1 Formulation of Boundary Element Method on a Circle

In this section, we define a differential equation which satisfies Laplace equation defined on a circle γ as follows:

- **Domain** : A Disk of radius 1.
- **Boundary Condition** : **Outer** normal vectors.
- **Analytic solution** : $u(x, y) = x^2 - y^2$.
- **Orientation of Integral** : Counter clock wise.

Note that, regardless of direction of normal vectors, mathematically this formulation should gives an unique numerical solution.

According to boundary element method, we can set up the following relation:

$$\frac{1}{2}u(x) = -\int_{\gamma} u(z)\nabla G(z,x) \cdot \mathbf{n}dz + \int_{\gamma} \frac{\partial u}{\partial n}(z)G(z,x)dz. \tag{1}$$

By discretization with element E_j on γ, $j = 1 \ldots N$,

$$\frac{1}{2}u(x) = - \sum_{j=1}^{N} \int_{E_j} u(z)\nabla G(z,x) \cdot \mathbf{n}dz + \sum_{j=1}^{N} \int_{E_j} \frac{\partial u}{\partial n}(z)G(z,x)dz. \tag{2}$$

Let $x_1, \ldots, x_N$ be the center points of elements $E_1, \ldots, E_N$. The we can approximate above equation by

$$\frac{1}{2}u(x_i) = -\sum_{j=1}^{N} u(x_j) \int_{E_j} \nabla G(z,x_i) \cdot \mathbf{n}dz + \sum_{j=1}^{N} \frac{\partial u}{\partial n}(x_j) \int_{E_j} G(z,x_i)dz. \tag{3}$$

Define

$$\alpha_j^i = \int_{E_j} G(z,x_i)dz \tag{4}$$

$$\beta_j^i = \int_{E_j} \nabla G(z,x_i) \cdot \mathbf{n}dz. \tag{5}$$

Then we can rearrange above equation by putting unknowns to left and known values to right side as follows:

$$\frac{1}{2}u(x_i) + \sum_{j=1}^{N} \beta_j^i u(x_j) = \sum_{j=1}^{N} \alpha_j^i \frac{\partial u}{\partial n}(x_j), \tag{6}$$

for all $i, j = 1 \ldots N$.

This enables to set up the following system of linear equations for $u(x_j), j = 0, \dot{;} N$.

$$\begin{bmatrix} \frac{1}{2}+\beta_1^1 & \beta_2^1 & \cdots & \beta_N^1 \\ \beta_1^2 & \frac{1}{2}+\beta_2^2 & \cdots & \beta_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1^N & \beta_2^N & \cdots & \frac{1}{2}+\beta_N^N \end{bmatrix} \begin{bmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_N) \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^N \alpha_j^1 \frac{\partial u}{\partial n}(x_j) \\ \sum_{j=1}^N \alpha_j^2 \frac{\partial u}{\partial n}(x_j) \\ \vdots \\ \sum_{j=1}^N \alpha_j^N \frac{\partial u}{\partial n}(x_j) \end{bmatrix}. \tag{7}$$

Note that $\{\alpha_j^i, \beta_j^i\}$ in (7) assume the condition defined in the beginning(CCW, Outer normal).

Mathematically, the original problem has same result as the case when we have both clockwise orientation and inward normal vector. In boundary element formulation, this fact can be seen through the change in (7), since $\{\alpha_j^i, \frac{\partial u}{\partial n}(x_j)\}$ have opposite signs but $\{\beta_j^i\}$ has no sign change.

- Radius 1cm
- Number of elements: $2^6 \cdots 2^{12}$
- Slope: 1.45xx

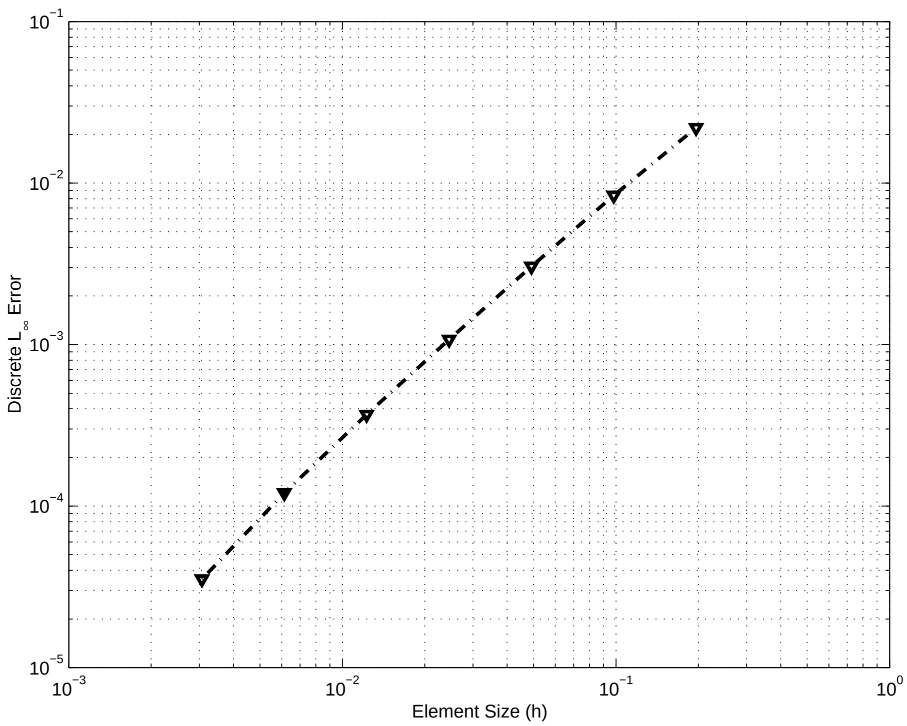

Figure 1: This figure is showing the h-convergence test for above problem

2 Case: Clockwise Orientation with Outer normal

In this section we define shortly a slightly different example based on the initial definition of example at previous section.

- **Domain** : A Disk of radius 1.
- **Boundary Condition** : **Outer** normal vectors.
- **Analytic solution** : $u(x, y) = x^2 - y^2$.
- **Orientation of Integral** : Clock wise.

This cause the changes to each element of formulation above:

- α_j^i : Sign Change
- β_j^i : Sign Change
- $\frac{\partial u}{\partial n}(x_j)$: No Change.

With the same meaning of notation as (7), we can set up the system of linear equation for this problem by modifying signs:

$$\begin{bmatrix} \frac{1}{2} - \beta_1^1 & -\beta_2^1 & \cdots & -\beta_N^1 \\ -\beta_1^2 & \frac{1}{2} - \beta_2^2 & \cdots & -\beta_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ -\beta_1^N & -\beta_2^N & \cdots & \frac{1}{2} - \beta_N^N \end{bmatrix} \begin{bmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_N) \end{bmatrix} = \begin{bmatrix} -\sum_{j=1}^N \alpha_j^1 \frac{\partial u}{\partial n}(x_j) \\ -\sum_{j=1}^N \alpha_j^2 \frac{\partial u}{\partial n}(x_j) \\ \vdots \\ -\sum_{j=1}^N \alpha_j^N \frac{\partial u}{\partial n}(x_j) \end{bmatrix}. \tag{8}$$

3 Question

Above two cases (7 and 8) has only difference in its orientation. But two solutions from 7 and 8 are definitely different. Is there anything wrong on the formulation of second problem(clockwise orientation)? I wonder if Gauss theorem presume a type of orientation or direction of normal vectors.

4 Computation of $\alpha_j(x_i), \beta_j(x_i)$

We set

$$(x_1, x_2) = R(\cos(\theta_1), \sin(\theta_1)) \tag{9}$$
$$(z_1, z_2) = r(\cos(\theta_2), \sin(\theta_2)) \tag{10}$$

4.1 Nonsingular α

$$\alpha_j(x) = \int_{E_j} -\frac{1}{2\pi}\log|x-z|dx \tag{11}$$
$$= -\frac{1}{2\pi}\int_{E_j}\frac{1}{2}\log\left[(x_1-z_1)^2+(x_2-z_2)^2\right]dx \tag{12}$$
$$= -\frac{1}{2\pi}r|J_j|\frac{1}{2}\left[\sum_k \log\{(R\cos(\theta_1)-r\cos(\theta_2(\xi_k))^2+(R\sin(\theta_1)-r\sin(\theta_2(\xi_k))^2\}\cdot w_k\right] \tag{13}$$

4.2 Singular α

In this case we have $R=r$. This enables:

$$|x-z| = 2R\sin(\frac{|\theta_1-\theta_2|}{2}). \tag{14}$$

$$\alpha_j(x_i) = -\frac{R}{2\pi}\int_{\theta_j}^{\theta_{j+1}}\log\left[2R\sin(\frac{|\theta-\theta_i|}{2})\right]d\theta \tag{15}$$
$$= -\frac{R}{2\pi}\int_{\theta_j}^{\theta_{j+1}}\log\left[\frac{\sin(\frac{|\theta-\theta_i|}{2})}{\frac{|\theta-\theta_i|}{2}}\right]+\log(R|\theta-\theta_i|)d\theta \tag{16}$$
$$= -\frac{R}{2\pi}|J_j|\left[\sum_k\log(\frac{\sin(\frac{|\theta-\theta_i|}{2})}{\frac{|\theta-\theta_i|}{2}})\right]-\frac{R}{2\pi}\left[\int_{\theta_j}^{\theta_{j+1}}\log(R|\theta-\theta_i|)d\theta\right]. \tag{17}$$

Note that

$$-\frac{R}{2\pi}\left[\int_{\theta_j}^{\theta_{j+1}}\log(R|\theta-\theta_i|)d\theta\right] \tag{18}$$
$$= -\frac{R}{2\pi}\left[\int_{\theta_j}^{\theta_i}\log\{R(\theta_i-\theta)\}d\theta+\int_{\theta_i}^{\theta_{j+1}}\log\{R(\theta-\theta_i)\}d\theta\right] \tag{19}$$
$$= -\frac{R}{2\pi}\left[\int_{R(\theta_i-\theta_j}^{0}\log\psi(-\frac{1}{R})d\psi+\int_0^{R(\theta_{j+1}-\theta_i}\log\psi\frac{1}{R}d\psi\right] \tag{20}$$
$$= -\frac{R}{2\pi}\frac{1}{R}\left[\int_0^{R(\theta_i-\theta_j)}\log\psi d\psi+\int_0^{R(\theta_{j+1}-\theta_i)}\log\psi d\psi\right] \tag{21}$$
$$= -\frac{1}{2\pi}\left[(\theta_i-\theta_j)\{\log\{R(\theta_i-\theta_j)\}-1\}+(\theta_{j+1}-\theta_i)\{\log\{R(\theta_{j+1}-\theta_i)\}-1\}\right] \tag{22}$$
$$\tag{23}$$

4.3 Nonsingular β

$$\begin{aligned} G(z,x) &= -\frac{1}{2\pi}\log|z-x| = -\frac{1}{2\pi}\frac{1}{2}\log\left[(z_1-x_1)^2+(z_2-x_2)^2\right] && (24)\\ G_{z_1} &= -\frac{1}{2\pi}\frac{z_1-x_1}{(z_1-x_1)^2+(z_2-x_2)^2} && (25)\\ G_{z_2} &= -\frac{1}{2\pi}\frac{z_2-x_2}{(z_1-x_1)^2+(z_2-x_2)^2} && (26)\\ & && (27) \end{aligned}$$

$$\begin{aligned} \nabla_z G(z,x)\cdot\mathbf{n} &= G_{z_1}\cos\theta_z + G_{z_2}\sin\theta_z && (28)\\ &= -\frac{1}{2\pi}\frac{(r\cos\theta_z - R\cos\theta_x)\cos\theta_z + (r\sin\theta_z - R\sin\theta_x)\sin\theta_z}{(r\cos\theta_z - R\cos\theta_x)^2 + (r\sin\theta_z - R\sin\theta_x)^2}. && (29) \end{aligned}$$

$$\beta_j(x_i) = R\int_{\theta_j}^{\theta_{j+1}} \nabla G(z,x)\cdot\mathbf{n}d\theta \tag{30}$$

$$= R|J_j|(-\frac{1}{2\pi})\sum_k \frac{(r\cos\theta_z(\xi_k) - R\cos\theta_x)\cos\theta_z(\xi_k) + (r\sin\theta_z(\xi_k) - R\sin\theta_x)\sin\theta_z(\xi_k)}{(r\cos\theta_z(\xi_k) - R\cos\theta_x)^2 + (r\sin\theta_z(\xi_k) - R\sin\theta_x)^2}\cdot w_k \tag{31}$$

4.4 Singular β

$$\begin{aligned} \beta_j(x_i) &= R\int_{\theta_j}^{\theta_{j+1}} -\frac{1}{2\pi}\frac{R(\cos\theta_z-\cos\theta_x)\cos\theta_z + R(\sin\theta_z-\sin\theta_x)\sin\theta_z}{R^2(\cos\theta_z-\cos\theta_x)^2+R^2(\sin\theta_z-\sin\theta_x)^2}d\theta && (32)\\ &= -\frac{1}{2\pi}\frac{1}{2}(\theta_{j+1}-\theta_j) && (33) \end{aligned}$$

5 Check list for debugging Bem with constant basis on a circle

- alpha - nonsing

  1. direct testing - Test passed

- alpha - sing

  1. direct testing - On testing

1. 1) trepezoidal / simson's a) b)
2. 2) trepezoidal / my quadrature a) b)
3. 3) simson's / my quadrature a) b)

- beta - nonsing
  - direct testing - Test passed, yet the case with radC=radT untested.
  - circuit testing - needless
- beta - sing
  - direct testing - Test passed
  - circuit testing - Test

6 Formulation of BEM on Annulus

The original formulation for any boundary γ:

$$\frac{1}{2}u(x) = -\int_{\gamma} u(z)\nabla G(z,x)\cdot \mathbf{n}dz + \int_{\gamma} \frac{\partial u}{\partial n}(z)G(z,x)dz. \tag{34}$$

In annulus domain we set two circle γ_1 and γ_2 to compose γ, where γ_1 is outer circle and γ_2 is inner circle respectively. Then we have

$$\frac{1}{2}u(x) = -\int_{\gamma_1} u(z)\nabla G(z,x)\cdot \mathbf{n}dz - \int_{\gamma_2} u(z)\nabla G(z,x)\cdot \mathbf{n}dz + \int_{\gamma_1} \frac{\partial u}{\partial n}(z)G(z,x)dz + \int_{\gamma_2} \frac{\partial u}{\partial n}(z)G(z,x)dz. \tag{35}$$

for all x on γ.

By discretization with element E_j^1 for γ_1, and E_j^2 for γ_2, $j = 1 \dots N$,

$$\begin{aligned}
\frac{1}{2}u(x) \quad &= \qquad (36)\\
&- \sum_{j=1}^{N}\int_{E_j^1} u(z)\nabla G(z,x)\cdot \mathbf{n}dz - \sum_{j=1}^{N}\int_{E_j^2} u(z)\nabla G(z,x)\cdot \mathbf{n}dz \qquad (37)\\
&+ \sum_{j=1}^{N}\int_{E_j^1} \frac{\partial u}{\partial n}(z)G(z,x)dz + \sum_{j=1}^{N}\int_{E_j^2} \frac{\partial u}{\partial n}(z)G(z,x)dz. \qquad (38)
\end{aligned}$$

Let $x_1^k, \dots, x_N^k$ be the center points of elements $E_1^k, \dots, E_N^k$ for $k = 1, 2$. The we can approximate above equation by

$$\begin{aligned}
\frac{1}{2}u(x_i^1) \quad &= \qquad (39)\\
&- \sum_{j=1}^{N} u(x_j^1)\int_{E_j^1} \nabla G(z,x_i^1)\cdot \mathbf{n}dz - \sum_{j=1}^{N} u(x_j^2)\int_{E_j^2} \nabla G(z,x_i^1)\cdot \mathbf{n}dz \qquad (40)\\
&+ \sum_{j=1}^{N} \frac{\partial u}{\partial n}(x_j^1)\int_{E_j^1} G(z,x_i^1)dz + \sum_{j=1}^{N} \frac{\partial u}{\partial n}(x_j^2)\int_{E_j^2} G(z,x_i^1)dz. \qquad (41)
\end{aligned}$$

Define

$$
\begin{aligned}
\alpha_j^k(x) &= \int_{E_j^k} G(z,x)dz && (42)\\
\beta_j^k(x) &= \int_{E_j^k} \nabla G(z,x)\cdot \mathbf{n} dz && (43)
\end{aligned}
$$

where $k = 1, 2$. Then we can rearrange above equation by putting unknowns to left and known values to right side as follows:

$$
\begin{aligned}
\frac{1}{2}u(x_i^1) &+ \sum_{j=1}^{N}\beta_j^1(x_i^1)u(x_j^1) - \sum_{j=1}^{N}\alpha_j^2(x_i^1)\frac{\partial u}{\partial n}(x_j^2) && (44)\\
&= -\sum_{j=1}^{N}\beta_j^2(x_i^1)u(x_j^2) + \sum_{j=1}^{N}\alpha_j^1(x_i^1)\frac{\partial u}{\partial n}(x_j^1). && (45)
\end{aligned}
$$

Let F_i^1 be right side of above equation, $i = 1, \dots, N$.
With same way, by applying $x = x_i^2, i = 1 \dots N$, we obtain the following

$$
\begin{aligned}
& \sum_{j=1}^{N}\beta_j^1(x_i^2)u(x_j^1) - \sum_{j=1}^{N}\alpha_j^2(x_i^2)\frac{\partial u}{\partial n}(x_j^2) && (46)\\
= & -\frac{1}{2}u(x_i^2) - \sum_{j=1}^{N}\beta_j^2(x_i^2)u(x_j^2) + \sum_{j=1}^{N}\alpha_j^1(x_i^2)\frac{\partial u}{\partial n}(x_j^1). && (47)
\end{aligned}
$$

Let F_i^2 be right side of above equation, $i = 1, \dots, N$.
Based on these two equations we can setup system of linear equation as follows:

$$
\begin{bmatrix}
\frac{1}{2}+\beta_1^1(x_1^1) & \beta_2^1(x_1^1) & \cdots & \beta_N^1(x_1^1) & -\alpha_1^2(x_1^1) & \cdots & \cdots & -\alpha_N^2(x_1^1)\\
\beta_1^1(x_2^1) & \frac{1}{2}+\beta_2^1(x_2^1) & \cdots & \beta_N^1(x_2^1) & -\alpha_1^2(x_2^1) & \cdots & \cdots & -\alpha_N^2(x_2^1)\\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots\\
\beta_1^1(x_N^1) & \beta_2^1(x_N^1) & \cdots & \frac{1}{2}+\beta_N^1(x_N^1) & -\alpha_1^2(x_N^1) & \cdots & \cdots & -\alpha_N^2(x_N^1)\\
\beta_1^1(x_1^2) & \beta_2^1(x_1^2) & \cdots & \beta_1^N(x_1^2) & -\alpha_1^2(x_1^2) & \cdots & \cdots & -\alpha_N^2(x_1^2)\\
\beta_1^1(x_2^2) & \beta_2^1(x_2^2) & \cdots & \beta_1^N(x_2^2) & -\alpha_1^2(x_2^2) & \cdots & \cdots & -\alpha_N^2(x_2^2)\\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots\\
\beta_1^1(x_N^2) & \beta_2^1(x_N^2) & \cdots & \beta_1^N(x_N^2) & -\alpha_1^2(x_N^2) & \cdots & \cdots & -\alpha_N^2(x_N^2)
\end{bmatrix}
\begin{bmatrix}
u(x_1^1)\\ u(x_2^1)\\ \vdots\\ u(x_N^1)\\ \frac{\partial u}{\partial n}(x_1^2)\\ \frac{\partial u}{\partial n}(x_2^2)\\ \vdots\\ \frac{\partial u}{\partial n}(x_N^2)
\end{bmatrix}
=
\begin{bmatrix}
F_1^1\\ F_2^1\\ \vdots\\ F_N^1\\ F_1^2\\ F_2^2\\ \vdots\\ F_N^2
\end{bmatrix}.
$$