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Disintegration theory for von Neumann algebras

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Contents

Abstract

The main theorem is the central decomposition of a von Neumann algebra into factors. Disintegration of Hilbert spaces is studied in order to be able for formulate the disintegration theory for von Neumann algebras afterwards. In order to reach the main theorem it will be necessary to investigate maximal abelian algebras and more generally the commutant of abelian algebras. Some technical measure theoretic arguments will be needed, but the underlaying theory is defered to the appendix. The concepts of Polish spaces and analytic sets are also explained there.

Due to a number of measure theoretic arguments the disintegration theory is limited to a countable setting in a certain sense. In effect, all Hilbert spaces studied will be separable, and the main theorem about the central decomposition only applies to von Neumann algebras acting on separable Hilbert spaces.

Resumé på dansk (abstract in Danish)

Hovedsætningen er opspaltningen af en von Neumann algebra som et direkte integral af faktorer. Teorien om disintegration af hilbertrum opbygges først, hvorefter disintegration af von Neumann -algebraer studeres. For at nå hovedresultatet er det nødvendigt at undersøge maksimale abelske algebraer og mere generelt kommutanten af en abelsk algebra. De målteoretiske finurligheder, som behøves, hives ind fra siden, mens en grundigere forklaring udskydes til appendikset. Samme sted finder man også en introduktion til begreberne polske rum og analytiske mængder.

Som en konsekvens af de mange målteoretiske argumenter har disintegrationsteorien en naturlig begrænsning i form af en række forudsætninger om tællelighed. Konkret udmøntes dette i at hilbertrummene, som studeres, forudsættes at være separable. Hovedsætningen om den centrale opspaltning er dermed også kun anvendelig på von Neumann algebraer, der virker på separable hilbertrum.

Introduction

The reader is assumed to be familiar with the basic theory of von Neumann algebras, so this introduction will only be a brief summary of what is considered known and a motivation for the theory presented in the project.

The usual definitions of types can be formulated in the following way.

Definition 0.1 (Types). A von Neumann algebra with unit I is said to be of

> type I, if it has an abelian projection with central support I,
>
> type I_n, if I is the sum of n (orthogonal) equivalent abelian projections (for some cardinal number n),
>
> type II, if it has no non-zero abelian projections, but it has a finite projection with central support I, further diveded into
>
> > type II_1, if also I is finite,
> >
> > type II_∞, if also I is properly infinite,
>
> type III, if it has no non-zero finite projections.

In order for this definition to be appropriate, one should of course verify, that the types exclude each other, that a von Neumann algebra of type I_n is also of type I, and that if a von Neumann algebra is of type I_n and of type I_m then $n = m$. All of this is true, but will not be proved here.

A von Neumann algebra need not be one of the above, but a von Neumann algebra, which is a factor, is exactly one of the types I, II_1, II_∞ or III. Also, any von Neumann algebra may be split into a direct sum of von Neumann subalgebras of the types I, II_1, II_∞ and III. But these subalgebras need not be factors - or direct sums of factors. So the study of von Neumann algebras cannot be reduced to the study of factors in this way. But something along these lines *can* be done. If the focus is narrowed down to von Neumann algebras acting only on separable Hilbert spaces, and one is willing to generalize the definition of a direct sum to that of a direct integral, one reaches the main result of this project: a von Neumann algebra acting on a separable Hilbert spaces is a direct integral of factors. And this result is in fact the main motivation for introducing the concept of factors in the study of von Neumann algebras. It originates from von Neumann and Murray, who studied von

Neumann algebras intensively in the 1930s and 1940s. The background for disintegration theory is the paper *On rings of operators. Reduction theory* written by von Neumann in 1949.

It is not hard to produce examples of von Neumann algebras which are not direct sums of factors. The only abelian factor is the algebra of complex numbers $\mathbb{C}$ (up to isomorphism), and thus any direct sum of abelian factors is of the form $\bigoplus_{i\in I}\mathbb{C}$. Such abelian algebras will always minimal projections (corresponding to each copy of $\mathbb{C}$). It is however easy to come up with abelian algebras with no minimal projections. An example is the abelian von Neumann algebra $L^\infty([0,1],\lambda)$ viewed as multiplication operators on the Hilbert space $L^2([0,1],\lambda)$, where λ denotes the usual Lebesgue measure. The line of thought behind this example is that the measure space $([0,1],\lambda)$ is "continuous" in some sense, where as the index set I is discrete. The continuity property is due to the fact that the measure space $([0,1],\lambda)$ has no atoms, i.e. any set of positive measure has a subset of strictly smaller, positive measure.

Along the lines of type decomposition, we will need the following theorems, which will not be proved here. Proofs can be found in [K&R II].

Theorem 0.2. *If $\mathcal{M}$ is a von Neumann algebra, then the type of the commutant (*I, II *or* III*) is the same as the type of $\mathcal{M}$.*

Theorem 0.3 (Type decomposition). *If $\mathcal{M}$ is a von Neumann algebra acting on a Hilbert space H, there are pairwise orthogonal central projections P_n (where $n \leq \dim H$), P_{II_1}, P_{II_∞} and P_{III} with sum I, maximal with respect to the properties that $\mathcal{M}P_n$ is of type I_n or $P_n = 0$, $\mathcal{M}P_{\mathrm{II}_1}$ is of type II_1 or $P_{\mathrm{II}_1} = 0$, $\mathcal{M}P_{\mathrm{II}_\infty}$ is of type II_∞ or $P_{\mathrm{II}_\infty} = 0$ and $\mathcal{M}P_{\mathrm{III}}$ is of type* III *of $P_{\mathrm{III}} = 0$.*

Every type I *von Neumann algebra can be decomposed as a direct sum of algebras of type I_m. It can also be decomposed as a direct sum of algebras of type I_m with commutants of type I_n.*

Notation. Throughout the project the set of natural numbers (excluding zero), rational, real and complex numbers are denoted by $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, respectively. The notation $B(H)$ will mean the bounded operators on a Hilbert space H.

CHAPTER 1

The commutant of an abelian von Neumann algebra

1.1 Maximal abelian algebras acting on separable spaces

The following section describes all maximal abelian von Neumann algebras acting on separable Hilbert spaces up to unitary equivalence. First we consider some examples, and then we move on to show that those are the only ones.

Example 1.1. Let n be a natural number or $\aleph_0$, and let S_n denote a set with n elements. The bounded functions $\ell^\infty(S_n)$ act on the separable Hilbert space $\ell^2(S_n)$ as multiplication operators M_f. They form a maximal abelian von Neumann algebra $\mathcal{A}_n$. The projections correspond to the characteristic functions of subsets of S_n, and the ordering on the projections correspond to the natural ordering of the subsets of S_n by inclusion. Thus the minimal projections correspond to the single point sets, and $\mathcal{A}_n$ contains exactly n of those. Each $M_f \in \mathcal{A}_n$ has the form $\sum_{y\in S_n} f(y) M_{\chi_{\{y\}}}$ where $\chi_{\{y\}}$ is the characteristic function of $\{y\}$, and the sum is convergent in the strong operator topology. It follows that the minimal projections generate $\mathcal{A}_n$ as a von Neumann algebra.

Notice the pleasant rule $\mathcal{A}_n \oplus \mathcal{A}_m \simeq \mathcal{A}_{n+m}$ for any $n, m = 1, 2, \ldots, \aleph_0$.

Example 1.2. Consider the separable Hilbert space $L^2([0,1])$ with the Lebesgue measure. The essentially bounded, measurable functions $L^\infty([0,1])$ act on $L^2([0,1])$ as multiplication operators, and it is well-known that the algebra of multiplication operators $\mathcal{A}_c$ is a maximal abelian von Neumann subalgebra of the bounded operators on $L^2([0,1])$ (see for instance chap. 4 in [Douglas]). The algebra $\mathcal{A}_c$ has no minimal projections, because the Lebesgue measure has no atoms (i.e. every subset of positive measure has a subset of smaller, positive measure). As remarked in the introduction this should be thought of as a kind of continuity property as opposed to the preceding example. This explains the suffix c.

Of course, direct sums of the examples above yield more examples. We have now produced examples $\mathcal{A}_n$, $\mathcal{A}_c$, $\mathcal{A}_n \oplus \mathcal{A}_c$ $(1 \le n \le \aleph_0)$ of maximal abelian von Neumann algebras acting on separable Hilbert spaces. None of these are *-isomorphic, since $\mathcal{A}_n$ has precisely n minimal projections and is generated by them, $\mathcal{A}_c$ has no minimal projections, and $\mathcal{A}_n \oplus \mathcal{A}_c$ has precisely n minimal projections, but is not generated by them. The following theorem tells us that these are the only examples.

Theorem 1.3. *Let $\mathcal{A}$ be a maximal abelian von Neumann algebra acting on a separable Hilbert space. Then $\mathcal{A}$ is unitarily equivalent to exactly one of the algebras $\mathcal{A}_n$, $\mathcal{A}_c$, $\mathcal{A}_n \oplus \mathcal{A}_c$ ($1 \leq n \leq \aleph_0$).*

We prove the theorem in steps.

Lemma 1.4. *An abelian C^*-algebra $\mathcal{A}$ acting on a separable Hilbert space H has a separating vector.*

Proof. See for example [Zhu].

□

Corollary 1.5. *A maximal abelian von Neumann algebra acting on a separable Hilbert space has a cyclic and separating vector.*

Proof. Note that a separating vector for $\mathcal{A}$ is cyclic for $\mathcal{A}'$, and $\mathcal{A} = \mathcal{A}'$. □

Proposition 1.6. *If $\mathcal{A}$ is a maximal abelian von Neumann algebra acting on a separable Hilbert space H, and $\mathcal{A}$ has no minimal projections, then $\mathcal{A}$ is unitarily equivalent to $\mathcal{A}_c$.*

Before the proof of the proposition we comment on the algebra $\mathcal{A}_c$. In $L^2([0,1])$ the constant function with value 1, denoted χ_1, is a cyclic and separating vector. If G_t denotes the projection corresponding to characteristic function χ_t of $[0,t]$, $0 \leq t \leq 1$, then $\{G_t\}_{t\in[0,1]}$ forms a totally ordered subset of the projections which is order isomorphic to $[0,1]$. Further, since $G_t\chi_1 = \chi_t$, the span of $\{G_t\chi_1 \mid 0 \leq t \leq 1\}$ consists of all step functions, which is dense in $L^2([0,1])$. Notice also the rule

$$\langle G_s\chi_1, G_t\chi_1\rangle = \langle \chi_s, \chi_t\rangle = \min\{s,t\}, \qquad s,t \in [0,1].$$

Proof of Proposition 1.6. The idea of the proof is to find a vector and projections with properties similar to the ones just described for $\mathcal{A}_c$. This will give rise to the unitary we are seeking.

First we prove that $\mathcal{A}$ is generated (as a von Neumann algebra) by a sequence of projections. Let $\mathcal{P}$ denote the set of projections on $\mathcal{A}$, partially ordered by the usual order relation $\leq$. According to Corollary 1.5 there is a cyclic and separating vector x for $\mathcal{A}$. Since H is separable, the set $\{Px \mid P \in \mathcal{P}\}$ has a countable dense subset $\{P_1x, P_2x, \ldots\}$. Given any $P \in \mathcal{P}$ we can find a subsequence $P_{n_k}x$ such that $Px = \lim_k P_{n_k}x$. Then for any $A \in \mathcal{A}$, since $\mathcal{A}$ is abelian,

$$PAx = APx = \lim_k AP_{n_k}x = \lim_k P_{n_k}Ax.$$

Since $\mathcal{A}x$ is dense in H, and since the projections lie in the unit ball of $\mathcal{A}$, we see that $P_{n_k} \to P$ in the strong operator topology. So the von Neumann algebra generated by $\{P_1, P_2, \ldots\}$ contains $\mathcal{P}$ and hence $\mathcal{A}$.

Inductively, we may construct finite, totally ordered subsets $\mathcal{F}_1, \mathcal{F}_2, \ldots$ of $\mathcal{P}$ such that $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$, and P_n is in the span of $\mathcal{F}_n$. First, let $\mathcal{F}_1 = \{0, P_1, I\}$. Having defined $\mathcal{F}_n = \{E_0, E_1, \ldots, E_m\}$, where

$0 = E_0 < E_1 < \cdots < E_m = I$, we have for $0 \leq r < m$

$$E_r + (E_{r+1} - E_r)P_{n+1} \leq E_{r+1},$$

since E_r and $(E_{r+1} - E_r)P_{n+1}$ are orthogonal and both majorized by E_{r+1}. Hence

$$E_r \leq E_r + (E_{r+1} - E_r)P_{n+1} \leq E_{r+1}.$$

Note also that

$$P_{n+1} = \sum_{r=0}^{m-1} (E_{r+1} - E_r)P_{n+1},$$

since the sum is telescoping with $E_0 = 0$, $E_m = I$. So it suffices to define

$$\mathcal{F}_{n+1} = \mathcal{F}_n \cup \{E_r + (E_{r+1} - E_r)P_{n+1} \mid 0 \leq r < m\}.$$

Now, let $\mathcal{F}_\infty = \bigcup_{n=1}^{\infty} \mathcal{F}_n$. Then $\mathcal{F}_\infty$ is a totally ordered subset of $\mathcal{P}$, and the linear span of $\mathcal{F}_\infty$ contains each P_n. Let $\mathscr{C}$ denote the family of all totally ordered subsets of $\mathcal{P}$ which contain $\mathcal{F}_\infty$. Then $\mathscr{C}$ is ordered by inclusion, and if $\mathscr{C}_0$ is a totally ordered subset of $\mathscr{C}$, the union of the sets in $\mathscr{C}_0$ is an upper bound for $\mathscr{C}_0$ in $\mathscr{C}$. By Zorn's lemma $\mathscr{C}$ contains a maximal element $\mathcal{F}$. The set $\mathcal{F}$ should resemble the projections G_t mentioned earlier.

Suppose $\mathcal{F}_0 \subseteq \mathcal{F}$. If $E \in \mathcal{F}$ then either $E \leq F$ for every $F \in \mathcal{F}_0$, in which case also $E \leq \bigwedge \mathcal{F}_0$, or $E \geq F$ for some F, in which case $E \geq \bigwedge \mathcal{F}_0$. Thus, $\mathcal{F} \cup \{\bigwedge \mathcal{F}_0\}$ is totally ordered, and by maximality of $\mathcal{F}$, the we conclude $\bigwedge \mathcal{F}_0 \in \mathcal{F}$. In the same way, $\bigvee \mathcal{F}_0 \in \mathcal{F}$. In other words, $\mathcal{F}$ is a totally ordered, complete lattice. Notice that since $\mathcal{F}_0$ is totally ordered, $\bigwedge \mathcal{F}_0$ and $\bigvee \mathcal{F}_0$ lie in the strong operator closure of $\mathcal{F}_0$.

We now define a strong operator continuous map $\varphi : \mathcal{F} \to [0, 1]$ by $\varphi(F) = \langle Fx, x\rangle$, where x is the cyclic and separating vector from before. If $E < F$ are distinct elements of $\mathcal{F}$, then

$$\varphi(F) - \varphi(E) = \langle (F - E)x, x\rangle = \|(F - E)x\|^2 > 0,$$

since x is separating. Since $\mathcal{F}$ is totally ordered, we infer that φ is order-preserving and injective. The map φ is also surjective as may be seen as follows (this is were it is important that $\mathcal{A}$ has no minimal projections). The range of φ contains 0 and 1, since $0, I \in \mathcal{F}$. If $0 < t < 1$, then let

$$\mathcal{F}_0 = \{F \in \mathcal{F} \mid \varphi(F) < t\}, \quad \mathcal{F}_1 = \{F \in \mathcal{F} \mid \varphi(F) \geq t\}.$$

Let $F_0 = \bigvee \mathcal{F}_0$ and $F_1 = \bigwedge \mathcal{F}_1$. Then $F_0, F_1 \in \mathcal{F}$, and since F_0 is in the strong operator closure of $\mathcal{F}_0$ we get by continuity of φ that $\varphi(F_0) \leq t$. Similarly, $\varphi(F_1) \geq t$. Suppose $F_0 \neq F_1$. Then since $\mathcal{A}$ has no minimal projections, there is a projection E in $\mathcal{P}$ such that $0 < E < F_1 - F_0$. Since $\mathcal{F}$ is the disjoint union of $\mathcal{F}_0$ and $\mathcal{F}_1$, and for $E_0 \in \mathcal{F}_0$, $E_1 \in \mathcal{F}_1$ we have

$$E_0 \leq F_0 < F_0 + E < F_1 \leq E_1,$$

it follows that $\mathcal{F} \cup \{F_0 + E\}$ is totally ordered, a contradiction to the maximality of $\mathcal{F}$. Hence $F_0 = F_1$ and $\varphi(F_0) = t$. This proves surjectivity of φ.

The inverse of φ is a stricly increasing bijection $t \mapsto F_t$. For any $s, t \in [0,1]$ we have

$$\langle F_s x, F_t x\rangle = \langle F_t F_s x, x\rangle = \langle F_{\min\{s,t\}} x, x\rangle = \min\{s,t\} = \langle G_s \chi_1, G_t \chi_1\rangle.$$

It follows that for any $t_1, \ldots, t_n \in [0,1]$ and scalars $a_1, \ldots, a_n \in \mathbb{C}$

$$\left\| \sum_{k=1}^{n} a_k F_{t_k} x \right\| = \left\| \sum_{k=1}^{n} a_k G_{t_k} \chi_1 \right\|.$$

The linear span of $\mathcal{F}$ is strong operator dense in $\mathcal{A}$, and therefore linear combinations of vectors of the form $F_t x$ is dense in $\mathcal{A}x$, and hence also in H. Since we already noticed that the linear span of $G_t \chi_1$ $(0 \le t \le 1)$ is dense in $L^2([0,1])$, there is a unitary $U : H \to L^2([0,1])$ such that $U(F_t x) = G_t \chi_1$. For each $s, t \in [0,1]$

$$UF_s F_t x = UF_{\min\{s,t\}} x = G_{\min\{s,t\}} \chi_1 = G_s G_t \chi_1 = G_s U F_t x,$$

and since vectors of the form $F_t x$ generate H, we get $UF_s = G_s U$, or $UF_s U^{-1} = G_s$. Thus $U\mathcal{F}U^{-1} \subseteq \mathcal{A}_c$. Remember that $\mathcal{F}$ generates $\mathcal{A}$, and so $U\mathcal{A}U^{-1} \subseteq \mathcal{A}_c$. And since $\mathcal{A}$ is maximal abelian, $U\mathcal{A}U^{-1} = \mathcal{A}_c$ (the property of being maximal abelian is preserved under unitary equivalence). This proves that $\mathcal{A}$ is unitarily equivalent to $\mathcal{A}_c$, when $\mathcal{A}$ has no minimal projections. □

Proof of Theorem 1.3. If $\mathcal{A}$ has no minimal projections, then Proposition 1.6 tells us that $\mathcal{A}$ is unitarily equivalent to $\mathcal{A}_c$. Suppose next that $\mathcal{A}$ has minimal projections, and let $\mathcal{E}$ denote the set of such. Let again x be a cyclic and separating vector for $\mathcal{A}$.

If $E \in \mathcal{E}$, then since E is minimal $E\mathcal{A} = \mathbb{C}E$. When $A \in \mathcal{A}$, then $EAx = aEx$ for some scalar $a \in \mathbb{C}$. Since vectors of the form Ax, $A \in \mathcal{A}$ are dense in H, we see that E has one-dimensional range spanned by Ex (note that $Ex \neq 0$, since x is separating).

If E and F are two distinct minimal projections, then EF is a projection majorized by both E and F, and so by minimality of the projections EF must be zero. So the projections in $\mathcal{E}$ are one-dimensional and orthogonal.

Since H is separable, $\mathcal{E}$ must be countable. We write $\mathcal{E} = \{E_k \mid k \in S_n\}$ where n is a cardinal number $1 \le n \le \aleph_0$ and S_n is a set with n elements.

Let $Q = \sum_{E \in \mathcal{E}} E$, and for for each $k \in S_n$ choose e_k to be a unit vector in the range of E_k. Then $\{e_k\}$ is an orthonormal basis for $Q(H)$, and the equation

$$Uy = \sum_{k \in S_n} y(k) e_k$$

defines a unitary operator $U : \ell^2(S_n) \to Q(H)$.

For $A \in \mathcal{A}$, we define $f(k)$ as the scalar such that $E_k A = f(k)E_k$. Since $|f(k)| \leq \|A\|$, the function f belongs to $\ell^\infty(S_n)$. Also, $AQ = \sum_k AE_k = \sum_k f(k)E_k$. It follows that $\mathcal{A}Q$ consists of all operators of the form $\sum_k f(k)E_k$, where $f \in \ell^\infty(S_n)$.

For $y \in \ell^2(S_n)$ and $f \in \ell^\infty(S_n)$ we have

$$\begin{aligned} UM_f y &= \sum_{k \in S_n} (M_f y)(k) e_k = \sum_{k \in S_n} f(k) y(k) E_k e_k \\ &= \left(\sum_{k \in S_n} f(k) E_k \right) \left(\sum_{k \in S_n} y(k) e_k \right) = \left(\sum_{k \in S_n} f(k) E_k \right) Uy \end{aligned}$$

Thus $UM_f U^{-1} = \sum_k f(k)E_k$, and therefor $U\mathcal{A}_n U^{-1} = \mathcal{A}Q$, and $\mathcal{A}Q$ is unitarily equivalent to $\mathcal{A}_n$.

If $Q = I$, then $\mathcal{A}$ is unitarily equivalent to $\mathcal{A}_n$. If $Q \neq I$, then $\mathcal{A}(I - Q)$ is a maximal abelian algebra acting on a separable Hilbert space and has no minimal projections. By Proposition 1.6, $\mathcal{A}(I - Q)$ is unitarily equivalent to $\mathcal{A}_c$, and then $\mathcal{A}$ is unitaily equivalent to $\mathcal{A}Q \oplus \mathcal{A}(I - Q)$ and hence to $\mathcal{A}_n \oplus \mathcal{A}_c$. This almost completes the proof.

To show uniqueness recall that none of the possibilities listed are *-isomorphic. □

1.2 The commutant of an abelian von Neumann algebra

In order to use the disintegration theory to be described later it will be necessary to know the structure of commutants of abelian von Neumann algebras. Theorem 1.3 is a big step on the way, and together with Theorem 1.9 it provides almost all the information needed.

Definition 1.7 (Inflation algebra). Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space H, and let n be a cardinal number. The *n'th inflation algebra* of $\mathcal{M}$ is the von Neumann algebra acting on the direct sum of n copies of H consisting of operators, whose matrix representation has all diagonal entries equal to the same operator in $\mathcal{M}$ and zero entries elsewhere. The n'th inflation algebra of $\mathcal{M}$ is denoted $\mathcal{M}^{(n)}$. If $n = \aleph_0$, one often writes $\mathcal{M}^{(\infty)}$ for the inflation algebra.

Remark 1.8. Note that $\mathcal{M}$ and $\mathcal{M}^{(n)}$ are *-isomorphic, but not necessarily unitarily equivalent.

Theorem 1.9. *Suppose $\mathcal{A}$ is an abelian von Neumann algebra with commutant of type I_n. Then $\mathcal{A}$ is unitarily equivalent to the n'th inflation algebra of a maximal abelian von Neumann algebra.*

Proof. Let H denote the Hilbert space on which $\mathcal{A}$ acts. Since $\mathcal{A}'$ is of type I_n, there is a family $(P_j)_{j \in J}$ of n orthogonal, equivalent, abelian projections in $\mathcal{A}'$ whose sum is I. Let P be one of the projections in the family, and let K be the range of P. Define $\mathcal{M}$ to be the von Neumann algebra $\mathcal{M} = P\mathcal{A}'P$ acting on K.

Since P is abelian in $\mathcal{A}'$, $\mathcal{M}$ is abelian. Also, $\mathcal{M}' = P\mathcal{A}''P = P\mathcal{A}P$ is a subalgebra of the abelian algebra $\mathcal{A}$, and hence $\mathcal{M}'$ is abelian. Thus $\mathcal{M} = \mathcal{M}' = P\mathcal{A}P$ and $\mathcal{M}$ is maximal abelian.

For each $j \in J$, let V_j be the partial isometry in $\mathcal{A}'$ with initial projection P_j and final projection P guaranteed by the equivalience $P \sim P_j$. Since $(P_j)_{j \in J}$ has sum I, the operator $V : H \to \bigoplus_{j \in J} K$

defined by

$$Vx = (V_j x)_{j \in J}, \quad x \in H$$

is unitary. It simply corresponds to the decompositions of H into n copies of K. Notice also that for $T \in \mathcal{A}$ (using $V_j, P \in \mathcal{A}'$) we have

$$VTx = (V_j Tx)_j = (TV_j x)_j = (TPV_j x)_j = (PTPV_j x)_j = (\bigoplus_{j \in J} PTP)(Vx).$$

Thus $VTV^{-1} = \bigoplus_{j \in J} PTP$, and $V\mathcal{A}V^{-1} = (P\mathcal{A}P)^{(n)} = \mathcal{M}^{(n)}$. We conclude that $\mathcal{A}$ is unitarily equivalent to $\mathcal{M}^{(n)}$. □

Combining this theorem with Theorem 1.3 and Theorem 0.3 we obtain a complete description of the commutant of any abelian von Neumann algbera acting on a separable Hilbert space. This will be used in the proof of Theorem 3.1.

CHAPTER 2

Disintegration theory

2.1 Disintegration of Hilbert spaces

In this section we describe what it means for a Hilbert space to be a direct integral of Hilbert spaces, and we give some basic, but important, examples of Hilbert spaces which are direct integrals.

The motivation for direct integrals of Hilbert spaces is the concept of a direct sum. A direct integral should be a generalization of a direct sum, just as a sum of numbers can be viewed as an integral with respect to the counting measure on the index set. The idea is to replace the 'discrete' index set of a direct sum by a measure space. Since measure spaces always come with certain countability restrictions, the theory build in the following will also have naturally arising countability restrictions. In practice, it is reflected in the assumption that our Hilbert spaces are separable. For instance the separability assumption implies that the weak and strong operator topologies on the unit ball of $B(H)$ are metrizable, and hence we can deal with sequences in stead of nets. It will also be necessary to take care of some measurability issues along the way.

In all of the following (X,μ) will denote a measure space. The measure μ is assumed to be complete, i.e. subsets of (measurable) null sets are again measurable sets, of course of measure zero. This is not a strong assumption, since any measure can be completed, i.e. extended to a complete measure. It will however be useful, since we sometimes only know the behaviour of a function outside a set of measure zero, and this will then be enough to show the required measurability (see Lemma A.2). When it is also assumed that X is a topological space, the σ-algebra on X will be assumed to be *completion of the Borel algebra (generated by the open sets) with respect to some Borel measure*.

There are two natural approaches to the theory of direct integrals of Hilbert spaces. One is to start with a collection of Hilbert spaces and try to define what is meant by the direct integral of those. This approach could be termed 'integration'. The other approach is to consider a given Hilbert space and ask whether it can be decomposed into smaller Hilbert spaces which 'add up' to the given one. This approach could be termed 'disintegraion' or 'decomposition', and it is the latter approach that we shall deal with.

Definition 2.1 (Direct integral decomposition of a Hilbert space). Let (X,μ) be a complete measure space, and let $\{H_p\}_{p\in X}$ be a set of non-zero Hilbert spaces indexed by X. A Hilbert space H is called the *direct integral* of $\{H_p\}$ (with respect to μ), if to each $x \in H$ there corresponds an element of $\prod_p H_p$ which we will also denote by x, i.e. a map $p \mapsto x(p) \in H_p$, such that

(i) for all $x, y \in H$ the map $p \mapsto \langle x(p), y(p)\rangle$ is measurable and integrable (with respect to μ) and

$$\langle x, y\rangle = \int_X \langle x(p), y(p)\rangle \, d\mu,$$

(ii) if $z_p \in H_p$ for all p in X and for every $y \in H$ the map $p \mapsto \langle z_p, y(p)\rangle$ is (measurable and) integrable, then there is a $z \in H$ such that $z(p) = z_p$ for μ-almost all p.

We write

$$H = \int_X H_p \, d\mu(p) \quad \text{and} \quad x = \int_X x(p)\, d\mu(p).$$

Part (ii) of the preceding definition should be thought of as a way to ensure that all of H is contained in the direct integral decomposition. The assumption that each H_p is non-zero is more or less superfluous, since we might as well reduce the disintegration to the subset where each H_p is non-zero and still get the same out of it. But it will simplify an argument or two along the way to have this assumption.

Some familiar Hilbert spaces are seen to be direct integrals.

Proposition 2.2. *Let (X, μ) be a σ-finite measure space. Then $L^2(X, \mu)$ is a direct integral of one-dimensional Hilbert spaces over X.*

Proof. Let $H_p = \mathbb{C}$ for each p in X. To each $f \in L^2(X, \mu)$ choose a representative for f, also denoted by f. Part (i) of Definition 2.1 is obvious by the definition of $L^2(X, \mu)$. Part (ii) amounts to checking that if $fg \in L^1(X, \mu)$ for each $g \in L^2(X, \mu)$ then $f \in L^2(X, \mu)$. This will now be done.

Since X is σ-finite we may write $X = \bigcup_{n=1}^{\infty} X_n$, where $\{X_n\}$ is an increasing sequence of sets of finite measure. Define

$$f_n(x) = \begin{cases} f(x), & x \in X_n, |f(x)| \le n \\ 0, & \text{else} \end{cases}$$

Since f_n is measurable, bounded and vanishes outside a set of finite measure, $f_n \in L^2(X, \mu)$. For each n let φ_n be the functional on $L^2(X, \mu)$ corresponding to f_n, i.e. given by

$$\varphi_n(g) = \int_X f_n g \, d\mu, \quad g \in L^2(X, \mu).$$

For each $x \in X$, $f_n(x) \to f(x)$. Since $|f_n g| \le |fg|$, and by assumption $|fg|$ is integrable, Lebesgue's dominated convergence theorem implies that

$$\varphi_n(g) = \int_X f_n g \, d\mu \to \int_X fg \, d\mu.$$

Hence $\sup_n |\varphi_n(g)| < \infty$ for each $g \in L^2(X, \mu)$, and so by the uniform boundedness principle, $\sup_n \|\varphi_n\| < \infty$. Since $|f_n(x)| \nearrow |f(x)|$, it follows from Lebesgue's monotone convergence theorem that

$$\|\varphi_n\|^2 = \|f_n\|_2^2 = \int_X |f_n|^2 \, d\mu \to \int_X |f|^2 \, d\mu,$$

and using $\sup_n \|\varphi_n\|^2 < \infty$, we reach the conclusion $f \in L^2(X, \mu)$. $\square$

Lemma 2.3. *Let A be a set and suppose $f : A \to [0, \infty[$ satisfies $\sum_{a \in B} f(a) < \infty$ for every countable subset $B \subseteq A$. Then $\sum_{a \in A} f(a) < \infty$.*

Proof. For each natural number n, let $A_n = \{a \in A \mid f(a) > \frac{1}{n}\}$. If A_n were infinite, it would have some countably infinite subset B_n, and then $\sum_{a \in B_n} f(a) = \infty$, contrary to the assumption. Hence each A_n is finite, and f has countable support $\operatorname{supp}(f) = \bigcup_{n=1}^{\infty} A_n$. Then $\sum_{a \in A} f(a) = \sum_{a \in \operatorname{supp}(f)} f(a) < \infty$. $\square$

Proposition 2.4. *A direct sum is also a direct integral.*

Proof. Suppose H is the direct sum of $\{H_p\}_{p \in X}$. We equip X with the counting measure τ defined on all subsets $\mathcal{P}(X)$ of X. If x is in H then x is a map $p \mapsto x(p) \in H_p$, and by the definition of the inner product structure on a direct sum

$$\langle x, y \rangle = \sum_{p \in X} \langle x(p), y(p) \rangle.$$

This sum is also the integral with respect to τ of the map $p \mapsto \langle x(p), y(p) \rangle$. So part (i) of Definition 2.1 is satisfied.

To see part (ii), suppose $z_p \in H_p$ for each p in X and that $p \mapsto \langle z_p, y(p) \rangle$ is integrable (i.e. summable) for every $y \in H$ (measurability is not an issue here). We should show that $p \mapsto \|z_p\|$ is square-summable, because then $p \mapsto z_p$ is in the direct sum. By the lemma, it is enough to prove that $p \mapsto \|z_p\|$ is square-summable over any countable subset of X.

Let X_0 be a countable subset of X. Note that $(X_0, \mathcal{P}(X_0), \tau)$ is a σ-finite measure space, and we can use an argument similar to the one produced in the preceding proposition. Therefore, let f be any element of $\ell^2(X_0)$. When $p \in X_0$ and $z_p \neq 0$, we define $y(p) = f(p)\|z_p\|^{-1} z_p$, and $y(p) = 0$ otherwise. Then clearly $(y(p))_{p \in X}$ is in H and

$$\sum_{p \in X_0} \|z_p\| \, |f(p)| = \sum_{p \in X_0} \langle z_p, y(p) \rangle = \sum_{p \in X} \langle z_p, y(p) \rangle < \infty$$

From the argument in the preceding proposition it follows that $p \mapsto \|z_p\|$ is in $\ell^2(X_0)$. This verifies part (ii). $\square$

Remark 2.5. If x and y are in H, and $x(p) = y(p)$ almost everywhere then

$$\langle x - y, x - y \rangle = \int_X \langle x(p) - y(p), x(p) - y(p) \rangle \, d\mu = 0$$

by using (i) in Definition 2.1 and (sesqui)linearity of $\langle \cdot, \cdot \rangle$ and the integral. Hence $x = y$.

If $x, y \in H$ and a is scalar, it will be reasonable to think that the function corresponding to $ax + y$ agrees with $p \mapsto ax(p) + y(p)$ almost everywhere. From (ii) there is an element z in H such that $ax(p) + y(p) = z(p)$ almost everywhere. And for any u in H we have

$$\langle ax + y - z, u\rangle = \int_X \langle ax(p) + y(p) - z(p), u(p)\rangle \, d\mu = 0.$$

Hence $z = ax + y$. This says that the decomposition of $ax + y$ agrees with $p \mapsto ax(p) + y(p)$ almost everywhere.

The following measure theoretic lemma is important and will be used over and over without mentioning. For a proof see [Hansen] Corollary 1.20.

Lemma 2.6. *Suppose (f_n) is a sequence of integrable functions converging in 1-norm to 0. Then there is a subsequence (f_{n_k}) that converges to 0 pointwise almost everywhere.*

Notation. If S is a set of vectors in H, we will use the notation $[S]$ to denote the closed linear subspace spanned by S. If $[S] = H$ we say that S is *total* in H.

Lemma 2.7. *If S is total in H, then for almost all p the set $\{x(p) \in H_p \mid x \in S\}$ is total in H_p.*

Proof. Let $S_p = \{x(p) \mid x \in S\}$ and put $N = \{p \in X \mid [S_p] \neq H_p\}$. We must prove that N is a null set. If $p \in N$ let z_p be a unit vector in $[S_p]^\perp$ (the orthogonal complement), and if $p \notin N$ simply let $z_p = 0$. Let $y \in H$ be arbitrary. Since S is total we can find a sequence (y_n) in the span of S such that $y_n \to y$. If we write $y_n = a_1x_1 + \cdots + a_kx_k$ where a_i is scalar and $x_i \in S$, $i = 1, \ldots, k$, then from Remark 2.5, $y_n(p) = a_1x_1(p) + \cdots a_kx_k(p)$ except for p in some null set N_n. Thus $\langle z_p, y_n(p)\rangle = 0$, when $p \notin N_n$. Since

$$\int_X \|y(p) - y_n(p)\|^2 \, d\mu = \|y - y_n\|^2 \to 0,$$

by Lemma 2.6 there is a subsequence y_{n_j} such that $\|y(p) - y_{n_j}(p)\| \to 0$ except for p in a null set N_0. So for p not in the null set $\bigcup_{n=0}^\infty N_n$ we have

$$\langle z_p, y(p)\rangle = \lim_j \langle z_p, y_{n_j}(p)\rangle = 0.$$

In particular, $p \mapsto \langle z_p, y(p)\rangle$ is integrable (measurability follows from Lemma A.2, since μ is complete). By part (ii) of Definition 2.1 there is a $z \in H$ such that $z(p) = z_p$ almost everywhere. Hence

$$0 = \langle z(p), z_p\rangle = \langle z_p, z_p\rangle$$

almost everywhere. Since z_p is a unit vector when $p \in N$, the set N must be a null set. □

Note 2.8. In the following we will often need to construct countable dense subsets with certain properties. Therefore the following terminology will be useful: by the set of complex rationals we mean the set $\mathbb{Q} + i\mathbb{Q} = \{q_1 + iq_2 \mid q_1, q_2 \in \mathbb{Q}\}$. Also from now on

all Hilbert spaces are assumed to be separable.

So from now on we may throw around sentences like “Choose a countable dense subset of the Hilbert space” without any fear of being expelled from the math department. We may also choose a countable dense subset which is a vector space over the complex rationals, if we please, by simply forming finite linear combinations of elements in the countable dense subset using only complex rational coefficients. It is almost as if we were in heaven now. Almost...

Lemma 2.9. *If $\{x_j\}$ is a countable, dense subset of H, then $\{x_j(p)\}$ is dense in H_p for almost all p.*

Proof. Let $\{x_j\}$ be a countable, dense subset of H, and assume that $\{x_j\}$ is a vector space over the complex rationals. By Lemma 2.7 there is a null set N_0 such that $\{x_j(p)\}$ is total in H_p whenever $p \notin N_0$. A complex rational linear combination $a_1x_1 + \cdots + a_kx_k$ is equal to some x_j and hence $a_1x_1(p) + \cdots + a_kx_k(p) = x_j(p)$ except for p in some null set. If we enumerate all possible rational complex (finite) linear combinations of elements of $\{x_j\}$ there are corresponding null sets $N_1, N_2, \ldots$ such that $a_1x_1(p) + \cdots + a_kx_k(p) = x_j(p)$ except for p in the null set $\bigcup_{m=1}^{\infty} N_m$. Hence for p not in the null set $N = \bigcup_{m=0}^{\infty} N_m$, the set $\{x_j(p)\}$ is dense in H_p. □

Lemma 2.10. *Let H be a direct integral of $\{H_p\}$ over (X,μ). The sets $X_n = \{p \in X \mid \dim H_p = n\}$, $n \in \{0, 1, \ldots, \aleph_0\}$ are measurable.*

Proof. First we make a general observation. Let $r_1, r_2, \ldots$ be an enumeration of the complex rationals, and suppose $r_1 = 1$. If $\{x_1, x_2, \ldots\}$ is a sequence of vectors in a normed space, n is a natural number and $M = \operatorname{span}\{x_1, x_2, \ldots\}$, then $\dim M < n$ if and only if the following condition holds:

> For all distinct natural numbers $k_1, \ldots, k_n$ and all $m \in \mathbb{N}$, there are $j_1, \ldots, j_n \in \mathbb{N}$ with some $j_h = 1$ such that
> $$\|r_{j_1}x_{k_1} + \cdots + r_{j_n}x_{k_n}\| < m^{-1}.$$

To see this suppose first M has dimension less than n. Given $k_1, \ldots, k_n$ distinct and $m \in \mathbb{N}$, there is h such that $x_{k_h} \in \operatorname{span}\{x_{k_l} \mid l \neq h\}$. Thus we may find a linear combination

$$x_{k_h} + \sum_{l \neq h} s_l x_{k_l} = 0.$$

Making a small pertubation of the s_l's we may obtain complex rationals $r_{j_1}, \ldots, r_{j_n}$ with $j_h = 1$ so

$$\|r_{j_1}x_{k_1} + \cdots + r_{j_n}x_{k_n}\| < m^{-1}.$$

For the converse, suppose we are given n vectors $x_{k_1}, \ldots, x_{k_n}$. For each $m \in \mathbb{N}$ it is possible to find h and $j_1, \ldots, j_n$ so

$$\|x_{k_h} + \sum_{l \neq h} r_{j_l} x_{k_l}\| < m^{-1}.$$

Notice that h depends on m, but as m varies over $\mathbb{N}$, the some h must appear an infinite number of times. This implies that x_{k_h} is in the closure of $\operatorname{span}\{x_{k_l} \mid l \neq h\}$. And as this is a finite dimensional subspace, it is closed. Thus x_{k_h} belongs to $\operatorname{span}\{x_{k_l} \mid l \neq h\}$. This proves that $\dim M < n$.

Now we return to the proof of the lemma. Let $\{x_a\}_{a=1}^{\infty}$ be an orthonormal basis for H. From Lemma 2.7 there is a null set X_0 such that $\{x_a(p)\}$ is total in H_p whenever $p \notin X_0$. Let

$$X_{k,m,j} = \{p \in X \mid \|r_{j_1} x_{k_1}(p) + \cdots + r_{j_n} x_{k_n}(p)\| < m^{-1}\},$$

where $k = (k_1, \ldots, k_n)$ and $j = (j_1, \ldots, j_n)$, $\{k_1, \ldots, k_n\}$ are distinct and some $j_h = 1$. The criterion above shows that for $p \notin X_0$ the space H_p has dimension less than n if and only if $p \in \bigcap_{k,m} \bigcup_j X_{k,m,j}$. Since each $X_{k,m,j}$ is measurable, we may use the completeness of μ to conclude that the set of points p where H_p has dimension less than n is measurable. Therefore also X_n is measurable. □

Corollary 2.11. *The map $p \mapsto \dim H_p$ is measurable.*

2.2 Decomposable operators

This section discusses the concept of decomposable operators relative to a direct integral decomposition. The main result here is that the decomposable operators form a von Neumann algebra, and that the commutant of this is the set of diagonalizable operators.

Definition 2.12 (Decomposable and diagonalizable operators). Let H be a direct integral of $\{H_p\}$ over (X, μ). An operator T on H is called *decomposable* (with respect to the given direct integral decomposition) if there is a map $p \mapsto T(p)$ defined on X such that $T(p) \in B(H_p)$ for every p, and for every $x \in H$, $T(p)x(p) = (Tx)(p)$ for almost all p. If, in addition, each $T(p)$ is scalar, i.e. $T(p) = f(p)I_p$ for some $f(p)$, where I_p is the identity operator on H_p, then T is called *diagonalizable*.

Note some uniqueness properties of a decomposition of T.

Lemma 2.13. *If $p \mapsto T(p)$ and $p \mapsto T'(p)$ are decompositions of T, then $T(p) = T'(p)$ almost everywhere. Conversely, if $p \mapsto T(p)$ and $p \mapsto S(p)$ are decompositions of operators T and S that agree almost everywhere, then $T = S$.*

Proof. Suppose $p \mapsto T(p)$ and $p \mapsto T'(p)$ are decompositions of T. Since H is separable, there is a countable set $\{x_j\}$, dense in H. By Lemma 2.9 there is a null set N_0 such that $\{x_j(p)\}$ is dense in H_p, when $p \in X \setminus N_0$. There are also null sets N_j such that $T(p)x_j(p) = (Tx_j)(p)$, whenever $p \in X \setminus N_j$. And similarly $T'(p)x_j(p) = (Tx_j)(p)$ except for p in a null sets N'_j. Let N be the unioun of all these null sets. Then N is a null set, and for $p \notin N$, $T(p)x_j(p) = T'(p)x_j(p)$ where $\{x_j(p)\}$ is dense in H_p. Hence $T(p) = T'(p)$ when $p \notin N$.

Suppose decompositions of S and T are given, and $S(p) = T(p)$ for almost all p. Given $x, y \in H$ we find

$$\begin{aligned}
\langle Sx, y\rangle &= \int_X \langle (Sx)(p), y(p)\rangle \, d\mu = \int_X \langle S(p)x(p), y(p)\rangle \, d\mu \\
&= \int_X \langle T(p)x(p), y(p)\rangle \, d\mu = \int_X \langle (Tx)(p), y(p)\rangle \, d\mu = \langle Tx, y\rangle
\end{aligned}$$

Hence $S = T$. □

Remark 2.14. Let f be a measurable, essentially bounded function on X. Then for any $x, y \in H$ the map $p \mapsto \langle f(p)x(p), y(p)\rangle$ is measurable and integrable. Hence there is a $z \in H$ such that for almost all p we have $f(p)x(p) = z(p)$. Defining $M_f x$ to be z, then it is readily seen that M_f is diagonalizable with decomposition $p \mapsto f(p)I_p$. In corollary 2.19 we will see that every diagonalizable operator arises in this form, i.e. that f automatically is measurable and essentially bounded when coming from a diagonalizable operator.

Definition 2.15 (Diagonalizable projection). If f is the characteristic function of a measurable subset X_0, then the operator M_f constructed above is a projection called the *diagonalizable projection of* X_0.

Now, we slowly move on to proving that the decomposable operators on H form a von Neumann algebra. Showing that they form a unital *-algebra is quite easy.

Proposition 2.16. *Let H be a direct integral of $\{H_p\}$ over (X, μ). The decomposable operators form a *-algebra containing the unit of $B(H)$. Furthermore, if S and T are decomposable operators and α is a scalar, then the following equalities hold for almost all p:*

(i) $(\alpha S + T)(p) = \alpha S(p) + T(p)$,

(ii) $(ST)(p) = S(p)T(p)$,

(iii) $S^*(p) = S(p)^*$,

(iv) $I(p) = I_p$,

where I denotes the identity on H and I_p denotes the identity on H_p.

Proof. Define $(\alpha S + T)(p)$ to be $\alpha S(p) + T(p)$. Then for $x \in H$ it is easy, using Definition 2.12 and Remark 2.5, to check that $(\alpha S + T)(p)x(p) = ((\alpha S + T)x)(p)$ for almost all p. Hence $\alpha S + T$ is decomposable with decomposition $p \mapsto \alpha S(p) + T(p)$. Using Lemma 2.13 we find that any decomposition of $\alpha S + T$ equals $p \mapsto \alpha S(p) + T(p)$ almost everywhere, and hence (i) holds.

The assertion about the product is similar to prove. Define $S^*(p)$ to be $S(p)^*$. Then for $x, y \in H$

$$\langle S(p)^* x(p), y(p)\rangle = \langle x(p), S(p)y(p)\rangle = \langle x(p), (Sy)(p)\rangle$$

almost everywhere. Since $p \mapsto \langle x(p), (Sy)(p)\rangle$ is integrable, there is by part (ii) of Definition 2.1 a z in H such that $z(p) = S(p)^* x(p)$ almost everywhere. Then

$$\langle S^* x - z, y\rangle = \langle x, Sy\rangle - \langle z, y\rangle = \int_X \langle x(p), (Sy)(p)\rangle \, d\mu - \int_X \langle S(p)^* x(p), y(p)\rangle \, d\mu = 0.$$

Since this holds for every $y \in H$, $S^* x = z$, and thus $(S^* x)(p) = z(p) = S(p)^* x(p)$ almost everywhere. Hence S^* is decomposable with decomposition $p \mapsto S(p)^*$.

Defining $I(p)$ to be I_p, we have $I(p)x(p) = I_p x(p) = x(p) = (Ix)(p)$ for all p. Hence I is decomposable with decomposition $p \mapsto I_p$. □

Proposition 2.17. *Let H be a direct integral of $\{H_p\}$ over (X, μ), and let S and T be decomposable, self-adjoint operators on H. Then $S \leq T$ if and only if $S(p) \leq T(p)$ almost everywhere.*

Proof. Suppose first $S(p) \leq T(p)$ almost everywhere and $x \in H$. Then

$$\begin{aligned} \langle Sx, x\rangle &= \int_X \langle (Sx)(p), x(p)\rangle \, d\mu = \int_X \langle S(p)x(p), x(p)\rangle \, d\mu \\ &\leq \int_X \langle T(p)x(p), x(p)\rangle \, d\mu = \int_X \langle (Tx)(p), x(p)\rangle \, d\mu = \langle Tx, x\rangle. \end{aligned}$$

hence $S \leq T$.

Suppose on the other hand that $S \leq T$. Then in the light of the previous proposition $P = T - S$ is a positive, decomposable operator with decomposition $p \mapsto T(s) - S(p)$. So it will suffice to prove that $P(p) \geq 0$ almost everywhere.

Let $\{x_j\}$ be a countable, dense subset of H, and let N be a null set such that $\{x_j(p)\}$ is dense in H_p for $p \notin N$, cf. Lemma 2.9.

To reach a contradiction, suppose there is a set X_0 of positive measure such that for $p \in X_0$ it holds that $\langle P(p)x_j(p), x_j(p)\rangle < 0$. We may assume there is some $\alpha < 0$ such that $\langle P(p)x_j(p), x_j(p)\rangle \leq \alpha$ for all $p \in X_0$ by replacing X_0 by a subset, still of positive measure. Note that since the map $p \mapsto \langle P(p)x_j(p), x_j(p)\rangle$ is integrable, X_0 has finite measure, but this will not be important.

Let M_f be the diagonalizable projection of X_0. Since $p \mapsto \langle (M_f x_j)(p), y(p)\rangle$ is integrable for each $y \in H$, there is a $z \in H$ such that $(M_f x_j)(p) = z(p)$ almost everywhere. Now,

$$\langle Pz, z\rangle = \int_X \langle P(p)M_f(p)x_j(p), M_f(p)x_j(p)\rangle \, d\mu = \int_{X_0} \langle P(p)x_j(p), x_j(p)\rangle \leq \alpha\mu(X_0) < 0,$$

a contradiction, since P is positive. Hence $\langle P(p)x_j(p), x_j(p)\rangle \geq 0$ except for p in a null set M_j. Let $M = \bigcup_{j=1}^{\infty} M_j$. Then for $p \notin M \cup N$, $\langle P(p)x_j(p), x_j(p)\rangle \geq 0$, and since $\{x_j(p)\}$ is dense in H_p, it follows that $P(p)$ is positive whenever p is not in the null set $M \cup N$. □

Proposition 2.18. *Let H be a direct integral of $\{H_p\}$ over (X, μ). If T is a decomposable operator, then $p \mapsto \|T(p)\|$ is measurable and essentially bounded with essential bound $\|T\|$.*

Proof. From Proposition 2.16, T^*T is decomposable with decomposition $p \mapsto T^*(p)T(p)$, so using the C*-identity, $\|T(p)\|^2 = \|T^*(p)T(p)\|$, we see that it is enough to proof the corollary in the case where T is positive.

First we establish measurability. Let $\{x_j\}$ and N be as in the proof of the previous proposition. Let $s > 0$ be a rational number, and let X_s denote the set of those p in $X \setminus N$ where $T(p) \leq sI_p$. Since $\{x_j(p)\}$ is dense in H_p it follows that $q \in X_s$ if and only if

$$q \in \{p \in X \setminus N \mid \langle T(p)x_j(p), x_j(p)\rangle \leq s\langle x_j(p), x_j(p)\rangle\} \quad \text{for all } j.$$

In other words

$$X_s = \bigcap_{j=1}^{\infty} \{p \in X \setminus N \mid \langle T(p)x_j(p), x_j(p)\rangle \leq s\|x_j(p)\|^2\}.$$

The maps $p \mapsto \langle T(p)x_j(p), x_j(p)\rangle$ and $p \mapsto s\|x_j(p)\|^2$ are measurable (the first due to completeness of μ), and hence each

$$\{p \in X \setminus N \mid \langle T(p)x_j(p), x_j(p)\rangle \leq s\|x_j(p)\|^2\}$$

is measurable, so it follows that X_s is measurable.

Now, let $]a, b[$ be an open interval. $\|T(p)\|$ lies in $]a, b[$ if and only if there are rationals r, s in that interval such that $\|T(p)\| \leq s$ and $\|T(p)\| > r$, or in other words $T(p) \leq sI_p$ and $T(p) \nleq rI_p$. Hence

$$\{p \in X \setminus N \mid a < \|T(p)\| < b\} = \bigcup_{r,s \in]a,b[\cap\mathbb{Q}} X_s \setminus X_r,$$

is a measurable set, and this shows that $p \mapsto \|T(p)\|$ is measurable.

Since $0 \leq T \leq \|T\|I$, it follows from the previous proposition that $0 \leq T(p) \leq \|T\|I_p$ almost everywhere. Hence $p \mapsto T(p)$ is essentially bounded with essential bound at most $\|T\|$. If $\|T(p)\| \leq a$ almost everywhere, then $T(p) \leq aI_p$ almost everywhere. Again by the previous proposition, $T \leq aI$. Hence $\|T\| \leq a$. This proves that $\|T\|$ is the essential bound. □

Corollary 2.19. *If T is diagonalizable, then T has the form M_f (see Remark 2.14) for some measurable, essentially bounded function f. If T is a projection, then f can be chosen as a characteristic function.*

Proof. It is enough to prove the corollary, when T is positive, and then use the fact that every operator is a linear combination of (four) positives. Since T is diagonalizable, T has decomposition $p \mapsto f(p)I_p$. We must show that f is measurable and essentially bounded. By Proposition 2.17 it follows that $T(p)$ is positive almost everywhere. Hence $f(p) = \|T(p)\|$ almost everywhere, and the first result follows by Proposition 2.18.

If T is a projection, then $f(p)^2 = f(p)$ almost everywhere. Hence f equals a (measurable) characteristic function g almost everywhere, and $T = M_f = M_g$. □

Lemma 2.20. *Let H be a direct integral of $\{H_p\}$ over (X, μ). If (T_n) is a sequence of decomposable operators converging strongly to an operator A, then A is decomposable and some subsequence (T_k) of (T_n) is such that $(T_k(p))$ converges to $A(p)$ almost everyhwhere.*

Proof. We may suppose A has norm 1, and that each T_n lies in the unit ball of $B(H)$. Let $\{x_j\}$ be a countable, dense subset of H. Since $T_n \to A$ strongly, $T_n x_j \to Ax_j$, and thus

$$\|T_n x_j - Ax_j\|^2 = \int_X \|T_n(p)x_j(p) - (Ax_j)(p)\|^2 \, d\mu \to 0$$

for each j.

From Lemma 2.6 there is a subsequence $(T_n^{(1)})$ of (T_n) such that $T_n^{(1)}(p)x_1(p) \to (Ax_1)(p)$ for almost all p. Again, there is a subsequence $(T_n^{(2)})$ of $(T_n^{(1)})$ such that $T_n^{(2)}(p)x_2(p) \to (Ax_2)(p)$ for almost all p, and so on. The diagonal $(T_n^{(n)})$ of these subsequences, i.e. the sequence $(T_1^{(1)}, T_2^{(2)}, \dots)$ then satisfies $T_n^{(n)}(p)x_j(p) \to (Ax_j)(p)$ almost everywhere for every j.

From this, Lemma 2.9 and Proposition 2.18 there is a null set N such that when $p \in X \setminus N$, $\{x_j(p)\}$ is dense in H_p, $\|T_n^{(n)}(p)\| \leq 1$ for all n, and $T_n^{(n)}(p)x_j(p) \to (Ax_j)(p)$. It follows that there is an operator $A(p)$ in the unit ball of $B(H_p)$ such that $A(p)x_j(p) = (Ax_j)(p)$ for all j. When $p \in N$ we define $A(p)$ arbitrarily, for instance just zero.

Let $x \in H$ be given and choose a sequence (x_k) from $\{x_j\}$ converging to x. Since

$$\|x - x_k\|^2 = \int_X \|x(p) - x_k(p)\|^2 \, d\mu \to 0$$

and

$$\|Ax - Ax_k\|^2 = \int_X \|(Ax)(p) - (Ax_k)(p)\|^2 \, d\mu \to 0,$$

using Lemma 2.6 twice implies that there is a subsequence $(x_{k'})$ of (x_k) such that $x_{k'}(p) \to x(p)$ and $(Ax_{k'})(p) \to (Ax)(p)$ except for p in some null set M. When $p \notin M \cup N$, it holds that

$$(Ax)(p) = \lim_{k'} (Ax_{k'})(p) = \lim_{k'} A(p)x_{k'}(p) = A(p) \lim_{k'} x_{k'}(p) = A(p)x(p).$$

Hence A is decomposable with decomposition $p \mapsto A(p)$. □

Corollary 2.21. *If (T_n) is a monotone sequence of decomposable operators and bounded (and hence has a limit A in the strong operator topology) then $T_n(p) \to A(p)$ almost everywhere. In particular, if $\{P_n\}$ is an orthogonal family of (decomposable) projections with sum P, then $\{P_n(p)\}$ has sum $P(p)$ almost everywhere.*

We are now ready to prove the main result of this section.

Theorem 2.22. *Let H be a direct integral of $\{H_p\}$ over (X, μ). The set $\mathcal{D}$ of decomposable operators on H with respect to the direct integral decomposition is a von Neumann algebra with abelian commutant coinciding with the diagonalizable operators, denoted C.*

Proof. It was already proved in Proposition 2.16 that $\mathcal{D}$ is a *-algebra of operators on H containing I. Suppose A is an operator in the strong operator closure of $\mathcal{D}$. Recall that, since H is separable, the strong operator topology on the unit ball of $B(H)$ is metrizable. By the Kaplansky Density Theorem there is sequence (T_n) of decomposable operators converging strongly to A. So by Lemma 2.20 A is decomposable, and hence $\mathcal{D}$ is strongly closed, and $\mathcal{D}$ is a von Neumann algebra. Using an analouge of Lemma 2.20 for diagonalizable operators, it may be proved that C is a von Neumann algebra.

It remains to prove that $\mathcal{D}' = C$, or equivalently $\mathcal{D} = C'$. Suppose first A is diagonalizable with decomposition $f(p)I_p$ and T is decomposable with decomposition $T(p)$. Then it follows from Proposition 2.16 that AT and TA are decomposable with decompositions $f(p)I_pT(p)$ and $T(p)f(p)I_p$, respectively. Thus, AT and TA have the same decompositions and $AT = TA$ from Lemma 2.13. This proves $\mathcal{D} \subseteq C'$.

To prove the other inclusion, it suffices to prove that every projection P in C' is decomposable.

First we prove that $\langle u(p), v(p)\rangle = 0$ almost everywhere, when $Pu = u$ and $Pv = 0$. Let Q denote the diagonalizable projection corresponding to X_0, where X_0 is a measurable subset of X. Then, since $P \in C'$ and $Q \in C$, P and Q commute and thus

$$\int_{X_0} \langle u(p), v(p)\rangle \, d\mu = \langle Qu, v\rangle = \langle QPu, v\rangle = \langle Qu, Pv\rangle = 0.$$

Since this holds for every measurable subset X_0, it follows that $\langle u(p), v(p)\rangle = 0$ almost everywhere (see Lemma A.3).

Let $\{u_j\}$ and $\{v_j\}$ be orthonormal bases for $P(H)$ and $(I - P)(H)$, respectively. Let $\{x_j\}$ be an enumeration of the countable set of finite, complex rational linear combinations of elements in $\{u_j, v_j\}$. As in the proof of Proposition 2.9 there is a null set N such that if $x_j = a_1x_1 + \cdots a_kx_k$ then $x_j(p) = a_1x_1(p) + \cdots a_kx_k(p)$ whenever p is not in N, and $\{x_j(p)\}$ is dense in H_p.

For p not in N define $P(p)$ to be the orthogonal projection onto the space spanned by $\{u_j(p)\}$. Suppose u is a finite, complex rational linear combination of elements in $\{u_j\}$. Then $(Pu)(p) = u(p) = P(p)u(p)$ when $p \notin N$. Suppose v is a finite, complex rational linear combination of elements in $\{v_j\}$. By the above there is a null set M_j such that $\langle u_j(p), v(p)\rangle = 0$ for every p not in M_j. Let $M = \bigcup_{j=1}^{\infty} M_j$. Then for $p \notin M$, it holds that $\langle u_j(p), v(p)\rangle = 0$ for every j, and hence $P(p)v(p) = 0 = (Pv)(p)$ almost everywhere. Hence for $p \notin N \cup M$, $P(p)x_j(p) = (Px_j)(p)$ for every j.

For a general $x \in H$, choose a sequence (x_k) in $\{x_j\}$ converging to x. Then using the subsequence argument of Lemma 2.20, we may assume $x_k(p) \to x(p)$ and $(Px_k)(p) \to (Px)(p)$ almost everywhere and thus

$$(Px)(p) = \lim_k (Px_k)(p) = \lim_k P(p)x_k(p) = P(p) \lim_k x_k(p) = P(p)x(p)$$

almost everywhere. This proves that P is decomposable, and the proof is complete. □

Let us return to the example $L^2(X, \mu)$ from Proposition 2.2 and describe the decomposable and diagonalizable operators.

Proposition 2.23. *Let (X, μ) be a σ-finite measure space. If $L^2(X, \mu)$ is considered as the direct integral of one-dimensional spaces (see Proposition 2.2), then the algebra of decomposable operators coincides with the diagonalizable operators and is the maximal abelian multiplication algebra $\mathcal{A} = \{M_f \mid f \in L^\infty(X, \mu)\}$.*

Proof. Since $\mathcal{A}$ is maximal abelian, and $\mathcal{D}' = C$, it will suffice to show that $\mathcal{A} \subseteq C$. Let $M_f \in \mathcal{A}$ be given and choose a representative of f, also denoted by f. Then for $g \in L^2(X, \mu)$ with representative also denoted by g, it follows that $M_f(p)g(p) = f(p)g(p) = fg(p) = (M_f g)(p)$ almost everywhere, and hence M_f is decomposable with decomposition $p \mapsto f(p)$. Since each component $f(p)$ is scalar, M_f is diagonalizable. This completes the proof. □

Example 2.24. Another example is the following. Suppose H is a direct sum of the Hilbert spaces $\{H_p\}$. As seen earlier this is just a special case of a direct integral, and a decomposable operator T is a direct sum of operators $T_p \in B(H_p)$ such that $\|T\| = \sup \|T_p\| < \infty$. The diagonalizable operators correspond to the situation where each T_p is a scalar multiple of the identity I_p on H_p.

2.3 Decomposable representations

We now turn to describe decomposable representations, since these will be essential in the description of decomposable algebras.

Definition 2.25 (Decomposable representation). Let H be a direct integral of $\{H_p\}$ over (X, μ), and let $\mathcal{A}$ be a C*-algebra. A representation φ of $\mathcal{A}$ on H is called *decomposable* with decomposition $p \mapsto \varphi_p$, if $\varphi(\mathcal{A})$ consists of decomposable operators, and there are representations φ_p of $\mathcal{A}$ on H_p such that $\varphi(a)(p) = \varphi_p(a)$ for almost all $p \in X$.

Similar to Lemma 2.13 the following uniqueness conditions hold.

Lemma 2.26. *Let H be a direct integral of $\{H_p\}$ over (X, μ), and let $\mathcal{A}$ be a separable C*-algebra. If φ, a representation of $\mathcal{A}$ on H, is decomposable with decompositions $p \mapsto \varphi_p$ and $p \mapsto \varphi'_p$, then $\varphi_p = \varphi'_p$ almost everywhere. Conversely, if φ and ψ are decomposable representations of $\mathcal{A}$ whose decompositions agree almost everywhere, then $\varphi = \psi$.*

Proof. Suppose $p \mapsto \varphi_p$ and $p \mapsto \varphi'_p$ decompositions of φ. Let $\{a_k\}$ be a countable, dense subset of $\mathcal{A}$. There are null sets N_k such that $\varphi(a_k)(p) = \varphi_p(a_k)$ whenever $p \notin N_k$ and null sets N'_k such that $\varphi(a_k)(p) = \varphi'_p(a_k)$ whenever $p \notin N'_k$. Now, when p lies in none of these null sets, $\varphi'_p(a_k) = \varphi_p(a_k)$ for all k, and hence $\varphi'_p = \varphi_p$, since $\{a_k\}$ is dense in $\mathcal{A}$.

Suppose φ and ψ are decomposable representations of $\mathcal{A}$ whose decompositions agree almost everywhere. Let $a \in \mathcal{A}$, $x, y \in H$ be given. Then

$$\begin{aligned}
\langle \varphi(a)x, y \rangle &= \int_X \langle (\varphi(a)x)(p), y(p) \rangle \, d\mu = \int_X \langle \varphi(a)(p)x(p), y(p) \rangle \, d\mu \\
&= \int_X \langle \varphi_p(a)x(p), y(p) \rangle \, d\mu = \int_X \langle \psi_p(a)x(p), y(p) \rangle \, d\mu \\
&= \int_X \langle \psi(a)(p)x(p), y(p) \rangle \, d\mu = \int_X \langle (\psi(a)x)(p), y(p) \rangle \, d\mu = \langle \psi(a)x, y \rangle
\end{aligned}$$

Hence $\varphi(a) = \psi(a)$. This proves $\varphi = \psi$. □

Theorem 2.27. *Let H be a direct integral of $\{H_p\}$ over (X, μ), and let $\mathcal{A}$ be a separable C*-algebra. Any representation of $\mathcal{A}$ on H with image contained in the algebra of decomposable operators is decomposable.*

Proof. Let $\mathcal{A}_0$ be a countable *-algebra over the complex rationals, dense in $\mathcal{A}$. Such an algebra exists, since $\mathcal{A}$ is separable. For $a_1, a_2 \in \mathcal{A}_0$ and r_1, r_2 complex rationals

$$\begin{aligned} (r_1\varphi(a_1) + r_1\varphi(a_2))(p) &= r_1\varphi(a_1)(p) + r_2\varphi(a_2)(p) \\ (\varphi(a_1)\varphi(a_2))(p) &= \varphi(a_1)(p)\varphi(a_2)(p) \\ \varphi(a_1)^*(p) &= \varphi(a_1)(p)^* \end{aligned}$$

almost everywhere, by Proposition 2.16. Since $\mathcal{A}_0$ is countable, there is a countable union N_1 of null sets such that these relations hold on $\mathcal{A}_0$, when $p \notin N_1$. In other words, $p \mapsto \varphi(a)(p)$ is a *-homomorphism of $\mathcal{A}_0$ into $B(H_p)$ for $p \notin N_1$.

By Proposition 2.18 there is a null set N_2 such that $\|\varphi(a)(p)\| \leq 1$ for every a in the unit ball of $\mathcal{A}_0$ when $p \notin N_2$. When $p \notin N_1 \cup N_2$, it follows that $p \mapsto \varphi(a)(p)$ is bounded on $\mathcal{A}_0$ and hence extends (uniquely) to a representation φ_p of $\mathcal{A}$ on H_p. Define φ_p to be zero elsewhere.

Now, let $a \in \mathcal{A}$ be given and choose a sequence (a_n) in $\mathcal{A}_0$ converging to a. Since this implies that $\varphi(a_n)$ converges to $\varphi(a)$, it follows from Lemma 2.20 that $\varphi(a_k)(p) \to \varphi(a)(p)$ for some subsequence (a_k) of (a_n) and p not in some null set N_3. Let $N = N_1 \cup N_2 \cup N_3$. When $p \notin N$, $\varphi(a_k)(p) = \varphi_p(a_k) \to \varphi_p(a)$, and hence $\varphi(a)(p) = \varphi_p(a)$. This proves that φ is decomposable with decomposition $p \mapsto \varphi_p$. □

2.4 Decomposable von Neumann algebras

The time has finally come to define what it means for a von Neumann algebra $\mathcal{M}$ acting on H to be decomposable with respect to a given decomposition of H as a direct integral of $\{H_p\}$. It will also follow quite easily from Theorem 2.27, that any von Neumann algebra contained in the von Neumann algebra of decomposable operators is decomposable.

Definition 2.28 (Decomposable von Neumann algebra). Let H be a direct integral of $\{H_p\}$ over (X, μ). Let $\mathcal{M}$ be a von Neumann algebra on H, and suppose $\mathcal{A}$ is a (norm) separable, strong operator dense C*-subalgbra of $\mathcal{M}$. Then $\mathcal{M}$ is said to be *decomposable* with decomposition $p \mapsto \mathcal{M}_p$, if the identity representation ι of $\mathcal{A}$ is decomposable ($p \mapsto \iota_p$), and $\iota_p(\mathcal{A})$ is strong operator dense in $\mathcal{M}_p$ almost everywhere. This is sometimes written

$$\mathcal{M} = \int_X \mathcal{M}_p \, d\mu,$$

and $\mathcal{M}$ is called the direct integral of $\{\mathcal{M}_p\}$. The image $\iota_p(\mathcal{A})$ is usually denoted $\mathcal{A}_p$.

Note that since ι is decomposable, $\mathcal{A}$ necessarily consists of decomposable operators, and by Theorem 2.22 so does $\mathcal{M}$. By Lemma 2.20, $T(p) \in \mathcal{M}_p$ for almost all p, when T is an operator in $\mathcal{M}$.

Two things are unsatisfactory with the definition just given. One is the assumption that there should be a separable, strong operator dense C*-algebra in $\mathcal{M}$. That this is always the case is the proved in Proposition 2.30 and is interesting in its own right. The other is that the decomposition of $\mathcal{M}$ might depend on the choice of the subalgebra $\mathcal{A}$. Lemma 2.31 takes care of this.

Lemma 2.29. *If $\varphi : \mathcal{M} \to \mathcal{N}$ is a *-isomorphism between von Neumann algebras, and $\mathcal{A}$ is a strong operator dense, norm separable C*-subalgebra of $\mathcal{M}$, then $\varphi(\mathcal{A})$ is a strong operator dense, separable C*-subalgebra of $\mathcal{N}$.*

Proof. The lemma is a consequence Kaplansky's density theorem and the fact that *-isomorphisms are strong operator continuous on bounded sets. Details are left to the reader. □

Proposition 2.30. *A von Neumann algebra $\mathcal{M}$ acting on a separable Hilbert space H has a strong operator dense, norm separable C*-subalgebra.*

Proof. Let $\iota : \mathcal{M} \to B(H)$ denote the identity representation, and let $\{x_n\}$ be a countable, dense subset of H consisting of non-zero vectors. The inflation of ι, $\iota^{(\infty)} = \bigoplus_{n=1}^{\infty} \iota$ is a faithful representation of $\mathcal{M}$ on the separable Hilbert space $\bigoplus_{n=1}^{\infty} H$, and the image $\iota^{(\infty)}(\mathcal{M})$ has a separating vector, $(\|x_1\|^{-1}x_1, \|x_2\|^{-2}x_2, \ldots)$. Thus, by the previous lemma we may assume that $\mathcal{M}$ has a separating vector.

Then every normal functional on $\mathcal{M}$ is of the form $T \mapsto \omega_{x,y}(T) = \langle Tx, y\rangle$ (see for instance Theorem 7.3.3 in [K&R II]). In the following $\|\omega_{x,y}\|$ denotes the norm of the functional viewed as a functional on $\mathcal{M}$ (as opposed to a functional on all of $B(H)$). For each $m, n \in \mathbb{N}$ choose A_{mn} in $(\mathcal{M})_1$ such that $\omega_{x_m,x_m}(A_{mn}) \geq \|\omega_{x_m,x_n}\| - \frac{1}{4}$, and let $\mathcal{A}$ be the C*-algebra generated by $\{A_{mn}\}$. Notice that $\mathcal{A}$ is norm separable. We prove that $\mathcal{A}$ is strong operator dense in $\mathcal{M}$.

Suppose $\|\omega_{x,y}|\mathcal{A}\| = 0$, but $\|\omega_{x,y}\| = 1$ for some $x, y \in H$. Let $\varepsilon > 0$ be given, and choose x_m, x_n such that $\|x_m - x\|$ and $\|x_n - y\|$ are less that ε. Then for $\|T\| \leq 1$

$$\begin{aligned}
|\omega_{x,y}(T) - \omega_{x_m,x_n}(T)| &= |\langle Tx, y\rangle - \langle Tx_m, x_n\rangle| \\
&= |\langle Tx, y - x_n\rangle + \langle T(x - x_m), x_n\rangle| \\
&\leq \|T\|\,\|x\|\,\|y - x_n\| + \|T\|\,\|x - x_m\|\,\|x_n\| \\
&\leq (\|x\| + \|y\| + \varepsilon)\varepsilon.
\end{aligned}$$

So if we let ε be such that $(\|x\| + \|y\| + \varepsilon)\varepsilon < \frac{1}{4}$, then $\|\omega_{x,y} - \omega_{x_m,x_n}\| < \frac{1}{4}$.

But also

$$\begin{aligned}
\|\omega_{x,y} - \omega_{x_m,x_n}\| &\geq |(\omega_{x,y} - \omega_{x_m,x_n})(A_{mn})| = \omega_{x_m,x_n}(A_{mn}) \\
&\geq \|\omega_{x_m,x_n}\| - \frac{1}{4} \geq \|\omega_{x,y}\| - \frac{1}{2} = \frac{1}{2}.
\end{aligned}$$

This gives a contradiction, and so each normal functional annihilating $\mathcal{A}$ also annihilates $\mathcal{M}$.

Since the normal functionals are the ultraweakly continuous functionals, we get by the Hahn-Banach theorem ([K&R I] theorem 1.2.13), that the closure of $\mathcal{A}$ in the ultraweak topology is $\mathcal{M}$.

Since the weak operator topology is weaker than the ultraweak topology, this implies that the weak operator closure of $\mathcal{A}$ is $\mathcal{M}$, and hence the strong operator closure of $\mathcal{A}$ is also $\mathcal{M}$. $\square$

Lemma 2.31. *Let H be a direct integral of $\{H_p\}$ over (X, μ), and let $\mathcal{A}$ and $\mathcal{B}$ be norm separable C^*-subalgebras of the algebra of decomposable operators, and suppose their strong operator closures coincide. If $\mathcal{A}_p$ and $\mathcal{B}_p$ denote the images in $B(H_p)$ of the decomposition of the identity representations of $\mathcal{A}$ and $\mathcal{B}$, then $\overline{\mathcal{A}_p} = \overline{\mathcal{B}_p}$ (strong operator closure) for almost all $p \in X$.*

Proof. Let $\mathcal{A}_0$ and $\mathcal{B}_0$ be countable, norm dense subsets of $\mathcal{A}$ and $\mathcal{B}$, respectively. By Theorem 2.27 the identity representation ι of $\mathcal{B}$ is decomposable ($p \mapsto \iota_p$), so it means that there is a null set N such that $B(p) = \iota_p(B)$ for all $p \notin N$ and all $B \in \mathcal{B}_0$. Notice that $\{B(p) \mid B \in \mathcal{B}_0\}$ is norm dense in $\mathcal{B}_p$, when $p \notin N$.

Let $\{x_j\}$ be a countable, dense subset of H, and let $B \in \mathcal{B}_0$ be given. Then since $\mathcal{A}_0$ is strong operator dense in $\overline{\mathcal{B}}$, it follows from the Kaplansky Density Theorem (and the fact that H is separable) that there is a sequence (A_n) in the ball of radius $\|B\|$ in $\mathcal{A}_0$ converging strongly to B. In particular, $A_n x_j \to B x_j$ for every x_j. Using the diagonal subsequence argument from Lemma 2.20, we obtain a subsequence (A_k) such that $A_k(p)x_j(p) \to B(p)x_j(p)$ almost everywhere for all x_j. By Lemma 2.9 we may suppose $\{x_j(p)\}$ is dense in H_p. Thus, $B(p) \in \overline{\mathcal{A}_p}$, except for p in some null set (depending on B). Repeating this procedure for each $B \in \mathcal{B}_0$, forming the countable union of all the attained null sets, and using that $\{B(p) \mid B \in \mathcal{B}_0\}$ is norm dense in $\mathcal{B}_p$ almost everywhere we obtain $\overline{\mathcal{B}_p} \subseteq \overline{\mathcal{A}_p}$ almost everywhere.

In a similar fasion it may be proved that $\overline{\mathcal{A}_p} \subseteq \overline{\mathcal{B}_p}$ almost everywhere, and hence $\overline{\mathcal{A}_p} = \overline{\mathcal{B}_p}$ almost everywhere. $\square$

Combining the results of Lemma 2.31 and Proposition 2.30 with Theorem 2.27 the following theorem is immediate.

Theorem 2.32. *Let H be a direct intetral of $\{H_p\}$ over (X, μ). If $\mathcal{M}$ is a von Neumann algebra of decomposable operators, then $\mathcal{M}$ is decomposable with (almost everywhere) unique decomposition $p \mapsto \mathcal{M}_p$.*

2.5 Decompositions and the commutant

In this section we will describe, how the commutant behaves under disintegration. This will eventually lead to the central decomposition of a von Neumann algebra into factors, but for now we will focus on proving Theorem 2.39, which says that the commutant of a decomposable von Neumann algebra is the direct integral of the commutants of the components. The proof of the theorem will require some measure theoretic aspects, we have not encountered yet. For some of the measure theoretic details we refer to the appendix. The concepts of analytic sets and Polish spaces are developed there.

The following proposition describes how some of the properties of projections in $\mathcal{M}$ are carried into the components $\mathcal{M}_p$.

Proposition 2.33. *Let H be a direct integral of $\{H_p\}$ over (X, μ). Let $\mathcal{M}$ be a von Neumann algebra of decomposable operators on H, and let E be a projection in $\mathcal{M}$. Then the following holds for almost all $p \in X$.*

(i) *$E(p)$ is a projection in $\mathcal{M}_p$.*

(ii) *If E belongs to the center of $\mathcal{M}$, then $E(p)$ belongs to the center of $\mathcal{M}_p$.*

(iii) *If E is abelian in $\mathcal{M}$, then $E(p)$ is abelian in $\mathcal{M}_p$.*

(iv) *If $E \sim F$ in $\mathcal{M}$, then $E(p) \sim F(p)$ in $\mathcal{M}_p$.*

(v) *If $\{x_j\}$ is a countable, total subset of $E(H)$, then $\{x_j(p)\}$ is total in $E(p)(H_p)$.*

Proof. (i) Since $E = E^2 = E^*$, we have $E(p) = E(p)^2 = E(p)^*$ almost everywhere from Proposition 2.16. Hence $E(p)$ is a projection in $\mathcal{M}_p$ for almost all p.

(ii) Suppose E belongs to the center of $\mathcal{M}$, and let $\mathcal{A}$ be a norm separable, strong operator dense C*-subalgebra of $\mathcal{M}$ with a countable dense subset $\{A_n\}$. Then $E(p)A_n(p) = A_n(p)E(p)$ for any n almost everywhere by Proposition 2.16. Thus, $E(p)$ commutes with $\mathcal{A}_p$, and is then in the center of $\mathcal{M}_p$ for almost all p.

(iii) The proof is similar to the proof of (ii), using Proposition 2.16 and the countable set $\{A_n\}$.

(iv) If V is a partial isometry witnessing $E \sim F$, then from Proposition 2.16, $V(p)$ will be a partial isometry witnessing $E(p) \sim F(p)$ almost everywhere.

(v) Let $\{y_k\}$ be a countable, total subset of $(I - E)(H)$, so that $\{x_j, y_k\}$ is total in H. Then by Lemma 2.7, the set $\{x_j(p), y_k(p)\}$ is total in H_p almost everywhere, and so $\{E(p)x_j(p), E(p)y_k(p)\}$ is total in $E(p)(H_p)$. Since $Ey_k = 0$, $E(p)y_k(p) = 0$ for every k almost everywhere, and since $E(p)x_j(p) = x_j(p)$ almost everywhere, it follows that $\{x_j(p)\}$ is total in $E(p)(H_p)$ for almost all p.

□

The next lemma is about transferring a direct integral deceomposition to a subspace or subalgebra.

Lemma 2.34. *Let H be a direct integral of $\{H_p\}$ over (X, μ). Suppose $\mathcal{M}$ be a von Neumann algebra of decomposable operators with decomposition $p \mapsto \mathcal{M}_p$, and suppose P is a diagonalizable projection. Then*

$$P(H) = \int_{X_0} P(p)(H_p)\, d\mu(p) \quad \text{and} \quad P\mathcal{M}P = \int_{X_0} P(p)\mathcal{M}_p P(p)\, d\mu(p).$$

for some measurable subset X_0 of X.

Proof. By Corollary 2.19 there is a measurable subset X_0 of X such that P is the diagonalizable projection corresponding to X_0. It is now straightforward to check the above equalities using the definitions. □

Lemma 2.35. *If E is an abelian projection in a von Neumann algebra $\mathcal{M}$ acting on a separable Hilbert space H, then E is a cyclic projection. In other words, there is a vector x (in the range of E) such that the range of E is $[\mathcal{M}'x]$.*

Proof. Since E is abelian, the algebra $E\mathcal{M}E$ is abelian acting on the separable Hilbert space $E(H)$. From Lemma 1.4, there is a cyclic vector $x \in E(H)$ for $E\mathcal{M}E$. This means that $E(H) = [E\mathcal{M}Ex]$. Since $E\mathcal{M}E$ is abelian, it follows that $E\mathcal{M}E$ is contained in its commutant $(E\mathcal{M}E)'$ inside $B(E(H))$. Since $(E\mathcal{M}E)' = E\mathcal{M}'E$, it follows that $[E\mathcal{M}Ex] \subseteq [E\mathcal{M}'Ex] \subseteq E(H)$. Also,

$$[\mathcal{M}'x] = [\mathcal{M}'Ex] = [E\mathcal{M}'x] \subseteq E(H).$$

Putting this together gives

$$E(H) = [E\mathcal{M}Ex] = [E\mathcal{M}'Ex] = [\mathcal{M}'x].$$

□

Lemma 2.36. *Let H be a direct integral of $\{H_p\}$ over (X, μ). The algebra $\mathcal{D}$ of decomposable operators is of type I_n if and only if H_p has dimension n almost everywhere.*

If $\mathcal{D}$ is of type I_n, and $\{E_j\}_{j=1}^n$ is an orthogonal family of equivalent, abelian projections with sum I, and x_j is a cyclic vector for E_j under $\mathcal{D}' = C$, then $x_j(p) \neq 0$ and $\{x_j(p)\}_{j=1}^n$ is an orthogonal, total family of vectors in H_p almost everywhere.

Proof. Suppose first that $\mathcal{M}$ has type I_n and $\{E_j\}_{j=1}^n$ is an orthogonal family of equivalent abelian projections with sum I. By Propositions 2.16 and 2.33 the family $\{E_j(p)\}$ consists of orthogonal, equivalent abelian projections with sum $I(p) = I_p$ almost everywhere (if n is $\aleph_0$, we apply Corollary 2.21 for the conclusion about the sum). Since $N_{jk} = \{p \in X \mid E_j(p) \nsim E_k(p)\}$ is a null set, the set $N = \bigcup_{j,k} N_{jk}$ is a null set, and outside N it holds that $E_j(p) \sim E_k(p)$ for every j, k. Thus either $E_j(p) = 0$ for all j or for none. Since they sum up to I_p, they are non-zero, and hence H_p has dimension at least n almost everywhere. (This is the place where we use that H_p is non-zero – see the comment after Definition 2.1).

To show that the dimension of H_p does not exceed n, we prove that $\{x_j(p)\}$ is total in H_p for almost all p, where x_j is a cyclic vector corresponding to E_j as guaranteed by Lemma 2.35. Let $\{C_m\}$ be a countable, strong operator dense subset of C. As $E_j(H) = [Cx_j]$, it follows that $\{C_m x_j\}_m$ is dense in $E_j(H)$. And as $\{E_j\}$ has sum I, it follows that the set $\{C_m x_j\}_{m,j}$ is total in H. Hence $\{C_m(p)x_j(p)\}_{m,j}$ is total in H_p almost everywhere. Since C_m is diagonalizable, $C_m(p)$ is scalar almost everywhere, and thus $\{x_j(p)\}$ is total in H_p. This proves that H_p has dimension n almost everywhere.

Conversely, suppose that H_p has dimension n for almost all p. As $\mathcal{D}' = C$ is abelian, it is of type I, and hence $\mathcal{D}$ is of type I according to Theorem 0.2. Using the type decomposition (Theorem 0.3) we find pairwise orthogonal central projections P_m with sum I such that $\mathcal{D}P_m$ is of type I_m or $P_m = 0$. We claim that $P_m \neq 0$ only if $m = n$. If $P_m \neq 0$ then, since P_m is central, it is a diagonalizable projection corresponding to some measurable subset X_0 of X, and X_0 must have positive measure.

Then $P_m(H)$ is the direct integral of $\{H_p\}$ over (X_0, μ), $\mathcal{D}P_m$ is the algebra of decomposable operators relative to this decomposition of $P_m(H)$, and CP_m is the algebra of diagonalizable operators. Since $\mathcal{D}P_m$ is of type I_m, the previous part of the proof yields that H_p has dimension m almost everywhere in X_0. Then by our assumption $m = n$. This proves that $\mathcal{D}$ is of type I_n.

Suppose now that $\mathcal{M}$ has type I_n and $\{E_j\}$ and $\{x_j\}$ are as in the lemma. To prove orthogonality of $\{x_j\}$ let j and k be different. If Y is a measurable subset of X and Q is the corresponding diagonalizable projection in $C = \mathcal{D}'$, then since E_j and E_k are orthogonal, and they commute with Q,

$$0 = \langle E_j Q x_j, E_k x_k \rangle = \langle Q E_j x_j, E_k x_k \rangle = \langle Q x_j, x_k \rangle = \int_Y \langle x_j(p), x_k(p) \rangle \, d\mu.$$

This holds for every measurable subset Y of X, and hence $\langle x_j(p), x_k(p) \rangle = 0$ almost everywhere (see Lemma A.3). So $\{x_j(p)\}$ is an orthogonal family almost everywhere.

If n is finite, then of course $x_j(p) \neq 0$ almost everywhere, because the set $\{x_j(p)\}$ spans an n-dimensional space. If n is infinite ($n = \aleph_0$) we must use another argument. Using the set $\{C_m\}$ from before we know that $\{C_m x_j\}_m$ is dense in $E_j(H)$. By Proposition 2.33 (v) we get that $\{C_m(p)x_j(p)\}_m$ is total in $E_j(p)(H_p)$ for almost all p. Then, arguing as before, since $E_j(p) \neq 0$ almost everywhere, $x_j(p) \neq 0$ almost everywhere. □

2.5.1 Measurable maps into H and $B(H)$

At this point it will be relevant to discuss measurability of maps $X \to H$ and $X \to B(H)$. This is a concept we have not yet dealt with, and we will only give a summary of the relevant parts and refer the reader to §52 in [Conway] for details.

A map $f : X \to H$ is called *weakly measurable*, if to each $y \in H$ the map $p \mapsto \langle f(p), y \rangle$ is measurable into $\mathbb{C}$. Since a countable (convergent) sum of measurable functions is measurable, and H is (assumed) separable, it follows that it suffices to demand that $p \mapsto \langle f(p), e_n \rangle$ is measurable for each e_n in an orthonormal basis for H. The map f is weakly measurable if and only if it is measurable, when we equip H with the σ-algebra generated by the norm topology on H. The measurable functions into H is a vector space, when the operations are defined pointwise.

A map $g : X \to B(H)$ is called *weakly measurable*, if to each $x, y \in H$ the map $p \mapsto \langle g(p)x, y \rangle$ is measurable into $\mathbb{C}$. Again, it is enough to require x and y to be elements in an orthonormal basis. The map g is weakly measurable if and only if it is measurable, when we equip $B(H)$ with σ-algebra generated by the strong operator topology. The measurable functions into $B(H)$ forms a *-algebra, when the operations are defined pointwise.

From now on weakly measurable maps are simply called measurable.

Lemma 2.37. *Let (Y, Ω) be a measurable space, and let H be a separable Hilbert space. Let $f, g : Y \to B(H)$ be measurable maps. The subset of Y where f and g agree is measurable.*

Proof. Let $\{e_n\}$ be an orthonormal basis for H. Observe that $f(y)$ and $g(y)$ are equal if and only if $\langle f(y)e_m, e_n \rangle = \langle g(y)e_m, e_n \rangle$ for every pair (m, n). The maps $y \mapsto \langle f(y)e_m, e_n \rangle$ and $y \mapsto \langle g(y)e_m, e_n \rangle$

are measurable maps into $\mathbb{C}$, and it is well-known that the subset of Y where such two maps agree is measurable. From the observation then follows that the subset of Y where f and g agree is a countable intersection measurable sets. □

2.5.2 The proof of Theorem 2.39

The following lemma contains the setup for the proof of Theorem 2.39.

Lemma 2.38. *Let H be a direct integral of $\{H_p\}$ over (X, μ), and let X_n denote the set of points p where H_p has dimension n. Let $M = \{n \mid \mu(X_n) > 0\}$, and put $X_0 = \bigcup_{n \in M} X_n$. If $n \in M$, then the diagonalizable projection P_n corresponding to X_n is the maximal central projection in $\mathcal{D}$, the algebra of decomposable operators, such that $\mathcal{D}P_n$ is of type I_n. There is a family $\{U_p\}_{p \in X_n}$ such that U_p is a unitary transformation of H_p onto a fixed Hilbert space K_n of dimension n. With K the direct sum of those K_n where $n \in M$ the entire family $\{U_p\}_{p \in X_0}$ is a family of isometries such that U_p maps H_p into K (with image K_n).*

The map $p \mapsto U_p x(p)$ is measurable $X_0 \to K$ for each $x \in H$, and the map $p \mapsto U_p A(p) U_p^*$ is measurable $X_0 \to B(K)$ for each $A \in \mathcal{D}$.

Let P_n be the diagonalizable projection corresponding to X_n in case $n \in M$. Since the diagonalizable operators C are the center of $\mathcal{D}$, each P_n is central. Note that X is the union of $\bigcup_{n \in M} X_n$ and a null set, so $\sum_{n \in M} P_n$ is the diagonalizable projection corresponding to X. In other words $\sum_{n \in M} P_n = I$. Using Lemma 2.34 the algebra $\mathcal{D}P_n$ acts on the Hilbert space $P_n(H)$ which has the direct integral decomposition inherited from H, and in this decomposition almost all Hilbert spaces have dimension n. From Lemma 2.36 follows now, that $\mathcal{D}P_n$ is of type I_n, and using $\sum_{n \in M} P_n = I$ we get that P_n is the maximal central projection such that $\mathcal{D}P_n$ has type I_n.

It remains to construct the unitaries U_p and show measurability of the maps in the statement of the lemma. It suffices to show measurability when we restrict ourselves to the subset X_n, since pasting a countable number of measurable functions together yields a measurable function. Thus we only work in the subspace $P_n(H)$ so we might as well suppose that each H_p has dimension n. We fix a Hilbert space K_n of dimension n with an orthonormal basis $\{y_j\}$. By Lemma 2.36 there are n vectors $\{x_j\}$ and a null set N such that $x_j(p) \neq 0$ for every j and $\{x_j(p)\}$ is an orthogonal family of vectors, total in H_p, when p is not in N. Let U_p be the unique unitary transformation of H_p onto K_n that maps $x_j(p)$ to $\|x_j(p)\| y_j$, $j = 1, \ldots, n$.

If $x \in H$ and $p \notin N$, then

$$x(p) = \sum_{j=1}^{n} \langle x(p), x_j(p) \rangle \|x_j(p)\|^{-2} x_j(p)$$

and

$$U_p x(p) = \sum_{j=1}^{n} \langle x(p), x_j(p) \rangle \|x_j(p)\|^{-1} y_j.$$

Since the map $p \mapsto \langle x(p), x_j(p) \rangle \|x_j(p)\|^{-1}$ is measurable for each j, the map $p \mapsto U_p x(p)$ is measurable for each x.

Similarly, if $p \notin N$ and $A \in \mathcal{D}$ then

$$\langle U_p A(p) U_p^* y_j, y_k\rangle = (\|x_j(p)\| \, \|x_k(p)\|)^{-1} \langle A(p)x_j(p), x_k(p)\rangle$$

for any j, k, and so the map $p \mapsto U_p A(p) U_p^*$ is measurable.

Theorem 2.39. *Let H be a direct integral of $\{H_p\}$ over (X, μ), where X is a Polish space. Assume further that X is a countable union of compact sets with finite measure. Let $\mathcal{M}$ be a von Neumann algebra of decomposable operators on H, and suppose $\mathcal{M}$ contains the diagonalizable operators C. Then $\mathcal{M}'$ is decomposable, and $(\mathcal{M}')_p = (\mathcal{M}_p)'$ almost everywhere. This may also be written*

$$\left(\int_X \mathcal{M}_p \, d\mu\right)' = \int_X (\mathcal{M}_p)' \, d\mu.$$

Proof. Since $C \subseteq \mathcal{M}$, we have $\mathcal{M}' \subseteq C' = \mathcal{D}$ (the algebra of decomposable operators), so by Theorem 2.32 the algebra $\mathcal{M}'$ is decomposable. To get an idea of how the proof will proceed we sketch the proof first and then fill in the hard details afterwards. Let $\{A_j\}$ and $\{A'_j\}$ be countable, strong operator dense subsets of $\mathcal{M}$ and $\mathcal{M}'$ respectively. Then $A_j(p)$ commutes with $A'_k(p)$ almost everywhere, and from this $(\mathcal{M}')_p \subseteq (M_p)'$ for almost all p. If $(M_p)' = (\mathcal{M}')_p$, we let $A'_p = 0$, and else we choose $A'_p \in (M_p)' \setminus (\mathcal{M}')_p$. If this selection can be done in a measurable manner, i.e. if there is $A' \in \mathcal{M}'$ such that $A'(p) = A'_p$ almost everywhere, then $A'(p)$ will commute with $A_j(p)$ almost everywhere. Then $A'_p = A'(p)$ belongs to $(\mathcal{M}')_p$ almost everywhere, and the set of those p where $(M_p)' \setminus (\mathcal{M}')_p \neq \emptyset$ is a null set. This would complete the proof. The key of the proof is to envoke the measurable selection theorem (Theorem A.27) to do the selection.

Let $\{y_k\}$ be a countable dense subset of H. Before applying the measurable selection theorem we introduce the Hilbert space K of Lemma 2.38 and the family of isometries $\{U_p\}$. Let $F_n : K \to K$ be the projection onto K_n. Notice that $U_p^* U_p = I_p$ and $U_p U_p^* = F_n$.

As our Polish spaces we use X and the unit ball $\mathcal{B}$ of $B(K)$ provided with its strong operator topology (see Example A.6). For a pair (p, T) in $X \times \mathcal{B}$ we may consider the following conditions.

(i) $p \in X_n$, $F_n T F_n = T$ and $T U_p A_j(p) U_p^* = U_p A_j(p) U_p^* T$ for all $j \in \mathbb{N}$.

(ii) $p \in X_n$, $F_n T F_n = T$ and there are $m, h \in \mathbb{N}$ such that for every $j \in \mathbb{N}$ there is a natural number $k \leq h$ such that

$$\|(T - U_p A'_j(p) U_p^*) U_p y_k(p)\| \geq \frac{1}{m}. \tag{2.1}$$

Since $\{A_j(p)\}$ is strong operator dense in $\mathcal{M}_p$ for almost all p, it is easy to see that except for p in a null set, the pair (p, T) satisfies (i) if and only $p \in X_n$ and $T \in U_p(\mathcal{M}_p)' U_p^*$. We may even choose the null set to be a Borel set N_1. Similarly, the pair satisfies (ii) if and only if $p \in X_n$, $F_n T F_n = T$ and $T \notin U_p(\mathcal{M}')_p U_p^*$ except for p in a null set. To see this, note first that $\{A'_j(p)\}$ is strong operator dense in $(\mathcal{M}')_p$ and $\{U_p y_j(p)\}$ is dense in K_n (almost everywhere). Suppose (p, T) satisfies $p \in X_n$

and $F_nTF_n = T$. If $T \in U_p(\mathcal{M}')_pU_p^*$, then using the fact that $\{U_pA_j'(p)U_p^*\}$ is dense in $U_p(\mathcal{M}')_pU_p^*$ we get that for every $m \in \mathbb{N}$ and every finite subset $\{y_k(p)\}_{k=1}^h$ of $\{y_k(p)\}$ there is $A_j'(p)$ such that

$$\|(T - U_pA_j'(p)U_p^*)U_py_k(p)\| < \frac{1}{m}, \quad k = 1, \ldots, h. \tag{2.2}$$

If on the other hand for every $m, h \in \mathbb{N}$ there is a $j \in \mathbb{N}$ such that (2.2) holds, then since $\{U_py_j(p)\}$ is dense in the Hilbert space K_n, it follows that F_nTF_n is in the strong operator closure of $\{U_pA_j'(p)U_p^*\}$. Since $T = F_nTF_n$, the claim follows. All of this happens, of course, only almost everywhere. So it happens everywhere outside a Borel null set N_2.

The maps $p \mapsto U_pA_j(p)U_p^*$, $p \mapsto U_pA_j'(p)U_p^*$ and $p \mapsto U_py_j(p)$ are measurable for each j according to Lemma 2.38. Due to Lemma A.1 there is a Borel null set N_3 such that, when restricted to $X \setminus N_3$ all of these maps are Borel maps. Let $X_0 = X \setminus (N_1 \cup N_2 \cup N_3)$. Then the maps are also Borel when restricted to X_0. Of course the identity map $T \mapsto T$ is a Borel map on $\mathcal{B}$, and hence the product map $(p, T) \mapsto (U_pA_j(p)U_p^*, T)$ is Borel, when we equip $\mathcal{B} \times \mathcal{B}$ with its product topological Borel structure (the same as the σ-algebra generated by the projection maps onto each of its two factors). Further, multiplication is a (jointly) strong operator continuous map $\mathcal{B} \times \mathcal{B} \to \mathcal{B}$, and hence the composed maps $(p, T) \mapsto U_pA_j(p)U_p^*T$ and $(p, T) \mapsto TU_pA_j(p)U_p^*$ are Borel maps. The subset of $(X_0 \cap X_n) \times (F_n\mathcal{B}F_n)$ where they agree is a Borel set $\mathscr{S}_{jn}'$ (see Lemma 2.37), and the intersection $\mathscr{S}_n' = \bigcap_j \mathscr{S}_{jn}'$ is the Borel set consisting of those pairs (p, T) in $X_0 \times \mathcal{B}$ satisfying (i). The union $\mathscr{S}' = \bigcup_n \mathscr{S}_n'$ is then the Borel set of pairs (p, T) in $X_0 \times \mathcal{B}$ where $T \in U_p(\mathcal{M}_p)'U_p^*$.

Along the same lines, let $\mathscr{S}_{jkmn}''$ be the pairs (p, T) in $(X_0 \cap X_n) \times (F_n\mathcal{B}F_n)$ fulfilling (2.1) for given j, k, m, n. Arguing as before one can show that the map $(p, T) \to \|(T - U_pA_j'(p)U_p^*)U_py_k(p)\|$ is Borel $X_0 \times \mathcal{B} \to \mathbb{R}$. The set

$$\mathscr{S}'' = \bigcup_{n,m,h=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{k=1}^{h} \mathscr{S}_{jkmn}''$$

is then the set of pairs (p, T) in $X_0 \times \mathcal{B}$ such that there is $n \in \mathbb{N}$ with $F_nTF_n = T$, $p \in X_n$ and $A \notin U_p(\mathcal{M}')_pU_p^*$. The set $\mathscr{S}''$ is a Borel set.

Let $\mathcal{F}_p$ denote the set $(U_p(\mathcal{M}_p)'U_p^*) \setminus (U_p(\mathcal{M}')_pU_p^*)$. Since $T \in U_p(\mathcal{M}_p)'U_p^*$ already implies that $F_nTF_n = T$, a pair (p, T) in $X_0 \times \mathcal{B}$ is in the intersection $\mathscr{S} = \mathscr{S}' \cap \mathscr{S}''$ if and only if $T \in \mathcal{F}_p$.

We have now proved that the set $\mathscr{S}$ of pairs (p, T) in $X_0 \times \mathcal{B}$ such that $T \in \mathcal{F}_p$ is a Borel subset of $X \times \mathcal{B}$. By Theorem A.17 the set $\mathscr{S}$ is an analytic subset of $X \times \mathcal{B}$. Let X_1 be the projection of $\mathscr{S}$ in X. Then X_1 is the subset of X consisting of those points p in X_0 where $\mathcal{F}_p \neq \emptyset$. The set X_1 is analytic being the image of an analytic set under the (continuous) projection map, and hence X_1 is measurable by Theorem A.19. Let $\eta : X_1 \to \mathcal{B}$ be a measurable such that $(p, \eta(p)) \in \mathscr{S}$ for every p – as garuanteed by Theorem A.27. The operator $\eta(p)$ has the form $U_pA_p'U_p^*$ for some $A_p' \in (\mathcal{M}_p)'$. Since $\eta(p) \notin U_p(\mathcal{M}')_pU_p^*$, we must have $A_p' \notin (\mathcal{M}')_p$.

For p in $X \setminus X_1$ we define A_p' to be zero. We hope to find some decomposable operator $A' \in B(H)$

with decomposition $p \mapsto A'_p$. For $x, y \in H$ we have

$$\langle A'_p x(p), y(p)\rangle = \langle U_p A'_p U_p^* U_p x(p), U_p y(p)\rangle,$$

and since η is meassurable, $p \mapsto \langle A'_p x(p), y(p)\rangle$ is measurable. Since $U_p A'_p U_p^*$ is in the unit ball $\mathcal{B}$, we have the following bound

$$|\langle U_p A'_p U_p^* U_p x(p), U_p y(p)\rangle| \leq \|x(p)\| \, \|y(p)\|,$$

and a simple application of Hölder's inequality then shows that the map $p \mapsto \langle A'_p x(p), y(p)\rangle$ is integrable. Going back to Definition 2.1 we see that there is a vector in H, called $A'x$, such that $(A'x)(p) = A'_p x(p)$ almost everywhere. So we get a map $A' : H \to H$, and since each A'_p is linear, it follows easily (use Remark 2.5) that A' is linear. Also,

$$\|A'x\|^2 = \int_X \|(A'x)(p)\|^2 \, d\mu = \int_X \|A'_p x(p)\|^2 \, d\mu \leq \int_X \|x(p)\|^2 \, d\mu = \|x\|^2,$$

so $\|A'\| \leq 1$, and in particular A' is a bounded operator on H. By construction A' is decomposable, and $A'(p) = A'_p \in (\mathcal{M}_p)' \setminus (\mathcal{M}')_p$ for every p in X_1. And since $A'(p) = 0$ when $p \notin X_1$, it follows that $A'(p)A_j(p) = A_j(p)A'(p)$ for every j almost everywhere. Hence A' commutes with every A_j, and $A' \in \mathcal{M}'$. As noted in connection with Definition 2.28, $A'(p) \in (\mathcal{M}')_p$ almost everywhere, and then X_1 must be a null set. The proof is now complete, since then $(\mathcal{M}_p)' = (\mathcal{M}')_p$ almost everywhere. $\square$

CHAPTER 3

The central decomposition

3.1 Decomposition relative to an abelian algebra

We have now reached the point where the main theorems can be proved.

Theorem 3.1. *Let $\mathcal{A}$ be an abelian von Neumann algbera on a separable Hilbert space H. There is a measure space (X, μ) (where X is Polish and is a countable union of compact sets of finite measure, and μ is the completion of a Borel measure), and there are Hilbert spaces $\{H_p\}_{p \in X}$ such that H is the direct integral of $\{H_p\}$ over X, and $\mathcal{A}$ is the algebra of diagonalizable operators relative to the direct integral decomposition of H.*

As is seen from the proof below a lot more can be said about the space X, but we are only interested in exactly these properties in order to be able to apply Theorem 2.39 later on. Before the proof it is convenient with the following definition.

Definition 3.2 (Direct sum of measure spaces). If $(X_i, \mathcal{S}_i, \mu_i)$ are measure spaces for all i in some index set I, and the sets X_i are disjoint, then the *direct sum* of these measure spaces is defined by taking $X = \bigcup_i X_i$, letting

$$\mathcal{S} = \{A \subseteq X \mid A \cap X_i \in \mathcal{S}_i \text{ for all } i\},$$

and

$$\mu(A) = \sum_i \mu_i(A \cap X_i), \quad A \in \mathcal{S}.$$

It must of course be proven that $(X, \mathcal{S}, \mu)$ is in fact a measure space, but this is easy and therefore omitted. If the sets X_i are not disjoint, then one uses the disjoint union in stead and corresponding definitions of the σ-algebra and measure.

Proof of Theorem 3.1.

Step 1.

We prove the theorem in steps according to the structure of $\mathcal{A}'$. Since $\mathcal{A}$ is abelian, $\mathcal{A}$ is of type I, and hence $\mathcal{A}'$ is of type I. If $\mathcal{A}'$ is of type I_1, then $\mathcal{A}'$ is abelian and $\mathcal{A} = \mathcal{A}'$, and so $\mathcal{A}$ is maximal abelian. By the characterization of maximal abelian algebras acting on separable spaces (Theorem

1.3), $\mathcal{A}'$ is unitarily equivalent to one of the multiplication algebras $\mathcal{A}_n$, $\mathcal{A}_c$ or $\mathcal{A}_n \oplus \mathcal{A}_c$. Let X denote the corresponding measure space: $X = S_n$ in the case where $\mathcal{A}' \simeq \mathcal{A}_n$, $X = [0, 1]$ if $\mathcal{A}' \simeq \mathcal{A}_c$ and X is the direct sum the measure spaces S_n and $[0, 1]$ in the last case. In any case, X is a σ-finite measure space, and H is isomorphic to $L^2(X, \mu)$. By Proposition 2.2, $L^2(X, \mu)$ is the direct integral of one-dimensional Hilbert spaces, and by Proposition 2.23, the diagonalizable operators relative to this decomposition is exactly the multiplication algebra of $L^2(X, \mu)$ identified as $L^\infty(X, \mu)$. Then H is also a direct integral of one-dimensional spaces with $\mathcal{A}'$ being the diagonalizable operators.

Step 2.

If $\mathcal{A}'$ is of type I_n, where n is a natural number, then by Theorem 1.9 there is a (necessarily separable) Hilbert space K and a maximal abelian von Neumann algebra $\mathcal{B}$ on K such that $\mathcal{A}$ is unitarily equivalent to $\mathcal{B}^{(n)}$. We can assume that $K = L^2(X, \mu)$ and $\mathcal{B}$ is the multiplication algebra arising from $L^\infty(X, \mu)$ acting on $L^2(X, \mu)$, where X is one of the measure spaces mentioned in Step 1. Then there is a unitary transformation of H onto n copies of $L^2(X, \mu)$ that maps $\mathcal{A}$ onto the set of operators $C = \{\bigoplus_{i=1}^n M_f \mid f \in L^\infty(X, \mu)\}$.

A vector in H corresponds to an n-tuple $(f_1, \ldots, f_n)$ of functions in $L^2(X, \mu)$. If we let each H_p be the n-dimensional Hilbert space $\mathbb{C}^n$, then it is easily seen that H is the direct integral of $\{H_p\}$, if we let $(f_1, \ldots, f_n)$ have component $(f_1(p), \ldots, f_n(p))$ in H_p. It is also a simple matter to check that the diagonalizable operators on $L^2(X, \mu) \oplus \cdots \oplus L^2(X, \mu)$ relative to this decomposition is exactly $\mathcal{A}$.

Step 3.

If $\mathcal{A}'$ is of type I_n, where $n = \aleph_0$, then the same procedure as in step 2 carries over with only small changes. Again, by Theorem 1.9 there is a separable Hilbert space $K = L^2(X, \mu)$ and a maximal abelian von Neumann algebra $\mathcal{B}$ on K such $\mathcal{A}$ is unitarily equivalent to $\mathcal{B}^{(\infty)}$. A vector in H corresponds to a sequence $(f_1, f_2, \ldots,)$ of functions in $L^2(X, \mu)$ such that $\sum_{n=1}^\infty \|f_n\|_2^2$ is finite. It follows that $\sum_{n=1}^\infty |f_n(p)|^2$ is finite almost everywhere. Hence for p outside a null set it holds that $(f_1(p), f_2(p), \ldots)$ belongs to $H_p = \ell^2$. Using this decomposition, H becomes a direct integral of $\{H_p\}$ over (X, μ), and the diagonalizable operators on H are exactly $\mathcal{A}$.

Step 4. In general, $\mathcal{A}'$ is the direct sum of algebras of type I_n, $n = 1, 2, \ldots, \aleph_0$. In other words, it is possible to write $\mathcal{A}' = \bigoplus_{n \in I} \mathcal{A}' P_n$, where $I = \{1, 2, \ldots, \aleph_0\}$ and P_n is a central projection in $\mathcal{A}'$ such that $\mathcal{A}' P_n$ is of type I_n or $P_n = 0$. Using the above approach to the algebra $\mathcal{A}' P_n$ acting on $P_n(H)$ we get a measure space (X_n, μ_n) such that $P_n(H)$ is unitarily equivalent to the sum of n copies of $L^2(X_n, \mu_n)$, denoted K_n, and the algebra $\mathcal{A}' P_n$ is unitarily equivalent to diagonalizable operators on K_n with respect to the decomposition described above.

Let (X, μ) be the direct sum of the measure spaces (X_n, μ_n). If we write

$$K_n = \int_{X_n} H_p \, d\mu_n(p)$$

then

$$K = \bigoplus_n K_n = \int_X H_p \, d\mu(p)$$

with the convention that $p \mapsto x_n(p) = 0 \in H_p$, when $p \notin X_n$. The direct sum of the unitaries mapping $P_n(H)$ onto K_n is a unitary transformation $H \to K$ that carries $\mathcal{A}$ onto the diagonalizable operators on K relative to the decomposition just described. □

Theorem 3.3. *Let H be a separable Hilbert space and $\mathcal{M}$ a von Neumann algebra acting on H with center C. Suppose $\mathcal{A}$ is an (abelian) subalgebra of C, and $\{H_p\}$ is the direct integral decomposition of H relative to $\mathcal{A}$ given by Theorem 3.1. Then $\mathcal{M}$ is decomposable and C_p is the center of $\mathcal{M}_p$ for almost everywhere.*

Proof. Using the direct integral decomposition of H given in Theorem 3.1, the decomposable operators are precisely $\mathcal{A}'$ by Theorem 2.22. By assumption $\mathcal{A} \subseteq C \subseteq \mathcal{M}$, and hence $\mathcal{M}' \subseteq C' \subseteq \mathcal{A}'$. So from Theorem 2.32 it follows that $\mathcal{M}'$ and C' are decomposable. Since $C \subseteq \mathcal{M}'$ it follows that $\mathcal{M} \subseteq C' \subseteq \mathcal{A}'$, and hence also C and $\mathcal{M}$ are decomposable according to Theorem 2.32.

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be strong operator dense, norm separable C*-subalgebras of $\mathcal{M}$ and $\mathcal{M}'$, respectively. Let $\mathcal{B}$ be the C*-algebra generated by $\mathcal{B}_1$ and $\mathcal{B}_2$. Then $\mathcal{B}$ is also separable.

Observe that for any von Neumann algebra $\mathcal{N}$, the commutant of the center of $\mathcal{N}$ is generated by $\mathcal{N} \cup \mathcal{N}'$. Hence C' is the von Neumann algebra generated by $\mathcal{M}$ and $\mathcal{M}'$, and it follows that $\mathcal{B}$ is strong operator dense in C'. Also using the observation, it follows from Theorem 2.39 that commutant of the center of $\mathcal{M}_p$ is generated by $\mathcal{M}_p \cup (\mathcal{M}')_p$ almost everywhere.

By Lemma 2.31, $(\mathcal{B}_1)_p$, $(\mathcal{B}_2)_p$ and $\mathcal{B}_p$ are strong operator dense in $\mathcal{M}_p$, $(\mathcal{M}')_p$ and $(C')_p$, respectively, almost everywhere. So $(\mathcal{B}_1)_p$ and $(\mathcal{B}_2)_p$ generate the commutant of the center of $\mathcal{M}_p$ almost everywhere. Notice that $(\mathcal{B}_1)_p$ and $(\mathcal{B}_2)_p$ generate $\mathcal{B}_p$ as well, and $\mathcal{B}_p$ is strong operator dense in C'_p almost everywhere (using Theorem 2.39 once again). Putting all of this together yields that C'_p is the commutant of the center of $\mathcal{M}_p$ almost everywhere. Hence C_p is the center of $\mathcal{M}_p$ almost everywhere. This completes the proof. □

As a special case of the theorem, if $\mathcal{A} = C$, we note the following, which is the main theorem of this project.

Theorem 3.4. *Let $\mathcal{M}$ be a von Neumann algebra acting on a separable Hilbert space. Then there is a decomposition $p \mapsto \mathcal{M}_p$ of $\mathcal{M}$ such that $\mathcal{M}_p$ is a factor almost everywhere.*

Proof. Use Theorem 3.3 with $\mathcal{A} = C$. Since C are then the diagonalizable operators, it follows that C_p consists of scalar operators almost everywhere. In other words, the center of $\mathcal{M}_p$ is the scalars, so $\mathcal{M}_p$ is a factor (almost everywhere). □

3.2 Type of the components

The central decomposition in Theorem 3.4 tells us that it is possible to write a von Neumann algebra $\mathcal{M}$ as a direct integral of factors. However, it does not say anything about the structure of $\mathcal{M}$ compared to the structure of the components. It is possible to prove that the type of a von Neumann algebra is preserved under disintegration. To be precise the following theorem holds.

Theorem 3.5. *If a von Neumann algebra $\mathcal{M}$ acting on a separable Hilbert space is decomposed as $p \mapsto \mathcal{M}_p$, then $\mathcal{M}$ is of type* I_n, II_1, II_∞ *or* III *if and only if $\mathcal{M}_p$ has type* I_n, II_1, II_∞ *or* III*, respectively, almost everywhere.*

In particular, if $\mathcal{M}$ is a von Neumann algebra of type I_n, II_1, II_∞ *or* III *acting on a separable Hilbert space, then the components in its direct integral decomposition relative to its center are factors of type* I_n, II_1, II_∞ *or* III*, respectively, almost everywhere.*

We will not prove the theorem, since by now this project has reached a suitable length. The part of the proof concerning the type III case is particularly harder than the rest. A proof can go via the measurable selection theorem, which we have already used to prove Theorem 2.39. Another way to prove the theorem is based on the Tomita-Takesaki theory concerning the modular automorphism group.

APPENDIX A

Measure theoretic results

A.1 Definitions

A σ-algebra (sometimes called a Borel structure or a Borel field) on a set X is a family Ω of subsets of X containing the empty set $\emptyset$, containing the complement of any set in Ω and the union of countable families of sets from Ω. The sets in Ω will be referred to as measurable sets. The intersection of σ-algebras on X is again a σ-algebra, and so to any collection $\mathcal{F}$ of subsets of X there is a smallest σ-algebra on X containing $\mathcal{F}$. This σ-algebra is called the σ-algebra generated by $\mathcal{F}$. In most cases relevant in this project the space X is a topological space, and the *Borel algebra* on X will then mean the σ-algebra generated by the open sets. The sets in Ω will then also be called *Borel sets*. The pair (X, Ω) is referred to as a *measurable space*, and in case Ω is the Borel algebra of X the pair is called a *Borel measure space*.

A (positive) *measure* on (X, Ω) is a map $\mu : \Omega \to [0, \infty]$ (notice that the value ∞ is allowed) such that $\mu(\emptyset) = 0$ and $\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$ for any countable family $(A_n)_{n=1}^{\infty}$ of disjoint, measurable sets. The triple (X, Ω, μ) will be referred to as a *measure space*. Often the measure space is written (X, μ) or simply X.

A set $N \in \Omega$ such that $\mu(N) = 0$ is called a *null set*. Subsets M of N, which are not necessarily measurable, are also called null sets. A property is said to hold *almost everywhere* in X, if there is a null set M such that for every $x \in X \setminus M$ the property holds. For instance a function $f : X \to \mathbb{R}$ is (stricly) positive almost everywhere, if $f(x) > 0$ for every $x \in X \setminus M$, and M is a null set.

If μ is a measure on (X, Ω), where X is a topological space, and Ω is the Borel algebra, μ is called a *Borel measure*.

A measure is called *complete*, if every null set is measurable, i.e. if $N \in \Omega$, $\mu(N) = 0$ and $M \subseteq N$ implies that $M \in \Omega$. Complete measures have nice properties such as the one stated in Lemma A.2. If (X, Ω, μ) is a measure space, then the family consisting of sets of the form $A \cup N$ where $A \in \Omega$ and N is a (not necessarily measurable) null set forms a σ-algebra $\overline{\Omega}$ on X. Defining $\overline{\mu}(A \cup N) = \mu(A)$ gives a well-defined measure on $(X, \overline{\Omega})$, which extends μ. The measure $\overline{\mu}$ is complete and is called the *completion* of μ. If (X, Ω, μ) is a Borel measure space, then $(X, \overline{\Omega}, \overline{\mu})$ is called a *complete Borel measure space*. The sets in Ω are called Borel sets to be distinguished from the measurable sets, i.e.

the sets in $\overline{\Omega}$. Of course, Borel sets are measurable, but the converse is not always true.

If (X, Ω) and (Y, Σ) are measurable spaces, a map $f : X \to Y$ is called *measurable*, if the preimage of any set from Σ is in Ω, i.e $f^{-1}(B) \in \Omega$ for any $B \in \Sigma$. In the case where (X, Ω, μ) is a Borel measure space, the map f is called a *Borel map* if $f^{-1}(B) \in \Omega$ for any $B \in \Sigma$, and a *measurable map* if $f^{-1}(B) \in \overline{\Omega}$ for any $B \in \Sigma$.

To check measurability of a map $f : (X, \Omega) \to (Y, \Sigma)$ it suffices to check that $f^{-1}(B) \in \Omega$ for any B in a *generating family* for the σ-algebra Σ.

A.2 Measure theoretic technicalities

Lemma A.1. *Let (X, μ) be a complete Borel measure space (so that μ is the completion of a Borel measure on X), and let (Y, Σ) be a measurable space, where Σ is a countably generated σ-algebra on Y. If $f : X \to Y$ is a measurable map, there is a Borel null set N of X such that $f|(X \setminus N)$ is a Borel map.*

Proof. Let $\{Y_n\}$ be a countable family of measurable sets of Y generating Σ. Then $f^{-1}(Y_n)$ is measurable in X and is therefore the union of a Borel set X_n and a null set M_n. Let N_n be a Borel null set containing the null set. Let $N = \bigcup_n N_n$. Of course N is a Borel null set, and also

$$f^{-1}(Y_n) \cap (X \setminus N) = X_n \cap (X \setminus N).$$

The set $X_n \cap (X \setminus N)$ is a Borel subset of $X \setminus N$, and since $\{Y_n\}$ generates Σ it follows that $f|(X \setminus N)$ is a Borel map. □

Lemma A.2. *Let (X, μ) be a complete measure space, $f : X \to \mathbb{C}$ and $g : X \to \mathbb{C}$ complex-valued functions on X. Suppose f is measurable, and f and g agree almost everywhere. Then g is also measurable.*

Proof. Let N denote the null set $\{x \in X \mid f(x) \neq g(x)\}$. Without loss of generality, it may be assumed that f is constantly zero. Let $U \subset \mathbb{C}$ be a measurable set. If $0 \notin U$, then $g^{-1}(U) \subseteq N$ is measurable, because μ is complete. If $0 \in U$, let $V = U \setminus \{0\}$. Then $g^{-1}(V)$ is measurable and

$$g^{-1}(U) = g^{-1}(V) \cup g^{-1}(\{0\}) = g^{-1}(V) \cup (X \setminus N).$$

Hence $g^{-1}(U)$ is measurable in this case as well. □

Lemma A.3. *Let (X, μ) be a measure space, and let $f : X \to \mathbb{C}$ be a measurable, integrable function. If*

$$\int_Y f \, d\mu = 0$$

for each measurable subset Y of X, then $f(x) = 0$ almost everywhere.

Proof. Suppose first that f is real-valued. Define measurable sets $A_n = \{x \in X \mid f(x) \geq \frac{1}{n}\}$ where $n \in \mathbb{N}$. Since

$$0 = \int_{A_n} f \, d\mu \geq \int_{A_n} \frac{1}{n} \, d\mu = \frac{\mu(A_n)}{n}$$

it follows that each A_n is a null set. Hence the union $A = \bigcup_n A_n = \{x \in X \mid f(x) > 0\}$ is a null set. Similarly, $B = \{x \in X \mid f(x) < 0\}$ is a null set, and then $f(x) = 0$ almost everywhere.

If f is arbitrary, then write $f = g + ih$ for some real-valued, measurable functions g, h. Then $f(x) = 0$ if and only if $g(x) = h(x) = 0$. It follows by the previous case, that $g(x) = h(x) = 0$ almost everywhere, and hence $f(x) = 0$ almost everywhere. □

A.3 Polish spaces

Definition A.4 (Polish space). A topological space X is *topologically complete*, if it is homoemorphic to a complete metric space or, alternatively, if X admits a metric for the topology with which it is complete. A topological space is called a *Polish space*, if it is separable and topologically complete.

Example A.5. The real line $\mathbb{R}$ and the closed unit interval $[0, 1]$ are standard examples of Polish spaces. The open unit interval $]0, 1[$ is also Polish, since it it homeomorphic to $\mathbb{R}$. Any closed subspace of a Polish space is itself a Polish space, and any countable direct sum (disjoint union) of Polish spaces is Polish.

Example A.6. If H is a separable Hilbert space, the closed unit ball in $B(H)$ endoved with its weak operator, strong operator or strong* operator topology is a Polish space. For example, the strong operator topology on the unit ball is induced by the complete metric $d(S, T) = \sum_{n=1}^{\infty} 2^{-n} \|(S - T)e_n\|$, where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis for H. The details of verifying this are left to the reader.

Remark A.7. Whenever d is a metric for X, the standard bounded metric

$$\overline{d}(x, y) = \min\{d(x, y), 1\}$$

induces the same topology, and X is complete under d if and only if X is complete under $\overline{d}$. So in our context we can always assume that our metrics are bounded with bound 1.

Proposition A.8. *Any countable product of Polish spaces is a Polish space.*

Proof. Let $(X_n, d_n)_{n=1}^{\infty}$ be Polish spaces. We assume each d_n is a bounded metric, bounded by 1. Let d be the metric on the product space $X = \prod X_n$ defined by

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} d_n(x_n, y_n), \qquad \text{where } x = (x_n),\ y = (y_n).$$

The metric d induces the product topology on X, and (X, d) is complete. To see that X is separable, choose a point p_n in each X_n, and let $\{p_1^{(n)}, p_2^{(n)}, \ldots\}$ be a countable dense subset of X_n. Then

$$D = \{(p_{k_1}^{(1)}, p_{k_2}^{(2)}, \ldots, p_{k_n}^{(n)}, p_{n+1}, p_{n+2}, \ldots) \mid n, k_1, \ldots, k_n \in \mathbb{N}\}$$

is countable and dense in X. □

Lemma A.9. *Let (X, d) be a metric space, and let A be a subspace. The map $f : X \to \mathbb{R}$ defined by $f(x) = d(x, A)$ is continuous, where $d(x, A) = \inf\{d(x, a) \mid a \in A\}$.*

Proof. Notice first that if $a \in A$, $x, y \in X$ then $d(x, y) + d(y, a) \geq d(x, a) \geq d(x, A)$, whence

$$d(x, y) + d(y, A) \geq d(x, A).$$

Suppose

$$x \in f^{-1}(]a, b[) = \{z \in A \mid a < d(z, A) < b\},$$

let $r = \min\{d(x, A) - a, b - d(x, A)\} > 0$. Then for $y \in B(x, r)$ we have

$$\begin{aligned} a = a + r - r \leq d(x, A) - r \leq d(x, y) + d(y, A) - r < d(y, A) \\ \leq d(y, x) + d(x, A) < r + d(x, A) \leq b. \end{aligned}$$

This shows that $y \in f^{-1}(]a, b[)$, and hence $f^{-1}(]a, b[)$ is open. This proves continuity. □

Proposition A.10. *Any open subset of a Polish space is a Polish space.*

Proof. If G is an open subset of a Polish space X, then $d(x, X \setminus G) > 0$ for every $x \in G$. Define the map $\varphi : G \to \mathbb{R}$ by $\varphi(x) = \frac{1}{d(x, X \setminus G)}$. By the previous lemma, φ is continuous. Let $f : G \to X \times \mathbb{R}$ be the graph of φ, that is $f(x) = (x, \varphi(x))$. Then $f(G)$ is a closed subset of the Polish space $X \times \mathbb{R}$. It is also clear, that f is a homeomorphism from G onto $f(G)$, and hence G is Polish. □

Corollary A.11. *A locally compact, second countable, Hausdorff topological space is a Polish space.*

Proof. Suppose X is locally compact, second countable and Hausdorff. The one-point compactification of X is then compact Hausdorff and second countable. Hence it is metrizable by Urysohn's metrization theorem, and due to compactness any metrization is complete. So the one-point compactification of X is a Polish space, and since X is an open subset of its one-point compactification, it is itself a Polish space. □

A subset of a topological space is called G_δ if it is the countable intersection of open sets.

Proposition A.12. *A G_δ set in a Polish space is itself a Polish space.*

Proof. Let X be a Polish space, and let A be the intersection of a sequence of open sets $G_n \subseteq X$. Let $f : A \to \prod G_n$ be the diagonal imbedding $f(a) = (a, a, \ldots,)$ of A into $\prod G_n$.

We show that $f(A)$ is closed in $\prod G_n$. Suppose (x_n) is a sequence in $f(A)$ converging to some $x \in \prod G_n$. Since $x_n = (a_n, a_n, \ldots)$ for some $a_n \in A$, and $x_n \to x = (b_1, b_2, \ldots)$ in the product topology, we get that $a_n \to b_1$, $a_n \to b_2$, etc. Hence x is of the form $(b, b, \ldots)$. It remains to show that $b \in A$. But this is trivial, since $b \in G_n$ for every n, and hence in $A = \bigcap G_n$.

We now know that $f(A)$ is Polish. The proposition now follows, since it is obvious that f is a homeomorphism of A onto $f(A)$. □

Corollary A.13. *The irrationals in $\mathbb{R}$ is a Polish space.*

Proof. The set of irrational numbers is the countable intersection of the open sets $G_q = \mathbb{R} \setminus \{q\}$, where $q \in \mathbb{Q}$. □

A.3.1 The Baire space

Consider the topological space $\mathbb{N}^{\mathbb{N}} = \prod_{n=1}^{\infty} \mathbb{N}$ consisting of all sequences of natural numbers endowed with the product topology, and where each copy of $\mathbb{N}$ has the standard discrete topology induced by the discrete metric $\delta(m, n) = \delta_{mn}$. The space $\mathbb{N}^{\mathbb{N}}$ is sometimes called the Baire space, but this should not be confused with the topological property of being a Baire space. We equip $\mathbb{N}^{\mathbb{N}}$ with the metric

$$d((x_n), (y_n)) = \sum_{n=1}^{\infty} 2^{-n} \delta(x_n, y_n).$$

The space $\mathbb{N}^{\mathbb{N}}$ is the universal Polish space in the following sense.

Theorem A.14. *Every Polish space is a continuous image of $\mathbb{N}^{\mathbb{N}}$.*

Proof. Let X be a Polish space. For any $\varepsilon > 0$ we can find a countable covering of X by non-empty, closed subsets of diameter not exceeding ε. For example we may take the closed balls of radius $\varepsilon/2$ and centers in a countable, dense subset of X. Let $X_1, X_2, \ldots$ be such a covering with sets of diameter not exceeding $\frac{1}{2}$. Now, do the same for each X_n to get a covering $X_{n1}, X_{n2}, \ldots$ of closed, non-empty sets of diameter not exceeding $\frac{1}{4}$. Repeat the same procedure to get closed sets $X_{k_1 k_2 \cdots k_n}$ with diameter not exceeding 2^{-n} and such that $X_{k_1 k_2 \cdots k_n} \subseteq X_{k_1 k_2 \cdots k_{n-1}}$ for each finite sequence $(k_1, \ldots, k_n)$ of natural numbers.

For any $(k_1, k_2, \ldots)$ in $\mathbb{N}^{\mathbb{N}}$ we may consider the intersection $\bigcap_{n=1}^{\infty} X_{k_1 k_2 \cdots k_n}$. Since the sets in the intersection form a descending chain of closed, non-empty sets, whose diameters tend to zero, as n tends to infinity, and X is complete, the intersection consists of exactly one point. Define $f(k_1, k_2, \ldots)$ to be this point. This defines a map $f : \mathbb{N}^{\mathbb{N}} \to X$.

Let us show that f is continuous. If $(k_1, k_2, \ldots) \in \mathbb{N}^{\mathbb{N}}$ and $\varepsilon > 0$ is given, we may choose a natural number n such that $2^{-n} < \varepsilon$. Then if $d((k_1, k_2, \ldots), (j_1, j_2, \ldots)) < 2^{-n}$, we must have $k_1 = j_1, \ldots, k_n = j_n$. Hence both $f(k_1, k_2, \ldots)$ and $f(j_1, j_2, \ldots)$ are in $X_{k_1 \cdots k_n}$, which has diameter not exceeding 2^{-n}, and hence the distance between $f(k_1, k_2, \ldots)$ and $f(j_1, j_2, \ldots)$ is less than ε. This proves continuity.

It remains to prove surjectivity of f. Let $p \in X$ be given. Since $X_1, X_2, \ldots$ covers X, the point p is in some X_{k_1}. In the same way, since $X_{k_1 1}, X_{k_1 2}, \ldots$ covers X_{k_1}, we get that p is in some $X_{k_1 k_2}$. Continuing in this way we construct a sequence $(k_1, k_2, \ldots)$ such that p is in every $X_{k_1 \cdots k_n}$. Hence $f(k_1, k_2, \ldots) = p$.

This proves that f is a continuous map from $\mathbb{N}^{\mathbb{N}}$ onto X. □

Remark A.15. The space $\mathbb{N}^{\mathbb{N}}$ should be contrasted with the Cantor set consisting of all (infinite) binary sequences. The proof of the previous theorem resembles that of proving that every compact metrizable space is a continuous image of the Cantor space.

A.4 Analytic sets

Definition A.16 (Analytic set). A subset of a Polish space is called *analytic*, if it is the continuous image of a Polish space.

Any Polish space is of course an analytic subset of itself, and the image of an analytic set under a continuous map is again an analytic set. By Theorem A.14 any analytic set is the continuous image of the space $\mathbb{N}^{\mathbb{N}}$.

Theorem A.17. *Let X be a Polish space. The family of analytic subsets of X whose complements are also analytic subsets form a σ-algebra on X containing the Borel sets. In particular every Borel set is analytic.*

Proof. Let (X_n) be a sequence of Polish spaces with continuous maps $f_n : X_n \to X$. The direct sum of these Polish spaces is a Polish space Y and defining $f : Y \to X$ to f_n on X_n gives a continuous map with image $\bigcup_n f(X_n)$. Hence the countable union of analytic sets is again an analytic set.

Forming the product of the spaces X_n gives a Polish space Z according to Proposition A.8. Let $\pi_n : Z \to X_n$ be the projection map. For any pair m, n the set Z_{mn} where the maps $f\pi_m$ and $f\pi_n$ agree is closed in Z (since X is Hausdorff). Hence $Z_0 = \bigcap_{m,n} Z_{mn}$ is closed, and all the maps $f\pi_n$ agree on Z_0, so there is a well-defined continuous map $g : Z_0 \to X$ with image $\bigcap_n f(X_n)$. Since Z_0 is closed in Z, it is Polish. Thus the countable intersection of analytic sets is again an analytic set.

It follows that the family Σ of analytic subsets of X whose complements are also analytic form a σ-algebra on X. Since any open subset of a Polish space is Polish by Proposition A.10, it is analytic. The complement is closed, and hence also (Polish and) analytic. The open sets generate the Borel algebra, and hence the Borel algebra is contained in Σ. It follows that any Borel set must be analytic. This completes the proof. □

Remark A.18. For this lemma to be true we should all agree that the empty set $\emptyset$ is a Polish space and is the image of itself under the identity map.

A partial converse of the previous theorem is true, if a few extra assumptions are added.

Theorem A.19. *Let X be a Polish space, and suppose that μ is the completion of a Borel measure on X. Suppose in addition that there is a sequence (K_n) of compact sets in X with finite μ-measure and union X. Then each analytic set is measurable.*

Note the difference between the words *measurable* and *Borel* in the two theorems.

Before the proof of the theorem, it will be convenient to prove a few lemmas. The following notation will be usefull. If (X, Ω, μ) is a measure space, define μ^* on *all* subsets A of X by

$$\mu^*(A) = \inf\{\mu(S) \mid A \subseteq S,\ S \in \Omega\}.$$

It is obvious that μ^* is monotone in the sense that if $A \subseteq B \subseteq X$ then $\mu^*(A) \leq \mu^*(B)$.

Lemma A.20. *For any $A \subseteq X$ there is $S \in \Omega$ containing A such that $\mu^*(A) = \mu(S)$.*

Proof. Let S_n be a measurable set containing A such that $\mu(S_n) \leq \mu^*(A) + \frac{1}{n}$, and let $S = \bigcap_n S_n$. Then $S \in \Omega$ and $\mu(S) \leq \mu^*(A)$. Since $A \subseteq S$, it is clear that $\mu^*(A) \leq \mu(S)$. □

Lemma A.21. *If $A_1 \subseteq A_2 \subseteq \cdots$ is an ascending chain with union A, then $\mu^*(A) = \lim \mu^*(A_n)$.*

Proof. Obviously $\lim \mu^*(A_n) \leq \mu^*(A)$, since each $A_n \subseteq A$. Choose measurable sets S_n such that $A_n \subseteq S_n$ and $\mu^*(A_n) = \mu(S_n)$ as garuanteed by the previous lemma. Let $T_n = \bigcap_{k \geq n} S_k$. Then $A_n \subseteq T_n \subseteq S_n$, and hence $\mu(T_n) = \mu(S_n) = \mu^*(A_n)$. Notice that (T_n) is an ascending chain of measurable sets with union T containing A. Thus $\mu^*(A) \leq \mu(T)$. And also $\mu(T) = \lim \mu(T_n) = \lim \mu^*(A_n)$. This proves the lemma. □

Lemma A.22. *Suppose μ is complete and finite. If $A \subseteq X$, and for each $m \in \mathbb{N}$ there is a measurable subset C_m of A such that $\mu^*(A) - \frac{1}{m} \leq \mu(C_m)$, then A is measurable.*

Proof. First choose $S \in \Omega$ such that $A \subseteq S$ and $\mu^*(A) = \mu(S)$. Let $C = \bigcup_m C_m$. Then C is a measurable subset of A and $\mu(C) \geq \mu^*(A)$ by the assumptions on C_m. Also $\mu(C) \leq \mu(S) = \mu^*(A)$ and $C \subseteq A \subseteq S$, and hence $\mu(S \setminus C) = 0$, since μ is finite. So A is the union of the measurable set C and the null set $A \setminus C$. □

We are almost ready to prove Theorem A.19, but first we will observe that the proof can be reduced to a simpler case. Let A be an analytic subset of X. Since K_n is analytic (being compact) the set $A \cap K_n$ is analytic. Since $A = \bigcup_n (A \cap K_n)$, it will suffice to prove the theorem in the case where X is compact and of finite measure. These assumptions are in force during the following two proofs.

Lemma A.23. *Let A be an analytic subset of X. There is a compact Polish space Y, a descending sequence (Y_n) of σ-compact subsets of Y with intersection B and a continuous map $f : Y \to X$ such that $f(B) = A$.*

Proof. Let $\overline{\mathbb{R}}_n$ denote a copy of the one-point compactification of $\mathbb{R}$ for each $n \in \mathbb{N}$. Let $Z = \prod_{n=1}^{\infty} \overline{\mathbb{R}}_n$ be the countable product of the one-point compactifications. We may view $\mathbb{N}^{\mathbb{N}}$ as a subset of Z. Let $Y = Z \times X$. Then Y is a compact Polish space (X is assumed to be compact).

As mentioned after Definition A.16 there is a continuous map $g : \mathbb{N}^{\mathbb{N}} \to X$ with image A. Let B denote the graph of g in $Z \times X$, and let $f : Z \times X \to X$ be the projection. Then f is continuous and $f(B) = A$.

For $n \in \mathbb{N}$ consider the subset F_{nm} of $\overline{\mathbb{R}}_n$ consisting of the union of all closed intervals of length $1/m$ and centers in a natural number in $\overline{\mathbb{R}}_n$. Let F_n be the σ-compact subset of Y

$$F_n = (F_{1n} \times F_{2n} \times \cdots \times F_n \times \overline{\mathbb{R}}_{n+1} \times \overline{\mathbb{R}}_{n+2} \times \cdots) \times X.$$

The sequence (F_n) is descending with intersection $\mathbb{N}^{\mathbb{N}} \times X$.

The map g is continuous $\mathbb{N}^{\mathbb{N}} \to X$, and hence the graph B is closed in $\mathbb{N}^{\mathbb{N}} \times X$. So if $\overline{B}$ denotes the closure of B in $Z \times X$, then $B = \overline{B} \cap (\mathbb{N}^{\mathbb{N}} \times X) = \overline{B} \cap F_1 \cap F_2 \cap \cdots$. The set $\overline{B}$ is compact, since it is closed, and if Y_n denotes $\overline{B} \cap F_n$ then Y_n is σ-compact. This concludes the proof. □

Proof of Theorem A.19. We continue with the notation from Lemma A.23. The aim is the use Lemma A.22 to conclude measurability of A, and the way to get there is by using the setup from Lemma A.23. We fix an $m \in \mathbb{N}$.

Since Y_n is σ-compact there is an ascending sequence of compact sets (K_{nj}) with union Y_n. Since $B \subseteq Y_1$, $(B \cap K_{1j})_j$ is an ascending chain with union B, and so $f(B)$ is the union of the ascending chain $f(B \cap K_{1j})$. From Lemma A.21 it follows that $\mu^*(f(B)) = \lim \mu^*(f(B \cap K_{1j}))$. Hence it is possible to find j_1 so

$$\mu^*(f(B)) - \frac{1}{m} \leq \mu^*(f(B \cap K_{1j_1})) \leq \mu(f(K_{1j_1}))$$

Since $B \cap K_{1j_1} \subseteq Y_2$, we have $B \cap K_{1j_1} = \bigcup_j (B \cap K_{1j_1} \cap K_{2j})$. Again, there is j_2 so

$$\mu^*(f(B)) - \frac{1}{m} \leq \mu^*(f(B \cap K_{1j_1} \cap K_{2j_2})) \leq \mu(f(K_{1j_1} \cap K_{2j_2}))$$

Continue in this way to find j_n such that

$$\mu^*(f(B)) - \frac{1}{m} \leq \mu(f(K_{1j_1} \cap \cdots \cap K_{nj_n}))$$

The sequence $(K_{1j_1} \cap \cdots \cap K_{nj_n})_n$ is descending an consists of compact sets. Hence the intersection is a compact set K and $(f(K_{1j_1} \cap \cdots \cap K_{nj_n}))_n$ is a descending sequence of compact sets with compact intersection L. Clearly $f(K) \subseteq L$. We claim that $L \subseteq f(K)$. Suppose $p \in L$. There is $q_n \in K_{1j_1} \cap \cdots \cap K_{nj_n}$ such that $f(q_n) = p$. The closure of the sets $\{q_j \mid j = n, n+1, \ldots\}$ form a descending sequence of non-empty compact sets, and hence their intersection is non-empty. If q is any element of the intersection, then $q \in K$. It follows by continuity of f, and since $f(q_n) = p$ for every n, that $f(q) = p$. Hence $p \in f(K)$.

We conclude that

$$\mu^*(f(B)) - \frac{1}{m} \leq \mu(f(K)).$$

Since $K \subseteq K_{nj_n} \subseteq Y_n$ for every n, and $B = \bigcap_n Y_n$, it follows that $K \subseteq B$. Therefore $f(K) \subseteq f(B)$, and also $f(K)$ is measurable, since it is compact. It is now a consequence of Lemma A.22 that $f(B) = A$ is measurable.

□

A.5 Measurable selections

The purpose of this section is to prove the measurable selection theorem, which is used in the course of proving Theorem 2.39. Before proving the measurable selection theorem (Theorem A.27) it will be relevant to introduce an order on the space $\mathbb{N}^{\mathbb{N}}$ and get a few lemmas in place.

Definition A.24. The *lexicographic order* (sometimes called the *dictionary order*) on $\mathbb{N}^{\mathbb{N}}$ is defined by

$$(n_1, n_2, \dots) < (m_1, m_2, \dots)$$

if and only there is a $k \in \mathbb{N}$ such that

$$n_j = m_j \text{ when } j < k \quad \text{and} \quad n_k < m_k.$$

It is clear, that $<$ is an order on $\mathbb{N}^{\mathbb{N}}$ and that $\mathbb{N}^{\mathbb{N}}$ becomes totally ordered with this order.

Lemma A.25. *Every non-empty, closed subset of* $\mathbb{N}^{\mathbb{N}}$ *has a smallest element in the lexicographical order.*

Proof. Let C be a non-empty, closed subset of $\mathbb{N}^{\mathbb{N}}$. First pick an element c_1 of C with smallest first coordinate x_1. Then among the elements in C with first coordinate x_1, we pick an element c_2 with smallest second coordinate x_2. And we continue in this to pick an element c_3 among the elements with first coordinate x_1 and second coordinate x_2 which has smallest third coordinate x_3, and so on. Obviously the element $x = (x_1, x_2, \dots)$ is a lower bound for C. Also, the sequence (c_n) in $\mathbb{N}^{\mathbb{N}}$ converges to x. Since C is closed, x is the smallest element in C. □

A cross-section of a surjective map $f : A \to B$ is a right inverse, i.e. a map $g : B \to A$ such that $f \circ g = \mathrm{id}_B$. In other words, to each $b \in B$ the map g *selects* an element $a \in A$ such that $f(a) = b$. The measurable selection theorem is related to the possibility of making the selection in a measurable manner.

Let X be a Polish space, and suppose that μ is the completion of a Borel measure on X. Suppose in addition that there is a sequence (K_n) of compact sets in X with finite μ-measure and union X.

Lemma A.26. *A continuous map* $f : \mathbb{N}^{\mathbb{N}} \to X$ *with image* A *has a measurable cross-section. In other words, there is a measurable map* $g : A \to \mathbb{N}^{\mathbb{N}}$ *such that* $f \circ g = \mathrm{id}_A$.

Proof. The idea is to use the fact that a closed set in $\mathbb{N}^{\mathbb{N}}$ has a smallest element to do the picking. Since f is continuous, $f^{-1}(p)$ is closed in $\mathbb{N}^{\mathbb{N}}$. Let $g(p)$ be the smallest element of $f^{-1}(p)$. Clearly, g is a cross-section of f. To show measurability of g it will suffice to prove that $g^{-1}(B)$ is measurable for all B in a basis for the topology on $\mathbb{N}^{\mathbb{N}}$. The sets of the form

$$B = \{n_1\} \times \{n_2\} \times \cdots \times \{n_k\} \times \mathbb{N} \times \mathbb{N} \times \cdots$$

form a basis. If

$$s = (n_1, n_2, \dots, n_k, 1, 1, \dots) \quad \text{and} \quad t = (n_1, n_2, \dots, n_k + 1, 1, 1, \dots)$$

then $B = \{x \in \mathbb{N}^{\mathbb{N}} \mid s \leq x < t\}$, and we notice that $p \in g^{-1}(B)$ if and only if $f^{-1}(p)$ has its smallest element in B. This happens if and only if there is an $x < t$ such that $f(x) = p$ but there are no $y < s$ such that $f(y) = p$. If B_s denotes the set $\{x \in \mathbb{N}^{\mathbb{N}} \mid x < s\}$, then we observe that $g^{-1}(B) = f(B_t) \setminus f(B_s)$.

If we write $s = (s_1, s_2, \dots)$ then for any $x = (x_1, x_2, \dots) \in B_s$ it holds for some k that $x_j = s_j$ for all $j < k$ and $x_k < s_k$. Then the set $U_x = \{x_1\} \times \cdots \times \{x_k\} \times \mathbb{N} \times \mathbb{N} \times \cdots$ is open in $\mathbb{N}^{\mathbb{N}}$, contains x and is contained in B_s. This proves that B_s is open. Hence B_s is analytic, and so is $f(B_s)$. Similarly, $f(B_t)$ is analytic, and it follows from Theorem A.19 that $f(B_t) \setminus f(B_s)$ is measurable. This proves that g is measurable. ◻

Theorem A.27 (Measurable selection theorem). *Let X and Y be a Polish spaces, and let μ be the completion of a Borel measure on X. Let $\pi_X : X \times Y \to X$ denote the projection. Suppose there is a sequence (K_n) of compact subsets of X with finite μ-measure and union X. If B is an analytic subset of $X \times Y$ and $A = \pi_X(B)$, there is a measurable map $\eta : A \to Y$ such that $(p, \eta(p)) \in B$ for every $p \in A$.*

Proof. Let $Y_p = \{q \in Y \mid (p, q) \in B\}$. Then since A is the projection of B, it follows that A is the set of points p such that Y_p is non-empty. The aim is to find a *measurable* map $\eta : A \to Y$ such that $\eta(p) \in Y_p$ for every p.

Since B is analytic, there is a continuous map $h : \mathbb{N}^{\mathbb{N}} \to X \times Y$ with image B. Composing with π_X we get a continuous map $\pi_X \circ h$ of $\mathbb{N}^{\mathbb{N}}$ onto A. From Lemma A.26 there is a measurable map $g : A \to \mathbb{N}^{\mathbb{N}}$ such that $\pi_X \circ h \circ g = \mathrm{id}_A$. Let $\eta = \pi_Y \circ h \circ g$, where $\pi_Y : X \times Y \to Y$ is the projection onto Y. The map η is measurable $A \to Y$.

It remains to check that $\eta(p) \in Y_p$ for each $p \in A$. Let $h(g(p))$ be denoted $(p', q) \in B$. Then $\eta(p) = q$, and since $p' = \pi_X(h(g(p))) = \mathrm{id}_A(p) = p$, it follows that $q \in Y_p$. This completes the proof. ◻

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