Fall 2003, Semester research;

Development of Numerical Solver using Spectral and Fourier Method

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Aug 28th, 2003

Abstract

This document is for specifying details of Spectral methods and Fourier method.

Contents

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1 A Steady-State Diffusion Problem

According to a report[1]

1.1 Diffusion Problem

1.2 Galerkin Method with Weak Solution

1.3 Boundary Conditions

2 Spectral Polynomial Methods in 1 Dimensional Space

Our problem is the Poisson equation

$$L(u) \equiv \nabla^2 u - f = 0, \tag{1}$$

for all $x \in \Omega$. In one dimensional case, 1 is written as

$$L(u)(x) \equiv \frac{d^2}{dx^2}u(x) - f(x) = 0, \tag{2}$$

for all x in $[a, b]$.

We solve this equation in weak sense. That is to say, we define a functional $(\cdot) : C^0 \to \Re$ such that

$$\left(\nabla^2 u, \nu\right) = \int_\Omega \frac{d^2}{dx^2}u(x) \cdot \nu(x) d\mu(x), \tag{3}$$

for each ν in C^0, where C^0 is a set of continuous functions. Then we find the solution u by solving the equation as follows:

$$\left(\nabla^2 u, \nu\right) = (f, \nu), \tag{4}$$

for each ν in C^0.

2.1 Basis Functions

The spectral approximation of solution u is generally represented as

$$u(x) = \sum_{i=0}^{N_{dof-1}} \hat{u}_i(x)\Psi_i(x) \tag{5}$$

on $[a, b]$. To construct this the global basis functions $\{\Psi_i(x)\}_{i=0}^{N_{dof}-1}$, each Ψ is represented by the linear combination of local basis functions ψ_i on each element in $[a, b]$, say Ω^e.

We define a basis functions ψ_i on Ω^{st} to be a real valued function with the Legendre polynomial $\{P_i^{1,1}\}$ as follows:

$$\psi_i(\xi) = \{ \begin{array}{ll} \frac{1-\xi}{2}, & i = 0 \\ \frac{1+\xi}{2}, & i = 1 \\ \left(\frac{1-\xi}{2}\right)\left(\frac{1+\xi}{2}\right) P_{i-2}^{1,1}(\xi), & i \geq 2 \end{array} \tag{6}$$

for all ξ in $[-1, 1]$.

Then on a single standard element Ω^{st}, the approximation $u(\xi)$ is represented as

$$u(\xi) = \sum_{i=0}^{N^e} \hat{u}_i^e \psi_i(\xi), \tag{7}$$

for ξ in Ω^{st}.

2.2 Spectral Polynomial Method in A Element

We apply the basis representation 7 to weak formulation 4 with the same test function $\{\psi_q\}$, then we obtain the following:

$$\sum_{p=0}^{N^e} \hat{u}_p^e \left(\nabla^2 \psi_p, \psi_q\right) = \left(\nabla^2 \sum_{p=0}^{N^e} u_p^e \psi_p, \psi_q\right) = (f, \psi_q) \tag{8}$$

for $q = 0, \cdots, N^e$.

With this relation we can setup a system of linear equations for the coefficient $\{\hat{u}_p^e\}_{p=0}^{N^e}$ with $N^e+1 \times N^e+1$ matrix $\mathbf{L}_{N^e}$ defined as follows:

$$\mathbf{L}_{N^e} \cdot \hat{\mathbf{u}} = \mathbf{f}, \tag{9}$$

where

$$\mathbf{L}_{N^e}(p,q) = \int_{\Omega^e} \frac{d^2}{d\xi^2} \psi_p(\xi) \psi_q(\xi) d\xi, \tag{10}$$

$$\hat{\mathbf{u}} = \left[\hat{u}_p\right]_{p=0}^{N^e}, \tag{11}$$

$$\mathbf{f} = \left[\int_{\Omega^e} f(\xi) \psi_q(\xi) d\xi\right]_{q=0}^{N^e}. \tag{12}$$

2.3 H/P Refinement using Global Assembly

3 Results

In section 1, 2, we present the result of convergence in both h refinement and p refinement with the following steady-state Poisson differential equation:

$$\frac{d^2}{dx^2}u(x) = \sin(\pi x),$$

for all x in $[0, 1]$.

3.1 H-Convergence of 1-D Spectral Method

This test is to validate the relation between size of element and the accuracy of approximation. We apply equidistance element and investigate the movement of error scale. As shown in Figure 1, the smaller are the elements, the more exact is the solution. Moreover by testing with different order of basis, we also could see the fact that the higher are orders, the faster do they converge.

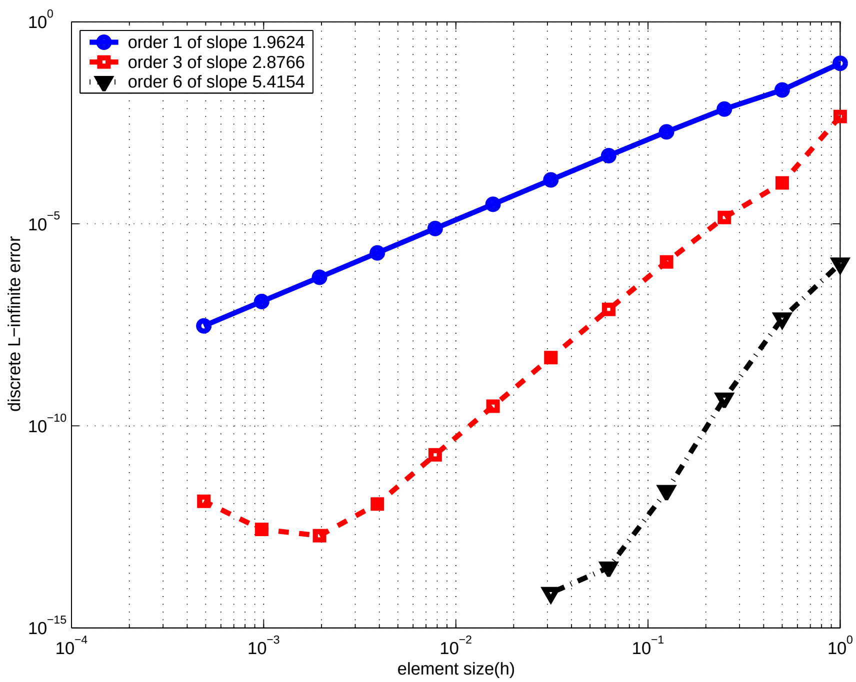

Figure 1: Graph showing change of errors by the size of elements

Figure 2 is showing the convergence of error bound when the order of basis goes close to 10. With the higher order than it, the error shows to value small enough of order -15.

Table 1: Specification of Figure1 and their errors

Polynomial order	Color	Err(h=1/32)	Err(h=1/512)
1	Blue	$1.2179e-004$	$1.9073e-006$
4	Red	$1.6362e-011$	$2.4647e-014$
7	Magenta	$7.3275e-015$	$3.6859e-014$

3.2 P-Convergence of 1-D Spectral Method

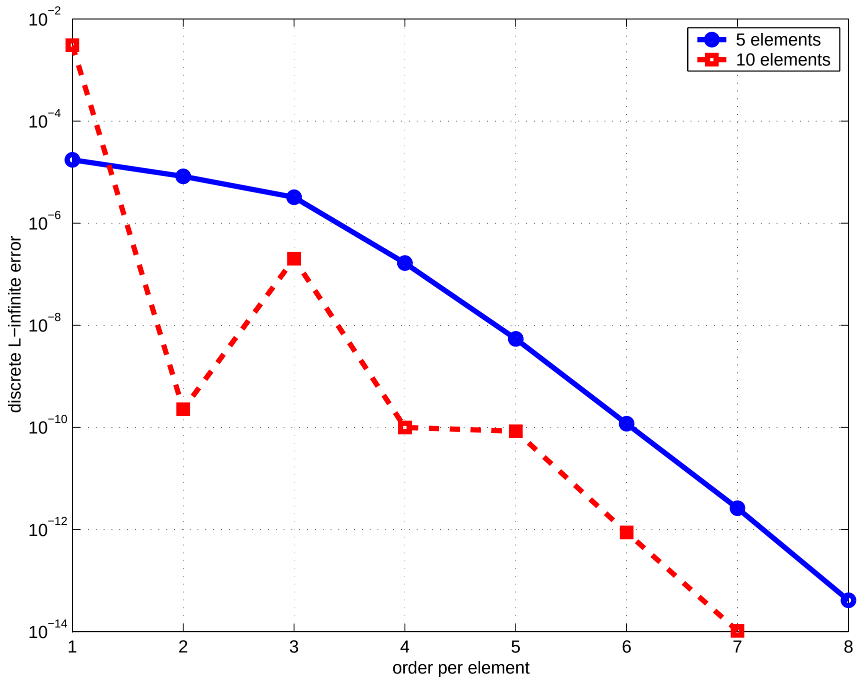

Figure 2: Graph showing change of errors by the increase of order Dirichlet-Neumann

Table 2: Specification of Figure2 and its error

Element Size	Err
0.2	$6.6613e-016$
0.1	4.9960e-016

Table 3: Specification of Figure3 and its error

Element Size	Err
0.2	$7.0777e-016$
0.1	$1.1796e-016$

3.3 Approximation of High order Polynomial solving 1-D Poisson Equation

In this section we construct a polynomial P_n of order n defined on $[0,1]$, which satisfies the following.

$$P_n(0) = 0, \qquad P_n(1) = 1$$
$$\frac{d^k}{dx^k}P_n(0) = 0, \quad \tfrac{d^k}{dx^k}P_n(1) = 0$$

for all $k = 1, \cdots, n-2$.
Then for each n, we obtain a polynomial P_n by solving a system of linear equations having unique

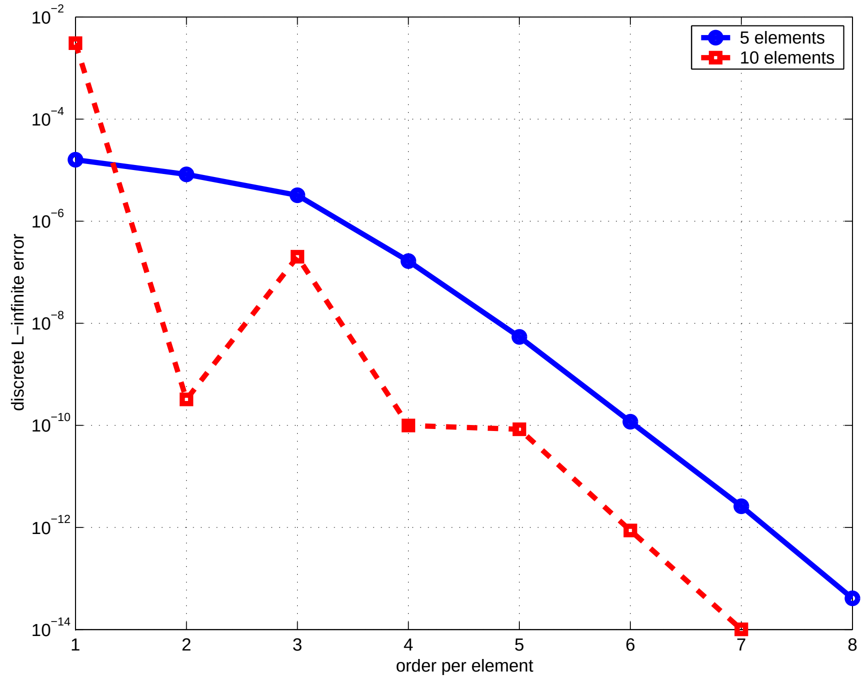

Figure 3: Graph showing change of errors by the increase of order Dirichlet-Dirichlet

solution which determines the set of coefficients of P_n. We will apply the spectral polynomial solver to approximate the second derivative Q_{n-2} of P_n.

Figure 4 is showing some samples of solution of order(n) 21.

Problem 3.1 *Consider the following differential equation for $u(x)$ such that*

$$\frac{d^2}{dx^2}u(x) = Q_{n-2},$$

for all x in $[0, 1]$. Then the problem is to find approximation $p(x)$ of $u(x)$ using spectral polynomial method.

3.3.1 Existence of Approximate Solution

The Figure 5 and 6 are showing the result of spectral polynomial method approximating the solution of Problem 3.1. The maximum error value in Table 4 is showing the approximation is within numerically exact solution tolerance.

Table 4: Specification of Figure 5 and its error

Element Size	Num. of Element	Orders	Err
0.2	5	5, 5, 5, 5, 5	$5.7732e-015$

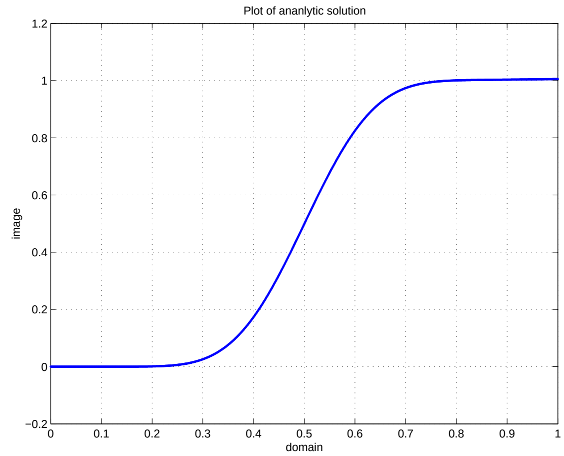

Figure 4: Solution polynomial of order 21

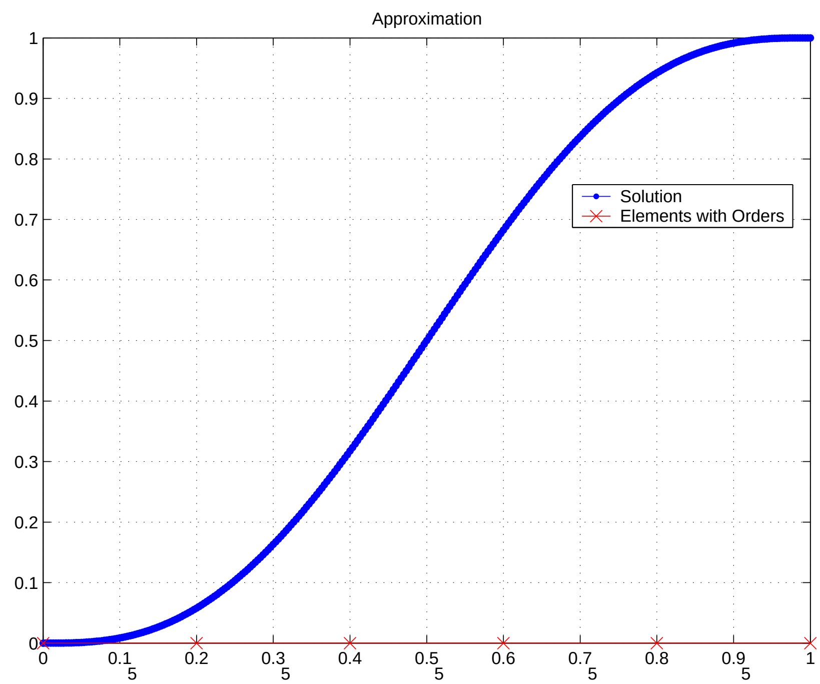

Figure 5: Graph showing spectral approximation satisfying Problem 3.1

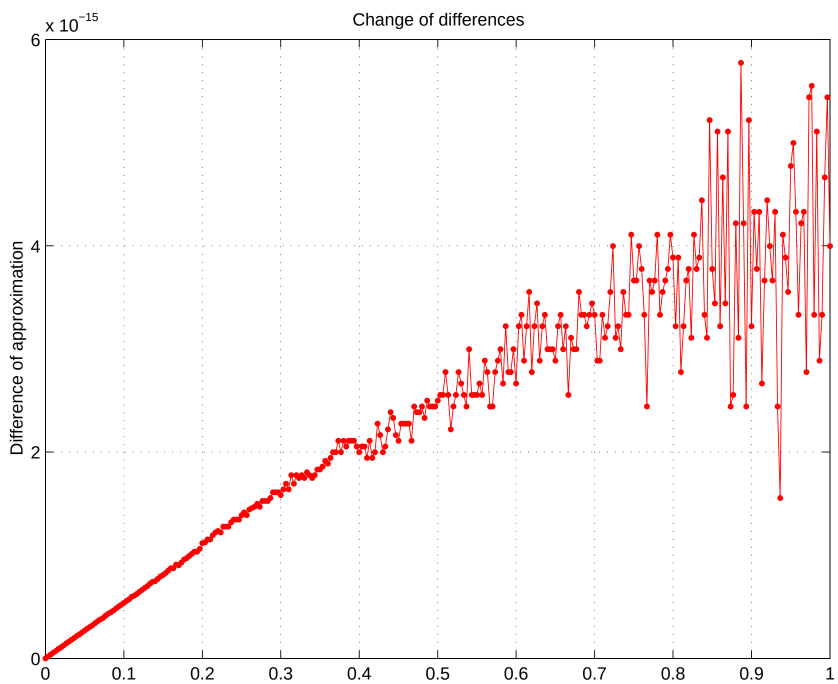

Figure 6: Graph showing the error of approximation in Figure 5

3.3.2 Convergence of Solution in Equidistance and Uniformly Ordered Elements

In this section we show the convergence of solutions obtained by handling orders of basis on each element. We fix the element to be same size(length) and divide the domain $[0, 1]$ by 5 elements.

Figure **??** is the result of convergence to approximating to a solution of order 5 in Problem 3.1. It shows monotonic decreasing with same similar slope until the order reaches from 1 to 4, and the slope get stiff between order 4 and 5.

Figure 8 is that of solution of order 7. In this case, the error stops to decrease after the order is larger than 7. This part should be considered carefully and need to be made sure the applicable range of the numerical method.

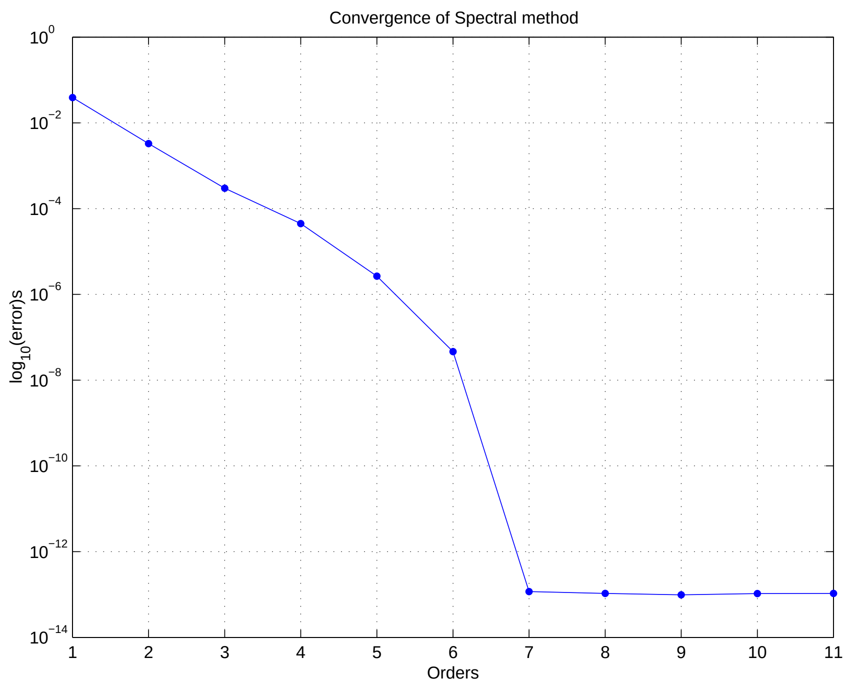

Figure 7: Graph showing convergence of order 7 problem

3.3.3 Test of Convergence of Solution in Variable Ordered Elements

According to the idea that the solution need to be carefully approximated specially in the center of the curve, we assign different orders by the position of elements. I tested 2 cases. The one is to variate 3 elements in center among 5 elements. The other is to variate 1 element in the exact center of all elements.

Figure 9 is the first case with 3 different orders at the 2 ends of elements. Since it is based on the solution of order 5, the orders in 3 center elements moves from 1 to 5.

Figure 10 is the same as Figure 9 except that it is based on solution of order 7 problem and the orders at the center elements varies from 1 to 7.

Figure 11 and Figure 12 are the same as 9 and 10 except that these have the element that varies only a center element. The 2 different control of orders on each element doesn't give out much different error movement. This means choosing wise orders in each element can save time of computing since the lower is the order, the faster does the system solve.

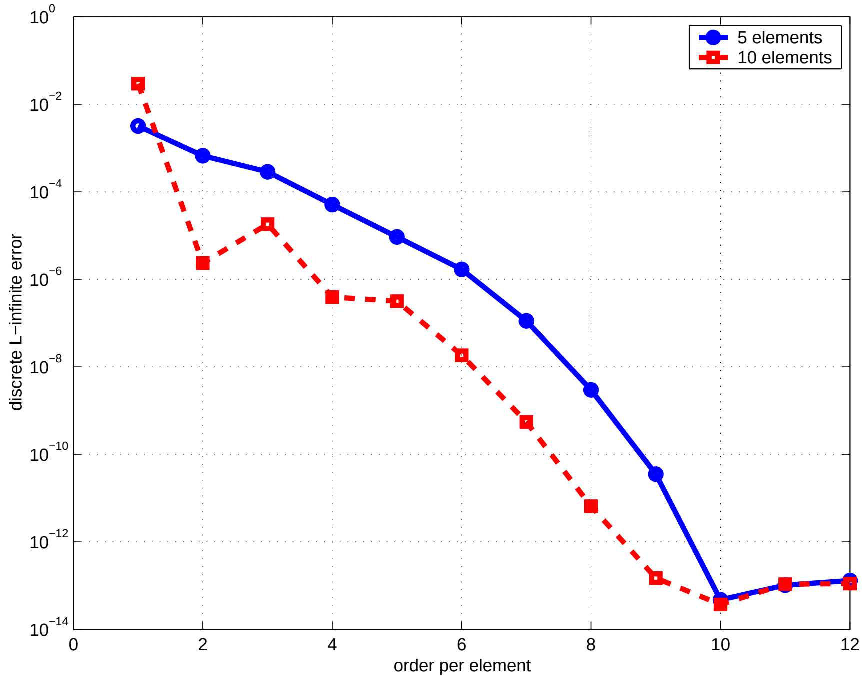

Figure 8: Graph showing convergence of order 10(explicit) problem with 5/10 elements

References

[1] **Spectral/hp Element Method** Karniadarkis, Wat. Res., 24, 97-101. S.A. 1990.

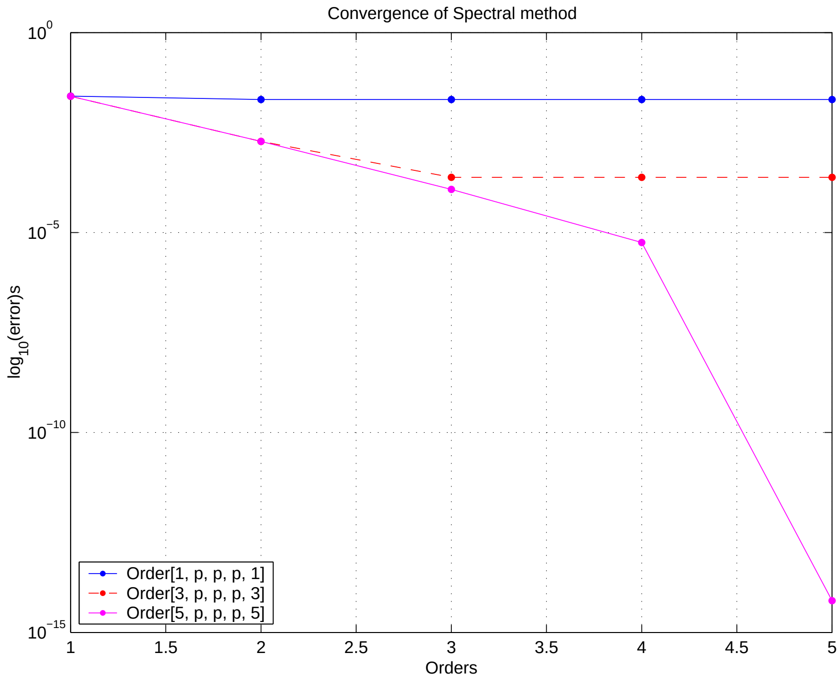

Figure 9: Graph showing convergence of order 5 problem

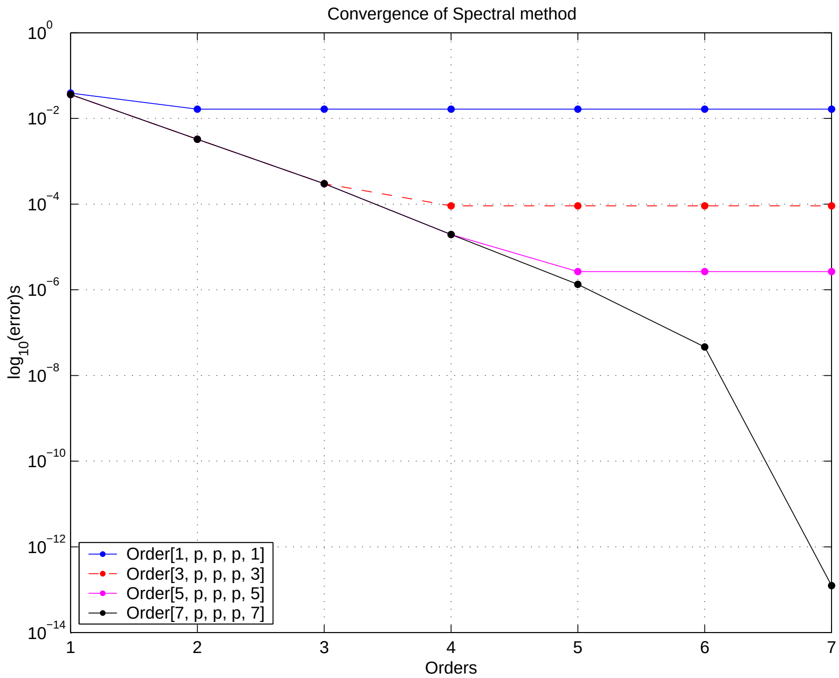

Figure 10: Graph showing convergence of order 7 problem

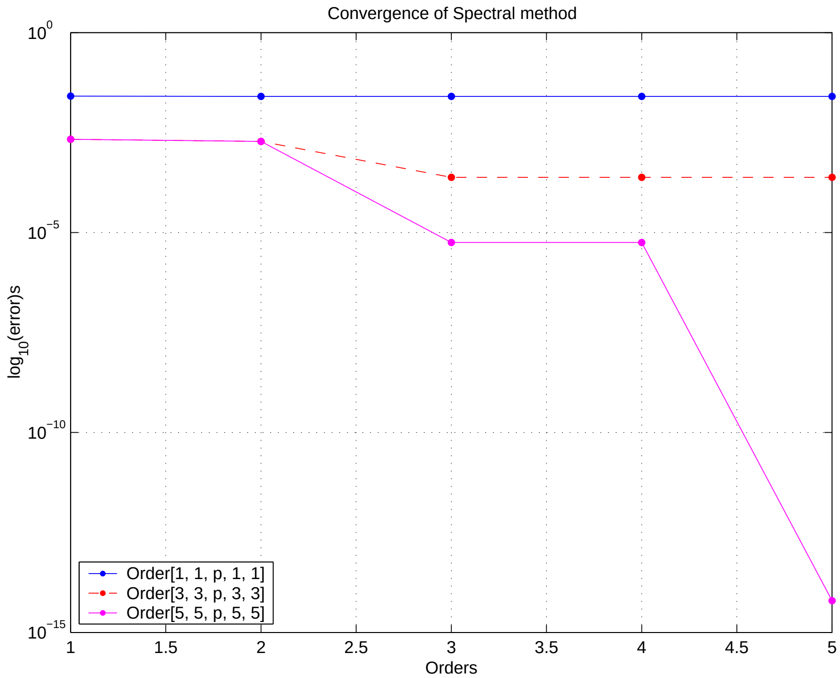

Figure 11: Graph showing convergence of order 5 problem

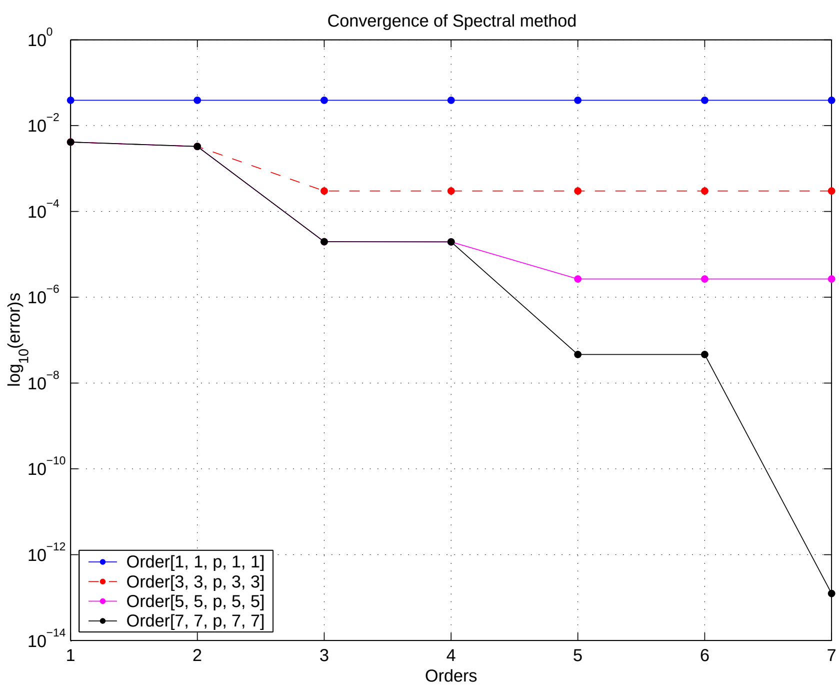

Figure 12: Graph showing convergence of order 7 problem