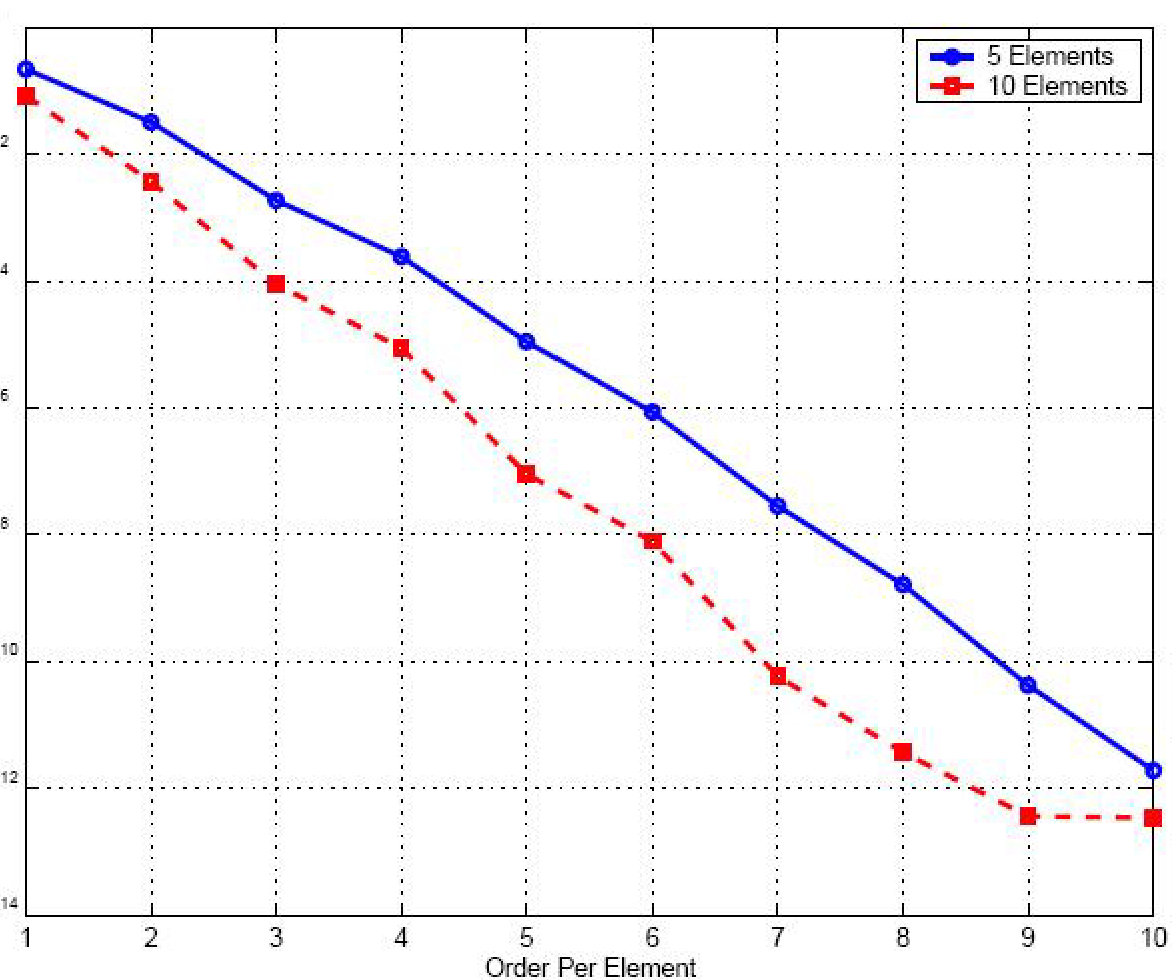
H-test for 2D Spectral Methods II

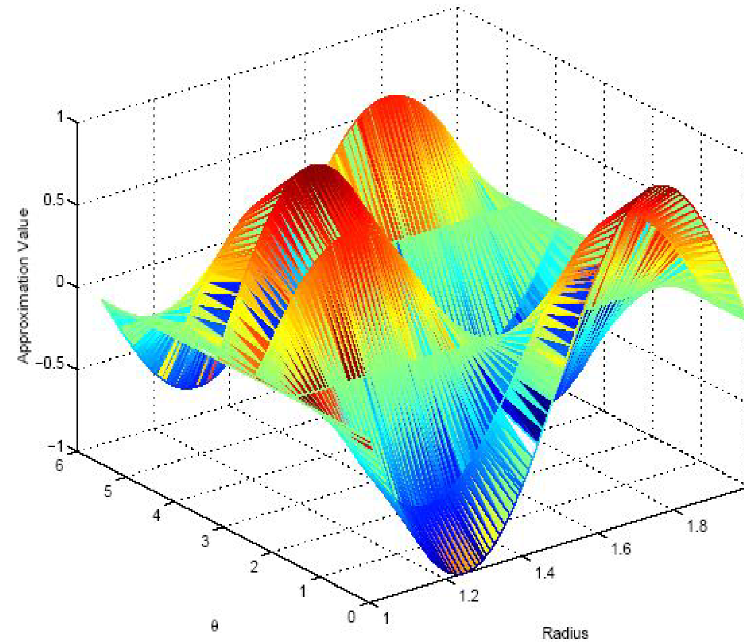




Polynomial order	Error(L^∞)	Slope
5	$8.5305e-012$	5.755
6	$4.7180e-012$	6.833

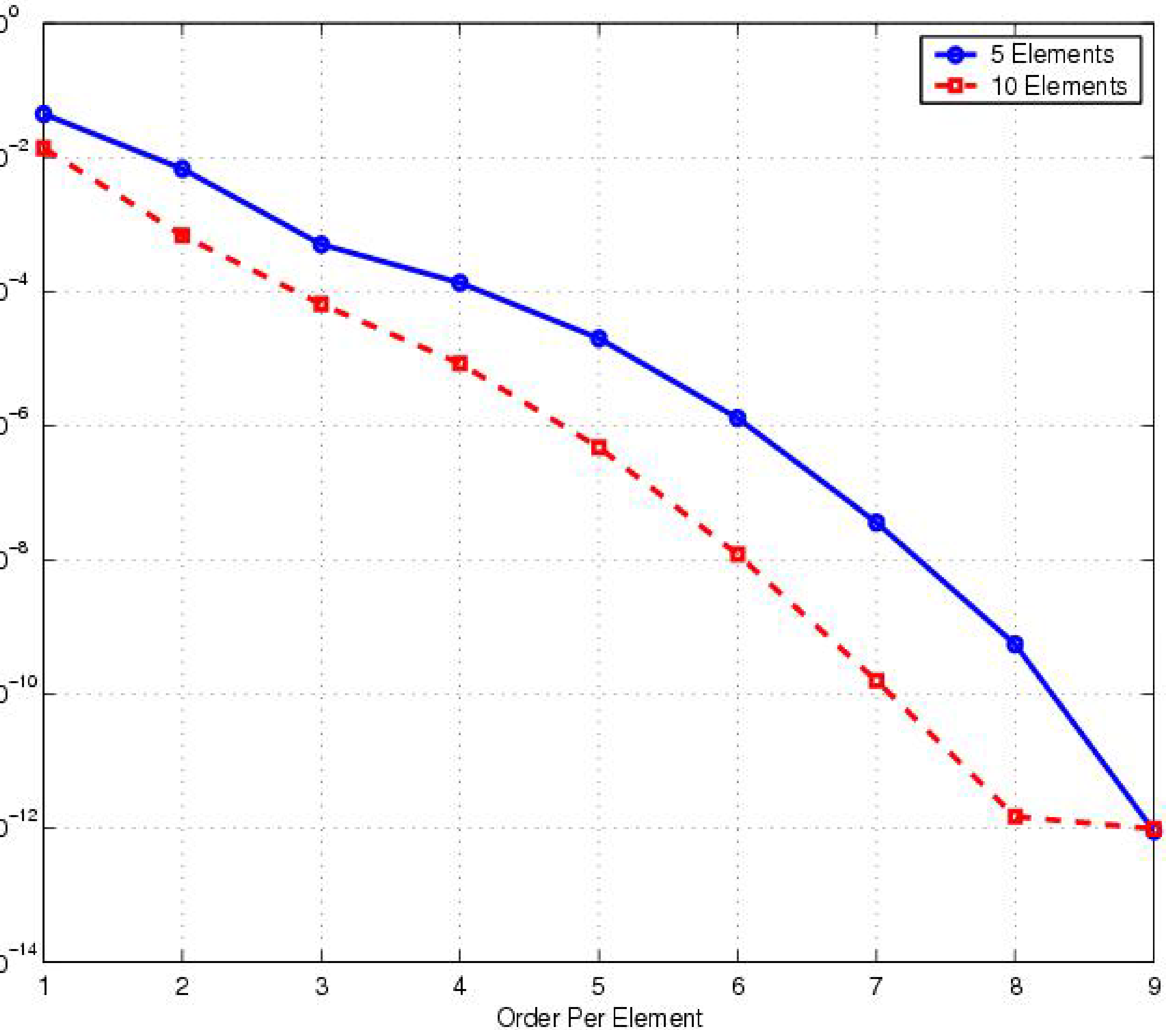
P-test for 2D Spectral Methods I

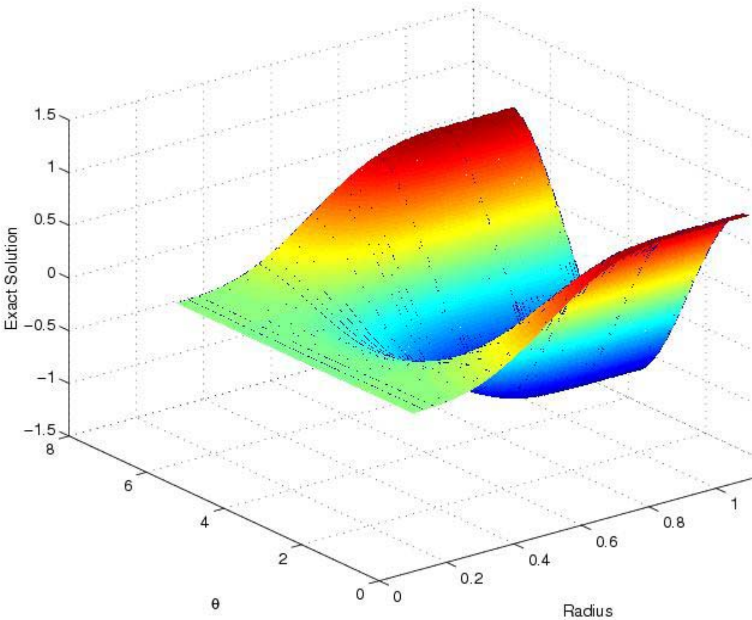




Element Size	Error(L^∞)
0.2	$1.9385e-01$
0.1	$3.4611e-01$

P-test for 2D Spectral Methods II





Element Size	Error(L^∞
0.2	$9.0616e -$
0.1	$9.4747e -$

Future Research

- Developing 2,3-dimensional solver with continuous / discontinuous conductivity.
- Developing solvers for the problem defined on boundary of general geometry.
- Computational inverse problem.

Questions and Answers