For angular direction we have mode $k = 0, \cdots, N-1$, azimuthal direction we have mode $p = 0, \cdots, M$. We have bases for each of direction

$$\{\psi(r)\}_{r=0}^{M}, \tag{1}$$
$$\{e^{ik\theta}\}_{k=0}^{N-1}. \tag{2}$$

Then in 2D circular domain, the solution u satisfies following

$$L(u)(x,y) \equiv \frac{d^2}{dx^2}u(x) + \frac{d^2}{dy^2}u(y) - f(x,y) = 0, \tag{3}$$

for all (x, y) in a circular domain with a hole in the middle.

We change the coordinate to polar so that the variables are r, θ. We denote $u(x, y), f(x, y)$ in polar coordinate as $\tilde{u}(r, \theta), \tilde{f}(r, \theta)$ such that

$$\frac{\partial^2}{\partial r^2}\tilde{u} + \frac{1}{r}\frac{\partial}{\partial r}\tilde{u} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\tilde{u} = \tilde{f} \tag{4}$$

To erase the fraction term $\frac{1}{r^2}$, we use the fact the domain has hole at the center, then $r \neq 0$. This enables to multiply r^2 at both sides as follows

$$r^2\frac{\partial^2}{\partial r^2}\tilde{u} + r\frac{\partial}{\partial r}\tilde{u} + \frac{\partial^2}{\partial \theta^2}\tilde{u} = r^2\tilde{f} \tag{5}$$

By using the above bases functions, we represent u as

$$\tilde{u}(r,\theta) = \sum_{p=0,q=0}^{M,N-1} \hat{u}_{p,q}\psi_p(r)e^{iq\theta} \tag{6}$$

$$\tag{7}$$

We have the following linear combination:

$$\left(r^2\nabla^2\tilde{u}, \tilde{\nu}\right) = \left(r^2\tilde{f}, \tilde{\nu}\right), \tag{8}$$

for each $\tilde{\nu} =$ which has same representation as $\tilde{u}$.

For any two (p, q), (j, k) components in basis $\{\psi_p(r)e^{iq\theta}\}_{p=0,q=0}^{M,N-1}$, we can define components of inner product block matrix:

$$\begin{aligned}
\left(r^2\nabla^2\left[\psi_p(r)e^{iq\theta}\right], \psi_j(r)e^{ik\theta}\right) &= \int_0^{2\pi}\int_a^b r^2\frac{\partial^2}{\partial r^2}\left[\psi_p(r)e^{iq\theta}\right]\left[\psi_j(r)e^{ik\theta}\right] dr d\theta && \cdots T_1 \\
&+ \int_0^{2\pi}\int_a^b r\frac{\partial}{\partial r}\left[\psi_p(r)e^{iq\theta}\right]\left[\psi_j(r)e^{ik\theta}\right] dr d\theta && \cdots T_2 \\
&+ \int_0^{2\pi}\int_a^b \frac{\partial^2}{\partial \theta^2}\left[\psi_p(r)e^{iq\theta}\right]\left[\psi_j(r)e^{ik\theta}\right] dr d\theta && \cdots T_3
\end{aligned}$$

$$
\begin{aligned}
T_1 \cdots & \int_0^{2\pi} \int_a^b r^2 \frac{\partial^2}{\partial r^2} \left[\psi_p(r) e^{iq\theta}\right] \left[\psi_j(r) e^{ik\theta}\right] dr d\theta \\
&= \left[\int_a^b r^2 \frac{\partial^2}{\partial r^2} \left[\psi_p(r)\psi_j(r)\right] dr\right] \left[\int_0^{2\pi} e^{iq\theta} e^{ik\theta} d\theta\right] \\
&= \left[r^2 \psi_j(r) \frac{\partial}{\partial r}\psi_p(r)]\|_a^b - \int_a^b \frac{\partial}{\partial r}\left[r^2\psi_j(r)\right] \frac{\partial}{\partial r}\psi_p(r) dr\right] \delta_{kq} \\
&= \left[r^2 \psi_j(r) \frac{\partial}{\partial r}\psi_p(r)]\|_a^b - \int_a^b (2r\psi_j(r)\frac{\partial}{\partial r}\psi_p(r) + r^2 \frac{\partial}{\partial r}\psi_j(r) \frac{\partial}{\partial r}\psi_q(r) dr)\right] \delta_{kq}
\end{aligned}
$$

$$
\begin{aligned}
T_2 \cdots & \int_0^{2\pi} \int_a^b r \frac{\partial}{\partial r} \left[\psi_p(r) e^{iq\theta}\right] \left[\psi_j(r) e^{ik\theta}\right] dr d\theta \\
&= \left[\int_a^b r \frac{\partial}{\partial r} \psi_p(r)\psi_j(r) dr\right] \left[\int_0^{2\pi} e^{iq\theta} e^{ik\theta} d\theta\right] \\
&= \left[\int_a^b r \frac{\partial}{\partial r} \psi_p(r)\psi_j(r) dr\right] \delta_{qk}
\end{aligned}
$$

$$
\begin{aligned}
T_3 \cdots & \int_0^{2\pi} \int_a^b \frac{\partial^2}{\partial \theta^2} \left[\psi_p(r) e^{iq\theta}\right] \left[\psi_j(r) e^{ik\theta}\right] dr d\theta \\
&= \left[\int_a^b \psi_p(r)\psi_j(r) dr\right] \left[\int_0^{2\pi} \frac{\partial^2}{\partial \theta^2} e^{iq\theta} e^{ik\theta} d\theta\right] \\
&= \left[\int_a^b \psi_p(r)\psi_j(r) dr\right] \delta_{qk}
\end{aligned}
$$