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Free probability theory and the interpolated free group factors

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NOTE: This is not the original version. A few minor mistakes have been corrected in this version. The original version is available upon request.

Abstract

We introduce the very basics of non-commutative probability theory and freeness. Thereafter the focus is on creation operators. They are used to prove results about matrices with mixed families of semicircular and circular elements instead of the usual approach using Gaussian random matrices and asymptotic freeness. We use the matrix results to prove isomorphisms of various free products such as $R * R \simeq L(\mathbb{Z}) * R \simeq L(\mathbb{Z}) * L(\mathbb{Z})$.

Finally the interpolated free group factors are introduced to give a partial answer to the isomorphism question concerning the free group factors. We prove that either $L(\mathbb{F}_n)$ are all isomorphic for different $n \geq 2$ or they are all pairwise non-isomorphic. As a corollary of the existence of the interpolated free group factors we prove that the fundamental group of $L(\mathbb{F}_\infty)$ is the group of positive reals.

Resumé på dansk (abstract in Danish)

Først gennemgås grundlæggende begreber inden for fri sandsynlighedsteori. Dernæst lægges vægten på skabelsesoperatorerne. Disse udnyttes til at vise resultater om matricer med blandede familier af semicirkulære og cirkulære elementer i stedet for den gængse fremgangsmåde, hvor gaussiske stokastiske matricer benyttes i samspil med asymptotisk frihed. Matrixresultaterne udnyttes til at vise diverse isomorfier, hvor frie produkter indgår, såsom $R * R \simeq L(\mathbb{Z}) * R \simeq L(\mathbb{Z}) * L(\mathbb{Z})$.

Slutteligt introduceres de interpolerede faktorer hørende til de frie grupper, og disse bruges til at give et delvist svar på isomorfiproblemet angående implikationen $L(\mathbb{F}_n) \simeq L(\mathbb{F}_m) \implies m = n$. Det vises, at alle faktorerne $L(\mathbb{F}_n)$ enten er isomorfe eller parvist ikke isomorfe, hvor $n \geq 2$. Som et korollar til eksistensen af de interpolerede faktorer vises, at fundamentalgruppen for $L(\mathbb{F}_\infty)$ er gruppen af positive, reelle tal.

Contents

Preliminaries

These preliminaries summarize terms, notational conventions and basic results of the theory of von Neumann algebras. Readers already familiar with von Neumann algebras will find nothing new here. At the end we shortly explain the structure of the thesis.

Notation and conventions

The letters $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{C}$ will denote the natural numbers $\{1, 2, 3, \ldots\}$, the non-negative integers, the integers, the reals, the positive reals (*without* zero) and the complex numbers, respectively.

The characteristic function of a set A will be denoted χ_A. It will always be clear from the context on which set χ_A is defined, usually $\mathbb{C}$. Along the same lines we use the Kronecker delta δ_{ab} which equals 1, if $a = b$, and 0 otherwise.

We use M_n to denote the set of complex $n \times n$ matrices and tr_n to denote the usual, *normalized* tracial state.

An algebra will always mean an associative algebra over $\mathbb{C}$, and we will only be concerned with unital algebras. We denote the unit by 1. A subalgebra is always assumed to share the same unit as the larger algebra, unless otherwise stated.

By a von Neumann algebra we always mean a weakly closed *-subalgebra of the bounded operators $B(H)$ acting on a Hilbert space H and containing the identity operator. Corner algebras $p\mathcal{A}p$, where $\mathcal{A}$ is a von Neumann algebra, and $p \in \mathcal{A}$ is a projection, are von Neumann algebras considered as acting on $p(H)$.

A trace on an algebra $\mathcal{A}$ is a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ such that $\varphi(ab) = \varphi(ba)$ for every $a, b \in \mathcal{A}$. In the context of C*-algebras and von Neumann algebras a trace will mean a tracial state, i.e. a trace which is also a positive linear functional of norm 1.

A state on a von Neumann algebra is called normal, if it is continuous in the ultraweak topology. Alternatively, normal states are weakly (and strongly) continuous on bounded subsets of the von Neumann algebra.

A system of $n \times n$ matrix units in an algebra $\mathcal{A}$ is a family $(e_{ij})_{i,j \in I}$ of elements in $\mathcal{A}$ satisfying the multiplication rule $e_{ij}e_{kl} = \delta_{jk}e_{il}$ and the condition that $\sum_{i \in I} e_{ii} = 1$. The index set I has cardinality n and should be finite, unless $\mathcal{A}$ is a von Neumann algebra, in which case we also allow I to be infinite, and convergence of the sum is in the strong operator topology. If

$\mathcal{A}$ is a *-algebra, and $e_{ij}^* = e_{ji}$ for each i and j, we say that the system of matrix units is self-adjoint. Whenever we work with a system of matrix units in a *-algebra we will always mean a self-adjoint system. Note that then $e_{ii}e_{ii} = e_{ii}$ and $e_{ii}^* = e_{ii}$, so that each e_{ii} is a projection, and e_{ij} is a partial isometry from e_{jj} to e_{ii}. The standard system of matrix units in M_n is the system, where e_{ij} has the (i,j)'th entry equal 1, and the rest equals zero.

We will often work with algebras given by generating sets. When S is a subset of an algebra $\mathcal{A}$, then we denote by $\mathrm{alg}(S)$ the algebra generated by S, i.e. the smallest subalgebra of $\mathcal{A}$ containing S and the unit 1. When T is another subset of $\mathcal{A}$ we take $\mathrm{alg}(S,T)$ to mean $\mathrm{alg}(S \cup T)$, and similarly with more sets. When $S = \{s_1, \dots, s_n\}$ is finite, we also write $\mathrm{alg}(s_1, \dots, s_n)$ for this algebra. The mix of sets and elements should cause no confusion.

In the context of C*-algebras and von Neumann algebras we use the notation $C^*(S)$ and $W^*(S)$ for the generated C*-algebra and generated von Neumann algebra. A note should be made here. If $\mathcal{A}$ is a von Neumann algebra with a projection p, and $S \subseteq p\mathcal{A}p$, we can view S as a subset of the von Neumann algebra $p\mathcal{A}p$, and the notation $W^*(S)$ could also mean a subalgebra of $p\mathcal{A}p$. When this situation occurs, it will be noted which of the two meanings is given to $W^*(S)$.

Basic facts

Given a discrete group G we can represent the group algebra $\mathbb{C}[G]$ faithfully on the Hilbert space $\ell^2(G)$ using the left regular representation, and the group von Neumann algebra of G is the weak (or strong) closure of $\mathbb{C}[G]$ inside $B(\ell^2(G))$. It is denoted $L(G)$. The group von Neumann algebra comes naturally equipped with a faithful trace τ given by $\tau(x) = \langle x\delta_e, \delta_e \rangle$. Here e denotes the neutral element in G, and δ_e is the unit vector $\chi_{\{e\}}$ in $\ell^2(G)$.

Of particular interest are the groups with the infinite conjugacy class property, i.e. every conjugacy class is infinite, except of course the conjugacy class of the neutral element. The group von Neumann algebras of these groups are II_1 factors. Examples of such groups are the free groups $\mathbb{F}_n$ with n generators, when $n \geq 2$ (we allow $n = \infty$ and interpret ∞ as countably infinite).

The following theorem is much used when we pass from an algebraic proof to a proof concerning von Neumann algebras.

Theorem (Kaplansky's Density Theorem)**.** *Let $\mathcal{A}$ be a *-subalgebra of $B(H)$, and let $\overline{\mathcal{A}}$ be the strong operator closure of $\mathcal{A}$. Then the unit ball of $\mathcal{A}$ is strongly dense in the unit ball of $\overline{\mathcal{A}}$.*

Among all von Neumann algebras we will mostly be concerned with II_1 factors. They may described by the fact that they admit a unique trace under which the image of the projections is the interval $[0,1]$. On a II_1 factor the trace is always faithful and normal. Further, it implements the equivalence relation on the projections in the sense that $p \sim q$ if and only if p and q have the same trace.

We will need the spectral theorem for normal operators at some point. Before we state it we recall the definition of a spectral measure. Let K be a compact Hausdorff space and H a Hilbert space. A spectral measure on K is a map from the σ-algebra of Borel sets of K to the set of projections in $B(H)$ such that

(a) $E(\emptyset) = 0$,

(b) $E(K) = 1$,

(c) $E(S_1 \cap S_2) = E(S_1)E(S_2)$ for all Borel sets S_1 and S_2,

(d) for each $x, y \in H$ the function

$$S \mapsto E_{x,y}(S) = \langle E(S)x, y\rangle$$

is a finite, regular, complex Borel measure on K.

Theorem (Spectral theorem)**.** *Let T be a normal operator in $B(H)$. There is a unique spectral measure E on the spectrum $\sigma(T)$ of T such that*

$$\langle Tx, y\rangle = \int_{\sigma(T)} \lambda \,\mathrm{d}E_{x,y}(\lambda)$$

for every $x, y \in H$.

The spectral theorem gives rise to the Borel functional calculus $B^\infty(\sigma(T)) \to W^*(T)$, which is a *-homomorphism such that the identity map on $\sigma(T)$ is sent to the operator T. Here $B^\infty(\sigma(T))$ denotes the bounded Borel functions on $\sigma(T)$.

The hyperfinite II_1 factor

One of the earliest examples of a II_1 factor was produced by Murray and von Neumann and is called the hyperfinite II_1 factor. In general, a C*-algebra is called *hyperfinite* or *approximately finite dimensional*, if it is the direct limit of a sequence of finite dimensional C*-algebras. In other words, there should be an increasing sequence of finite dimensional C*-algebras with dense union. Murray and von Neumann showed [9] that there is exactly one hyperfinite factor of type II_1 up to isomorphism, and this of course justifies the term, *the* hyperfinite II_1 factor, though there are many constructions of it. One construction is as follows.

Let G be a non-trivial discrete, countable group, which is locally finite in the sense that every finite subset of G generates a finite subgroup. If G also has the infinite conjugacy class property, then the group von Neumann algebra $L(G)$ is a II_1 factor, and because G is locally finite, $L(G)$ is hyperfinite. There are many examples of groups with these properties, e.g. the group of all finite permutations of the natural numbers $\mathbb{N}$ (finite here means that each permutation only moves a finite number of elements and fixes the rest).

Another construction of the hyperfinite II_1 factor uses infinite tensor products. The hyperfinite II_1 factor is just a special case of a more general construction, and although we do not need the general construction in the thesis, it is more illuminating than the special case.

Suppose $(\mathcal{A}_n, \varphi_n)_{n=1}^{\infty}$ is a sequence of von Neumann algebra with normal, faithful states. We want to define the infinite tensor product of this seqeunce. From the sequence we construct the sequence of finite (algebraic) tensor products

$$\mathcal{A}_1, \quad \mathcal{A}_1 \otimes \mathcal{A}_2, \quad \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_3, \ldots,$$

which may in a natural way be considered as an increasing sequence of *-algebras, when we identify x with $x \otimes 1$ in the subsequent algebra. Let $\mathcal{B}$ be the direct limit with respect to these imbeddings,

$$\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n.$$

Then

$$\mathcal{B} = \operatorname{span}\{x_1 \otimes \cdots \otimes x_n \otimes 1 \otimes \cdots \mid n \in \mathbb{N}, x_i \in \mathcal{A}_i\},$$

and $\mathcal{B}$ is a unital *-algebra. To define the infinite tensor product we use the GNS-construction with respect to a certain positive, linear functional.

There is a unique positive, unit-preserving linear functional $\varphi : \mathcal{B} \to \mathbb{C}$ such that

$$\varphi(x_1 \otimes \cdots \otimes x_n \otimes 1 \otimes \cdots) = \varphi_1(x_1) \cdots \varphi_n(x_n).$$

Let $(\pi_\varphi, H_\varphi, \xi_\varphi)$ be the GNS-representation of $(\mathcal{B}, \varphi)$. Then we define the infinite tensor product of $(\mathcal{A}_n, \varphi_n)_{n=1}^{\infty}$, denoted $\mathcal{A}$, to be the strong operator closure of $\pi_\varphi(\mathcal{B})$ inside $B(H_\varphi)$. The representation π_φ is faithful, so we may identify $\mathcal{B}$ with a dense subalgebra of $\mathcal{A}$, and φ extends to a normal, faithful state on $\mathcal{A}$ by

$$\varphi(a) = \langle a\xi_\varphi, \xi_\varphi \rangle, \quad a \in \mathcal{A}.$$

We also write

$$(\mathcal{A}, \varphi) = \bigotimes_{n=1}^{\infty} (\mathcal{A}_n, \varphi_n).$$

The infinite tensor product does in fact depend on the choice of the states φ_n.

If each $\mathcal{A}_n$ is a factor, then $\mathcal{A}$ becomes a factor too. If we let each $\mathcal{A}_n = M_2$ and let $\varphi_n = \text{tr}_2$

be the normalized trace on M_2, then the infinite tensor product

$$(R,\tau) = \bigotimes_{n=1}^{\infty} (M_2, \mathrm{tr}_2)$$

is the hyperfinite II_1 factor, where τ is the unique trace on R.

The hyperfinite II_1 factor has many interesting properties. For instance R has a generating, increasing sequence of matrix algebras containing projections of every rational trace. Also, any corner of R is isomorphic to R. In other words, if $p \in R$ is a non-zero projection, then $pRp \simeq R$. This is sometimes expressed by saying that the fundamental group of R is $\mathbb{R}_+$.

Structure of the thesis

The first chapter gives a brief overview of the history of the subject treated in the thesis. The second chapter is mostly based on [13, 14] and descibes the most basic concepts of free probability. In the third chapter the creation operators are treated. The exposition given here is based on [6]. The fourth chapter is where the efforts from the previous chapters start to pay off. The theorem about the polar decomposition of a circular element is proved using creation operators as well as ideas from [1]. After that we prove the isomorphisms $R * R \simeq R * L(\mathbb{Z}) \simeq L(\mathbb{Z}) * L(\mathbb{Z})$ which are needed, when we eventually end up at the interpolated free group factors in the fifth chapter. This is based on [4, 5].

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CHAPTER 1

Introduction

The early study of von Neumann algebras (or rings of operators as they were called) was mostly shaped by Murray and von Neumann in a series of papers in the 1930s and 1940s. They distinguished between certain types of von Neumann algebras, some of which were easy to classify (type I), and some of which seemed artificial (type III). And in between were the type II algebras which had certain pleasant properties, but were not as easy to classify as the type I algebras. Murray and von Neumann also draw the attention to those von Neumann algebras which are furthest from being abelian in the sense that their centers are trivial, i.e. consisting only of the scalar multiples of the identity operator. Those von Neumann algebras were (and are) called factors, and in a certain sense all other von Neumann algebras are composed of factors, see [17, 8].

Since abelian von Neumann algebras, matrix algebras and direct sums of such are all of type I, the most typical examples of von Neumann algebras are all of type I. Murray and von Neumann managed to give examples of type II algebras by using the group construction to construct II_1 factors $L(G)$ from discrete groups G with the infinite conjugacy class property. Examples of such groups are the free groups $\mathbb{F}_n$ with $n \geq 2$. Another example is the group S_∞ of finite permutations of the natural numbers, i.e. permutations of $\mathbb{N}$ that leave all but a finite number of elements fixed.

Murray and von Neumann showed that the II_1 factor $L(S_\infty)$ was not isomorphic to any of the free group factors $L(\mathbb{F}_n)$. After that they tried to determined whether or not the free group factors $L(\mathbb{F}_n)$ were isomorphic for different values of n. This turned out to be a much more difficult problem. In fact, it is still unsolved.

To distinguish between different II_1 factors they introduced an invariant called the fundamental group [9]. The fundamental group of a II_1 factor $\mathcal{A}$ consists of classes of automorphisms of the II_∞ factor $\mathcal{A} \otimes B(H)$, where H is a separable, infinite dimensional Hilbert space. The automorphisms are classified according to an associated scaling factor, which is a positive, real number. So fundamental groups of II_1 factors are always subgroups of $\mathbb{R}_+$ (with multiplication).

Without to much difficulty one may show that the fundamental group of $L(S_\infty)$ is $\mathbb{R}_+$, the largest possible (see Theorem B.11). For a long time, however, this was the only known fundamental group. In 1980 Connes showed that the fundamental group of the group von Neumann algebra of a countable, discrete group with Kazhdan's property T is countable [3]. An example of such a group is $\mathrm{SL}_3(\mathbb{Z})$. Later Popa [10] showed that the fundamental group can be trivial as is the case for the group $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$.

The fundamental group of $L(\mathbb{F}_n)$ is still unknown, though some things can be said about it. In 1992 Rădulescu [11] showed that the fundamental group of $L(\mathbb{F}_\infty)$ is $\mathbb{R}_+$. Later he and Dykema showed (independently, and in slightly different ways) that the fundamental group of $L(\mathbb{F}_n)$ is either trivial or $\mathbb{R}_+$, see [12, 5]. They did so by introducing a continuum of II_1 factors (indexed by $]1, \infty]$) obeying certain composition rules, and such that for integer values they agree with the free group factors. Finally, they related the result to the isomorphism question of $L(\mathbb{F}_n)$. Their result is the main result of this thesis (Theorem 5.11).

The framework in which both Rădulescu and Dykema proved their results is free probability theory, which was developed by Voiculescu in the late 1980s and early 1990s. An overview of the theory is given in [14]. In some sense free probability theory is a non-commutative version of the usual probability theory. And the group von Neumann algebras provide excellent examples of non-commutative probability spaces with all the interesting distributions. In free probability the divine normal distribution from the usual probability theory is replaced by the semicircle distribution, and independence is replaced by the concept of freeness. It turns out that $L(\mathbb{F}_n)$ is generated by n free, semicircular variables, and then the whole theory of free probability can be applied to give results about the free group factors.

CHAPTER 2

Basics of free probability

2.1 Non-commutative probability spaces

Free (or non-commutative) probability theory is a non-commutative analouge of the usual probability theory. In the classical probability theory one studies probability spaces and random variables (measurable functions). So to follow the usual approach in the non-commutative world of mathematics we focus on algebras of functions, try to ignore the underlying space, and view the algebra as the actual space under consideration. And then we generalize the algebras of functions to include non-commutaive algebras.

A natural algebra on a probability space $(\Omega, \mathcal{F}, P)$ is the algebra $\bigcap_{p\geq 1} L^p(\Omega, \mathcal{F}, P)$ consisting of (classes of) all complex random variables with moments of all orders. This algebra has a unit (the constant function 1) and is naturally equipped with an expectation E given by integration,

$$E(X) = \int_\Omega X \,\mathrm{d}P, \qquad X \in \bigcap_{p\geq 1} L^p(\Omega, \mathcal{F}, P).$$

The expectation is a (positive) linear functional on $\bigcap_{p\geq 1} L^p(\Omega, \mathcal{F}, P)$, and the fact that P is a probability measure is demonstrated by the fact that $E(1) = 1$. This motivates the following definition of a non-commutative probability space.

Definition 2.1. A *non-commutative probability space* $(\mathcal{A}, \varphi)$ is an (associative) algebra $\mathcal{A}$ over $\mathbb{C}$ with unit $1 \in \mathcal{A}$ equipped with a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$, such that $\varphi(1) = 1$. The elements in $\mathcal{A}$ are called (non-commutative) *random variables*, or simply variables.

The definition of a non-commutative probability space is very general, and most of the time we will want to work with C*-algebras or von Neumann algebras and use some of their properties such as notions of positivity and convergence. So we have the following extension of the previous definition.

Definition 2.2. A non-commutative probability space $(\mathcal{A}, \varphi)$ is called

a *-*probability space*, if $\mathcal{A}$ is a *-algebra, and $\varphi(a^*) = \overline{\varphi(a)}$ for any $a \in \mathcal{A}$,

a *C^*-probability space*, if $\mathcal{A}$ is a C*-algebra, and φ is a state,
a *W^*-probability space*, if $\mathcal{A}$ is a von Neumann algebra, and φ is a normal state.

We think of φ is the expectation, and for a random variable $a \in \mathcal{A}$ we call the numbers $\varphi(a^n)$, $n \in \mathbb{N}$, the moments of a. We will often need to single out the centered variables in a set $S \subseteq \mathcal{A}$, and we will use the notation S^0 for this. In other words we use the notation

$$S^0 = \{a \in S \mid \varphi(a) = 0\} = S \cap \ker \varphi.$$

We will also use the notation $x^0 = x - \varphi(x)1$. In this way we may split x as $x = x^0 + \varphi(x)1$, where $x^0 \in \mathcal{A}^0 = \ker \varphi$.

Lemma 2.3. *Let $\{x_i\}_{i \in I}$ be a subset of a non-commutative probability space $(\mathcal{A}, \varphi)$, and suppose some $x_{i_0} = 1$. Then*

$$\operatorname{span}\{x_i \mid i \in I\}^0 = \operatorname{span}\{x_i^0 \mid i \in I\}.$$

Proof. Since some $x_{i_0} = 1$,

$$\operatorname{span}\{x_i^0 \mid i \in I\} \subseteq \operatorname{span}\{x_i \mid i \in I\},$$

and hence

$$\operatorname{span}\{x_i^0 \mid i \in I\} \subseteq \operatorname{span}\{x_i \mid i \in I\}^0,$$

For the reverse inclusion, write $x_i = x_i^0 + \varphi(x_i)1$. If

$$y = \sum \beta_i x_i = \sum (\beta_i x_i^0 + \beta_i \varphi(x_i)1) = \sum \beta_i x_i^0 + \Big(\sum \beta_i \varphi(x_i)\Big) 1 \quad \text{(a finite sum)},$$

and

$$0 = \varphi(y) = \sum \beta_i \varphi(x_i),$$

then $y = \sum \beta_i x_i^0$. This proves the reverse inclusion. □

In classical probability theory the moments of a random variable play an important role, and they reveal important properties of the distribution of the random variable. In some cases they even completely determine the distribution, for instance if the distribution is supported on a compact interval.

The approach in non-commutative probability theory is to think of the moments of a random variable X as its distribution, and using linearity we can think of the expectation of polynomials in X as the distribution of X. We have the following definition.

Definition 2.4. Let a be a random variable in a non-commutative probability space $(\mathcal{A}, \varphi)$. The *distribution* of a is the linear functional φ_a defined on the algebra $\mathbb{C}[X]$ of complex polynomials by

$$\varphi_a(P) = \varphi(P(a)), \qquad P \in \mathbb{C}[X].$$

Note that in the case where $(\mathcal{A}, \varphi)$ is a C*-probability space, and a is self-adjoint, the distribution φ_a extends to a linear functional on $C(\sigma(a))$, and by the Riesz Representation Theorem there is a unique, regular Borel measure μ_a on $\sigma(a)$ such that μ_a represents φ_a in the sense that

$$\varphi_a(f) = \int_{\sigma(a)} f \, \mathrm{d}\mu_a, \qquad f \in C(\sigma(a)).$$

We will often identify the distribution of a with the measure μ_a.

It will be of relevance to consider several variables at once, and we need the notion of a joint distribution as opposed to marginal distributions. In the following definition we will need the notion of non-commutative polynomials in several variables. We use the notation $\mathbb{C}\langle X_i \mid i \in I \rangle$ to denote the set of non-commutative polynomials over $\mathbb{C}$ with indeterminates $(X_i)_{i \in I}$. When $|I| = 1$, this is simply $\mathbb{C}[X]$. In some sense $\mathbb{C}\langle X_i \mid i \in I \rangle$ is the largest unital algebra with $|I|$ generators. More precisely, it is the free, associative, unital algebra over $\mathbb{C}$ with $|I|$ generators.

When $(a_i)_{i \in I}$ is a family of elements in a unital algebra $\mathcal{A}$, there is a unique unital homomorphism of algebras $\Lambda : \mathbb{C}\langle X_i \mid i \in I \rangle \to \mathcal{A}$ such that $\Lambda(X_i) = a_i$. When $P \in \mathbb{C}\langle X_i \mid i \in I \rangle$ we take $P((a_i)_{i \in I})$ to mean $\Lambda(P)$.

Definition 2.5. Let $(a_i)_{i \in I}$ be a family of random variables in a non-commutative probability space $(\mathcal{A}, \varphi)$. The *(joint) distribution* of $(a_i)_{i \in I}$ is the linear functional ψ on $\mathbb{C}\langle X_i \mid i \in I \rangle$ given by

$$\psi(P) = \varphi(P((a_i)_{i \in I})), \qquad P \in \mathbb{C}\langle X_i \mid i \in I \rangle.$$

If $(\mathcal{A}, \varphi)$ is a *-probability space, the *-distribution of a family $(a_i)_{i \in I}$ is simply the distribution of the family $(a_i)_{i \in I} \cup (a_i^*)_{i \in I}$.

2.2 Independence

The important concept of independence in probability theory has a natural extension to non-commutative probability spaces, which we will now decribe. However, it turns out that another concept, freeness, will be more important to us.

Definition 2.6. A family $(\mathcal{A}_i)_{i \in I}$ of subalgebras in a non-commutative probability space $(\mathcal{A}, \varphi)$ is called *independent*, if the algebras commute with each other and

$$\varphi(a_1 \cdots a_n) = \varphi(a_1) \cdots \varphi(a_n),$$

whenever $a_k \in \mathcal{A}_{i_k}$ and $k \neq l$ implies $i_k \neq i_l$.

Variables in $\mathcal{A}$ are called independent if the algebras they generate are independent.

The concept of *-independence in a *-probability space is defined similarly by replacing the word algebra with *-algebra. But we will only need independence of self-adjoint elements, where the two concepts agree.

Note that we do not require the algebras $\mathcal{A}_i$ to be commutative, simply that $[\mathcal{A}_i, \mathcal{A}_j] = 0$ for all $i \neq j$. It is clear from the definition that if an independent family $(\mathcal{A}_i)_{i\in I}$ generates $\mathcal{A}$ as an algebra, then φ is uniquely determined by its restrictions $\varphi|_{\mathcal{A}_i}$.

Independence arises naturally in connection with tensor products. It is obvious that the tensor product $(\mathcal{A} \otimes \mathcal{B}, \varphi \otimes \psi)$ of two non-commutative probability spaces contains an independent pair of subalgebras, $\mathcal{A} \otimes 1$ and $1 \otimes \mathcal{B}$.

2.3 Freeness

As it often happens when we move from a commutative setting to a non-commutative setting, the notion of a free object or a free product (i.e. coproduct) does not generalize – it changes. For instance, in the category of abelian groups the free objects are (possibly infinite) direct sums of the integers, and the coproduct is the direct sum. In the category of *all* groups these groups are no longer free (except of course for $\mathbb{Z}$ (and the trivial group)). The free objects in this larger category are the *free groups* – appropriate name, right? And the free groups are far from being abelian. Also, the coproduct changes from the direct sum to the free product of groups.

The requirement in the definition of independence that the algebras commute is rather strict, when we want to work with non-commutative probability spaces. But if we remove the condition from the definition, we can no longer determine the joint distribution of a family of variables from the marginal distributions alone. To remedy this flaw, we will come up with another concept to replace independence, but this will be non-commutative in nature and not a generalization of independece. The new concept is called *freeness*.

Definition 2.7. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space.

A *family of subalgebras* $(\mathcal{A}_i)_{i\in I}$ is called *free* if

$$\varphi(a_1 \cdots a_n) = 0$$

whenever $a_k \in \mathcal{A}_{i_k}$ with $i_1 \neq i_2 \neq \cdots \neq i_n$ and $\varphi(a_k) = 0$ for all k.

A *family of subsets* $(\Omega_i)_{i\in I}$ of $\mathcal{A}$ is called free, if the unital subalgebras $\mathcal{A}_i$ generated by Ω_i form a free family.

A *family of random variables* $(f_i)_{i\in I}$ is called free, if the family of subsets $(\{f_i\})_{i\in I}$ is free.

If $(\mathcal{A}, \varphi)$ is a *-probability space, a family of subsets is called **-free*, if $(\Omega_i \cup \Omega_i^*)_{i\in I}$ is free, and a family of random variables is called *-free, if the family of subsets $(\{f_i\})_{i\in I}$ is *-free.

Note that in the above definition we allow for instance $i_1 = i_3 = \cdots$ and $i_2 = i_4 = \cdots$. As we will see in Proposition 2.11 the joint distribution of a free family of random variables is determined by marginal distributions of the elements in the family.

We could also have made similar definitions of freeness for C*- and W*-probability spaces, but the following proposition shows that it is not necessary.

Proposition 2.8. *Let $(\mathcal{A}, \varphi)$ be a C*-probability space (resp. W*-probability space), and let $(\mathcal{A}_i)_{i\in I}$ be a free family of self-adjoint (unital) subalgebras of $\mathcal{A}$. Let $\overline{\mathcal{A}_i}$ be the C*-algebra (resp. von Neumann algebra) generated by $\mathcal{A}_i$. Then the family $(\overline{\mathcal{A}_i})_{i\in I}$ is free.*

Note. This is the first of many statements where we will move from an algebraic level to the level of C*-algebras and von Neumann algebras. The approach is always the same: approximation. In the C*-case we use the norm continuity of φ, and in the W*-case we apply Kaplansky's Density Theorem and strong operator continuity of φ on bounded sets.

Proof. Consider random variables $a_k \in \overline{\mathcal{A}_{i_k}}$ with $i_1 \neq \cdots \neq i_n$ and $\varphi(a_k) = 0$, $1 \leq k \leq n$.

In the C*-case we choose for each k a sequence $(a_k^m)_{m\in\mathbb{N}}$ in $\mathcal{A}_{i_k}$ such that $a_k^m \to a_k$ in norm. Then by continuity of φ we see that $(a_k^m)^0 = a_k^m - \varphi(a_k^m)1 \to a_k$, since $\varphi(a_k) = 0$. Hence $(a_1^m)^0 \cdots (a_n^m)^0 \to a_1 \cdots a_n$, and thus

$$\varphi(a_1 \cdots a_n) = \lim_{m\to\infty} \varphi((a_1^m)^0 \cdots (a_n^m)^0) = 0.$$

It suffices to show the W*-case under the additional assumption that each $||a_k|| \leq 1$. By Kaplansky's Density Theorem we may find a net $(a_k^\lambda)_{\lambda\in\Lambda_k}$ in the unit ball of $\mathcal{A}_{i_k}$ such that $a_k^\lambda \to a_k$ in the strong operator topology. By passing to a larger index set we may assume that each net has the same index set $\Lambda = \Lambda_1 = \cdots = \Lambda_n$. Since φ is strongly continuous on bounded sets we see as before that $(a_k^\lambda)^0 \to a_k$ strongly for each k. Since $||(a_k^\lambda)^0|| \leq ||a_k^\lambda|| + |\varphi(a_k^\lambda)| \leq 2$, and multiplication is strongly continuous on bounded sets, we infer that $(a_1^\lambda)^0 \cdots (a_n^\lambda)^0$ converges strongly $a_1 \cdots a_n$, and hence

$$\varphi(a_1 \cdots a_n) = \lim_{\lambda\to\infty} \varphi((a_1^\lambda)^0 \cdots (a_n^\lambda)^0) = 0.$$

□

The following definition will be useful later on, and it has a natural connection with freeness.

Definition 2.9. Let $\mathcal{A}$ be an algebra with unit 1, and let $(S_i)_{i\in I}$ be subsets of $\mathcal{A}$. A non-trivial *traveling product* in $(S_i)_{i\in I}$ is a product $a_1 a_2 \cdots a_n$ such that $a_j \in S_{i_j}$ $(1 \leq j \leq n)$ and $i_1 \neq \cdots \neq i_n$. The trivial traveling product is simply the identity element 1. We use the notation $\Lambda((S_i)_{i\in I})$ to denote the set of all traveling products in $(S_i)_{i\in I}$, including the trivial one. In the case where $|I| = 2$ we shall also refer to traveling products as *alternating products.*

We may rephrase the definition of freeness using traveling products by saying that a family $(\mathcal{A}_i)_{i\in I}$ is free if $\varphi(x) = 0$, whenever x is a non-trivial traveling product in $(\mathcal{A}_i^0)_{i\in I}$.

Our next goal is to show that the marginal distributions of a free family determine the joint distribution. But first we need to introduce some notation that will ease the proof of the following lemma.

If K is a non-empty subset of $\{1, \dots, n\}$ of size m, it has a natural ordering $k_1 < k_2 < \cdots < k_m$. Given random variables $a_1, \dots, a_n$ in $(\mathcal{A}, \varphi)$ we use the notation

$$\prod_{k \in K}^{\text{ord}} a_k = a_{k_1} \cdots a_{k_m}$$

for the *ordered product*. We also allow $K = \emptyset$, and by convention the empty ordered product is $1 \in \mathcal{A}$.

Lemma 2.10. *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space with a family $(\mathcal{A}_i)_{i \in I}$ of subalgebras that generate $\mathcal{A}$. Then*

$$\mathcal{A} = \mathbb{C}1 + \sum_{n \in \mathbb{N}} \sum_{\substack{i_1, \dots, i_n \in I \\ i_1 \neq \cdots \neq i_n}} \mathcal{A}_{i_1}^0 \cdots \mathcal{A}_{i_n}^0. \tag{2.1}$$

Proof. As $\mathcal{A}$ is generated by $(\mathcal{A}_i)_{i \in I}$, any element in $\mathcal{A}$ is a linear combination of products of the form $a_1 \cdots a_n$, where $a_k \in \mathcal{A}_{i_k}$ for suitable $i_1, \dots, i_n \in I$ such that $i_1 \neq \cdots \neq i_n$. So by linearity it suffices to prove the any such product is in the right-hand side of (2.1). This is done by induction over n. The basis step is trivial, so we assume $n \geq 2$. Writing $a_k = a_k^0 + \varphi(a_k)1$ and expanding the product we get

$$(a_1^0 + \varphi(a_1)1) \cdots (a_n^0 + \varphi(a_n)1) = \sum_{K \subseteq \{1, \dots, n\}} \left(\prod_{k \in K^c} \varphi(a_k) \right) \prod_{k \in K}^{\text{ord}} a_k^0. \tag{2.2}$$

It will suffice to prove that each term in the above sum is in the right-hand side of (2.1). The term with $K = \{1, \dots, n\}$ is already in the right-hand side of (2.1). If $K \neq \{1, \dots, n\}$, then the product

$$\prod_{k \in K}^{\text{ord}} a_k^0$$

has the form $b_1 \cdots b_m$ for some $m \leq n - 1$ and $b_k \in \mathcal{A}_{j_k}$ $(1 \leq k \leq m)$, where $\{j_1, \dots, j_m\}$ is a subset of $\{i_1, \dots, i_n\}$ satisfying $j_1 \neq \cdots \neq j_m$. Then the induction hypothesis implies that $b_1 \cdots b_m$ is in the right-hand side of (2.1), and the lemma follows. $\square$

Note that it follows directly from (2.2) that if $(\mathcal{A}_i)_{i \in I}$ is free, $i_1, \dots, i_n$ are all distinct indices in I, and $a_k \in \mathcal{A}_{i_k}$, then

$$\varphi(a_1 \cdots a_n) = \varphi(a_1) \cdots \varphi(a_n). \tag{2.3}$$

Proposition 2.11. *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space with a free family*

$(\mathcal{A}_i)_{i\in I}$ of subalgebras.

(i) *The restriction of φ to the subalgebra generated by $(\mathcal{A}_i)_{i\in I}$ is determined by the restrictions $\varphi|_{\mathcal{A}_i}$, $i \in I$.*

(ii) *If $(\mathcal{A}, \varphi)$ is a C*-probability space (resp. W*-probability space), and each $\mathcal{A}_i$ is self-adjoint, then the restriction of φ to the C*-subalgebra (resp. von Neumann algebra) generated by $(\mathcal{A}_i)_{i\in I}$ is determined by the restrictions $\varphi|_{\mathcal{A}_i}$, $i \in I$.*

Proof. Let $\mathcal{A}_0$ be the algebra generated by $(\mathcal{A}_i)_{i\in I}$. Looking at (2.1) we see that

$$\mathcal{A}_0 \cap \ker\varphi = \sum_{n\in\mathbb{N}} \sum_{\substack{i_1,\dots,i_n\in I\\ i_1\neq\cdots\neq i_n}} \mathcal{A}^0_{i_1}\cdots\mathcal{A}^0_{i_n},$$

and since $\varphi(1) = 1$, (i) follows.

To prove (ii) we let $\mathcal{A}_1$ be the C*-algebra (resp. von Neumann algebra) generated by $(\mathcal{A}_i)_{i\in I}$. Then $\mathcal{A}_0$ is dense in $\mathcal{A}_1$ in the norm topology (resp. strong operator topology). The C*-case now follows from (i) and norm continuity of φ. In the W*-case we apply Kaplansky's Density Theorem to see that the unit ball of $\mathcal{A}_0$ is strong operator dense in the unit ball of $\mathcal{A}_1$. Since φ is strong operator continuous on the unit ball of $\mathcal{A}_1$, it follows from (i) that the restriction of φ to the unit ball of $\mathcal{A}_1$ is determined. Of course, then φ is determined on all of $\mathcal{A}_1$. □

Proposition 2.12. *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space with a free family $(\mathcal{A}_i)_{i\in I}$ of subalgebras.*

(i) *Let $I = \bigcup_{j\in J} I_j$ be a partition of I, and let $\mathcal{B}_j$ denote the subalgebra of $\mathcal{A}$ generated by $(\mathcal{A}_i)_{i\in I_j}$. Then the family $(\mathcal{B}_j)_{j\in J}$ is free.*

(ii) *Suppose that for each $i \in I$ there is a free family $(C^i_j)_{j\in J_i}$ of subalgebras in the non-commutative probability space $(\mathcal{A}_i, \varphi|_{\mathcal{A}_i})$. Then the family $\{C^i_j \mid i \in I, j \in J_i\}$ is free in $(\mathcal{A}, \varphi)$.*

Proof.

(i) Suppose for $1 \le k \le n$ that $b_k \in \mathcal{B}^0_{j_k}$, and $j_1 \neq \cdots \neq j_n$. Since $\mathcal{B}_j$ is generated by $(\mathcal{A}_i)_{i\in I_j}$, it follows from Lemma 2.10 that we may write b_k as $b_k = \beta_k 1 + b'_k$, where

$$\beta_k \in \mathbb{C} \quad \text{and} \quad b'_k \in \sum_{m\geq 1} \sum_{\substack{i_1,\cdots,i_m\in I_{j_k}\\ i_1\neq\cdots\neq i_m}} \mathcal{A}^0_{i_1}\cdots\mathcal{A}^0_{i_m}.$$

Since $\varphi(b'_k) = 0$ by freeness, and $\varphi(b_k) = 0$ by assumption, we must have $\beta_k = 0$, and hence $b_k = b'_k$. From this it follows that the product $b_1\cdots b_n$ is a sum of terms, each of which is a product of the form $a_1\cdots a_l$, where for $1 \le k \le l$ we have $a_k \in \mathcal{A}^0_{i_k}$, and also $i_1 \neq \cdots \neq i_l$. Since $(\mathcal{A}_i)_{i\in I}$ is a free family, it follows that φ applied to each term is zero, and hence $\varphi(b_1\cdots b_n) = 0$.

(ii) For $1 \le k \le n$ we consider random variables $c_k \in C_{i_k}^{j_k}$, where (i_k, j_k) is in the set $\{(i,j) \mid i \in I,\ j \in J_i\}$, and $(i_1, j_1) \neq \cdots \neq (i_n, j_n)$. Suppose $\varphi(c_k) = 0$ for each k. Then we group those consecutive i_k's that are equal. More precisely, we find numbers $m_1, \ldots, m_l$ such that

$$i_1 = \cdots = i_{m_1}, \quad i_{m_1+1} = \cdots = i_{m_2}, \quad \cdots \quad i_{m_l+1} = \cdots = i_n,$$

and $i_{m_1} \neq i_{m_2} \neq \cdots \neq i_{m_l} \neq i_n$. Then we let

$$d_1 = c_1 \cdots c_{m_1}, \quad d_2 = c_{m_1+1} \cdots c_{m_2}, \quad \cdots \quad d_{l+1} = c_{m_l+1} \cdots c_n.$$

Since the family $(C_j^i)_{j \in J_i}$ is free in $(\mathcal{A}_i, \varphi|_{\mathcal{A}_i})$ for each i, we have $\varphi(d_k) = 0$ for all $1 \le k \le l+1$. Since the family $(\mathcal{A}_i)_{i \in I}$ is free, it follows that $\varphi(c_1 \cdots c_n) = \varphi(d_1 \cdots d_{l+1}) = 0$. $\square$

Proposition 2.13. *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space with a free family $(\mathcal{A}_i)_{i \in I}$ of subalgebras.*

(i) *If $\mathcal{A}$ is generated by $(\mathcal{A}_i)_{i \in I}$, and each $\varphi|_{\mathcal{A}_i}$ is a trace, then φ is a trace.*

(ii) *If $(\mathcal{A}, \varphi)$ is a C^*-probability space (resp. W^*-probability space), each $\mathcal{A}_i$ is self-adjoint, $\mathcal{A}$ is the C^*-algebra (resp. von Neumann algebra) generated by $(\mathcal{A}_i)_{i \in I}$, and each $\varphi|_{\mathcal{A}_i}$ is a trace, then φ is a trace.*

Proof. (ii) follows from (i) as usual, so we will focus on (i). By linearity it suffices to prove that $\varphi(xy) = \varphi(yx)$, when

$$x = a_1 \cdots a_n, \quad y = b_1 \cdots b_m,$$
$$a_1 \in \mathcal{A}_{i_1}, \ldots, a_n \in \mathcal{A}_{i_n} \quad b_1 \in \mathcal{A}_{j_1}, \ldots, b_m \in \mathcal{A}_{j_m} \quad i_1 \neq \cdots \neq i_n,\ j_1 \neq \cdots \neq j_m.$$

We can also assume $m \le n$. First we prove $\varphi(xy) = \varphi(yx)$ under the additional assumption that $\varphi(a_k) = 0$ $(1 \le k \le n)$ and $\varphi(b_k) = 0$ $(1 \le k \le m)$. Then if $i_n \neq j_1$ we get $\varphi(a_1 \cdots a_n b_1 \cdots b_m) = 0$ by freeness. If $i_n = j_1$, then we write $a_n b_1 = (a_n b_1)^0 + \varphi(a_n b_1)1$, and by freeness

$$\varphi(a_1 \cdots a_n b_1 \cdots b_m) = 0 + \varphi(a_n b_1)\varphi(a_1 \cdots a_{n-1} b_2 \cdots b_m).$$

We may combine these two cases as

$$\varphi(a_1 \cdots a_n b_1 \cdots b_m) = \delta_{i_n j_1}\varphi(a_n b_1)\varphi(a_1 \cdots a_{n-1} b_2 \cdots b_m).$$

Repeating the procedure, we find

$$\varphi(xy) = \delta_{mn}\delta_{i_n j_1}\delta_{i_{n-1} j_2} \cdots \delta_{i_{n-m+1} j_m}\varphi(a_n b_1) \cdots \varphi(a_{n-m+1} b_m).$$

Similarly,

$$\varphi(yx) = \delta_{mn}\delta_{i_1 j_m} \cdots \delta_{i_m j_1}\varphi(b_m a_1) \cdots \varphi(b_1 a_m).$$

If $m \neq n$, $\varphi(xy) = \varphi(yx) = 0$. Otherwise, if $i_k = j_{m-k+1}$ for some k we know by assumption that $\varphi(a_k b_k) = \varphi(b_k a_k)$, so we see that $\varphi(xy) = \varphi(yx)$.

Now for the general case. We prove this by induction over $m+n$, and we may always suppose $m > 0$ and $n > 0$. If $m + n = 2$ and $i_1 \neq j_1$, we simply use (2.3) to see that

$$\varphi(a_1 b_1) = \varphi(a_1)\varphi(b_1) = \varphi(b_1 a_1).$$

If $i_1 = j_1$, then we already know that $\varphi(a_1 b_1) = \varphi(b_1 a_1)$, since φ is a trace on $\mathcal{A}_{i_1}$.

If $m + n \geq 3$ and $m \geq 2$, then

$$\varphi(a_1 \cdots a_n b_1 \cdots b_m) = \varphi(b_m)\varphi(a_1 \cdots a_n b_1 \cdots b_{m-1}) + \varphi(a_1 \cdots a_n b_1 \cdots b_{m-1}(b_m)^0).$$

We can apply the induction hypothesis to the first term, and the last term may be expanded as the following sum:

$$\sum_{K \subseteq \{1,\ldots,m+n-1\}} \left(\prod_{\substack{k \in K^c \\ k \leq n}} \varphi(a_k) \prod_{\substack{k \in K^c \\ k > n}} \varphi(b_{k-n}) \right) \varphi \left(\left(\prod_{\substack{k \in K \\ k \leq n}}^{\text{ord}} a_k^0 \prod_{\substack{k \in K \\ k > n}}^{\text{ord}} b_{k-n}^0 \right) (b_m)^0 \right).$$

If $K = \{1, \ldots, m+n-1\}$, we are back in the special case above. If $K \neq \{1, \ldots, m+n-1\}$, the ordered product may be written in the form $a'_1 \cdots a'_{n'} b'_1 \cdots b'_{m'}$, where $n' \leq n$, $m' \leq m$ and $m' + n' < m + n$. Then we apply the induction hypothesis.

In the end we end up with

$$\varphi(b_m)\varphi(b_1 \cdots b_{m-1} a_1 \cdots a_n) + \varphi(b_1 \cdots b_{m-1}(b_m)^0 a_1 \cdots a_n) = \varphi(yx).$$

This completes the algebraic case, and we are done. □

2.4 Free products

At this point we have not yet seen any examples of free subalgebras in a non-commutative probability space. Of course the subalgebra $\mathbb{C}1$ is always free of anything else, but otherwise easy examples are hard to come by. A very important example is shown in Lemma 3.4. Otherwise most examples arise from the construction of free products of algebras. Free products come in various flavors, for instance a free product of unital algebras and the so-called *reduced free product* of von Neumann algebras. The free product should be contrasted with the tensor product in the commutative case. The free product contains images the original algebras as subalgebras, these images generate the whole free product, and there is a natural *free product state* that makes the images free.

We shall only be concerned with the reduced free product of von Neumann algebras, and from now on we will simply call it the free product. For the actual construction we refer the reader to for instance [13, 14] or for the more general *reduced amalgamated free product* [14] and [2, Section 4.7]. We will simply state the properties needed in this thesis and omit the proofs.

Let $(\mathcal{A}_i, \varphi_i)_{i\in I}$ be a family of W*-probability spaces. We denote the free product as

$$(\mathcal{A}, \varphi) = \mathop{\ast}_{i\in I}(\mathcal{A}_i, \varphi_i).$$

When it is clear from the context which states we use we simply write $\ast_{i\in I}\mathcal{A}_i$ instead.

Suppose that for all $i \in I$ the GNS representation $\pi_{\varphi_i} : \mathcal{A}_i \to B(H_{\varphi_i})$ is injective. Then there are normal *-monomorphisms $\psi_i : \mathcal{A}_i \to \mathcal{A}$ such that the images $(\psi(\mathcal{A}_i))_{i\in I}$ form a free family, which generates $\mathcal{A}$. Further, $\varphi \circ \psi_i = \varphi_i$. We shall always identify $\mathcal{A}_i$ with its image $\psi(\mathcal{A}_i)$.

If each $\mathcal{A}_i$ is a von Neumann algebra of operators on H_i, then the free product von Neumann algebra acts on the Hilbert space free product $(H, \xi) = \ast_i(H_{\varphi_i}, \xi_i)$, where H_{φ_i} is the GNS Hilbert space with cyclic vector ξ_i. And the free product state φ is the vector state given by the distinguished unit vector ξ, which is cyclic for the free product. In particular, the GNS representation associated to φ is also injective. If each state is a trace, then the free product state is a trace. This is actually a consequence of Proposition 2.13. Also, if each state is faithful, the free product state is also faithful (see [15]).

Taking free products is an associative and commutative operation. This actually follows from the properties stated so far and the Isomorphism Theorem (next section).

Free products of von Neumann algebras behave particularly well with respect to the group von Neumann algebra construction. If the group G is the free product of a family $(G_i)_{i\in I}$ of groups, then $L(G)$ is unitarily equivalent to the free product $\ast_i(L(G_i), \tau_i)$. Here τ_i denotes the canonical trace on $L(G_i)$, and the unitary identifies the free product state with the canonical trace on $L(G)$.

Since the free group $\mathbb{F}_n$ on n generators equals the free product $\ast_{i=1}^n \mathbb{Z}$, we see that $L(\mathbb{F}_n)$ contains n copies of $L(\mathbb{Z})$ which form a free, generating family.

2.5 The Isomorphism Theorem

The following theorem is fundamental in free probability. The content is that in sufficiently nice non-commutative probability spaces the isomorphism class of the algebra is determined by the *-distribution of a generating family. The theorem will be used over and over again in the rest of the thesis, with and without mentioning.

Theorem 2.14 (Isomorphism Theorem)**.** *Let $(\mathcal{A}, \varphi)$ and $(\mathcal{B}, \psi)$ be C*-probability spaces (respectively W*-probability spaces) such that the GNS representations associated to φ and*

ψ are injective. Let $(f_i)_{i\in I}$ and $(g_i)_{i\in I}$ be families in $\mathcal{A}$ and $\mathcal{B}$ that generate $\mathcal{A}$ and $\mathcal{B}$ as C-algebras (respectively W*-algebras). If $(f_i)_{i\in I}$ and $(g_i)_{i\in I}$ have the same *-distributions, then there is an isomorphism $\Gamma : \mathcal{A} \to \mathcal{B}$ such that $\varphi = \psi \circ \Gamma$ and $\Gamma(f_i) = g_i$ for all $i \in I$.*

Proof. First we recall the GNS construction. Associate to φ the left ideal

$$\mathcal{N}_\varphi = \{a \in \mathcal{A} \mid \varphi(a^*a) = 0\}.$$

The quotient space $\mathcal{A}/\mathcal{N}_\varphi$ has a well-defined inner product

$$\langle a_1 + \mathcal{N}_\varphi, a_2 + \mathcal{N}_\varphi\rangle = \varphi(a_2^* a_1), \quad a_1, a_2 \in \mathcal{A}. \tag{2.4}$$

Completing $\mathcal{A}/\mathcal{N}_\varphi$ with respect to this inner product gives a Hilbert space H_φ on which $\mathcal{A}$ acts by left multiplication. More precisely, we have the GNS representation $\pi_\varphi : \mathcal{A} \to B(H_\varphi)$ given by

$$\pi_\varphi(a)(a_1 + \mathcal{N}_\varphi) = aa_1 + \mathcal{N}_\varphi, \quad a, a_1 \in \mathcal{A},$$

which is well-defined, since $\mathcal{N}_\varphi$ is a left ideal.

Let $\mathcal{A}_0$ and $\mathcal{B}_0$ be the *-subalgebras of $\mathcal{A}$ and $\mathcal{B}$ generated by $(f_i)_{i\in I}$ and $(g_i)_{i\in I}$. We have assumed, that π_φ and π_ψ are injective, and hence they are isometries.

We consider the C*-case first. Then $\mathcal{A}_0$ is norm dense in $\mathcal{A}$, and we claim that $\mathcal{A}_0/\mathcal{N}_\varphi = \{a + \mathcal{N}_\varphi \mid a \in \mathcal{A}_0\}$ is dense in H_φ. Since $\mathcal{A}/\mathcal{N}_\varphi$ is dense in H_φ, it suffices to show that any element $a + \mathcal{N}_\varphi$ in $\mathcal{A}/\mathcal{N}_\varphi$ can be approximated by elements in $\mathcal{A}_0/\mathcal{N}_\varphi$ in the norm induced by (2.4). Since $\mathcal{A}_0$ is dense in $\mathcal{A}$, we may find a sequence (a_n) in $\mathcal{A}_0$ such that $a_n \to a$ in norm. Then $(a_n - a)^*(a_n - a) \to 0$ in norm, and hence

$$||(a_n + \mathcal{N}_\varphi) - (a + \mathcal{N}_\varphi)||^2 = \varphi((a_n - a)^*(a_n - a)) \to 0.$$

Of course, it also holds that $\mathcal{B}_0/\mathcal{N}_\psi$ is dense in H_ψ.

We will now define a unitary $V : H_\varphi \to H_\psi$ to implement the desired isomorphism Γ (together with the GNS representations). First we define V between the dense subsets $\mathcal{A}_0/\mathcal{N}_\varphi$ and $\mathcal{B}_0/\mathcal{N}_\psi$ as follows. Let P be any polynomial in $2n$ non-commuting variables, and let $i_1, \ldots, i_n \in I$ be given. Then we define

$$V(P(f_{i_1}, \ldots, f_{i_n}, f_{i_1}^*, \ldots, f_{i_n}^*) + \mathcal{N}_\varphi) = P(g_{i_1}, \ldots, g_{i_n}, g_{i_1}^*, \ldots, g_{i_n}^*) + \mathcal{N}_\psi.$$

Note that since $(f_i)_{i\in I}$ and $(g_i)_{i\in I}$ have the same *-distributions,

$$\begin{aligned}&\varphi(P(f_{i_1}, \ldots, f_{i_n}, f_{i_1}^*, \ldots, f_{i_n}^*)^* P(f_{i_1}, \ldots, f_{i_n}, f_{i_1}^*, \ldots, f_{i_n}^*))\\ &= \psi(P(g_{i_1}, \ldots, g_{i_n}, g_{i_1}^*, \ldots, g_{i_n}^*)^* P(g_{i_1}, \ldots, g_{i_n}, g_{i_1}^*, \ldots, g_{i_n}^*)),\end{aligned}$$

and hence V is a well-defined map $\mathcal{A}_0/\mathcal{N}_\varphi \to \mathcal{B}_0/\mathcal{N}_\psi$. It also shows that V is norm-preserving, and hence it extends to a unitary, also denoted V, from H_φ to H_ψ.

Now we will show that for any polynomial $Q \in \mathbb{C}\langle X_i \mid 1 \le i \le 2m\rangle$ and $j_1, \ldots, j_m \in I$ we have

$$V\pi_\varphi(Q(f_{j_1}, \ldots, f_{j_m}, f_{j_1}^*, \ldots, f_{j_m}^*))V^* = \pi_\psi(Q(g_{j_1}, \ldots, g_{j_m}, g_{j_1}^*, \ldots, g_{j_m}^*)). \tag{2.5}$$

It suffices to demonstrate that the two operators agree on the dense subspace $\mathcal{B}_0/\mathcal{N}_\psi$ of H_ψ, that is, on elements $P(g_{i_1}, \ldots, g_{i_n}, g_{i_1}^*, \ldots, g_{i_n}^*) + \mathcal{N}_\psi$ where P is any polynomial in $\mathbb{C}\langle X_i \mid 1 \le i \le 2n\rangle$ and $i_1, \ldots, i_n \in I$. This amounts to the straightforward calculation:

$$\begin{aligned}
&V\pi_\varphi(Q(f_{j_1}, \ldots, f_{j_m}, f_{j_1}^*, \ldots, f_{j_m}^*))V^* \left(P(g_{i_1}, \ldots, g_{i_n}, g_{i_1}^*, \ldots, g_{i_n}^*) + \mathcal{N}_\psi\right)\\
&= V\pi_\varphi(Q(f_{j_1}, \ldots, f_{j_m}, f_{j_1}^*, \ldots, f_{j_m}^*)) \left(P(f_{i_1}, \ldots, f_{i_n}, f_{i_1}^*, \ldots, f_{i_n}^*) + \mathcal{N}_\varphi\right)\\
&= V\left(Q(f_{j_1}, \ldots, f_{j_m}, f_{j_1}^*, \ldots, f_{j_m}^*)P(f_{i_1}, \ldots, f_{i_n}, f_{i_1}^*, \ldots, f_{i_n}^*) + \mathcal{N}_\varphi\right)\\
&= Q(g_{j_1}, \ldots, g_{j_m}, g_{j_1}^*, \ldots, g_{j_m}^*)P(g_{i_1}, \ldots, g_{i_n}, g_{i_1}^*, \ldots, g_{i_n}^*) + \mathcal{N}_\psi\\
&= \pi_\psi(Q(g_{j_1}, \ldots, g_{j_m}, g_{j_1}^*, \ldots, g_{j_m}^*)) \left(P(g_{i_1}, \ldots, g_{i_n}, g_{i_1}^*, \ldots, g_{i_n}^*) + \mathcal{N}_\psi\right).
\end{aligned}$$

The third equality follows from the definition of V and the fact that

$$Q(f_{j_1}, \ldots, f_{j_m}, f_{j_1}^*, \ldots, f_{j_m}^*)P(f_{i_1}, \ldots, f_{i_n}, f_{i_1}^*, \ldots, f_{i_n}^*)$$

can be thought of as a polynomial in elements of the set $\{f_i \mid i \in I\} \cup \{f_i^* \mid i \in I\}$.

We know that $\pi_\varphi(\mathcal{A}_0)$ and $\pi_\psi(\mathcal{B}_0)$ are norm dense in $\pi_\varphi(\mathcal{A})$ and $\pi_\psi(\mathcal{B})$. Moreover, by what we just argued, the map $\mathrm{Ad}V : B(H_\varphi) \to B(H_\psi)$ given by conjugation by V, i.e. $T \mapsto VTV^*$, maps $\pi_\varphi(\mathcal{A}_0)$ isometrically onto $\pi_\psi(\mathcal{B}_0)$, and by continuity it takes $\pi_\varphi(\mathcal{A})$ onto $\pi_\psi(\mathcal{B})$. Then with $\Gamma = \pi_\psi^{-1} \circ \mathrm{Ad}V \circ \pi_\varphi$ we have an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

It follows from (2.5) that $\Gamma(f_i) = g_i$ for every $i \in I$. Also, since $(f_i)_{i\in I}$ and $(g_i)_{i\in I}$ have the same *-distribution, this implies that φ and $\psi \circ \Gamma$ coincide on $\mathcal{A}_0$. By continuity $\varphi = \psi \circ \Gamma$.

For the W*-case we follow the same approach, but since we only assume that $\mathcal{A}_0$ is strongly dense in $\mathcal{A}$, we need to modify some of the arguments. As usual, the modifications rely on Kaplansky's Density Theorem.

First we show that $\mathcal{A}_0/\mathcal{N}_\varphi$ is dense in $\mathcal{A}/\mathcal{N}_\varphi$. Given any $a \in \mathcal{A}$ there is a net $(a_\lambda)_{\lambda\in\Lambda}$ in $\mathcal{A}_0$ such that $||a_\lambda|| \le ||a||$ for every λ, and $a_\lambda \to a$ strongly. For any $x, y \in H$

$$|\langle(a_\lambda - a)^*(a_\lambda - a)x, y\rangle| = |\langle(a_\lambda - a)x, (a_\lambda - a)y\rangle| \le ||(a_\lambda - a)x||\ ||(a_\lambda - a)y|| \to 0,$$

since $a_\lambda \to a$ strongly. Thus $(a_\lambda - a)^*(a_\lambda - a) \to 0$ weakly. Notice that the net $((a_\lambda - a)^*(a_\lambda - a))_{\lambda\in\Lambda}$ is bounded in $\mathcal{A} \subseteq B(H)$. Since φ is weakly continuous on bounded sets, it follows that

$$\varphi((a_\lambda - a)^*(a_\lambda - a)) \to 0.$$

This means exactly that $a_\lambda + \mathcal{N}_\varphi \to a + \mathcal{N}_\varphi$, and hence $\mathcal{A}_0/\mathcal{N}_\varphi$ is dense in $\mathcal{A}/\mathcal{N}_\varphi$ and then of course also dense in H_φ.

We define V in the same way as before, and (2.5) holds also in this case.

Since φ and ψ are normal, it follows that the unit balls of $\pi_\varphi(\mathcal{A})$ and $\pi_\psi(\mathcal{B})$ are weak operator closed. To prove that AdV extends to an isormorphism $\pi_\varphi(\mathcal{A}) \to \pi_\psi(\mathcal{B})$, we need only that $\pi_\varphi(\mathcal{A}_0)$ and $\pi_\psi(\mathcal{B}_0)$ are weakly dense in $\pi_\varphi(\mathcal{A})$ and $\pi_\psi(\mathcal{B})$. But this is obvious, since π_φ is continuous between the unit balls, when both the unit ball of $\mathcal{A}$ and $\pi_\varphi(\mathcal{A})$ are endowed with the weak operator topology.

Finally, to show that $\varphi = \psi \circ \Gamma$ it is enough to remark, that both functionals are strong operator continuous on bounded sets and coincide on the unit ball of $\mathcal{A}_0$ which is a strong operator dense subset of the unit ball $(\mathcal{A})_1$. □

We note the following immediate corollary, which will be useful later on when we deal with direct limits of von Neumann algebras.

Corollary 2.15. *Consider a W*-probability space $(\mathcal{M}, \varphi)$ with φ a faithful state. Let $(f_i)_{i\in I}$ and $(g_i)_{i\in I}$ be families with identical *-distributions. Let J be a subset of I. Then the isomorphism $\Phi_I : W^*(f_i \mid i \in I) \to W^*(g_i \mid i \in I)$ coming from Theorem 2.14 extends the isomorphism $\Phi_J : W^*(f_i \mid i \in J) \to W^*(g_i \mid i \in J)$ coming from Theorem 2.14.*

Proof. This follows from the fact that $\Phi_I(f_i) = \Phi_J(f_i) = g_i$ for each $i \in J$. □

The condition that the GNS representation of some C*-probability space is injective will be relevant in a number of places. Here we present of couple of simple criterias for this to happen. Of course, if φ is a faithful state, then π_φ is injective. This is true, because

$$\pi_\varphi(a) = 0 \implies \pi_\varphi(a^*a) = 0 \implies \varphi(a^*a) = \langle \pi_\varphi(a^*a)\xi_\varphi, \xi_\varphi\rangle = 0 \implies a = 0.$$

Lemma 2.16. *Let $(\mathcal{A}, \varphi)$ be a C*-probability space. Then the GNS representation corresponding to φ is injective if and only if to each non-zero $a \in \mathcal{A}$ there are $b, c \in \mathcal{A}$ such that $\varphi(c^*ab) \neq 0$.*

Proof. Let (π_φ, H_φ) be the GNS representation corresponding to φ. Then

$$\varphi(c^*ab) = \langle \pi_\varphi(a)[b], [c]\rangle,$$

where $b \mapsto [b]$ is the quotient map $\mathcal{A} \to H_\varphi$. Since vectors of the form $[b]$ with $b \in \mathcal{A}$ are dense in H_φ, the result follows. □

Remark 2.17. If $\mathcal{A} \subseteq B(H)$ is a C*-algebra, and if φ is the vector state $\varphi(a) = \langle a\xi, \xi \rangle$ associated to some vector $\xi \in H$, which is cyclic for $\mathcal{A}$ (i.e. $\mathcal{A}\xi$ is dense in H), then π_φ is injective. To see this, note that if $a \in \mathcal{A}$, $a \neq 0$, then by cyclicity of ξ there are operators $b, c \in \mathcal{A}$ such that $\langle ab\xi, c\xi \rangle \neq 0$. That is, $\varphi(c^*ab) \neq 0$.

Another way to think of this example, is that when ξ is cyclic, then the GNS representation associated to the vector state is unitarily equivalent to the identity representation, which is of course injective.

2.6 Important distributions

In this section we describe a few distributions that will play an important role in the rest of the thesis. In later chapters we will exhibit concrete examples of random variables with these distributions. The study of a concrete example will often help us to reveal important properties of any other variable with the same distribution, cf. the Isomorphism Theorem. In this section (Theorem 2.23) we reach the link between free probability theory and the free group factors.

Definition 2.19. Let $(\mathcal{A}, \varphi)$ be a *-probability space.

(i) An element $f \in \mathcal{A}$ is called *semicircular*, if it is self-adjoint and its distribution is the semicircle distribution given by

$$\varphi(f^n) = \frac{1}{2\pi} \int_{-2}^{2} t^n \sqrt{4 - t^2}\, \mathrm{d}t, \quad n \in \mathbb{N}.$$

(ii) A family $(f_i)_{i \in I}$ of elements in $\mathcal{A}$ is called a *semicircular family*, if it is free, and each f_i is semicircular.

(iii) A family $(g_i)_{i \in I}$ of elements in $\mathcal{A}$ is called a *circular family*, if the family $(x_i)_{i \in I} \cup (y_i)_{i \in I}$ is semicircular, where

$$x_i = \frac{g_i + g_i^*}{\sqrt{2}}, \qquad y_i = \frac{g_i - g_i^*}{i\sqrt{2}}.$$

Note that the joint distribution of a semicircular or circular family is uniquely determined, according to Proposition 2.11.

Some authors (Voiculescu, among others) use a different normalization of the semicircular elements, so that the support of μ_f is $[-1, 1]$, and the variance of f is $\frac{1}{4}$.

If $(x_i)_{i\in I} \cup (y_i)_{i\in I}$ is a semicircular family, then with $g_i = \frac{1}{\sqrt{2}}(x_i + iy_i)$ the family $(g_i)_{i\in I}$ is circular. Also, any circular family arises in this way. To see this, note simply that

$$\left.\begin{array}{l} x_i = \frac{g_i+g_i^*}{\sqrt{2}} \\ y_i = \frac{g_i-g_i^*}{i\sqrt{2}} \end{array}\right\} \iff \left\{\begin{array}{l} x_i, y_i \text{ self-adjoint} \\ g_i = \frac{1}{\sqrt{2}}(x_i + iy_i) \end{array}\right. .$$

In free probability theory the semicircle distribution is extremely important. It plays the same role as the normal (or Gaussian) distribution plays in classical probability theory. For instance there is a free analogue of the central limit theorem, and here the limit distribution is the semicircle distribution. It may come as a surprise to some that the normal distribution is not as important in free probaility theory as the semicircle distribution, but as we discussed earlier (before introducing the notion of freeness) this change is in fact quite natural.

We will not prove the free version of the central limit theorem. The interested reader is referred to [14, 15]. Nonetheless, the semicircular families will be very important to us. It turns out that the free group factors are generated by free semicircular families, and the semicircular elements will also be essential to us when we define the interpolated free group factors, which eventually lead to our main results.

Along the lines of semicircular and circular elements we should also mention the following, quartercircular elements, which will be relevant in a few places.

Definition 2.20. An element q in a *-probability space $(\mathcal{A}, \varphi)$ is called *quartercircular*, if $q = q^*$ and

$$\varphi(q^n) = \frac{1}{\pi}\int_0^2 t^n\sqrt{4-t^2}\,\mathrm{d}t, \quad n \in \mathbb{N}.$$

It is easily seen that if $(\mathcal{A}, \varphi)$ is a C*-probability space, and f is semicircular in $\mathcal{A}$, then $|f|$ is quartercircular. Perhaps more interesting is the fact (Theorem 4.11) that $|g|$ is also quartercircular, if g is circular.

Another important class of variables are the Haar unitaries.

Definition 2.21. Let $(\mathcal{A}, \varphi)$ be a *-probability space. A unitary $u \in \mathcal{A}$ is called a *Haar unitary*, if its *-distribution is given by

$$\varphi(u^n) = 0, \quad n \in \mathbb{Z} \setminus \{0\}.$$

The name Haar unitary comes from the fact that the distribution of u induces the Haar measure on the circle.

It is easy to give concrete examples of the four types of variables introduced. For instance the identity function on $[-2, 2]$ is a semicircular variable in the C*-probability space $(C([-2, 2]), \varphi)$, if φ is given by integration with respect to the semicircle measure. However, this is not very interesting. A little more interesting is the identity function on the unit circle

$\mathbb{T}$. Since

$$\frac{1}{2\pi}\int_{\mathbb{T}} z^n \,\mathrm{d}z = \delta_{0,n}, \quad n \in \mathbb{Z} \setminus \{0\},$$

this is a Haar unitary in the W*-probability space $(L^\infty(\mathbb{T}), \tau)$, where τ is integration with respect to the normalized Lebesgue measure. A more interesting example of a semicircular element is given in Proposition 3.5.

Lemma 2.22. *The von Neumann algebra $L(\mathbb{Z})$ is generated by a semicircular element. Here it is implicit that $L(\mathbb{Z})$ is equipped with its canonical trace τ.*

Proof. Let m denote the Lebesgue measure on $\mathbb{R}$, and let μ be the semicircle measure on $[-2, 2]$, i.e. μ has density $\frac{1}{2\pi}\sqrt{4-t^2}$ on $[-2, 2]$ with respect to m. The identity function $\mathrm{id}(t) = t$ on $[-2, 2]$ is semicircular in $\mathcal{A} = L^\infty(\mu)$, when $\mathcal{A}$ is equipped with the functional φ given by integration with respect to μ. Note that φ is the vector state $\varphi(x) = \langle x1, 1\rangle_\mu$. Thus, it will suffice to find a unitary $L^2(\mu) \to \ell^2(\mathbb{Z})$ that takes $\mathcal{A}$ to $L(\mathbb{Z})$ and the unit vector $1 \in L^2(\mu)$ to unit vector $\delta_0 \in \ell^2(\mathbb{Z})$. Using Fourier transform and the standard identification of $L^\infty(\mathbb{T})$ with $L^\infty([-\pi, \pi])$, it suffices to find a unitary $V : L^2(\mu) \to L^2([-\pi, \pi], \frac{m}{2\pi})$ that takes 1 to 1 and satisfies $V\mathcal{A}V^* = L^\infty([-\pi, \pi], \frac{m}{2\pi})$. We will now construct such a unitary.

Let $\psi : [-2, 2] \to \mathbb{R}$ be given by

$$\psi(t) = 2\arcsin(t/2) + (t/2)\sqrt{4-t^2}.$$

Then ψ has derivative $\psi'(t) = \sqrt{4-t^2}$, and so ψ is stricly increasing with range $[-\pi, \pi]$. Note that

$$\mu(\psi^{-1}([-\pi, t_0])) = \frac{1}{2\pi}\int_{-2}^{\psi^{-1}(t_0)} \sqrt{4-t^2}\,\mathrm{d}t = \frac{1}{2\pi} m([-\pi, t_0]).$$

It follows that ψ is a bi-measurable function that takes the semicircle measure to the Lebesgue measure. The desired unitary V is then given by composition with ψ^{-1}, i.e. $V(f) = f \circ \psi^{-1}$ for $f \in L^2(\mu)$.

To complete the proof we should show that $V1 = 1$ and $V\mathcal{A}V^* = L^\infty([-\pi, \pi])$. The first is obvious. For the last assertion, note that $VM_gV^* = M_{g\circ\psi^{-1}}$, where M_g denotes the multiplication operator. Since $\psi \in \mathcal{A}$ it follows that M_{id} is in $V\mathcal{A}V^*$, and since M_{id} generates $L^\infty([-\pi, \pi])$ the conclusion is now immediate. □

Theorem 2.23. *With $n \in \mathbb{N} \cup \{\infty\}$, the free group factor $L(\mathbb{F}_n)$ with canonical trace τ is generated by a semicircular family $(x_i)_{i=1}^n$ of size n.*

Also, if $(\mathcal{A}, \varphi)$ is a W-probability space, where the GNS representation π_φ is injective, and $\mathcal{A}$ is generated by a semicircular family $(f_i)_{i=1}^n$, then $\mathcal{A}$ is isomorphic to $L(\mathbb{F}_n)$ with an isomorphism Γ such that $\Gamma(f_i) = x_i$ for each i, and $\varphi = \tau \circ \Gamma$.*

Proof. The first assertion follows from the previous lemma and the fact that

$$L(\mathbb{F}_n) = L\left(\mathop{\ast}_{i=1}^{n} \mathbb{Z}\right) = \mathop{\ast}_{i=1}^{n} L(\mathbb{Z}).$$

Since the GNS representation associated to τ is injective, the second assertion follows immediately from the first and the Isomorphism Theorem. $\square$

The von Neumann algebra generated by a Haar unitary is (isomorphic to) the abelian von Neumann algebra $L(\mathbb{Z})$, and hence it contains only normal elements. But the *-distribution of any normal element in any W*-probability space may also be found as the *-distribution of an element in $L(\mathbb{Z})$. Since normality is not a strong assumption, the proof of this fact is a bit technical. But it is a nice result in itself, and we will also need it later on (in the proof of Proposition 4.12).

Lemma 2.24. *Let $(\mathcal{A}, \varphi)$ be a W*-probability space, and let $a \in \mathcal{A}$ be a normal element. Then there is an element $f \in L(\mathbb{Z})$ with the same *-distribution as a.*

Proof. We identify $L(\mathbb{Z})$ with $L^\infty([0,1])$, where $[0,1]$ is equipped with the Lebesgue measure, and hence it will suffice to prove that there is a bounded measurable function $f \in L^\infty([0,1])$ such that

$$\int_0^1 f(t)^k \overline{f(t)}^m \, \mathrm{d}t = \varphi(a^k (a^*)^m), \quad \text{for all } k, m \in \mathbb{N}_0. \tag{2.6}$$

Without loss of generality we may assume that

$$\sigma(a) \subseteq K^0 = \{\alpha + i\beta \in \mathbb{C} \mid 0 < \alpha \leq 1,\ 0 < \beta \leq 1\}.$$

We will approximate the identity function on K^0 uniformly by simple functions.

First we partition K^0 into four congruent squares K_j^1, $0 \leq j \leq 3$, such that K_j^1 contains the upper and right boundaries, but not the lower and left boundaries (just as K^0). The center of K_j^1 is denoted m_j^1. For the function g^1 given by $g^1 = \sum_{j=0}^3 m_j^1 \chi_{K_j^1}$ we have

$$||(g^1 - \mathrm{id})\chi_{\sigma(a)}||_\infty \leq \frac{\sqrt{2}}{2}.$$

We continue to partion each square into four congruent squares, and at the n'th step we have a partition of K^0 into sets $K^n_{j_1 \cdots j_n}$ with centers $m^n_{j_1 \cdots j_n}$, where $0 \leq j_1, \ldots, j_n \leq 3$ such that

$$\bigcup_{j_n=0}^{3} K^n_{j_1 \cdots j_n} = K^{n-1}_{j_1 \cdots j_{n-1}},$$

and the function g^n given by

$$g^n = \sum_{j_1=0}^{3} \cdots \sum_{j_n=0}^{3} m^n_{j_1\cdots j_n} \chi_{K^n_{j_1\cdots j_n}}$$

satisfies

$$||(g^n - \mathrm{id})\chi_{\sigma(a)}||_\infty \leq 2^{-n}\sqrt{2}, \qquad ||(g^{n+1} - g^n)\chi_{\sigma(a)}||_\infty \leq 2^{-(n-1)}\sqrt{2}.$$

We see that $g^n(a) \to a$ in norm. Let E be the spectral measure for a. Divide the interval $]0,1]$ into four half-open intervals $]d^1_j, d^1_{j+1}]$, $0 \leq j \leq 3$ such that $0 = d^1_0 < d^1_1 < d^1_2 < d^1_3 < d^1_4 = 1$, and

$$d^1_{j+1} - d^1_j = \varphi(E(K^1_j)).$$

We continue to partition each half-open interval into four disjoint pieces, and at the n'th step we have disjoint intervals $I^n_{j_1\cdots j_n}$, $0 \leq j_k \leq 3$ with union $]0,1]$ such that $I^n_{j_1\cdots j_n}$ has length $\varphi(E(K^n_{j_1\cdots j_n}))$, and

$$\bigcup_{j_n=0}^{3} I^n_{j_1\cdots j_n} = I^{n-1}_{j_1\cdots j_{n-1}}.$$

Now define the simple function $f_n \in L^\infty([0,1])$ by

$$f_n = \sum_{j_1=0}^{3} \cdots \sum_{j_n=0}^{3} m^n_{j_1\cdots j_n} \chi_{I^n_{j_1\cdots j_n}}.$$

Clearly,

$$||f_{n+1} - f_n||_\infty \leq 2^{-n-1}\sqrt{2}.$$

Also, for any $k, m \in \mathbb{N}_0$

$$\begin{aligned}
\int_0^1 f_n(t)^k \overline{f_n(t)}^m \,\mathrm{d}t &= \sum_{j_1=0}^{3} \cdots \sum_{j_n=0}^{3} (m^n_{j_1\cdots j_n})^k (\overline{m^n_{j_1\cdots j_n}})^m \varphi(E(K^n_{j_1\cdots j_n})) \\
&= \varphi\left(\sum_{j_1=0}^{3} \cdots \sum_{j_n=0}^{3} (m^n_{j_1\cdots j_n})^k (\overline{m^n_{j_1\cdots j_n}})^m E(K^n_{j_1\cdots j_n}) \right) \\
&= \varphi\left(g^n(a)^k (g^n(a)^*)^m \right) \to \varphi(a^k (a^*)^m).
\end{aligned}$$

Let $f = \lim f_n$ (uniformly). Then $f \in L^\infty([0,1])$, and f satisfies (2.6). $\square$

CHAPTER 3

Creation operators

3.1 Definition and basic properties

The most useful example of a semicircular family in this thesis is obtained using so-called creation operators acting on the tensor Hilbert space (to be defined below). One of the reasons why this example is so useful is that there are fairly simple criteria for recognizing creation operators. Another reason is that the creation operators behave well under the manipulations we need.

Definition 3.1 (Tensor Hilbert space)**.** Let H be a Hilbert space, and let $H^{\otimes n}$ denote the n-fold tensor product of H with itself. Then the *tensor Hilbert space* associated to H is the Hilbert space

$$\mathcal{T}(H) = \mathbb{C} \oplus \left(\bigoplus_{n=1}^{\infty} H^{\otimes n} \right),$$

Remark 3.2. The tensor Hilbert space is sometimes called the Fock space or just the tensor space. It is customary to write Ω for the unit vector $1 \in \mathbb{C} \subseteq \mathcal{T}(H)$, and we shall continue do so. The vector Ω is called the *vacuum vector*, and the vector state ω on $B(\mathcal{T}(H))$ given by $\omega(T) = \langle T\Omega, \Omega \rangle$ is called the *vacuum state*.

Definition 3.3 (Creation operator)**.** Let H be a Hilbert space with a unit vector h. The *creation operator* associated to h is the bounded operator ℓ_h acting on $\mathcal{T}(H)$ defined by

$$\ell_h \Omega = h$$

$$\ell_h x = h \otimes x, \quad x \in H^{\otimes n},$$

and extended by linearity and continuity to all of $\mathcal{T}(H)$.

Note that

$$||\ell_h(x)|| = ||h \otimes x|| = ||h||\,||x|| = ||x||,$$

so ℓ_h is an isometry from $H^{\otimes n}$ into $H^{\otimes (n+1)}$. And since $H^{\otimes n}$ is orthogonal to $H^{\otimes m}$ in $\mathcal{T}(H)$ when $n \neq m$, this shows that ℓ_h in fact defines a bounded operator on $B(\mathcal{T}(H))$, which is even an isometry.

Note too that if h and h' are orthogonal unit vectors, then

$$\ell_h(H^{\otimes n}) = h \otimes H^{\otimes n} \quad \text{and} \quad \ell_{h'}(H^{\otimes n}) = h' \otimes H^{\otimes n}$$

are orthogonal. This implies that

$$\ell_h(\mathcal{T}(H)) = 0 \oplus \left(\bigoplus_{n=0}^{\infty} h \otimes H^{\otimes n} \right)$$

and that the ranges of ℓ_h and $\ell_{h'}$ are orthogonal. In particular $\ell_h^* \ell_{h'} = 0$. As noted before, ℓ_h is an isometry, so $\ell_h^* \ell_h = 1$. We also remark, that $\ell_h^* \Omega = 0$.

In the special case where the Hilbert space has dimension one, i.e. $H = \mathbb{C}$, the tensor space is simply $\ell^2(\mathbb{N}_0)$, if we identify $\mathbb{C}^{\otimes n} \simeq \mathbb{C}$. Under this identification the creation operator ℓ_1 then corresponds to the unilateral shift operator.

Usually we will consider several creation operators at the same time, and when they arise from an orthonormal set $(e_i)_{i\in I}$ we will usually denote them $(\ell_i)_{i\in I}$ instead of $(\ell_{e_i})_{i\in I}$.

Lemma 3.4. *Let H be a Hilbert space with orthonormal basis $(e_i)_{i\in I}$, and let $(\ell_i)_{i\in I}$ be the corresponding creation operators. Then the family $(\ell_i)_{i\in I}$ is *-free in the non-commutative probability space $(B(\mathcal{T}(H)), \omega)$.*

Proof. Let $\mathcal{A}_i = \operatorname{alg}(1, \ell_i, \ell_i^*)$. Since $\ell_i^* \ell_i = 1$, we can write $\mathcal{A}_i = \operatorname{span}\{\ell_i^m (\ell_i^*)^n \mid m, n \in \mathbb{N}_0\}$. Recall that $\ell_i^* \Omega = 0$. This implies that

$$\mathcal{A}_i^0 = \operatorname{span}\{\ell_i^m (\ell_i^*)^n \mid m, n \in \mathbb{N}_0, (m,n) \neq (0,0)\},$$

because

$$\omega(\ell_i^m (\ell_i^*)^n) = \langle \ell_i^m (\ell_i^*)^n \Omega, \Omega \rangle = \langle (\ell_i^*)^n \Omega, (\ell_i^*)^m \Omega \rangle = \begin{cases} 1, & (m,n) = (0,0) \\ 0, & (m,n) \neq (0,0). \end{cases}$$

To prove the lemma it is then sufficient to prove that

$$\langle \ell_{i_1}^{m_1} (\ell_{i_1}^*)^{n_1} \ell_{i_2}^{m_2} (\ell_{i_2}^*)^{n_2} \cdots \ell_{i_k}^{m_k} (\ell_{i_k}^*)^{n_k} \Omega, \Omega \rangle = 0,$$

whenever $k \in \mathbb{N}$, $i_j \neq i_{j+1}$ for $1 \leq j < k$, and $(m_j, n_j) \neq (0,0)$ for $1 \leq j \leq k$.

Suppose

$$\langle \ell_{i_1}^{m_1} (\ell_{i_1}^*)^{n_1} \ell_{i_2}^{m_2} (\ell_{i_2}^*)^{n_2} \cdots \ell_{i_k}^{m_k} (\ell_{i_k}^*)^{n_k} \Omega, \Omega \rangle \neq 0.$$

Then $n_k = 0$ and it follows that $m_k \neq 0$. We know that $\ell_{i_{k-1}}^* \ell_{i_k} = 0$, since $i_{k-1} \neq i_k$. So we deduce that $n_{k-1} = 0$ and $m_{k-1} \neq 0$. Continuing in this fashion we conclude that $n_j = 0$ and $m_j \neq 0$ for all $1 \leq j \leq k$. But with $m_1 \neq 0$ we see

$$\langle \ell_{i_1}^{m_1} \cdots \ell_{i_k}^{m_k} \Omega, \Omega \rangle = \langle \ell_{i_2}^{m_2} \cdots \ell_{i_k}^{m_k} \Omega, (\ell_{i_1}^*)^{m_1} \Omega \rangle = 0,$$

and we have reached a contradiction. This shows that

$$\langle \ell_{i_1}^{m_1}(\ell_{i_1}^*)^{n_1}\ell_{i_2}^{m_2}(\ell_{i_2}^*)^{n_2}\cdots\ell_{i_k}^{m_k}(\ell_{i_k}^*)^{n_k}\Omega,\Omega\rangle = 0,$$

and concludes the proof of the lemma. □

3.2 Models for specific random variables

The main reason why we have introduced the creation operators is that it is easy to point out free semicircular elements in the algebra generated by these. And because of the Isomorphism Theorem (Theorem 2.14) we may study abstract semicircular elements by studying the concrete ones found using creation operators. Along with Lemma 3.4 the main interest is the following proposition.

Proposition 3.5. *Let H be a Hilbert space with a unit vector e, and let ℓ be the corresponding creation operator on the tensor space $\mathcal{T}(H)$. Then the element $\ell+\ell^*$ is semicircular (with respect to the usual vector state given by the vacuum vector Ω).*

Proof. Let μ denote the measure on $\mathbb{R}$ induced by $\ell+\ell^*$. Since $\ell+\ell^*$ has norm at most 2, the support of μ is contained in $[-2,2]$.

We must prove that $d\mu = \frac{1}{2\pi}\sqrt{4-t^2}\;dt$ on $[-2,2]$, or in other words that

$$\langle(\ell+\ell^*)^n\Omega,\Omega\rangle = \frac{1}{2\pi}\int_{-2}^{2} t^n\sqrt{4-t^2}\;dt, \quad n\in\mathbb{N}_0.$$

Define polynomials R_n recursively by

$$R_0(x)=1, \quad R_1(x)=x, \quad R_{n+1}(x) = xR_n(x)-R_{n-1}(x),\; n\geq 1.$$

This sequence of polynomials is a scaled version of the Chebyshev polynomials of second kind, which are the orthogonal polynomials belonging to the semicircular distribution.

One may easily prove, by induction, that $R_n(\ell+\ell^*)\Omega = e\otimes\cdots\otimes e$ (with n factors). From this is follows that

$$\langle R_n(\ell+\ell^*)\Omega,\Omega\rangle = 0,$$

when $n\geq 1$. Hence we know that (for $n\geq 1$)

$$\int_{-2}^{2} R_n(t)\;d\mu(t) = 0, \tag{3.1}$$

and it remains to see that this determines the semicircular measure. Note first that R_n has degree n, and hence the sequence $(R_n)_{n=1}^{\infty}$ spans the set of all polynomials, which is a norm dense set in $C([-2,2])$, so it suffices to prove that (3.1) holds when μ is replaced by the semicircular measure.

It is well-known (or easily established using induction and elementary trigonometry) that

$$R_n(2\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta},$$

and from this we find (substituting $t = 2\cos\theta$)

$$\int_{-2}^{2} R_n(t)\sqrt{4-t^2}\; dt = 4\int_0^{\pi} \sin((n+1)\theta)\sin\theta\; d\theta = \begin{cases} 0, & n \neq 0 \\ 2\pi, & n = 0 \end{cases}$$

which concludes the proof. □

Corollary 3.6. *Let $(\ell_i)_{i\in I}$ be a family of creation operators, and let $s_i = \ell_i + \ell_i^*$. Then $(s_i)_{i\in I}$ is a semicircular family.*

Proof. This is a direct consequence of the combination of Proposition 3.5 and Lemma 3.4 □

Corollary 3.7. *Let H be a Hilbert space with an orthogonal pair of unit vectors (e_1, e_2), and let (ℓ_1, ℓ_2) be the corresponding creation operators on the tensor space $\mathcal{T}(H)$. Then the element $\ell_1 + \ell_2^*$ is circular (with respect to the usual vector state given by the vacuum vector Ω).*

Proof. To prove that $c = \ell_1 + \ell_2^*$ is circular, we should prove that

$$z_1 = \frac{c + c^*}{\sqrt{2}}, \quad z_2 = \frac{c - c^*}{i\sqrt{2}}$$

is a semicircular family. Now,

$$z_1 = \frac{\ell_1 + \ell_2}{\sqrt{2}} + \left(\frac{\ell_1 + \ell_2}{\sqrt{2}}\right)^*, \quad z_2 = \frac{\ell_1 - \ell_2}{i\sqrt{2}} + \left(\frac{\ell_1 - \ell_2}{i\sqrt{2}}\right)^*.$$

If we let

$$g_1 = \frac{1}{\sqrt{2}}(e_1 + e_2), \quad g_2 = \frac{1}{i\sqrt{2}}(e_1 - e_2),$$

then (g_1, g_2) is an orthogonal pair of unit vectors in H with the same span as (e_1, e_2). The creation operators corresponding to (g_1, g_2) are precisely

$$k_1 = \frac{\ell_1 + \ell_2}{\sqrt{2}}, \quad k_2 = \frac{\ell_1 - \ell_2}{i\sqrt{2}}.$$

It now follows, that (z_1, z_2) is a semicircular family, and this completes the proof. □

Lemma 3.8. *The vacuum vector Ω is cyclic for the *-algebra generated by $(\ell_i)_{i\in I}$ and also for the algebra generated by $(s_i)_{i\in I}$, where $s_i = \ell_i + \ell_i^*$.*

Proof. Clearly, if suffices to prove only the last part. The tensor space $\mathcal{T}(H)$ has a basis consisting of

$$\Omega, \quad (e_i)_{i\in I}, \quad (e_{i_1}\otimes e_{i_2})_{(i_1,i_2)\in I\times I}, \quad (e_{i_1}\otimes e_{i_2}\otimes e_{i_3})_{(i_1,i_2,i_3)\in I\times I\times I}, \quad \dots$$

Let R_n be the polynomials introduced in the proof of Proposition 3.5. As noted in that proof, it is easily proved by induction that $R_n(s_i)\Omega = e_i\otimes\cdots\otimes e_i = e_i^{\otimes n}$. In the same way one may prove that $R_m(s_j)R_n(s_i)\Omega = e_j^{\otimes m}\otimes e_i^{\otimes n}$, if $i\neq j$. Continuing in this way it is clear that any basis vector can be written in the form

$$R_{n_1}(s_{i_1})R_{n_1}(s_{i_2})\cdots R_{n_r}(s_{i_r})\Omega,$$

where $i_1\neq i_2\neq\cdots\neq i_r$ and $n_1,\dots,n_r\geq 1$, $r\geq 0$. This proves that Ω is cyclic for the algebra generated by $(s_i)_{i\in I}$. ☐

Note that this implies that the GNS representation of $C^*(s_i \mid i\in I)$ with respect to ω is injective according to Remark 2.17.

Let us note an important consequence of the above results.

Proposition 3.9. *Let $\mathcal{A}$ be the von Neumann subalgebra in $B(\mathcal{T}(H))$ generated by $(s_i)_{i\in I}$. Then $(\mathcal{A},\omega|_{\mathcal{A}})\simeq(L(\mathbb{F}(I)),\tau)$, where τ is the canonical trace on $L(\mathbb{F}(I))$. In particular, ω is a faithful trace on $\mathcal{A}$.*

Proof. Both $\mathcal{A}$ and $L(\mathbb{F}(I))$ are generated by a semicircular family of size $|I|$, see Corollary 2.23. So if both GNS representations $\pi_{\omega|_{\mathcal{A}}}$ and π_τ are injective, then then proposition follows the Isomorphism Theorem. The canonical trace on any group von Neumann algebra is faithful, so π_τ is injective. The fact that $\pi_{\omega|_{\mathcal{A}}}$ is injective follows from Remark 2.17, since we just proved that Ω is cyclic for $\mathcal{A}$. ☐

Lemma 3.10. *Let $(e_i)_{i\in I}\cup(e'_j,e''_j)_{j\in J}$ be a basis for a Hilbert space H of dimension $|I|+2|J|$. Let $(\ell_i)_{i\in I}\cup(\ell'_j,\ell''_j)_{j\in J}$ be the corresponding creation operators in $(B(\mathcal{T}(H)),\omega)$. For $i\in I, j\in J$ let*

$$s_i=\ell_i+\ell_i^*,\quad c_j=\ell'_j+(\ell''_j)^*,$$

and put $\mathcal{C}=W^*(s_i,c_j\mid i\in I, j\in J)$. Then

1. *$(s_i)_{i\in I}$ is a semicircular family,*
2. *$(c_j)_{j\in J}$ is a circular family,*
3. *$(s_i)_{i\in I}\cup(c_j)_{j\in J}$ is *-free,*
4. *the vector Ω is cyclic for $\mathcal{C}$ and $\omega|_{\mathcal{C}}$ is a faithful normal trace on $\mathcal{C}$.*
5. *$(\mathcal{C},\omega|_{\mathcal{C}})\simeq(L(\mathbb{F}_{|I|+2|J|}),\tau)$, where τ is the canonical trace.*

Proof. We already know 1.– 3. from Lemma 3.4, Proposition 3.5 and Corollary 3.7. It follows from the proof of Corollary 3.7 that with

$$z'_j = \frac{c_j + c_j^*}{\sqrt{2}}, \quad z''_j = \frac{c_j - c_j^*}{i\sqrt{2}}$$

and

$$g'_j = \frac{e'_j + e''_j}{\sqrt{2}}, \quad g''_j = \frac{e'_j - e''_j}{i\sqrt{2}}$$

the set $\{e_i\}_{i\in I} \cup \{g'_j, g''_j\}_{j\in J}$ is a basis for H and $\{s_i\}_{i\in I} \cup \{z'_j, z''_j\}_{j\in J}$ are the corresponding semicircular operators. Since $c_j = \frac{1}{\sqrt{2}}(z'_j + iz''_j)$, it follows that

$$W^*(s_i, z'_j, z''_j \mid i \in I, j \in J) = W^*(s_i, c_j \mid i \in I, j \in J) = \mathcal{C}.$$

4. and 5. now follow from Lemma 3.8 and Proposition 3.9. □

3.3 Working with creation operators

We will eventually want to know the distribution of matrices with entries which are creation operators, semicircular elements, circular elements, quartercircular elements or more likely combinations of these. In order to be able to handle such matrices we need some ways of recognizing creation operators. This will be achieved in Proposition 3.13, which is taken from [7, Lemma 4.3].

Lemma 3.11. *Suppose the C^*-probability space $(\mathcal{A}, \varphi)$ is generated by two C^*-subalgebras $\mathcal{B}$ and $\mathcal{C}$. Then the set $\Lambda(\mathcal{B}^0, \mathcal{C}^0)$ of alternating products in $\mathcal{B}^0$ and $\mathcal{C}^0$ (including the trivial product) spans a dense subalgebra of $\mathcal{A}$. The same holds in the W^*-case, where dense then refers to the weak or strong operator topology.*

Proof. This is an immediate consequence of Lemma 2.10. □

Lemma 3.12. *Suppose the C^*-probability space $(\mathcal{A}, \varphi)$ is generated by two free C^*-subalgebras $\mathcal{B}$ and $\mathcal{C}$. If the GNS representation of $\mathcal{A}$ with respect to φ is injective, then the GNS representation of $\mathcal{B}$ with respect to φ is also injective. The same holds in the W^*-case.*

Proof. Suppose the GNS representation of $\mathcal{B}$ is not injective. Then by Lemma 2.16 there is a non-zero $b \in \mathcal{B}$ such that for every $x, y \in \mathcal{B}$ we have $\varphi(xby) = 0$. We claim now that $\varphi(xby) = 0$ for every $x, y \in \mathcal{A}$, which will then show that the GNS representation of $\mathcal{A}$ is not injective, and this will conclude the proof.

According to Lemma 3.11 it suffices to prove $\varphi(xby) = 0$ with $x, y \in \Lambda(\mathcal{B}^0, \mathcal{C}^0)$. Writing $x = x_n \cdots x_1$ and $y = y_1 \cdots y_m$ with x_i's and y_i's alternating between the sets $\mathcal{B}^0$ and $\mathcal{C}^0$ we have several cases to check according to whether $x = 1$, $x_1 \in \mathcal{B}^0$ or $x_1 \in \mathcal{C}^0$ and $y = 1$,

$y_1 \in \mathcal{B}^0$ or $y_1 \in \mathcal{C}^0$. Using the freeness assumption and that $\varphi(x_1 b) = \varphi(b y_1) = 0$ when $x_1, y_1 \in \mathcal{B}^0 \subseteq \mathcal{B}$, we see that $\varphi(xby) = 0$ in all nine cases. ☐

Proposition 3.13. *Let $(\mathcal{A}, \varphi)$ be a C^*-probability space containing a C^*-subalgebra $\mathcal{B}$ $(1 \in \mathcal{B})$ and a family $(\ell_i)_{i \in I}$.*

(1) *Suppose the GNS representation of $C^*(\mathcal{B}, (\ell_i)_{i\in I})$ associated to φ is injective. If the following two conditions hold:*

(A1) *$(\ell_i)_{i \in I}$ has the same *-distribution as a (free) family of creation operators.*

(A2) *$\mathcal{B}$ and $(\ell_i)_{i\in I}$ are free in $(\mathcal{A}, \varphi)$.*

Then the following two conditions also hold:

(B1) *$\ell_i^* b \ell_j = \delta_{ij}\varphi(b)1$ for $b \in \mathcal{B}$ and $i, j \in I$.*

(B2) *$\varphi(b^* \ell_i \ell_i^* b) = 0$ for $b \in \mathcal{B}$ and $i \in I$.*

(2) *Conversely,* (B1) *together with* (B2) *implies* (A1) *and* (A2)*.*

The same holds in the W^*-case.

Proof. We may as well assume that $\mathcal{A} = C^*(\mathcal{B}, (\ell_i)_{i\in I})$.

Proof of (1). It follows from Lemma 3.12 that the GNS representation of $C^*(\ell_i \mid i \in I)$ is injective, and hence (using Theorem 2.14) we see that $\ell_i^* \ell_j = \delta_{ij}$.

First we prove (B1). Let $b \in \mathcal{B}$, and write $b = b^0 + \varphi(b)1$ with $b^0 \in \mathcal{B}^0$. Since $\ell_i^*\ell_j = \delta_{ij}$ it will be enough to show that $\ell_i^* b^0 \ell_j = 0$ for all $i, j \in I$. The proof of this resembles the proof of Lemma 3.12.

Since the GNS representation of $\mathcal{A}$ is injective, we can use Lemma 2.16 to see that we should prove $\varphi(x\ell_i^* b^0 \ell_j y) = 0$ for every $x, y \in \mathcal{A}$, or even just for x and y in a set X that spans a dense subset of $\mathcal{A}$. We take as X the set $\Lambda(\mathcal{B}^0, \mathcal{C}^0)$, where $\mathcal{C} = C^*(\ell_i \mid i \in I)$. This set will do, according to Lemma 3.11. Again we have nine cases to check, but if we can show that $x_1\ell_i^*, \ell_j y_1 \in \mathcal{C}^0$ for any $x_1, y_1 \in \mathcal{C}^0$, then all nine cases will follow from the freeness of $\mathcal{B}$ and $\mathcal{C}$. And since φ is a state we need actually only check $x_1\ell_i^*$.

As noted earlier, the GNS representation of $C^*(\ell_i \mid i \in I)$ is injective, so it suffices to prove $x_1\ell_i^* \in \mathcal{C}^0$, when $(\ell_i)_{i\in I}$ are the creation operators in $(\mathcal{B}(\mathcal{T}(H)), \omega)$, and $x_1 \in \mathcal{C}^0$. And this is easy:

$$\omega(x_1\ell_i^*) = \langle x_1 \ell_i^* \Omega, \Omega\rangle = 0,$$

since $\ell_i^*\Omega = 0$.

This proves $\ell_i^* b^0 \ell_j = 0$, and hence $\ell_i^* b \ell_j = \varphi(b)\ell_i^*\ell_j = \delta_{ij}\varphi(b)1$.

To prove (B2) we write again $b = b^0 + \varphi(b)1$. Then

$$\varphi(b^*\ell_i\ell_i^* b) = \varphi((b^0)^*\ell_i\ell_i^* b^0) + \overline{\varphi(b)}\varphi(\ell_i\ell_i^* b^0) + \varphi(b)\varphi((b^0)^*\ell_i\ell_i^*) + |\varphi(b)|^2\varphi(\ell_i\ell_i^*).$$

Since $\varphi(\ell_i \ell_i^*) = 0$, it follows by freeness that all four terms are zero. This finishes the proof of (1).

Proof of (2). Let

$$\mathcal{N}_\varphi = \{a \in \mathcal{A} \mid \varphi(a^*a) = 0\}, \quad \mathcal{N}_\varphi^* = \{a^* \mid a \in N_\varphi\} = \{a \in \mathcal{A} \mid \varphi(aa^*) = 0\}.$$

Using Schwarz' inequality $|\varphi(b^*a))| \leq \varphi(a^*a)^{1/2}\varphi(b^*b)^{1/2}$ we see that

$$\varphi(ab) = 0, \quad \text{if } a \in \mathcal{N}_\varphi^* \text{ or } b \in \mathcal{N}_\varphi.$$

From (B2) we see that $\ell_i^* b \in \mathcal{N}_\varphi$ and $b\ell_i \in \mathcal{N}_\varphi^*$ whenever $b \in \mathcal{B}$ and $i \in I$. So

$$\varphi(a\ell_i^* b) = \varphi(b\ell_i a) = 0, \quad a \in \mathcal{A},\ b \in \mathcal{B}. \tag{3.2}$$

In particular,

$$\varphi(\ell_i^m (\ell_i^*)^n) = \begin{cases} 1, & (m,n) = (0,0) \\ 0, & (m,n) \neq (0,0) \end{cases}. \tag{3.3}$$

From (B1) it follows in particular that $\ell_i^* \ell_i = 1$, and so

$$\mathcal{A}_i = \operatorname{alg}(1, \ell_i, \ell_i^*) = \operatorname{span}\{\ell_i^m (\ell_i^*)^n \mid m, n \in \mathbb{N}_0\}.$$

From this and (3.3) it follows that $\mathcal{A}_i^0 = \operatorname{span}\{\ell_i^m (\ell_i^*)^n \mid (m,n) \neq (0,0)\}$, i.e $\mathcal{A}_i^0 = \operatorname{span} \mathcal{A}_i^{00}$, where $\mathcal{A}_i^{00} = \{\ell_i^m (\ell_i^*)^n \mid (m,n) \neq (0,0)\}$.

Now we set out to prove that $\{\mathcal{B}\} \cup \{\mathcal{A}_i\}_{i \in I}$ is a free family, which will prove (A2). It will be enough to prove that for any $m \geq 1$

$$\varphi(a_1' \cdots a_m') = 0$$

when each $a_k' \in \mathcal{B}^0 \cup (\bigcup_{i \in I} \mathcal{A}_i^{00})$ and neighbors come from different sets among $\mathcal{B}^0$ and $(\mathcal{A}_i^{00})_{i \in I}$. We can rewrite $a_1' \cdots a_m'$ as

$$a_1' \cdots a_m' = b_0 a_1 b_1 \cdots a_p b_p$$

where

(i) $a_k \in \mathcal{A}_{i_k}^{00}$, $1 \leq k \leq p$,

(ii) $b_k \in \mathcal{B}$,

(iii) $b_k \in \mathcal{B}^0$ if $1 \leq k \leq p-1$ and $i_k = i_{k+1}$.

The rewriting is obtained by placing the operator 1 between a_k' and a_{k+1}' if neither comes from $\mathcal{B}^0$, placing the operator 1 in front of a_1', if $a_1' \notin \mathcal{B}^0$, and placing the operator 1 at the end, if $a_m' \notin \mathcal{B}^0$. Each a_k has the form $a_k = (\ell_{i_k})^{m_k}(\ell_{i_k}^*)^{n_k}$ with $(m_k, n_k) \neq (0,0)$.

Now, assume $\varphi(b_0 a_1 b_1 \cdots a_p b_p) \neq 0$. The following argument resembles that from the proof of Lemma 3.4. If $n_p \geq 1$, then the product $b_0 a_1 b_1 \cdots a_p b_p$ ends with $\ell_{i_p}^* b_p$, so by (3.2) we get $\varphi(b_0 a_1 b_1 \cdots a_p b_p) = 0$ contrary to our assumption. Hence $n_p = 0$, and thus $m_p \neq 0$. If $n_{p-1} \geq 1$, then

$$b_0 a_1 b_1 \cdots a_p b_p = a' \ell_{i_{p-1}}^* b_{p-1} \ell_{i_p} a''$$

for some $a', a'' \in \mathcal{A}$. From (B1) we know that

$$\ell_{i_{p-1}}^* b_{p-1} \ell_{i_p} = \begin{cases} \varphi(b_{p-1})1, & i_{p-1} = i_p \\ 0, & i_{p-1} \neq i_p. \end{cases}$$

But if $i_{p-1} = i_p$, then $b_{p-1} \in \mathcal{B}^0$ by (iii), and so $\varphi(b_{p-1}) = 0$. So in any case $\ell_{i_{p-1}}^* b_{p-1} \ell_{i_p} = 0$, and thus $\varphi(b_0 a_1 b_1 \cdots a_p b_p) = 0$. It follows that $n_{p-1} = 0$ and $m_{p-1} \geq 1$.

Continuing this process we find after p steps that

$$n_1 = \cdots = n_p = 0 \quad \text{and} \quad m_1, \ldots, m_p \geq 1.$$

Then

$$\varphi(b_0 a_1 b_1 \cdots a_p b_p) = \varphi(b_0 \ell_1^{m_1} b_1 \cdots \ell_p^{m_p} b_p) = 0$$

according to (3.2). This proves that $\{\mathcal{B}\} \cup \{\mathcal{A}_i\}_{i \in I}$ is free.

Proving (A1) is now easy. Let $(k_i)_{i \in I}$ be the creation operators on $\mathcal{T}(H)$ corresponding to a basis $(e_i)_{i \in I}$ for a Hilbert space H. As $k_i^* k_i = \ell_i^* \ell_i = 1$ we see that

$$\begin{aligned} \operatorname{alg}(1, \ell_i, \ell_i^*) &= \operatorname{span}\{\ell_i^m (\ell_i^*)^n \mid m, n \in \mathbb{N}_0\} \\ \operatorname{alg}(1, k_i, k_i^*) &= \operatorname{span}\{k_i^m (k_i^*)^n \mid m, n \in \mathbb{N}_0\}. \end{aligned}$$

Then it follows from (3.3) that ℓ_i and k_i have the same *-distribution. Since $(k_i)_{i \in I}$ and $(\ell_i)_{i \in I}$ are *-free families, it follows that they have the same joint *-distribution. This proves (A1). $\square$

In the proof of (2) in Proposition 3.13 we did not make use of the assumption that the GNS representation of $C^*(\mathcal{B}, (\ell_i)_{i \in I})$ was injective. Hence we have the following corollary.

Corollary 3.14. *Let $(\mathcal{A}, \varphi)$ be a C^*-probability space, and let $(\ell_i)_{i \in I}$ be elements in $\mathcal{A}$ which satisfy*

(i) $\ell_i^* \ell_j = \delta_{ij} 1,\ i, j \in I,$

(ii) $\varphi(\ell_i \ell_i^*) = 0,\ i \in I.$

Then $(\ell_i)_{i\in I}$ has the same *-distribution as a free family of creation operators.

Lemma 3.15. *Let $(\mathcal{A}, \tau)$ be a W*-probability space. Suppose $\mathcal{B} \subseteq \mathcal{A}$ is a von Neumann subalgebra on which τ is a trace, and $(\ell_i)_{i\in I}$ is a family of creation operators inside $\mathcal{A}$ that is *-free of $\mathcal{B}$. Suppose further that the GNS representation of $W^*(\mathcal{B}, (\ell_i)_{i\in I})$ associated to φ is injective. Let $(u_i)_{i\in I}$ and $(v_i)_{i\in I}$ be families of unitaries in $\mathcal{B}$. Then $(u_i\ell_i v_i)_{i\in I}$ is *-distributed as a family of creation operators, and $\mathcal{B}$ and $(u_i\ell_i v_i)_{i\in I}$ are *-free.*

Proof. By Proposition 3.13 the following conditions hold for any $b \in \mathcal{B}$ and $i, j \in I$.

$$\ell_i^* b \ell_j = \delta_{ij}\tau(b)1, \qquad \tau(b^*\ell_i\ell_i^* b) = 0$$

Now, let $k_i = u_i\ell_i v_i$. We will prove that

$$k_i^* b k_j = \delta_{ij}\tau(b)1, \qquad \tau(b^* k_i k_i^* b) = 0.$$

With $b \in \mathcal{B}$ we have (since $u_i^*, u_j \in \mathcal{B}$)

$$k_i^* b k_j = v_i^*\ell_i^* u_i^* b u_j \ell_j v_j = \delta_{ij} v_i^* \tau(u_i^* b u_j) 1 v_j.$$

If $i \neq j$, then we are done. If $i = j$, then

$$k_i^* b k_i = v_i^* \tau(u_i^* b u_i) 1 v_i = \tau(b)1,$$

where we use that τ is a trace on $\mathcal{B}$. Further,

$$\tau(b^* k_i k_i^* b) = \tau(b^* u_i \ell_i v_i v_i^* \ell_i^* u_i^* b) = \tau(b^* u_i \ell_i \ell_i^* u_i^* b) = 0.$$

By Proposition 3.13 it follows that $(k_i)_{i\in I}$ has the same *-distribution with respect to τ as a family of creation operators. Also, $\mathcal{B}$ and $(k_i)_{i\in I}$ are *-free. The result now follows. □

Corollary 3.16. *Let $(\mathcal{A}, \tau)$ be a W*-probability space, where τ is faithful normal trace. Suppose $\mathcal{B} \subseteq \mathcal{A}$ is a von Neumann subalgebra and $(s_i)_{i\in I}$ is a semicircular family in $\mathcal{A}$ such that $\mathcal{B}$ and $(s_i)_{i\in I}$ are free. Let $(u_i)_{i\in I}$ be a family of unitaries in $\mathcal{B}$. Then $(u_i s_i u_i^*)_{i\in I}$ is a semicircular family, and $\mathcal{B}$ and $(u_i s_i u_i^*)_{i\in I}$ are free.*

Proof. It is enough to prove the corollary in the case where $s_i = \ell_i + \ell_i^*$, and $(\ell_i)_{i\in I}$ is a family of creation operators free of $\mathcal{B}$. It is then an immediate consequence of the previous lemma. □

Theorem 3.17. *Let $(\mathcal{A}, \tau)$ be a W*-probability space with τ faithful. Let $\mathcal{B}$ be a von Neumann subalgebra, and let $(s_i)_{i\in I}$ be a semicircular family such that $\mathcal{B}$ and $(s_i)_{i\in I}$ are free. Let $p \in \mathcal{B}$ be a non-zero projection. Then in $(p\mathcal{A}p, \tau(p)^{-1}\tau)$ the family $(\tau(p)^{-1/2} p s_i p)_{i\in I}$ is semicircular and free of $p\mathcal{B}p$.*

Proof. We assume that $\mathcal{A}$ is a subalgebra of $(\mathcal{B}, \tau|_{\mathcal{B}}) * (W^*(l_i \mid i \in I), \omega)$, where $(l_i)_{i\in I}$ is a free family of creation operators such that $s_i = l_i + l_i^*$. Let $h_i = \tau(p)^{-1/2} p l_i p$. We will use Proposition 3.13 to show that $(h_i)_{i\in I}$ is a family *-free of $p\mathcal{B}p$ with the same *-distribution as creation operators. With $b \in \mathcal{B}$ we find using (B1)

$$\begin{aligned} h_i^*(pbp)h_j &= (\tau(p)^{-1/2} p l_i^* p)(pbp)(\tau(p)^{-1/2} p l_j p) \\ &= \tau(p)^{-1} \delta_{ij} p \tau(pbp) p \\ &= \delta_{ij} \tau(p)^{-1} \tau(pbp) p. \end{aligned}$$

Notice that p is the unit in $p\mathcal{A}p$. Also,

$$\tau((pbp)^* h_i h_i^* (pbp)) = \tau(p)^{-1} \tau((pb^*p)(pl_i p)(pl_i^* p)(pbp))$$

and using Schwartz' inequality

$$|\tau(d^*c)| \leq \tau(c^*c)^{1/2} \tau(d^*d)^{1/2}$$

with $d^* = pb^*pl_i$ and $c = pl_i^* pbp$ we find

$$|\tau((pbp)^* h_i h_i^*(pbp))| \leq \tau(c^*c)^{1/2} \tau(pb^*pl_i l_i^* pbp)^{1/2} = 0$$

by (B2). So by Proposition 3.13 it follows that $(h_i)_{i=1}^k$ has the same *-distribution (wrt. $\tau(p)^{-1}\tau$) as a free family of creation operators and is free of $p\mathcal{B}p$. Thus $(\tau(p)^{-1/2} p s_i p)_{i=1}^k$ has the desired properties. $\square$

3.4 Matrices

Another application of Proposition 3.13 is to investigate the distribution of matrices with semicircular and circular entries (see Theorem 3.19). We start by looking at matrices with creation operators as entries.

Let $(\mathcal{A}, \varphi)$ be a C*-probability space. Consider then the C*-probability space $(M_n(\mathcal{A}), \varphi_n)$ of $n \times n$ matrices over $\mathcal{A}$ with φ_n given by

$$\varphi_n \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \frac{1}{n} \sum_{i=1}^{n} \varphi(a_{ii}).$$

If we identify $M_n(\mathcal{A}) \simeq \mathcal{A} \otimes M_n$, then $\varphi_n = \varphi \otimes \mathrm{tr}_n$. It is a simple matter to check that φ_n is in fact a state on $M_n(\mathcal{A})$. Also, φ_n is a trace, when φ is a trace, and φ_n is faithful, when φ is faithful. And the GNS representation $\pi_{\varphi_n} : M_n(\mathcal{A}) \to B(H_{\varphi_n})$ is injective if π_φ is injective.

If $(\mathcal{A}, \varphi)$ is a W*-probability space, then φ_n will be normal, so $(M_n(\mathcal{A}), \varphi_n)$ is also a W*-

probability space.

Proposition 3.18. *Let $(\mathcal{A},\varphi)$ be a C*-probability space with a C*-subalgebra $\mathcal{B}$ ($1\in\mathcal{B}$) and a family $\nu=\{\ell_{ij}^{(s)} \mid 1\le i,j\le n,\ s\in S\}$ such that $\mathcal{B}$ and ν are *-free, and ν has the same *-distribution as a family of creation operators. Here $n\in\mathbb{N}$ and S is just some set. Suppose the GNS representation of $C^*(\mathcal{B},\nu)$ with repspect to φ is injective.*

Let L_s denote the matrix in $M_n(\mathcal{A})$

$$L_s=\frac{1}{\sqrt{n}}\left(\ell_{ij}^{(s)}\right)_{i,j=1}^{n}.$$

Let $\mathcal{A}_s=\mathrm{alg}(1,L_s,L_s^*)$. Then $M_n(\mathcal{B})$ and $(\mathcal{A}_s)_{s\in S}$ are free subalgebras of $(M_n(\mathcal{A}),\varphi_n)$. Also, $(L_s)_{s\in S}$ has the same *-distribution as a family of creation operators. The same assertion holds also in the W-case.*

Proof. The proof is a straightforward application of Proposition 3.13. We will simply check that $M_n(\mathcal{B})$ and $(L_s)_{s\in S}$ satisfy the conditions (B1) and (B2) in the C*-probability space $(M_n(\mathcal{A}),\varphi_n)$. From Proposition 3.13 we know that for $b\in\mathcal{B}$

$$(\ell_{ij}^{(s)})^*b\ell_{kl}^{(t)}=\begin{cases}\varphi(b)1, & (s,i,j)=(t,k,l)\\ 0, & (s,i,j)\neq(t,k,l),\end{cases}\tag{3.4}$$

$$\varphi(b^*\ell_{ij}^{(s)}(\ell_{ij}^{(s)})^*b)=0,\quad 1\le i,j\le n,\ s\in S.\tag{3.5}$$

So with $B=(b_{ij})_{i,j=1}^n\in M_n(\mathcal{B})$ we see that the (i,j) entry in $L_s^*BL_t$ is

$$(L_s^*BL_t)_{ij}=\frac{1}{n}\sum_{p,q=1}^{n}(\ell_{pi}^{(s)})^*b_{pq}\ell_{qj}^{(t)}.$$

From (3.4) we see that if $s\neq t$, then $L_s^*BL_t=0$, and if $s=t$, then $L_s^*BL_s$ is diagonal with

$$(L_s^*BL_s)_{ii}=\frac{1}{n}\sum_{p=1}^{n}(\ell_{pi}^{(s)})^*b_{pp}\ell_{pi}^{(s)}=\frac{1}{n}\sum_{p=1}^{n}\varphi(b_{pp})1=\varphi_n(B)1.$$

Thus $L_s^*BL_s=\varphi_n(B)\mathbf{1}$, where $\mathbf{1}$ is the identity in $M_n(\mathcal{A})$. So we have proved

$$L_s^*BL_t=\delta_{st}\varphi_n(B)\mathbf{1},\quad B\in M_n(\mathcal{B}).$$

We continue with $B \in M_n(\mathcal{B})$ and get

$$(B^* L_s L_s^* B)_{ii} = \frac{1}{n} \sum_{p,q,r=1}^{n} b_{pi}^* \ell_{pq}^{(s)} (\ell_{rq}^{(s)})^* b_{ri},$$

$$\varphi_n(B^* L_s L_s^* B) = \frac{1}{n^2} \sum_{p,q,r,i=1}^{n} \varphi(b_{pi}^* \ell_{pq}^{(s)} (\ell_{rq}^{(s)})^* b_{ri}).$$

Using Schwarz' inequality $|\varphi(d^*c)| \leq \varphi(c^*c)^{1/2}\varphi(d^*d)^{1/2}$ with $c = (\ell_{rq}^{(s)})^* b_{ri}$ and $d = (\ell_{pq}^{(s)})^* b_{pi}$ we conclude using (3.5) that

$$\varphi_n(B^* L_s L_s^* B) = 0.$$

In the light of Proposition 3.13 this ends the proof. □

The following theorem shows that matrices with semicircular and circular entries gives semicircular elements in the matrix algebra. This is very useful when we want to investigate the situation when semicircular elements are cut down by projections. This is central in the definition of the interpolated free group factors.

Theorem 3.19. *Let $n \in \mathbb{N}$ and let $(\mathcal{A}, \varphi)$ be a W^*-probability space containing a von Neumann subalgebra $\mathcal{B}$ ($1 \in \mathcal{B}$), a semicircular family $\mu_1 = \{x_i^{(s)} \mid 1 \leq i \leq n, s \in S\}$ and a circular family $\mu_2 = \{y_{ij}^{(s)} \mid 1 \leq i < j \leq n, s \in S\}$, such that $\{\mathcal{B}, \mu_1, \mu_2\}$ is *-free. Suppose the GNS representation of $W^*(\mathcal{B}, \mu_1, \mu_2)$ with respect to φ is injective. Put*

$$X_s = \frac{1}{\sqrt{n}} \begin{pmatrix} x_1^{(s)} & y_{12}^{(s)} & \cdots & y_{1n}^{(s)} \\ (y_{12}^{(s)})^* & x_2^{(s)} & \cdots & y_{2n}^{(s)} \\ \vdots & \vdots & \ddots & \vdots \\ (y_{1n}^{(s)})^* & (y_{2n}^{(s)})^* & \cdots & x_n^{(s)} \end{pmatrix}, \tag{3.6}$$

and let as usual φ_n be the state $\varphi \otimes \mathrm{tr}_n$ on $M_n(\mathcal{A})$. Then $(X_s)_{s \in S}$ is a semicircular family, and $M_n(\mathcal{B})$ and $(X_s)_{s\in S}$ are free in $(M_n(\mathcal{A}), \varphi_n)$.

Proof. As usual we may appeal to the Isomorphism Theorem to see that we may as well assume that $\mathcal{A}$ contains creation operators $\nu = \{\ell_{ij}^{(s)} \mid 1 \leq i, j \leq n,\ s \in S\}$ such that

$$x_i^{(s)} = \ell_{ii}^{(s)} + (\ell_{ii}^{(s)})^*, \quad y_{ij}^{(s)} = \ell_{ij}^{(s)} + (\ell_{ji}^{(s)})^*,$$

and ν and $\mathcal{B}$ are *-free.

The theorem then follows from Proposition 3.18. With $L_s = \frac{1}{\sqrt{n}}(\ell_{ij}^{(s)})_{i,j=1}^n$ we get that $(L_s)_{s\in S}$ is a family in $(M_n(\widetilde{\mathcal{A}}), \widetilde{\varphi}_n)$ with the same *-distribution as a family of creation operators. Further, $M_n(\mathcal{B})$ and $(L_s)_{s\in S}$ are *-free. Hence $(L_s + L_s^*)_{s\in S}$ is a semicircular family free of $M_n(\mathcal{B})$. Now, notice that $L_s + L_s^* = X_s$. This completes the proof. □

There is statement similar to Theorem 3.19 about matrices with quartercircular elements in some of the entries. Since we will only need it, when we prove that $R * L(\mathbb{F}_n) = L(\mathbb{F}_{n+1})$ we postpone it until then.

Proposition 3.20. *Let $(\mathcal{M}, \varphi)$ be a W*-probability space with φ faithful. Let $\nu_1 = (X_s)_{s\in S}$ be a semicircular family and $\mathcal{B}$ be a von Neumann subalgebra of $\mathcal{M}$ containing a system of matrix units $\nu_2 = (e_{ij})_{i,j=1}^n$ such that ν_1 and $\mathcal{B}$ are free. Then in the W*-probability space $(e_{11}\mathcal{M}e_{11}, n\varphi)$ the family $\omega_1 = (n^{1/2}e_{1i}X_s e_{i1} \mid 1 \le i \le n, s \in S)$ is semicircular, the family $\omega_2 = (n^{1/2}e_{1i}X_s e_{j1} \mid 1 \le i < j \le n, s \in S)$ is circular, and $\{\omega_1, \omega_2, e_{11}\mathcal{B}e_{11}\}$ are *-free.*

Proof. Let $\mathcal{D} = e_{11}\mathcal{B}e_{11}$. We use Theorem 3.19 (and the notation from that theorem) and let $\mathcal{C} = W^*(\mu_1, \mu_2)$. Then from the theorem we get a semicircular family $(X_s)_{s\in S}$ in $M_n(\mathcal{A})$, where $\mathcal{A} = \mathcal{D} * \mathcal{C}$. And the family $(X_s)_{s\in S}$ is free of $\mathcal{B} = M_n(\mathcal{D})$. So in the case where $\mathcal{M} = M_n(\mathcal{A})$, the proposition is clearly true, because then $\omega_1 = \mu_1$, $\omega_2 = \mu_2$ and $e_{11}\mathcal{B}e_{11} = \mathcal{D}$.

In general we appeal to the Isomorphism Theorem. □

Proposition 3.21. *Let $m \in \mathbb{N}$ and let $(\mathcal{M}, \varphi)$ be a W*-probability space with φ a faithful. Let $\nu_1 = (X_s)_{s\in S}$ be a semicircular family and let $\nu_2 = (e_{ij})_{i,j=1}^{2m}$ be a $2m \times 2m$ system of matrix units in $\mathcal{M}$ such that ν_1 and ν_2 are free. Consider the 2×2 system of matrix units*

$$E_{11} = \sum_{k=1}^{m} e_{kk}, \quad E_{12} = \sum_{k=1}^{m} e_{k,k+m}, \quad E_{21} = E_{12}^*, \quad E_{22} = 1 - E_{11}.$$

Then in the W-probability space $(E_{11}\mathcal{M}E_{11}, 2\varphi)$ the families*

$$\{E_{11}X_sE_{11}\}_{s\in S}, \quad \{E_{12}X_sE_{21}\}_{s\in S}, \quad \{E_{11}X_sE_{21}\}_{s\in S}, \quad \{e_{ij} \mid 1 \le i, j \le m\}$$

are *-free.

Proof. First apply proposition 3.20 to split each X_s into semicircular and circular entries, and then apply Theorem 3.19 to paste each $E_{1a}X_sE_{b1}$ from these. We leave the details. □

CHAPTER 4

Applications on von Neumann algebras

In this chapter we apply the techniques developed in the previous chapters to the isomorphisms $R * R \simeq R * L(\mathbb{Z}) \simeq L(\mathbb{Z}) * L(\mathbb{Z})$. Along the way we need the result about the polar decomposition of a circular variable, and we have included a proof of this result as well. Since it will take only a minimal effort at this point to prove $L(\mathbb{F}_k) * M_n \simeq L(\mathbb{F}_{n^2k}) \otimes M_n$, we will also prove this result.

Before we can really get use of the matrix model for semicircular elements constructed in the previous chapter we need a few helpful results.

4.1 Generators, matrix algebras and corner algebras

This section has nothing to do really with free probability theory. It contains some useful lemmas concerning generators of various algebras. These lemmas will be useful from time to time, and we have collected them here for a better overview.

Lemma 4.1. *Let $\mathcal{A}$ be a C^*-algebra containing a system of matrix units $(e_{ij})_{i,j=1}^n$. Then*

$$\mathcal{A} \simeq M_n(e_{11}\mathcal{A}e_{11}).$$

If $\mathcal{B}$ and $\mathcal{C}$ are C^*-subalgebras of $\mathcal{A}$ with a common subalgebra $\mathcal{D}$, and $e_{11}\mathcal{B}e_{11}$ is isomorphic to $e_{11}\mathcal{C}e_{11}$ with an isomorphism that is the identity on $e_{11}\mathcal{D}e_{11}$, then $\mathcal{B} \simeq \mathcal{C}$ with an isomorphism that is the identity on $\mathcal{D}$.

Proof. Straightforward. $\square$

Lemma 4.2. *Let $\mathcal{A} \subseteq B(H)$ be a von Neumann algebra with a projection $p \in \mathcal{A}$. Suppose Ω is a subset of $B(H)$ such that $\Omega = p\Omega p$, and let $\mathcal{B}$ be the von Neumann algebra generated by $\mathcal{A}$ and Ω. Then*

$$p\mathcal{B}p = W^*(p\mathcal{A}p \cup \Omega).$$

Proof. Note that our assumptions imply that $\Omega^* = p\Omega^* p$. The linear span of elements of the form

$$a_0\omega_1 a_1\omega_2 a_2 \cdots \omega_n a_n$$

with $n \in \mathbb{N}, a_i \in \mathcal{A}, \omega_i \in \Omega \cup \Omega^*$ is weakly dense in $\mathcal{B}$. Hence the linear span of elements of the form

$$pa_0\omega_1 a_1\omega_2 a_2 \cdots \omega_n a_n p$$

is weakly dense in $p\mathcal{B}p$. Writing $\omega_i = p\omega_i' p$ (as we may by our assumptions) and moving parentheses we find that the linear span of elements of the form

$$(pa_0p)\omega_1'(pa_1p)\omega_2' \cdots \omega_n'(pa_np)$$

is weakly dense in $p\mathcal{B}p$. This proves the lemma. □

Lemma 4.3. *Let $\mathcal{A}$ be a *-algebra containing a system of matrix units $(e_{ij})_{i,j=1}^n$. Let $\Omega \subseteq \mathcal{A}$ be a set which generates a $\mathcal{A}$ as a *-algebra. Then $e_{11}\mathcal{A}e_{11}$ is generated as a *-algebra (with unit e_{11}) by the set $\widetilde{\Omega} = \bigcup_{i,j=1}^n e_{1i}\Omega e_{j1}$.*

The same assertion holds also in the C-case and W*-case.*

Proof. We consider the algebraic case first. We need only prove that, if $x = x_1 \cdots x_k$ is a product with each $x_j \in \Omega \cup \Omega^*$, then $e_{11}xe_{11}$ is a sum of elements of which are products of elements from $\widetilde{\Omega} \cup \widetilde{\Omega}^*$. For this, write $1 = \sum_i e_{i1}e_{1i}$, and insert this between the factors x_j and x_{j+1} for all $1 \le j \le n-1$. Now, distribute and regroup.

The C*-case and W*-case follow from the algebraic case. If $\mathcal{A}_0$ denotes the unital *-algebra generated by Ω, then one should simply show that $e_{11}\mathcal{A}_0e_{11}$ is dense in $e_{11}\mathcal{A}e_{11}$, and this follows immediately from continuity of the map $x \mapsto e_{11}xe_{11}$ (in norm and weak operator toplogy, respectively). □

The following lemma is dual to the previous in that it describes the generators of a matrix algebra instead of a corner algebra. We only consider the W*-case.

Lemma 4.4. *Let $\mathcal{A}$ be a von Neumann algebra, and let $\Omega = \{z_{ij}^{(s)} \mid 1 \le i,j \le n,\ s \in S\}$ be a subset of $\mathcal{A}$. Let $\mathcal{B} = W^*(\Omega)$. With $Z_s = (z_{ij}^{(s)})_{i,j=1}^n \in M_n(\mathcal{A})$ we let $\mathcal{C}$ denote the von Neumann subalgebra in $M_n(\mathcal{A})$ generated by the standard matrix units $\{e_{ij} \mid 1 \le i,j \le n\}$ and $\{Z_s\}_{s\in S}$. Then $\mathcal{C} = M_n(\mathcal{B})$.*

Proof. Obviously, $\mathcal{C} \subseteq M_n(\mathcal{B})$. Let $\mathcal{B}_0$ be the unital *-algebra generated by Ω so that $\mathcal{B}$ is the weak operator closure of $\mathcal{B}_0$. For all i, j we see that

$$e_{1i} Z_s e_{j1} = \begin{pmatrix} z_{ij}^{(s)} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

and it follows that for every $b \in \mathcal{B}_0$

$$\begin{pmatrix} b & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathcal{C}.$$

And since $\mathcal{C}$ is weakly closed the same holds for $b \in \mathcal{B}$. Now, observe that for any $B \in M_n(\mathcal{B})$ with $B = (b_{ij})_{i,j=1}^n$ we have

$$B = \sum_{i,j=1}^{n} e_{i1} \begin{pmatrix} b_{ij} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} e_{1j},$$

so $B \in \mathcal{C}$, and hence $M_n(\mathcal{B}) \subseteq \mathcal{C}$. $\square$

Remark 4.5. When dealing with W*-probability spaces we will use the following conventions.

- Group von Neumann algebras $L(G)$ are equipped with their canoncial trace, which is the vector state given by $\delta_e \in \ell^2(G)$, where e is the unit in G.
- Finite factors, such as the matrix algebras M_n and the hyperfinite II_1 factor R, will be equipped with their unique traces.
- A tensor product $\mathcal{A} \otimes \mathcal{B}$ will have the tensor product state $\varphi_{\mathcal{A}} \otimes \varphi_{\mathcal{B}}$ of the given states on $\mathcal{A}$ and $\mathcal{B}$.
- A free product $A * B$ will have the free product state $\varphi_{\mathcal{A}} * \varphi_{\mathcal{B}}$ of the given states on $\mathcal{A}$ and $\mathcal{B}$.
- For a W*-probability space $(\mathcal{A}, \varphi)$, the compression $p\mathcal{A}p$ of $\mathcal{A}$ by some non-zero projection $p \in \mathcal{A}$ will have the state $\varphi(p)^{-1}\varphi$ restricted to $p\mathcal{A}p$.

Also, an isomorphism of W*-probability spaces is an isomorphism of the von Neumann algebras that preserves the given states.

4.2 Free products of $L(\mathbb{F}_n)$ with matrix algebras

As an application of the matrix model from the previous chapter we are now able to prove the following theorem.

Theorem 4.6.

$$L(\mathbb{F}_k) * M_n(\mathbb{C}) \simeq L(\mathbb{F}_{n^2k}) \otimes M_n(\mathbb{C})$$

where $k \in \mathbb{N} \cup \{\infty\}$, *and* $n \in \mathbb{N}$.

Proof. From Lemma 3.10 we know that $L(\mathbb{F}_{n^2k})$ is generated by a free family of kn semicircular elements and $\frac{1}{2}(kn(n-1))$ circular elements. Let $\{x_i^{(s)} \mid 1 \leq i \leq n,\ 1 \leq s \leq k\}$ be the semicircular elements, and let $\{y_{ij}^{(s)} \mid 1 \leq i < j \leq n,\ 1 \leq s \leq k\}$ be the circular elements. From Lemma 4.4 we see that $L(\mathbb{F}_{n^2k}) \otimes M_n(\mathbb{C}) = M_n(L(\mathbb{F}_{n^2k}))$ is generated as a von Neumann algebra by the matrix units $\{e_{ij}\}_{i,j=1}^n$ together with the elements

$$X_s = \frac{1}{\sqrt{n}} \begin{pmatrix} x_1^{(s)} & y_{12}^{(s)} & \cdots & y_{1n}^{(s)} \\ (y_{12}^{(s)})^* & x_2^{(s)} & \cdots & y_{2n}^{(s)} \\ \vdots & \vdots & \ddots & \vdots \\ (y_{1n}^{(s)})^* & (y_{2n}^{(s)})^* & \cdots & x_n^{(s)} \end{pmatrix}, \quad 1 \leq s \leq k.$$

It follows from Theorem 3.19 (with $\mathcal{B} = \mathbb{C}$), that $M_n(\mathbb{C})$ and $\{X_1\}, \ldots, \{X_k\}$ are free, and $(X_1, \ldots, X_k)$ is a semicircular family with respect to the trace $\tau_n = \tau \otimes \mathrm{tr}_n$, where τ is the canonical trace on $L(\mathbb{F}_{n^2k})$. Since τ is faithful, τ_n is faithful, and it follows that $W^*(X_1, \ldots, X_k) \simeq L(\mathbb{F}_k)$. Since $M_n(\mathbb{C})$ and $W^*(X_1, \ldots, X_k)$ are free von Neumann subalgebras of $L(\mathbb{F}_{n^2k}) \otimes M_n(\mathbb{C})$ which generate $L(\mathbb{F}_{n^2k}) \otimes M_n(\mathbb{C})$, we have

$$L(\mathbb{F}_{n^2k}) \otimes M_n(\mathbb{C}) \simeq M_n(\mathbb{C}) * W^*(X_1, \ldots, X_k) \simeq M_n(\mathbb{C}) * L(\mathbb{F}_k).$$

This completes the proof. □

4.3 The polar decomposition of a circular element

As the headline suggests this section deals with the polar decomposition of a circular element. Theorem 4.11 about the polar decomposition of a circular variable was first proved by Voiculescu [16] using Gaussian random matrices. The proof given here avoids the random matrices and uses creation operators instead. This was first done by Banica [1].

In view of Theorem 2.14 the following lemma tells us that the *-distribution of the polar decomposition of a random variable x in a W*-probability space $(\mathcal{A}, \varphi)$ is determined by the *-distribution of x (if the GNS representation π_φ is injective).

Lemma 4.7. *Let* $\Gamma : \mathcal{A} \to \mathcal{B}$ *be a* **-isomorphism between von Neumann algebras. If* $a \in \mathcal{A}$

has polar decomposition $a = vp$, then $v, p \in \mathcal{A}$ and $\Gamma(a)$ has polar decomposition $\Gamma(a) = \Gamma(v)\Gamma(p)$.

We consider the lemma as well-known and omit the proof.

Proposition 4.8. *Let $(\mathcal{A}, \varphi)$ be a C*-probability space with a Haar unitary u and a semicircular element s. If the GNS representation associated to φ is injective, and $\{u, s\}$ is *-free, then us is circular.*

Proof. We may assume that $\mathcal{A}$ is generated by $\{u, s\}$, and by imbedding $(\mathcal{A}, \varphi)$ into a larger space $(\widetilde{\mathcal{A}}, \widetilde{\varphi})$ we may assume that $s = \ell + \ell^*$, where ℓ is a creation operator such that $\{\ell, u\}$ is *-free, and $\widetilde{\mathcal{A}}$ is generated by $\{\ell, u\}$. We may still assume that the GNS representation of $\widetilde{\mathcal{A}}$ with respect to $\widetilde{\varphi}$ is injective. We will use Corollary 3.14 to show that $(u\ell, \ell u^*)$ has distribution as a (free) pair of creation operators. This will conclude the proof since then

$$us = u\ell + u\ell^* = u\ell + (\ell u^*)^*$$

is circular according to Corollary 3.7.

According to Corollary 3.14 we should simply prove that

$$\ell^* u^* u\ell = u\ell^* \ell u^* = 1, \quad \ell^* u^* \ell u^* = 0, \quad \varphi(u\ell\ell^* u^*) = \varphi(\ell u^* u\ell^*) = 0. \tag{4.1}$$

That $\ell^* u^* u\ell = u\ell^* \ell u^* = 1$ is clear, since $\ell^*\ell = 1$ and u is unitary. Also, $\varphi(u\ell\ell^* u^*) = 0$ by freeness, and $\varphi(\ell u^* u\ell^*) = \varphi(\ell\ell^*) = 0$.

Let $z = \ell^* u^* \ell u^*$. To prove $z = 0$, we prove (see Lemma 2.16) that $\varphi(xzy) = 0$ for every $x, y \in \widetilde{\mathcal{A}}$. From Lemma 3.11 we see that we may assume $x, y \in \Lambda(\mathcal{B}^0, \mathcal{C}^0)$, where $\mathcal{B} = W^*(u)$, $\mathcal{C} = W^*(\ell)$. By linearity and continuity of φ we may even assume that $x, y \in \Lambda(\mathcal{B}_0, \mathcal{C}_0)$, where

$$\mathcal{B}_0 = \{u^n \mid n \in \mathbb{Z} \setminus \{0\}\}, \quad \mathcal{C}_0 = \{\ell^m (\ell^*)^n \mid (m, n) \neq (0, 0)\}.$$

Writing x and y as alternating products, $x = x_n \cdots x_1$ and $y = y_1 \cdots y_m$, we have to check different cases according to whether x_1 and y_1 are 1, in $\mathcal{B}_0$ or in $\mathcal{C}_0$. Since $x_1\ell^* \in \mathcal{C}_0$ and $\ell y_1 \in \mathcal{C}_0$ for every $x_1, y_1 \in \mathcal{C}_0$, we get $\varphi(xzy) = 0$ by freeness in all nine cases. This proves (4.1). □

Proposition 4.9. *Let $s = dq$ be the polar decomposition of a semicircular element in a W*-probability space $(\mathcal{A}, \varphi)$, where the GNS representation associated to φ is injective. Then (q, d) are *-independent (see Definition 2.6), q is quartercircular, and d is a unitary with *-distribution given by $\varphi(d^{2n}) = 1$ and $\varphi(d^{2n+1}) = 0$, $n \in \mathbb{Z}$.*

Proof. It is enough to look at a specific semicircular variable. Take $s(t) = t$ in the space $(L^\infty([-2, 2]), d\mu)$, where μ is the semicircle distribution. Then $q(t) = |t|$, and $d(t) = \mathrm{sign}(t)$. Clearly, these elements satisfy the conditions claimed in the proposition. □

Corollary 4.10. *Let (q, d) be *-independent in a W^*-probability space $(\mathcal{A}, \varphi)$, where the GNS representation associated to φ is injective. If q is quartercircular, and d is a unitary with *-distribution given by $\varphi(d^{2n}) = 1$ and $\varphi(d^{2n+1}) = 0$, then $s = dq$ is semicircular.*

Proof. Before we can apply Theorem 2.14 and the previous proposition we should simply note, that the joint *-distribution of (q, d) is determined by the *-distributions of q and d, since they are *-independent. □

Theorem 4.11. *Let $(\mathcal{A}, \varphi)$ be a W^*-probability space, where the GNS representation associated to φ is injective.*

(i) *If $x = vb$ is the polar decomposition of a circular random variable in $\mathcal{A}$, then v is a Haar unitary, b is quartercircular and $\{v, b\}$ is a *-free pair.*

(ii) *Conversely, if $\{v, b\}$ is a *-free pair in $\mathcal{A}$, where v is a Haar unitary, and b is quartercircular, then vb is circular.*

Proof. If suffices to prove (i) for a particular circular variable. Consider as $\mathcal{A}$ the group von Neumann algebra $L(G)$, where $G = \mathbb{Z} * (\mathbb{Z} \times \mathbb{Z}/2)$, and let φ be the canonical trace on $L(G)$. Let u, t, d denote the images of $1 \in \mathbb{Z}$, $(1, [0]) \in \mathbb{Z} \times \mathbb{Z}/2$ and $(0, [1]) \in \mathbb{Z} \times \mathbb{Z}/2$. We consider u, t, d as elements of G under the canonical imbeddings into G and also as elements of $L(G)$ by the imbedding of G into the unitary group of $L(G)$ using the left regular representation. Note that

$$L(G) = L(\mathbb{Z}) * L(\mathbb{Z} \times \mathbb{Z}/2) = L(\mathbb{Z}) * (L(\mathbb{Z}) \otimes L(\mathbb{Z}/2)).$$

Then $W^*(t) \simeq L(\mathbb{Z})$, and there is a quartercircular element $q \in W^*(t)$. Then (q, d) are independent, and d is a unitary with distribution given by $\varphi(d^{2n}) = 1$ and $\varphi(d^{2n+1}) = 0$. So by Corollary 4.10 the element dq is semicircular. Since u is a Haar unitary, it follows from Proposition 4.8, that $c = udq$ is circular. With this particular c we see that $|c| = q$, which is quartercircular. Also, the polar part of c is ud, which is a Haar unitary, by *-freeness of $\{u, d\}$.

It remains to be seen that $\{q, ud\}$ is *-free. It will suffice to prove that $\{t, ud\}$ is *-free, and so we should simply show that a non-trivial alternating product in $\{t^n \mid n \in \mathbb{Z} \setminus \{0\}\}$ and $\{(ud)^n \mid n \in \mathbb{Z} \setminus \{0\}\}$ has trace zero. Regrouping gives a non-trivial alternating product in $\{u, u^*\}$ and

$$S = \{d\} \cup \{t^n \mid n \in \mathbb{Z} \setminus \{0\}\} \cup \{dt^n \mid n \in \mathbb{Z} \setminus \{0\}\} \cup \{t^n d \mid n \in \mathbb{Z} \setminus \{0\}\}.$$

Since d and t are *-independent, any element of S has trace zero, it follows by freeness that a non-trivial alternating product in $\{u, u^*\}$ and S has trace zero.

(ii) This is an immediate consequence of the Isomorphism Theorem together with (i). □

4.4 Free products of $L(\mathbb{F}_n)$ with the hyperfinite II_1 factor

In this section we investigate the free product $L(\mathbb{F}_n) * R$. Recall Theorem 4.6

$$L(\mathbb{F}_{n^2k}) \otimes M_n \simeq L(\mathbb{F}_k) * M_n.$$

In [16] Voiculescu proved

$$L(\mathbb{F}_{n^2k+1}) \otimes M_n \simeq L(F_{k+1})$$

(and we will prove a more general result in Theorem 5.6), so one might expect something similar for $L(\mathbb{F}_{n^2k}) \otimes M_n$ when n is "large". Since R is the direct limit of M_{2^n} as $n \to \infty$, this leads us to expect $L(\mathbb{F}_k) * R \simeq L(\mathbb{F}_{k+1})$.

Also, in [4] Dykema proved that

$$L(\mathbb{F}_{n^2k}) \otimes M_n \simeq L(\mathbb{F}_k) * L^\infty(\{0\}\cup\,]1/n, 1[, \mu),$$

where μ is the probability measure having mass $1/n$ on $\{0\}$ and Lebesgue measure on the open interval. If $L^\infty(\{0\}\cup\,]1/2^n, 1[)$ "approaches" $L^\infty([0,1]) \simeq L(\mathbb{Z})$ as $n \to \infty$, then Dykema's result also suggests that $L(\mathbb{F}_k) * R \simeq L(\mathbb{F}_{k+1})$. As Dykema proved [4] this is indeed the case, although the proof does not follow this intuitive idea of letting $L^\infty(\{0\}\cup\,]1/2^n, 1[)$ approach $L(\mathbb{Z})$.

Before we prove $L(\mathbb{F}_k) * R \simeq L(\mathbb{F}_{k+1})$ we need the following proposition about matrices with semicircular and quartercircular elements. A more general version was proved by Dykema [4], but we only need the case with 2×2 matrices, so we will stick to that to ease notation.

Proposition 4.12. *Let $(\mathcal{M}, \varphi)$ be a W^*-probability space with φ a faithful normal trace, and suppose $\mathcal{M}$ contains a *-free family (s_1, s_2, q, a_1, a_2), where s_1 and s_2 are semicircular, q is quartercircular, and a_1 and a_2 are normal. Let*

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} s_1 & q \\ q & s_2 \end{pmatrix}, \qquad C = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

Then S is semicircular and the pair $\{S, C\}$ is *-free in $(M_2(\mathcal{M}), \varphi_2)$.

Proof. If $\{u, v\}$ is a set of generators for $\mathbb{F}_2$, and we use the identification $\mathbb{F}_2 \subseteq L(\mathbb{F}_2)$, then $\{v, u^*vu\}$ is a *-free family of Haar unitaries. Since $L(\mathbb{Z})$ is generated by a Haar unitary, it follows from Lemma 2.24 that there are bounded measurable functions h_1, h_2 on the unit circle such that $(h_1(v), h_2(u^*vu))$ has the same *-distribution as (a_1, a_2). By changing $\mathcal{M}$ if necessary and using the Isomorphism Theorem we can assume that $a_1 = h_1(v)$ and $a_2 = h_2(u^*vu)$, where u, v are Haar unitaries such that (s_1, s_2, q, u, v) is *-free.

Let U be the diagonal matrix $U = \mathrm{diag}(1, u)$. Clearly, U is a unitary, so it will suffice to prove that USU^* is semicircular and that $\{USU^*, UCU^*\}$ is *-free.

Notice that $ua_2u^* = uh_2(u^*vu)u^* = h_2(v)$. Doing the matrix multiplication we find

$$USU^* = \frac{1}{\sqrt{2}}\begin{pmatrix} s_1 & qu^* \\ uq & us_2u^* \end{pmatrix}, \qquad UCU^* = \begin{pmatrix} h_1(v) & 0 \\ 0 & h_2(v) \end{pmatrix}.$$

It follows from Corollary 3.16 that (s_1, us_2u^*) is a semicircular family free of $\{q, u, v\}$. By Theorem 4.11 we see that uq is circular, and from Theorem 3.19 we see that USU^* is semicircular and free of UCU^*. □

Theorem 4.13. *Let R denote the hyperfinite II_1 factor. For $1 \leq n \leq \infty$*

$$L(\mathbb{F}_n) * R \simeq L(\mathbb{F}_{n+1}).$$

Proof. By associativity of $*$ it suffices to prove the theorem when $n = 1$, i.e. that $L(\mathbb{Z}) * R$ is isomorhpic to $L(\mathbb{F}_2)$. Let X be a semicircular variable generating $L(\mathbb{Z})$. Let R be generated by $\bigcup_{k\geq 1} \Lambda_k$ where Λ_k is the system of $2^k \times 2^k$ matrix units

$$\Lambda_k = \{e(p_1 \cdots p_k, q_1 \cdots q_k; 2^k) \mid p_i, q_i \in \{0, 1\}\},$$

where $p_1 \cdots p_k$ and $q_1 \cdots q_k$ are binary expansions, such that

$$e(p_1 \cdots p_k, q_1 \cdots q_k; 2^k) = e(p_1 \cdots p_k 0, q_1 \cdots q_k 0; 2^{k+1}) + e(p_1 \cdots p_k 1, q_1 \cdots q_k 1; 2^{k+1}).$$

Let $\mathcal{M} = L(\mathbb{Z}) * R$ with faithful trace φ.

By $\mathcal{A}_k$ we will denote the algebra $e(0 \cdots 0, 0 \cdots 0; 2^k)\mathcal{M}e(0 \cdots 0, 0 \cdots 0; 2^k)$ containing

$$f(p_1 \cdots p_k; k) = 2^{k/2} e(0 \cdots 0, p_1 \cdots p_k; 2^k) X e(p_1 \cdots p_k, 0 \cdots 0; 2^k)$$
$$g(p_1 \cdots p_k, q_1 \cdots q_k; k) = 2^{k/2} e(0 \cdots 0, p_1 \cdots p_k; 2^k) X e(q_1 \cdots q_k, 0 \cdots 0; 2^k).$$

From Proposition 3.20

$$\Omega_1 = \{f(p_1 \cdots p_k; k) \mid p_i \in \{0, 1\}\}$$

is a semicircular family, and

$$\Omega_2 = \{g(p_1 \cdots p_k, q_1 \cdots q_k; k) \mid p_i, q_i \in \{0, 1\},\ p_1 \cdots p_k < q_1 \cdots q_k\}$$

is a circular family in $(\mathcal{A}_k, 2^k\varphi)$ such that $\{\Omega_1, \Omega_2\}$ is *-free. With $a \in \mathcal{A}_k$ we use the notation

$$a \otimes e(p_1 \cdots p_k, q_1 \cdots q_k; 2^k) = e(p_1 \cdots p_k, 0 \cdots 0; 2^k) a e(0 \cdots 0, q_1 \cdots q_k; 2^k).$$

Let $g(0 \cdots 0, 0 \cdots 01; k) = v(k)b(k)$ be the polar decomposition. It follows from Theorem 4.11

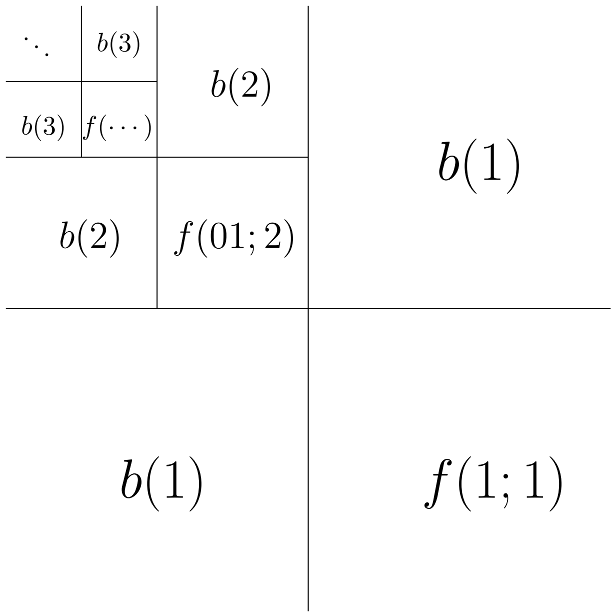

Figure 4.1: A picture of H in the proof of Theorem 4.13.

that $b(k)$ is quartercircular, $v(k)$ is a Haar unitary, and $(v(k), b(k))$ is *-free in $\mathcal{A}_k$. Define

$$\begin{aligned} H &= \sum_{k=1}^{\infty} 2^{-k/2} \Big(f(0\cdots 01;k) \otimes e(0\cdots 01, 0\cdots 01; 2^k) \\ &\quad + b(k) \otimes (e(0\cdots 0, 0\cdots 01; 2^k) + e(0\cdots 01, 0\cdots 0; 2^k)) \Big) \\ W &= \sum_{k=1}^{\infty} \frac{1}{k} v(k) \otimes e(0\cdots 01, 0\cdots 01; 2^k). \end{aligned}$$

(see Figure 4.1). The sums defining W and H are norm convergent.

We will show that $\{H, W\}$ is *-free and generate $\mathcal{M}$. Further, we show that H is semicircular and that W generates a copy of $L(\mathbb{Z})$. Then H and W will generate two free copies of $L(\mathbb{Z})$ inside $\mathcal{M}$, and this will complete the proof.

First we argue, that $\{H, W\}$ generates $\mathcal{M}$. Taking spectral projections of W we see that $\chi_{\{|z|=1/k\}}(W) = e(0\cdots 01, 0\cdots 01; 2^k)$, and adding these diagonal matrix units we get (as a strong operator limit) the operators $e(0\cdots 0, 0\cdots 0; 2^k)$. With all these diagonal matrix units we can extract from H the operators $b(k) \otimes e(0\cdots 0, 0\cdots 01; 2^k)$. As polar parts of these we get the operators $e(0\cdots 0, 0\cdots 01; 2^k)$. Together with the diagonal matrix units from before, these generate all the matrix units $\bigcup_k \Lambda_k$, hence R. With the matrix units we can decompose W and H and build X from these parts. Since X together with the matrix units generates $\mathcal{M}$, this shows that $\{H, W\}$ generates $\mathcal{M}$.

Next we argue that W generates a copy of $L(\mathbb{Z})$. Let g be the function on $[0,1]$ given by $g(t) = e^{2\pi i t}$. Then with $k \in \mathbb{N}$ the function g^{2^k} is a Haar unitary in $L^\infty([2^{-k}, 2^{-k+1}], 2^k\, \mathrm{d}m)$.

If we let $f \in L^\infty([0,1], \mathrm{d}m)$ be given as

$$f = \sum_{k=1}^{\infty} \frac{1}{k} g^{2^k} \chi_{[2^{-k}, 2^{-k+1}]},$$

then taking spectral projections of f and using them to split f, we see that f generates $L^\infty([0,1])$. It is easily established that W has the same *-distribution as f, and hence the von Neumann algebra generated by W is isomorphic to $L^\infty([0,1]) \simeq L(\mathbb{Z})$.

We now turn to prove that H is semicircular, and that $\{H, W\}$ is *-free. This is done by approximation. Let

$$\begin{aligned} H_n &= \sum_{k=1}^{n} 2^{-k/2} \Big(f(0\cdots 01; k) \otimes e(0\cdots 01, 0\cdots 01; 2^k) \\ &\quad + b(k) \otimes (e(0\cdots 0, 0\cdots 01; 2^k) + e(0\cdots 01, 0\cdots 0; 2^k)) \Big) \\ &\quad + 2^{-n/2} f(0\cdots 0; n) \otimes e(0\cdots 0, 0\cdots 0; 2^n), \\ W_n &= \sum_{k=1}^{n} \frac{1}{k} v(k) \otimes e(0\cdots 01, 0\cdots 01; 2^k) \end{aligned}$$

Then H_n and W_n converge to H and W in norm, and so it suffices to prove that, for each n, H_n is semicircular, and $\{H_n, W_n\}$ is *-free. This is done by induction over n. If $n = 1$, we have

$$H_1 = 2^{-1/2} \begin{pmatrix} f(0;1) & b(1) \\ b(1) & f(1;1) \end{pmatrix}, \quad W_1 = \begin{pmatrix} 0 & 0 \\ 0 & v(1) \end{pmatrix}.$$

A look at Proposition 4.12 shows that H_1 is semicircular, and $\{H_1, W_1\}$ is *-free.

For the inductive step we let

$$\begin{aligned} K_n &= 2^{1/2} \left(H_n - f(1;1) \otimes e(1,1;2) - b(1) \otimes (e(1,0;2) + e(0,1;2)) \right), \\ Y_n &= W_n - v(1) \otimes e(1,1;2), \end{aligned}$$

so that

$$H_n = 2^{-1/2} \begin{pmatrix} K_n & b(1) \\ b(1) & f(1;1) \end{pmatrix}, \quad W_n = \begin{pmatrix} Y_n & 0 \\ 0 & v(1) \end{pmatrix}.$$

By the inductive hypothesis applied to $\mathcal{A}_1$, we have that in $\mathcal{A}_1$ the element K_n is semicircular, and $\{K_n, Y_n\}$ is *-free. Let S be the set

$$S = \{ e(0p_2 \cdots p_k, 0q_2 \cdots q_k; 2^k) \mid p_i, q_i \in \{0,1\},\ 2 \le k \le n \}.$$

Then $S \subseteq \mathcal{A}_1$, and from Proposition 3.21 the family $\{S, f(0;1), g(1,0;1), f(1;1)\}$ is *-free. Since K_n and Y_n are build from S and $f(0;1)$ we see (keeping Theorem 4.11 in mind) that $\{K_n, Y_n, v(1), b(1), f(1;1)\}$ is *-free. It now follows from Proposition 4.12 that H_n is semicircular, and $\{H_n, W_n\}$ is *-free. $\square$

4.5 Two free projections

The von Neumann algebra generated by a single non-trivial projection is two-dimensional and hence isomorphic to $L(\mathbb{Z}_2)$, so a natural model for a probability space generated by two free projections (each of trace $1/2$) is $L(\mathbb{Z}_2) * L(\mathbb{Z}_2) = L(\mathbb{Z}_2 * \mathbb{Z}_2)$. In Proposition 4.14 we will identify this algebra with the algebra $\mathcal{M} = L^\infty([0, \frac{\pi}{2}]) \otimes M_2$, where $[0, \frac{\pi}{2}]$ is equipped with the normalized Lebesgue measure m, and $\mathcal{M}$ is equipped with the faithful trace $\varphi = \,\mathrm{d}m \otimes \mathrm{tr}_2$. But first we will investigate $\mathcal{M}$.

In $\mathcal{M}$ we consider the elements

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}, \tag{4.2}$$

where $\theta \in [0, \frac{\pi}{2}]$. It is easily verified that p and q are projections with trace $1/2$. With x the polar part of $(1-p)qp$ we have

$$x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and since $\cos^2\theta$ generates $L^\infty([0, \frac{\pi}{2}])$, it follows that $\mathcal{M} = W^*(pqp, p, x)$. Since $x \in W^*(p, q)$, it follows that p and q together generate $\mathcal{M}$.

Moreover, let us show that p and q are free. For this we let $a = 2p - 1$ and $b = 2q - 1$. Then a and b are unitaries of order two, and showing that $\{p, q\}$ is free is equivalent to showing that $\{a, b\}$ is free. By linearity it suffices to prove that any alternating product in a and b has trace zero, and since φ is a trace, it will actually suffice to prove that $\varphi((ab)^n) = 0$ for any $n \geq 1$. Observe, that

$$ab = \begin{pmatrix} -1 + 2\cos^2\theta & 2\cos\theta\sin\theta \\ -2\cos\theta\sin\theta & 1 - 2\sin^2\theta \end{pmatrix}$$

has eigenvalues $2\cos^2\theta - 1 \pm 2i\cos\theta\sin\theta = \cos 2\theta \pm i\sin 2\theta$, and hence $(ab)^n$ has eigenvalues $\cos 2n\theta \pm i\sin 2n\theta$. Thus,

$$\varphi((ab)^n) = \int_0^{\frac{\pi}{2}} 2\cos(2n\theta)\,\mathrm{d}m(\theta) = 0$$

for all $n \geq 1$. This shows that $\{p, q\}$ is free.

Collecting all the information in the above paragraphs we get the following from the Isomorphism Theorem.

Proposition 4.14. *Consider $L(\mathbb{Z}_2) * L(\mathbb{Z}_2)$ with canonical trace τ, and let p and q be projections of trace $1/2$ generating the first and second copy of $L(\mathbb{Z}_2)$, respectively. Then* *there is an isomorphism*

$$L(\mathbb{Z}_2) * L(\mathbb{Z}_2) \simeq L^\infty([0, \tfrac{\pi}{2}], m) \otimes M_2$$

such that p and q are given by (4.2). *Here m is the normalized Lebesgue measure, and the trace on $L^\infty([0, \frac{\pi}{2}], m) \otimes M_2$ is* $\mathrm{d}m \otimes \mathrm{tr}_2$.

Remark 4.15. As before, let x be the polar part of $(1-p)qp$. Then x is a partial isometry from p to $1-p$. Let y be the polar part of $(1-q)pq$, and let

$$w = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

Then w is a unitary of order two such that $wpw = q$, and hence also $wxw = y$. So y is a partial isometry from q to $1-q$.

4.6 The free product of the hyperfinite II_1 factor with itself

In this section we will prove that $R * R \simeq R * L(\mathbb{Z})$. It will follow easily from the following more general theorem.

Theorem 4.16. *Let $\mathcal{A}$ and $\mathcal{B}$ be von Neumann algebras with specified faithful traces - see Remark 4.5. Then*

(i) $(\mathcal{A} \otimes L(\mathbb{Z}_2)) * (\mathcal{B} \otimes L(\mathbb{Z}_2)) \simeq (\mathcal{A} * \mathcal{A} * \mathcal{B} * \mathcal{B} * L(\mathbb{Z})) \otimes M_2$,

(ii) $(\mathcal{A} \otimes M_2) * (\mathcal{B} \otimes L(\mathbb{Z}_2)) \simeq (\mathcal{A} * \mathcal{B} * \mathcal{B} * L(\mathbb{F}_2)) \otimes M_2$,

(iii) $(\mathcal{A} \otimes M_2) * (\mathcal{B} \otimes M_2) \simeq (\mathcal{A} * \mathcal{B} * L(\mathbb{F}_3)) \otimes M_2$.

Proof of (i). Let $\mathcal{M}$ be the von Neumann algebra on the left-hand side of (i) with trace τ. To ease notation we shall identify each algebra in a tensor/free product as a subset of the tensor/free product, so in this case for instance $\mathcal{A} \subseteq \mathcal{A} \otimes L(\mathbb{Z}_2) \subseteq \mathcal{M}$. Let p and q denote projections with trace $1/2$ generating each copy of $L(\mathbb{Z}_2)$ and commuting with $\mathcal{A}$ and $\mathcal{B}$, respectively. Since p and q are free, $W^*(p,q) = L(\mathbb{Z}_2) * L(\mathbb{Z}_2)$. Let $x, y, w \in W^*(p,q)$ be as in Remark 4.15.

Note that p is contained in the system of matrix units $(p, 1-p, x, x^*)$ in $\mathcal{M}$. Hence, to prove (i) it suffices to prove

$$p\mathcal{M}p \simeq \mathcal{A} * \mathcal{A} * \mathcal{B} * \mathcal{B} * L(\mathbb{Z}) \tag{4.3}$$

according to Lemma 4.1.

Since w is a unitary in $W^*(p,q)$ we see that

$$\mathcal{M} = W^*(p,q,\mathcal{A},\mathcal{B}) = W^*(p,q,\mathcal{A},w\mathcal{B}w). \tag{4.4}$$

From Lemma 4.3 we get that $p\mathcal{M}p$ is generated by the matrix units applied to the generators on the right-hand side of (4.4). Since $[q,\mathcal{B}]=0$, we have $[p,w\mathcal{B}w]=[wqw,w\mathcal{B}w]=0$. Also, $[p,\mathcal{A}]=0$, so the off-diagonal elements of $\mathcal{A}$ and $w\mathcal{B}w$ are zero, that is

$$p\mathcal{A}x = x^*\mathcal{A}p = pw\mathcal{B}wx = x^*w\mathcal{B}wp = 0.$$

Moreover, the elements pqx and x^*qx are seen to belong to $W^*(pqp)$, since

$$x^*qx = p - pqp, \quad pqx = (pqp)^{1/2}(p-pqp)^{1/2}.$$

Using all of this we conclude

$$p\mathcal{M}p = W^*(pqp, p\mathcal{A}p, x^*\mathcal{A}x, pw\mathcal{B}wp, x^*w\mathcal{B}wx). \tag{4.5}$$

Observe that pqp generates $L^\infty([0,\frac{\pi}{2}],m)\simeq L(\mathbb{Z})$ inside $p\mathcal{M}p$. Also $p\mathcal{A}=p\mathcal{A}p\simeq\mathcal{A}\simeq x^*\mathcal{A}x$, the first (trace-preserving) isomorphism being

$$\varphi:\mathcal{A}\to p\mathcal{A}p=\mathcal{A}\otimes p, \quad \varphi(a)=a\otimes p.$$

Similarly, $pw\mathcal{B}wp\simeq\mathcal{B}\simeq x^*w\mathcal{B}wx$.

The goal is to prove that the generating sets listed in (4.5) are free in $(p\mathcal{M}p,2\tau)$. Once we know this, (4.3) follows from what we have just argued.

First we prove that pqp, $p\mathcal{A}$ and $x^*\mathcal{A}x$ are free in $p\mathcal{M}p$. The elements of the form $f_k=(pqp)^k$ ($k\geq 0$) span a strongly dense subset of $W^*(pqp)$. Let $g_k=(f_k)^0=(pqp)^k-2\tau((pqp)^k)p$. Then according to Lemma 2.3 it suffices to prove that a non-trivial traveling product in $\{g_k \mid k\geq 1\}$, $(p\mathcal{A})^0$ and $(x^*\mathcal{A}x)^0$ has trace zero.

Note that $(p\mathcal{A})^0=p\mathcal{A}^0$, and $(x^*\mathcal{A}x)^0=x^*\mathcal{A}^0x$. The first is seen by

$$\tau(pa)=\tau(a\otimes p)=\tau(a)\tau(p)=\frac{1}{2}\tau(a), \quad a\in\mathcal{A},$$

and the second by

$$\tau(x^*ax)=\tau(axx^*)=\tau(a)\tau(1-p)=\frac{1}{2}\tau(a), \quad a\in\mathcal{A}.$$

Taking a non-trivial traveling product in $\{g_k \mid k\geq 1\}$, $p\mathcal{A}^0$ and $x^*\mathcal{A}^0x$ and regrouping gives a non-trivial traveling product in $\mathcal{A}^0$ and

$$\Omega_0=\{x,x^*\}\cup\{g_k, xg_k, g_kx^*, xg_kx^* \mid k\geq 1\}.$$

Now let $a = 2p - 1$ and $b = 2q - 1$ be the self-adjoint unitaries of order two coming from p and q. Then $W^*(p, q)$ is generated by a and b, and $\operatorname{span}\Lambda(\{a\}, \{b\})$ is a strongly dense *-subalgebra of $W^*(p, q)$. Since $\Omega_0 \subseteq W^*(p, q)$ it follows from Kaplansky's Density Theorem that any $z \in \Omega_0$ is the strong operator limit of a bounded net (in fact also a sequence) in $\operatorname{span}\Lambda(\{a\}, \{b\})$. Using that a and b both have trace zero and are free, we see that the trace of an element in $\operatorname{span}\Lambda(\{a\}, \{b\})$ equals the coefficient of 1. Since one can easily check that $\tau(z) = 0$ for any $z \in \Omega_0$ – for instance

$$\tau(xg_k) = \tau(x(pqp)^k - 2\tau((pqp)^k)xp) = \tau(x(pqp)^{k-1}qx^*x) - 2\tau((pqp)^k)\tau(x) = 0$$

using $\tau(x) = 0$ and $x^2 = 0$ – we may assume that each element in the approximating sequence from $\operatorname{span}\Lambda(\{a\}, \{b\})$ has the coefficient of 1 equal to zero. Using this assumption and that also $\tau(pz) = 0$ for any $z \in \Omega_0$ we may also assume that each coefficient of a is zero. In other words we have a bounded approximating sequence for z by elements of $\operatorname{span}(\Lambda(\{a\}, \{b\}) \setminus \{1, a\})$. Now it suffices to prove that a non-trivial alternating product in $\Lambda(\{a\}, \{b\}) \setminus \{1, a\}$ and $\mathcal{A}^0$ has trace zero. Regrouping gives a non-trivial alternating product in b and $\{a\} \cup \mathcal{A}^0 \cup a\mathcal{A}^0$ which by freeness has trace zero.

Let $\mathcal{N} = W^*(pqp, p\mathcal{A}, x^*\mathcal{A}x) \subseteq p\mathcal{M}p$. If we can show that $(\mathcal{N}, wq\mathcal{B}w, wy^*\mathcal{B}yw)$ is free in $p\mathcal{M}p$, then we have proved (i). We show instead that $(w\mathcal{N}w, q\mathcal{B}, y^*\mathcal{B}y)$ is free in $q\mathcal{M}q$, which will also suffice. Note first that

$$(w\mathcal{N}w)^0 = w\mathcal{N}^0w, \quad (q\mathcal{B})^0 = q\mathcal{B}^0, \quad (y^*\mathcal{B}y)^0 = y^*\mathcal{B}^0y.$$

Now take a non-trivial traveling product from $\Lambda(w\mathcal{N}^0w, q\mathcal{B}^0, y^*\mathcal{B}^0y)$ and regroup it to get a non-trivial alternating product in $\mathcal{B}^0$ and

$$\Omega_1 = \{y, y^*\} \cup w\mathcal{N}^0w \cup yw\mathcal{N}^0w \cup w\mathcal{N}^0wy^* \cup yw\mathcal{N}^0wy^*.$$

A few manipulations show that $\tau(z) = \tau(qz) = 0$ for any $z \in \Omega_1$. For instance

$$\tau(w\mathcal{N}^0wy^*) = \tau(w\mathcal{N}^0x^*w) = \tau(x^*\mathcal{N}^0) = 0,$$

since $x^*\mathcal{N} = 0$. Notice that $\Omega_1 \subseteq W^*(\mathcal{A}, a, b)$, and that $\operatorname{span}\Lambda(\{a\} \cup \mathcal{A}^0 \cup a\mathcal{A}^0, \{b\})$ is a strongly dense *-subalgebra of $W^*(\mathcal{A}, a, b)$. Arguing as above, each $z \in \Omega_1$ is the strong operator limit of a bounded sequence in the span of

$$\Lambda(\{a\} \cup \mathcal{A}^0 \cup a\mathcal{A}^0, \{b\}) \setminus \{1, b\}. \tag{4.6}$$

Now it suffices to show that a non-trivial alternating product in (4.6) and $\mathcal{B}^0$ has trace zero. Regrouping once again gives a non-trivial alternating product in $\{a\} \cup \mathcal{A}^0 \cup a\mathcal{A}^0$ and $\{b\} \cup \mathcal{B}^0 \cup b\mathcal{B}^0$ which by freeness has trace zero. This concludes to proof of (i). ■

Proof of (ii). The proof of (ii) resembles the proof of (i), so we will focus on the differences and leave out most of the parts that are identical.

Let $\mathcal{M}$ denote the left-hand side of (ii) with trace τ. As before we use the identifications concerning tensor and free products. Let p and q be projections in M_2 and $L(\mathbb{Z}_2)$ with trace $1/2$ and commuting with $\mathcal{A}$ and $\mathcal{B}$, respectively. Again, p and q are free, so that $W^*(p,q) = L(\mathbb{Z}_2) * L(\mathbb{Z}_2)$, and we let $x, y, w, a, b \in W^*(p,q)$ be as in the proof of (i).

As before it suffices to prove

$$p\mathcal{M}p \simeq \mathcal{A} * \mathcal{B} * \mathcal{B} * L(\mathbb{F}_2).$$

Let $u \in M_2$ be a partial isometry from p to $1-p$. Then u generates M_2 as a von Neumann algebra, and

$$\mathcal{M} = W^*(u, q, \mathcal{A}, w\mathcal{B}w).$$

Again we use Lemma 4.3 to find the generators for $p\mathcal{M}p$ using system of the matrix units $(p, 1-p, x, x^*)$. Note again that $[p, \mathcal{A}] = [p, w\mathcal{B}w] = 0$. Also, $pup = pux = x^*ux = 0$, and $x^*up = x^*u$. So

$$p\mathcal{M}p = W^*(pqp, x^*u, p\mathcal{A}, x^*\mathcal{A}x, wq\mathcal{B}w, wy^*\mathcal{B}yw).$$

Let $v = u + u^*$. Then v is a unitary, and $vx = u^*x$. Note that

$$x^*\mathcal{A}x = xv^*v\mathcal{A}x = x^*v^*\mathcal{A}vx = x^*u(p\mathcal{A})u^*x,$$

and hence we can remove $x^*\mathcal{A}x$ from the list of generators so that

$$p\mathcal{M}p = W^*(pqp, x^*u, p\mathcal{A}, wq\mathcal{B}w, wy^*\mathcal{B}yw). \tag{4.7}$$

We will prove that x^*u is a Haar unitary, and that the generators in (4.7) are *-free. This will prove (ii).

Note first that x^*u is a partial isometry from p to p, and hence a unitary in $p\mathcal{M}p$. For $n > 0$ the element $(x^*u)^n$ is an alternating product in x^* and u, and x^* is a strong limit of a bounded sequence in $\mathrm{span}(\Lambda(\{a\},\{b\}) \setminus \{1,a\})$. So it suffices to show that a non-trivial alternating product in $\Lambda(\{a\},\{b\}) \setminus \{1,a\}$ and $\{u\}$ has trace zero. Regrouping gives a non-trivial alternating product in $\{a, u, au, ua\} = \{a, \pm u\}$ and $\{b\}$, which by freeness has trace zero. This proves that x^*u is a Haar unitary in $p\mathcal{M}p$.

Next we prove that pqp, x^*u and $p\mathcal{A}$ are *-free in $p\mathcal{M}p$. It suffices to show that a non-trivial traveling product in $\{g_k \mid k \geq 1\}$, $\{(x^*u)^n \mid n \in \mathbb{Z} \setminus \{0\}\}$ and $p\mathcal{A}^0$ has trace zero. Taking such a traveling product and regrouping gives a non-trivial alternating product in Ω_0 and

$$\Omega_2 = \{u, u^*\} \cup \mathcal{A}^0 \cup u\mathcal{A}^0 \cup \mathcal{A}^0u^*.$$

As before, we can use $\operatorname{span}(\Lambda(\{a\},\{b\}) \setminus \{1,a\})$ to approximate elements of Ω_0 and then regroup. Then it suffices to see that a non-trivial alternating product in $a\mathcal{A}^0 \cup \Omega_2 \cup a\Omega_2$ and $\{b\}$ has trace zero, and this is a consequence of freeness. This proves that the family $(\{pqp\},\{x^*u\},p\mathcal{A})$ is *-free in $(p\mathcal{M}p, 2\tau)$.

Let $\mathcal{N} = W^*(pqp, x^*u, p\mathcal{A}) \subseteq p\mathcal{M}p$. Finally we show that $(w\mathcal{N}w, q\mathcal{B}, y^*\mathcal{B}y)$ is free in $q\mathcal{M}q$. This will conclude the proof of (ii).

The approach is exactly the same as in the proof of (i), except that $\Omega_1 \subseteq W^*(u,b,\mathcal{A})$, and the dense *-subalgebra is now spanned by the more complicated set $\Lambda(\Omega_3,\{b\})$, where

$$\Omega_3 = \{u,u^*,a\} \cup \mathcal{A}^0 \cup u\mathcal{A}^0 \cup u^*\mathcal{A}^0 \cup a\mathcal{A}^0. \qquad \blacksquare$$

Proof of (iii). Let $\mathcal{M}$ be the von Neumann algebra on the left-hand side of (iii). Take p and u in $1 \otimes M_2$ commuting with $\mathcal{A}$ as above, and take q and v in $1 \otimes M_2$ commuting with $\mathcal{B}$, where q is a projection of trace $1/2$, and v is a partial isometry from q to $1-q$. Let $x,y,w \in W^*(p,q)$ be as before. As before we may show that x^*u and y^*v are Haar unitaries. Notice that

$$\mathcal{M} = W^*(u,v,\mathcal{A},\mathcal{B}) = W^*(u,q,wvw,\mathcal{A},w\mathcal{B}w),$$

and so

$$p\mathcal{M}p = W^*(x^*u, pqp, wy^*vw, p\mathcal{A}, wq\mathcal{B}w).$$

Using the same approach as before we may prove that $(\{x^*u\},\{pqp\},\{wy^*vw\},p\mathcal{A},wq\mathcal{B}w)$ is *-free. This concludes the proof of (iii). $\square$

As an almost immediate consequence of the previous theorem we get the following.

Theorem 4.17. *Let R and $\widetilde{R}$ be copies of the hyperfinite II_1 factor. Then*

$$R * \widetilde{R} \simeq R * L(\mathbb{Z}),$$

with an isomorphism that maps R identically to R.

Proof. Let p and q be projections of trace $1/2$ in R and $\widetilde{R}$, respectively. Then $R = (pRp) \otimes M_2$ and $\widetilde{R} = (q\widetilde{R}q) \otimes M_2$. From the proof of Theorem 4.16 we see that

$$p(R * \widetilde{R})p \simeq (pRp) * (q\widetilde{R}q) * L(\mathbb{F}_3)$$

with an isomphism which maps $pRp \subseteq p(R * \widetilde{R})p$ onto pRp. Writing $L(\mathbb{Z}) \simeq L(\mathbb{Z}) \otimes L(\mathbb{Z}_2)$ we get from the proof of Theorem 4.16 that

$$p(R * L(\mathbb{Z}))p \simeq (pRp) * L(\mathbb{F}_4)$$

with an isomphism that is the identity from $pRp \subseteq p(R * L(\mathbb{Z}))p$ to pRp. From Theorem

4.13 we know that $(q\widetilde{R}q) * L(\mathbb{F}_3) \simeq L(\mathbb{F}_4)$, and so combining the two isomorphisms we obtain an isomorphism from $p(R * \widetilde{R})p$ to $p(R * L(\mathbb{Z}))p$ which is the identity on pRp. Finally, tensor with M_2. $\square$

CHAPTER 5

Interpolated free group factors

The group von Neumann algebras $L(\mathbb{F}_m)$ defined for $m \in \mathbb{N} \cup \{\infty\}$ satisfy

$$L(\mathbb{F}_m) * L(\mathbb{F}_n) = L(\mathbb{F}_{m+n}), \quad m, n \in \mathbb{N} \cup \{\infty\},$$

and when $m \geq 2$, they are II_1 factors with the relation

$$L(\mathbb{F}_m)_{1/n} = L(\mathbb{F}_{(m-1)n^2+1}), \quad n \in \mathbb{N}. \tag{5.1}$$

With $0 < t < \infty$ the notation $\mathcal{A}_t$ for a II_1 factor $\mathcal{A}$ means the algebra introduced in Definitions B.1 and B.6. For $0 < t \leq 1$ the algebra $\mathcal{A}_t$ is $p\mathcal{A}p$ where $p \in \mathcal{A}$ is a projection of trace t, for $t \in \mathbb{N}$ the algebra $\mathcal{A}_t$ is $\mathcal{A} \otimes M_t$, and we have the rule $(\mathcal{A}_s)_t = \mathcal{A}_{st}$. See Appendix B for more details as well as the definition of the *fundamental group* of a II_1 factor.

A priori the expression on the right-hand side of (5.1) has no meaning, when n is not an integer. But the left-hand side does have a meaning, so one could use the equation to *define* $L(\mathbb{F}_r)$ for non-integer $r = (m-1)n^2+1$. This gives rise to the *interpolated* free group factors. Of course, different combinations of $m \in \{2, 3, \ldots, \infty\}$ and $n \in \mathbb{R}_+$ can give the same r, so one should check that $L(\mathbb{F}_r)$ would be well-defined. Or one could simply define $L(\mathbb{F}_r)$ as $L(\mathbb{F}_2)_{1/\sqrt{r-1}}$. The purpose of this section is to prove that the interpolated free group factors are well-defined, but we will use an even more general definition than the two suggestions made here. The benefits of a more general definition come when we want to work with the factors afterwards.

We will define the interpolated free group factors $L(\mathbb{F}_r)$ for $r \in]1, \infty]$ in such a way that they are II_1 factors that equal the free group factors, when r is in $\{2, 3, \ldots, \infty\}$, and they satisfy the rules

$$L(\mathbb{F}_r) * L(\mathbb{F}_s) \simeq L(\mathbb{F}_{r+s}), \quad 1 < r, s \leq \infty, \tag{5.2}$$

$$L(\mathbb{F}_r)_t \simeq L(\mathbb{F}_{(r-1)t^{-2}+1}), \quad 1 < r \leq \infty, \ 0 < t < \infty. \tag{5.3}$$

5.1 Definition

We follow the approach due to Dykema [5] when defining the interpolated free group factors. For an alternative approach see [12].

Definition 5.1 (Interpolated free group factor)**.** Let $(\mathcal{M}, \tau)$ be a W*-probability space with τ a faithful normal trace. Let R be a copy of the hyperfinite II_1 factor inside $\mathcal{M}$, let $\omega = \{X_t \mid t \in T\}$ be an infinite semicircular family in $\mathcal{M}$, and suppose R and ω are free. Note that we can always find a W*-probability space $\mathcal{M}$ with the desired properties. Take for instance $\mathcal{M} = R * L(\mathbb{F}_\infty) = L(\mathbb{F}_\infty)$.

Fix some $r \in]1, \infty]$, and choose projections $p_t \in R$ such that

$$r = 1 + \sum_{t \in T} \tau(p_t)^2.$$

Note that this is always possible, since R is a II_1 factor. We let $L(\mathbb{F}_r)$ denote any von Neumann algebra isomorphic to

$$W^*(R \cup \{p_t X_t p_t \mid t \in T\}).$$

The von Neumann algebra $L(\mathbb{F}_r)$ is called an *interpolated free group factor*.

We prove below that $L(\mathbb{F}_r)$ is well-defined, always a II_1 factor, and that $L(\mathbb{F}_r)$ is the usual free group factor when $r = 2, 3, \ldots, \infty$. When r is not an integer only the symbol $L(\mathbb{F}_r)$ makes sense, not $\mathbb{F}_r$ alone.

Notation. While proving the subsequent lemmas and Proposition 5.4 we keep the notation from the preceding definition. Also, let $(f_n)_{n=1}^{\infty}$ be an orthogonal family of projections in R such that $\tau(f_n) = 2^{-n}$, and let $f_0 = 1$.

Lemma 5.2. *Fix $t_0 \in T$ and $m, n \in \mathbb{N}_0$ such that $m < n$. Then*

$$W^*(R \cup \{f_n X_{t_0} f_m\}) \simeq W^*(R \cup \{f_n X_{t_i} f_n \mid 1 \leq i \leq 2 \cdot 2^{n-m}\})$$

with an isomorphism that is the identity on R, whenever $t_0, t_1, \ldots, t_{2 \cdot 2^{n-m}}$ are distinct members of T.

Proof. The key idea of the proof is to use Proposition 3.20 to realize the semicircular elements X_t as matrices with circular and semicircular entries.

Notice that f_m is an orthogonal sum of $N = 2^{n-m}$ projections in R each of which is equivalent to f_n. Write f_m in this way

$$f_m = \sum_{i=1}^{N} g_i$$

where each $g_i \sim f_n$. First we prove that

$$W^*(R \cup \{f_n X_{t_0} f_m\}) \simeq W^*(R \cup \{f_n X_{t_i} g_i \mid 1 \le i \le N\}) \tag{5.4}$$

where $t_0, t_1, \dots, t_N$ are distinct members of T, and the isomorphism is the identity on R.

Let

$$\mathcal{N}_1 = W^*(R \cup \{f_n X_{t_0} f_m\}), \qquad \mathcal{N}_2 = W^*(R \cup \{f_n X_{t_i} g_i \mid 1 \le i \le N\})$$

Since the projections lie in R, we notice that

$$\mathcal{N}_1 = W^*(R \cup \{f_n X_{t_0} f_m\}) = W^*(R \cup \{f_n X_{t_0} g_i \mid 1 \le i \le N\}).$$

We may imbed $\{g_1, \dots, g_N, f_n\}$ in a system of matrix units $(e_{ij})_{i,j=1}^K$ in R where $K = 2^n$, $g_i = e_{ii}$ (for $1 \le i \le N$) and $f_n = e_{N+1,N+1}$.

Using Proposition 3.20 we find that each X_t has decomposition with respect to the matrix units as

$$X_t = \frac{1}{\sqrt{K}} \begin{pmatrix} x_1^{(t)} & y_{12}^{(t)} & \cdots & y_{1K}^{(t)} \\ (y_{12}^{(t)})^* & x_2^{(t)} & \cdots & y_{2K}^{(t)} \\ \vdots & \vdots & \ddots & \vdots \\ (y_{1K}^{(t)})^* & (y_{2K}^{(t)})^* & \cdots & x_K^{(t)} \end{pmatrix}, \tag{5.5}$$

where $\mu_1 = \{x_i^{(t)} \mid 1 \le i \le K, t \in T\}$ is semicircular and $\mu_2 = \{y_{ij}^{(t)} \mid 1 \le i < j \le K, t \in T\}$ is circular such that μ_1, μ_2 and $\widetilde{R} = e_{11} R e_{11}$ are *-free in $e_{11} \mathcal{M} e_{11}$.

Then $f_n X_{t_0} g_i = K^{-1/2} (y_{i,N+1}^{(t_0)})^* \otimes e_{N+1,i}$. Since R contains the matrix units,

$$\mathcal{N}_1 = W^*(R \cup \{y_{i,N+1}^{(t_0)} \otimes e_{11} \mid 1 \le i \le N\}).$$

To prove $\mathcal{N}_1 \simeq \mathcal{N}_2$ it will suffice to prove $e_{11} \mathcal{N}_1 e_{11} \simeq e_{11} \mathcal{N}_2 e_{11}$ according to Lemma 4.1, and the isomorphism should be the identity on $\widetilde{R}$.

Using Lemma 4.2 we find

$$e_{11} \mathcal{N}_1 e_{11} = W^*(e_{11} R e_{11} \cup \{y_{i,N+1}^{(t_0)} \otimes e_{11} \mid 1 \le i \le N\}) \simeq W^*(\widetilde{R} \cup \{y_{i,N+1}^{(t_0)} \mid 1 \le i \le N\}).$$

Since μ_2 is a free family of identically distributed variables, and μ_2 and $\widetilde{R}$ are free, it follows from Theorem 2.14 that

$$e_{11} \mathcal{N}_1 e_{11} \simeq W^*(\widetilde{R} \cup \{y_{i,N+1}^{(t_i)} \mid 1 \le i \le N\}).$$

where $t_1, \dots, t_N$ are distinct members of T, and the isomorphism is the identity on $\widetilde{R}$. And using the same approach as with $\mathcal{N}_1$ we may show that

$$e_{11}\mathcal{N}_2 e_{11} \simeq W^*(\widetilde{R} \cup \{y^{(t_i)}_{i,N+1} \mid 1 \leq i \leq N\}),$$

also with an isomorphism which is the identity on $\widetilde{R}$. This proves (5.4). Next let

$$\mathcal{N}_3 = W^*(R \cup \{f_n X_{t_i} f_n \mid 1 \leq i \leq 2N\}).$$

with $t_1, \ldots, t_{2N}$ distinct members of T. We must prove $\mathcal{N}_3 \simeq \mathcal{N}_1$, and in view of what we have already proved, it suffices to prove that $e_{11}\mathcal{N}_3 e_{11} \simeq e_{11}\mathcal{N}_2 e_{11}$.

Using (5.5) we see that $f_n X_{t_i} f_n = x^{(t_i)}_{N+1} \otimes e_{N+1,N+1}$, and hence

$$e_{11}\mathcal{N}_3 e_{11} \simeq W^*(\widetilde{R} \cup \{x^{(t_i)}_{N+1} \mid 1 \leq i \leq 2N\}),$$

where $\{x^{(t_i)}_{N+1} \mid 1 \leq i \leq 2N\}$ is a semicircular family, free of $\widetilde{R}$. We write $y_i = y^{(t_i)}_{i,N+1}$. Then y_i is circular, and if we let

$$x_i = \frac{y_i + y_i^*}{\sqrt{2}}, \quad x_i' = \frac{y_i - y_i^*}{i\sqrt{2}},$$

then $\{x_i, x_i'\}_{i=1}^N$ is a semicircular family free of $\widetilde{R}$. Since

$$e_{11}\mathcal{N}_2 e_{11} \simeq W^*(\widetilde{R} \cup \{x_i, x_i' \mid 1 \leq i \leq N\}),$$

it follows again from Theorem 2.14 that $e_{11}\mathcal{N}_2 e_{11} \simeq e_{11}\mathcal{N}_3 e_{11}$ with an isomorphism that is the identity on $\widetilde{R}$. This completes the proof of the lemma. □

Lemma 5.3. *Fix a natural number n and $t_0 \in T$. Then*

$$W^*(R \cup \{f_n X_{t_i} f_n \mid 1 \leq i \leq 4\}) \simeq W^*(R \cup \{f_{n-1} X_{t_0} f_{n-1}\})$$

with an isormophism which is the identity on R, whenever $t_1, \ldots, t_4 \in T$ are distinct.

Proof. The ideas in the proof are similar to those of the previous lemma. For that reason we will not include as many details as before. Let

$$\mathcal{N}_1 = W^*(R \cup \{f_{n-1} X_{t_0} f_{n-1}\}), \quad \mathcal{N}_2 = W^*(R \cup \{f_n X_{t_i} f_n \mid 1 \leq i \leq 4\}).$$

Since f_{n-1} is a sum of two orthogonal projections g_1, g_2 each equivalent to f_n, we imbed $\{g_1, g_2, f_n\}$ in a system of matrix units in R. It will suffice to prove $e_{11}\mathcal{N}_1 e_{11} \simeq e_{11}\mathcal{N}_2 e_{11}$ with an isomophism that is the identity on $e_{11} R e_{11}$. As before each X_t has the form (5.5), and so

$$f_{n-1}X_{t_0}f_{n-1} = \frac{1}{\sqrt{K}}\begin{pmatrix} x_1^{(t_0)} & y_{12}^{(t_0)} & 0 & \cdots & 0 \\ (y_{12}^{(t_0)})^* & x_2^{(t_0)} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Since R contains all the matrix units,

$$\mathcal{N}_1 = W^*(R \cup \{x_1^{(t_0)} \otimes e_{11},\ x_2^{(t_0)} \otimes e_{11},\ y_{12}^{(t_0)} \otimes e_{11}\})$$

and with $\widetilde{R} = e_{11}Re_{11}$ we have

$$e_{11}\mathcal{N}_1 e_{11} \simeq W^*(\widetilde{R} \cup \{x_1^{(t_0)},\ x_2^{(t_0)},\ y_{12}^{(t_0)}\}).$$

If we let

$$x_3 = \frac{y_{12}^{(t_0)} + (y_{12}^{(t_0)})^*}{\sqrt{2}}, \quad x_4 = \frac{y_{12}^{(t_0)} - (\ y_{12}^{(t_0)})^*}{i\sqrt{2}},$$

then $\{x_1^{(t_0)}, x_2^{(t_0)}, x_3, x_4\}$ is a semicircular family free of $\widetilde{R}$, and

$$e_{11}\mathcal{N}_1 e_{11} \simeq W^*(\widetilde{R} \cup \{x_1^{(t_0)}, x_2^{(t_0)}, x_3, x_4\}).$$

And since

$$e_{11}\mathcal{N}_2 e_{11} \simeq W^*(\widetilde{R} \cup \{x_3^{(t_1)}, \ldots, x_3^{(t_4)}\}),$$

where $\{x_3^{(t_1)}, \ldots, x_3^{(t_4)}\}$ is a semicircular family free of $\widetilde{R}$, the lemma now follows from Theorem 2.14. $\square$

Now we are in a position to prove that $L(\mathbb{F}_r)$ is well-defined.

Proposition 5.4. *The interpolated free group factors are well-defined. In other words, if*

$$\mathcal{A} = W^*(R \cup \{p_t X_t p_t \mid t \in T\}) \quad \textit{and} \quad \mathcal{B} = W^*(R \cup \{q_t X_t q_t \mid t \in T\}),$$

where

$$1 + \sum_{t \in T} \tau(p_t)^2 = r = 1 + \sum_{t \in T} \tau(q_t)^2,$$

then $\mathcal{A} \simeq \mathcal{B}$.

Proof. The strategy of the proof is to show that $\mathcal{A}$ (and hence also $\mathcal{B}$) is isomorphic to an algebra $\mathcal{C}$ of a certain standard form. We now describe the algebra $\mathcal{C}$.

If $r < \infty$, write $r-1$ in base 4 as

$$r - 1 = \sum_{l=0}^{\infty} N_l 4^{-l},$$

where N_l $(l \geq 0)$ are nonnegative integers, $N_l \leq 3$ when $l \geq 1$, and for every $l' \geq 0$

$$\sum_{l>l'} N_l 4^{-l} < 4^{-l'}.$$

If $r = \infty$, let $N_0 = \infty$ and $N_l = 0$ $(l \geq 1)$. The conditions ensure a unique choice of the numbers N_l.

Then choose $S \subseteq T$ and numbers $k_s \in \mathbb{N}_0$ for $s \in S$ such that $S(l) = \{s \in S \mid k_s = l\}$ has cardinality N_l. The algebra in the standard form is then

$$\mathcal{C} = W^*(R \cup \{f_{k_s} X_s f_{k_s} \mid s \in S\}),$$

and

$$1 + \sum_{s \in S} \tau(f_{k_s})^2 = 1 + \sum_{s \in S} 4^{-k_s} = 1 + \sum_{l=0}^{\infty} N_l 4^{-l} = r.$$

If we can show $\mathcal{A} \simeq \mathcal{C}$, this will prove the proposition, since $\mathcal{C}$ is uniquely determined (up to isomorphism) by the choices made.

First off we want to be able to replace the projections p_t by "standard" projections f_k. If u_t are unitaries in R, then by Corollary 3.16 the family $\{u_t X_t u_t^* \mid t \in T\}$ is again a semicircular family free of R. Each projection $p_t \in R$ is conjugate by a unitary u_t in R to a projection q_t that is a (possibly infinite) sum of projections in $\{f_k \mid k \geq 0\}$, and then

$$\begin{aligned} W^*(R \cup \{p_t X_t p_t \mid t \in T\}) &\simeq W^*(R \cup \{p_t u_t X_t u_t^* p_t \mid t \in T\}) \\ &= W^*(R \cup \{u_t q_t X_t q_t u_t^* \mid t \in T\}) \\ &= W^*(R \cup \{q_t X_t q_t \mid t \in T\}). \end{aligned}$$

So we may as well assume that each non-zero p_t is equal to such a sum and write

$$p_t = \sum_{k \in K_t} f_k,$$

where $K_t \subseteq \mathbb{N}_0$. Let $T' = \{t \in T \mid p_t \neq 0\}$. We claim that

$$\mathcal{A} = W^*(R \cup \{f_k X_t f_{k'} \mid t \in T',\ k, k' \in K_t,\ k \leq k'\}) \tag{5.6}$$

Since every f_k belongs to R, it is immediate that $f_k X_t f_{k'} = f_k(p_t X_t p_t) f_{k'}$ belongs to $\mathcal{A}$, and so the right-hand side is contained in the left-hand side. On the other hand, since

$f_{k'}X_tf_k = (f_kX_tf_{k'})^*$, the right-hand side also contains $f_kX_tf_{k'}$ when $k > k'$, and since

$$p_tX_tp_t = \left(\sum_{k\in K_t} f_k\right) X_t \left(\sum_{k'\in K_t} f_{k'}\right) = \sum_{k,k'\in K_t} f_kX_tf_{k'}$$

it follows that $p_tX_tp_t$ belongs to the right-hand side. This proves the equality in (5.6).

Claim. *If necessary we enlarge the set T, and then there is an injective map α from the set $G = \{(t,k,k') \mid t\in T,\, k,k'\in K_t,\, k\le k'\}$ into T such that*

$$\mathcal{A} \simeq W^*(R\cup\{f_kX_{\alpha(t,k,k')}f_{k'} \mid t\in T',\ k,k'\in K_t,\ k\le k'\}), \tag{5.7}$$

and the isomorphism is the identity on R.

Proof of Claim. Suppose first that G is finite. Then there is a largest number k_0 in $\bigcup_{t\in T'} K_t$, and each f_k appearing in (5.6) is an orthogonal sum of projections $g_j\in R$ equivalent (in R) to f_{k_0}. We may imbed these projections g_j and f_{k_0} into an $n\times n$ system of matrix units in R, where $n = 2^{k_0}$. Using the matrix model (Proposition 3.20) we see that each X_t $(t\in T)$ has the form

$$X_t = \frac{1}{\sqrt{n}}\begin{pmatrix} x_1^{(t)} & y_{12}^{(t)} & \cdots & y_{1n}^{(t)} \\ (y_{12}^{(t)})^* & x_2^{(t)} & \cdots & y_{2n}^{(t)} \\ \vdots & \vdots & \ddots & \vdots \\ (y_{1n}^{(t)})^* & (y_{2n}^{(t)})^* & \cdots & x_n^{(t)} \end{pmatrix},$$

where the family $\omega_1 = \{x_i^{(t)} \mid 1\le i\le n,\ t\in T\}$ is semicircular in $f_{k_0}\mathcal{M}f_{k_0}$, the family $\omega_2 = \{y_{ij}^{(t)} \mid 1\le i<j\le n,\ t\in T\}$ is circular in $f_{k_0}\mathcal{M}f_{k_0}$, and ω_1, ω_2 and $f_{k_0}Rf_{k_0}$ are *-free. If $(t,k,k')\ne(s,j,j')$ then $f_kX_tf_{k'}$ and $f_jX_sf_{j'}$ are build up of different parts of $\omega_1\cup\omega_2$, and then the same approach as used in Lemma 5.2 and 5.3 can be used to prove (5.7).

If G is infinite, then we write it as a union of an increasing sequence of finite sets $G = \bigcup_{n=1}^{\infty} G_n$, and for each n we let $\mathcal{A}_n$ be

$$\mathcal{A}_n = W^*(R\cup\{f_kX_tf_{k'} \mid (t,k,k')\in G_n\}).$$

Then $(\mathcal{A}_n)_{n=1}^{\infty}$ forms an increasing sequence of subalgebras of $\mathcal{A}$ whose union is weakly dense, and we have compatible isomorphisms $\mathcal{A}_n\to\mathcal{B}_n$ (see Corollary 2.15), where

$$\mathcal{B}_n = W^*(R\cup\{f_kX_{\alpha(t,k,k')}f_{k'} \mid (t,k,k')\in G_n\}).$$

Using direct limits (see Theorem A.3) we find that (5.7) holds also when G is infinite. ■

If $t\in T$, $k',k\in K_t$ and $k'<k$ then an application of Lemma 5.2 yields

$$W^*(R\cup\{f_kX_tf_{k'}\}) \simeq W^*(R\cup\{f_kX_{t_i}f_k \mid 1\le i\le 2\cdot 2^{k-k'}\})$$

where $t, t_1, \ldots, t_{2\cdot 2^{k-k'}}$ are distinct members of T, and the isomorphism is the identity on R. Using direct limits again we find that

$$\mathcal{A} \simeq \widetilde{\mathcal{C}} = W^*(R \cup \{f_{k_s} X_s f_{k_s} \mid s \in S'\})$$

for some subset S' of T, $k_s \in \mathbb{N}_0$ for each $s \in S'$.

Claim. *We have* $1 + \sum_{s\in S'} \tau(f_{k_s})^2 = r$.

Proof of Claim. Recall that $r = 1 + \sum_{t\in T} \tau(p_t)^2$. And we also have

$$\sum_{t\in T} \tau(p_t)^2 = \sum_{t\in T} \tau\left(\sum_{k\in K_t} f_k\right)^2 = \sum_{t\in T}\left(\sum_{k\in K_t} \tau(f_k)\right)^2.$$

Multiplying out the square, we find

$$\sum_{t\in T}\left(\sum_{k\in K_t}\tau(f_k)\right)^2 = \sum_{t\in T}\sum_{k,k'\in K_t} \tau(f_k)\tau(f_{k'}) = \sum_{t\in T}\left(\sum_{k'<k} 2\tau(f_k)\tau(f_{k'}) + \sum_k \tau(f_k)^2\right).$$

Using that $\tau(f_{k'}) = 2^{k-k'}\tau(f_k)$ we continue

$$\sum_{t\in T}\left(\sum_{k'<k} 2\tau(f_k)\tau(f_{k'}) + \sum_k \tau(f_k)^2\right) = \sum_{t\in T}\left(\sum_{k'<k} 2\cdot 2^{k-k'}\tau(f_k)^2 + \sum_k \tau(f_k)^2\right)$$

and this last sum is the same sum as $\sum_{s\in S'} \tau(f_{k_s})^2$. So we have $1 + \sum_{s\in S'} \tau(f_{k_s})^2 = r$. ■

Suppose now that $r < \infty$, and that r is not a dyadic rational. We saw before that $r - 1$ may be written as

$$r - 1 = \sum_{s\in S'} \tau(f_{k_s})^2 = \sum_{s\in S'} 4^{-k_s},$$

but this is probably not the canonical base 4 expansion of $r-1$. The idea is now to use (5.9) repeatedly to show $\widetilde{\mathcal{C}} \simeq \mathcal{C}$.

Claim. *There is an increasing sequence $\widetilde{S}(l)$ of finite subsets of S' with union S' such that*

$$\sum_{s\in\widetilde{S}(l)} 4^{-k_s} = \sum_{k=0}^{l} N_k 4^{-k}.$$

Proof of Claim. Fix $l \geq 0$. Each $S'(j) = \{s \in S' \mid k_s = j\}$ is finite, since $r - 1 < \infty$. Also, $|S'| = \infty$, since $r-1$ is not dyadic. Let $S''(m) = \bigcup_{j\leq m} S'(j)$. Consider

$$b_m = \sum_{s\in S''(m)} 4^{-k_s}.$$

Then $(b_m)_{m=0}^{\infty}$ is an increasing sequence with limit $r-1$. Also

$$b_m < r-1 = \sum_{k\leq m} N_k 4^{-k} + \sum_{k>m} N_k 4^{-k} < \sum_{k\leq m} N_k 4^{-k} + 4^{-m},$$

so that

$$b_m \leq \sum_{k\leq m} N_k 4^{-k}. \tag{5.8}$$

Put $\varepsilon = (r-1) - \sum_{k=0}^{l} N_k 4^{-k}$. Then $\varepsilon > 0$, since r is not dyadic. Hence there is a smallest k_0 such that $(r-1) - b_{k_0} < \varepsilon$. So

$$b_{k_0-1} \leq \sum_{k\leq l} N_k 4^{-k} < b_{k_0}.$$

By (5.8) we must have $l \leq k_0$. Then there must be a subset $\overline{S}_l \subseteq S'(k_0)$ such that

$$\sum_{k\leq l} N_k 4^{-k} - b_{k_0-1} = \sum_{s\in\overline{S}_l} 4^{-k_s}.$$

Now put $\widetilde{S}_l = \bigcup_{j<k_0} S'(j) \cup \overline{S}_l$. Then

$$\sum_{s\in\widetilde{S}(l)} 4^{-k_s} = b_{k_0-1} + \sum_{s\in\overline{S}_l} 4^{-k_s} = \sum_{k=0}^{l} N_k 4^{-k}.$$

Finally we should mention that if the k_0's corresponding to different l's happen to coincide, then we can choose $\overline{S}_l \subseteq \overline{S}_{l+1}$, and in this way we obtain that $\widetilde{S}(l)$ becomes an increasing sequence. It is clear that we must have $\bigcup \widetilde{S}(l) = S'$. ■

For $l \geq 0$ we let

$$\begin{aligned} \mathcal{C}(l) &= W^*(R \cup \{f_{k_s} X_s f_{k_s} \mid s \in S,\ k_s \leq l\}) \subseteq \mathcal{C}, \\ \widetilde{\mathcal{C}}(l) &= W^*(R \cup \{f_{k_s} X_s f_{k_s} \mid s \in \widetilde{S}(l)\}) \subseteq \widetilde{\mathcal{C}}. \end{aligned}$$

We recall the result of Lemma 5.3.

$$W^*(R \cup \{f_n X_{t_i} f_n \mid 1 \leq i \leq 4\}) \simeq W^*(R \cup \{f_{n-1} X_t f_{n-1}\}) \tag{5.9}$$

by an isrmophism which is the identity on R, whenever $n \geq 1$, $t_1, \ldots, t_4 \in T$ are distinct, and $t \in T$.

Using (5.9) repeatedly we find a compatible family of isomorphisms $\varphi_l : \widetilde{\mathcal{C}}(l) \to \mathcal{C}(l)$, and taking direct limits implies $\widetilde{\mathcal{C}} \simeq \mathcal{C}$.

If r is dyadic, then S is finite. If $|S'|$ is finite, then a finite number of applications of (5.9) yields $\widetilde{\mathcal{C}} \simeq \mathcal{C}$. It may happen that S' is infinite, and then we have to work some more to show $\widetilde{\mathcal{C}} \simeq \mathcal{C}$.

Let l be the largest such that $N_l \neq 0$. In much the same way as we proved the previous claim, we may also prove

Claim. *There is an increasing sequence $\widetilde{S}(m)$ $(m > l)$ of finite subsets of S' with union S' such that*

$$\sum_{s \in \widetilde{S}(m)} 4^{-k_s} = \left(\sum_{k=0}^{l} N_k 4^{-k} \right) - 4^{-m}.$$

We omit the details of the proof of the claim, but remark that one should choose $\varepsilon = 4^{-m}$ instead of the choice made in the previous proof.

Let $\sigma \in S$ be such that $k_\sigma = l$. Also, choose projections $f_l \geq g_{l+1} \geq g_{l+2} \geq \cdots$ in R such that $\tau(g_m) = 2^{-m}$. For each $m > l$ let

$$\mathcal{C}(m) = W^*(R \cup \{f_{k_s} X_s f_{k_s} \mid s \in S \setminus \{\sigma\}\} \cup \{f_{k_\sigma} X_\sigma f_{k_\sigma} - g_m X_\sigma g_m\}).$$

Using the matrix model, we see from the claim that we may find subalgebras $\widetilde{\mathcal{C}}(m)$ and compatible isomorphisms $\varphi_m : \widetilde{\mathcal{C}}(m) \to \mathcal{C}(m)$, and taking direct limits yields $\widetilde{\mathcal{C}} \simeq \mathcal{C}$.

If r is infinite, and $S'(l) = \{s \in S' \mid k_s = l\}$, then we know that $\sum_{l=0}^{\infty} |S'(l)| 4^{-l} = \infty$. We claim that we can use (5.9) to transform the situtaion so that some $|S'(l)| = \infty$ (with an isomorphism that is the identity on R). If some $S'(l)$ is already infinite this is of course trivial. Suppose all $|S'(l)| < \infty$. There is an l_1 so that

$$\sum_{l=1}^{l_1} |S'(l)| 4^{-l} > 1.$$

Then we may find l_2 so that

$$\sum_{l=l_1+1}^{l_2} |S'(l)| 4^{-l} > 1.$$

Continuing in this way we find inductively for each $M \in \mathbb{N}$ an l_M so that

$$\sum_{l=l_{M-1}+1}^{l_M} |S'(l)| 4^{-l} > 1.$$

(The formula also holds for $M = 1$, if we define $l_0 = 0$). Using (5.9) we may inductively transform the situation so that $|S'(0)| = M$, and taking direct limits yields $|S'(0)| = \infty$.

Then we may of course ensure that all $|S'(l)| = \infty$, and then finally we can collect things together so that $|S'(0)| = \infty$ and $|S'(l)| = 0$ when $l \geq 1$. Thus

$$\widetilde{\mathcal{C}} \simeq R * L(\mathbb{F}_\infty) = \mathcal{C},$$

where the last isomorphism follows from Theorem 4.13. $\square$

As we mentioned at the beginning of this section we would argue that $L(\mathbb{F}_r)$ equals the usual free group factor, when r is an integer or ∞. If $r \in \{2, 3, \ldots, \infty\}$, then we simply choose $r-1$ projections p_t as 1 and the rest as 0. It then follows from the fact $R * L(\mathbb{F}_n) = L(\mathbb{F}_{n+1})$ (Theorem 4.13) that the interpolated free group factor equals the usual free group factor.

We should also explain, why $L(\mathbb{F}_r)$ is always a II_1 factor. Since $L(\mathbb{F}_2)$ is a II_1 factor, this will follow once we show that $L(\mathbb{F}_r) = L(\mathbb{F}_2)_{1/\sqrt{r-1}}$ in the next section.

5.2 Product and compression formulas

In this section we prove the formulas (5.2) and (5.3). All the hard work has already been done. And as a corollary of the compression formula (5.3) we are actually able to determine the fundamental group of $L(\mathbb{F}_\infty)$.

Theorem 5.5 (Product formula)**.**

$$L(\mathbb{F}_r) * L(\mathbb{F}_{r'}) = L(\mathbb{F}_{r+r'}), \quad 1 < r, r' \leq \infty.$$

Proof. Let $(\mathcal{M}, \tau)$ be a W*-probability space with τ a faithful trace, such that $\mathcal{M}$ contains copies R and $\widetilde{R}$ of the hyperfinite II_1 factor and a semicircular family $\nu = \{X_t \mid t \in T\}$ such that $(R, \widetilde{R}, \nu)$ is free. Let

$$\begin{aligned} L(\mathbb{F}_r) &= \mathcal{A} = W^*(R \cup \{p_s X_s p_s \mid s \in S\}), \\ L(\mathbb{F}_{r'}) &= \mathcal{B} = W^*(\widetilde{R} \cup \{q_s X_s q_s \mid s \in S'\}), \end{aligned}$$

where S and S' are disjoint subsets of T, $p_s \in R$, $q_s \in \widetilde{R}$ are projections and

$$r = 1 + \sum_{s \in S} \tau(p_s)^2, \quad r' = 1 + \sum_{s \in S'} \tau(q_s)^2.$$

Then $\mathcal{A}$ and $\mathcal{B}$ are free in $\mathcal{M}$, and so

$$L(\mathbb{F}_r) * L(\mathbb{F}_{r'}) = \mathcal{C} = W^*(R \cup \widetilde{R} \cup \{p_s X_s p_s \mid s \in S\} \cup \{q_s X_s q_s \mid s \in S'\}).$$

Let $\mathcal{C}_0 = W^*(R \cup \widetilde{R})$. By Theorem 4.17 there is a semicircular element $Y \in \mathcal{C}_0$ such that R and Y are free and generate $\mathcal{C}_0$. Since $\mathcal{C}_0$ is a II_1 factor (being isomorphic to $L(\mathbb{F}_2)$), there are unitaries $u_s \in \mathcal{C}_0$ such that $u_s q_s u_s^* = f_s \in R$. Then

$$\mathcal{C} = W^*(R \cup \{Y\} \cup \{p_s X_s p_s \mid s \in S\} \cup \{f_s(u_s X_s u_s^*) f_s \mid s \in S'\}.$$

It follows from Corollary 3.16, that $\{X_s\}_{s\in S} \cup \{u_s X_s u_s^*\}_{s\in S'} \cup \{Y\}$ is a semicircular family free of R, and counting shows $\mathcal{C} = L(\mathbb{F}_{r+r'})$. $\square$

Theorem 5.6 (Compression formula). *$L(\mathbb{F}_r)$ is a II_1 factor, and*

$$L(\mathbb{F}_r)_\gamma = L\left(\mathbb{F}_{(r-1)\gamma^{-2}+1}\right) \tag{5.10}$$

for $1 < r \leq \infty$ and $0 < \gamma < \infty$.

Proof. To prove that $L(\mathbb{F}_r)$ is a II_1 factor we first show (5.10) in the case $r = 2$. But since the proof is identical to the proof with $r > 1$ arbitrary, we will not make a distinction between the two cases.

Case 1. Suppose $0 < \gamma < 1$. Let $L(\mathbb{F}_r)$ be generated by R and $(p_t X_t p_t)_{t\in T}$ as in Definition 5.1, so $1 + \sum_t \tau(p_t)^2 = r$. Let $p \in L(\mathbb{F}_r)$ be a projection from R with trace γ. Without loss of generality we may assume that $p_t \leq p$ for each t. From Lemma 4.2 it follows that

$$pL(\mathbb{F}_r)p = W^*(pRp \cup \{p_t X_t p_t \mid t \in T\}) = W^*(pRp \cup \{p_t(\tau(p)^{-1/2} p X_t p) p_t \mid t \in T\}),$$

and since pRp is again the hyperfinite II_1 factor, Theorem 3.17 shows that $pL(\mathbb{F}_r)p$ is a free group factor $L(\mathbb{F}_s)$ with

$$s = 1 + \sum_t (\tau(p)^{-1}\tau(p_t))^2 = 1 + \gamma^{-2} \sum_t \tau(p_t)^2 = 1 + \gamma^{-2}(r-1).$$

Case 2. If $\gamma = n \in \{2, 3, \ldots\}$, then $L(\mathbb{F}_r)_n = L(\mathbb{F}_{(r-1)n^{-2}+1})$ is equivalent to proving

$$L(\mathbb{F}_r) = pL(\mathbb{F}_{(r-1)n^{-2}+1})p,$$

where p is a projection in an $n \times n$ system of matrix units in $L(\mathbb{F}_{(r-1)n^{-2}+1})$. Such a p exists (in R), and the equality follows the case $\gamma < 1$.

Case 3. If $\gamma > 1$ is non-integer, just use the cases proved so far and the rule $\mathcal{A}_{st} = (\mathcal{A}_s)_t$. $\square$

The following result, first proved by Rădulescu [11], is an immediate consequence of (5.10).

Corollary 5.7. *The fundamental group of $L(\mathbb{F}_\infty)$ is $\mathbb{R}_+$.*

5.3 Main results

A consequence of the formulas (5.2) and (5.3) is that the isomorphism problem concerning the free group factors $L(\mathbb{F}_r)$ must be one of two extremes. But first we state a result about the fundamental group.

Theorem 5.8. *One (and only one) of the following holds*

(i) *The fundamental group $\mathcal{F}(L(\mathbb{F}_r))$ is $\mathbb{R}_+$ for every finite $r > 1$.*

(ii) *The fundamental group $\mathcal{F}(L(\mathbb{F}_r))$ is $\{1\}$ for every finite $r > 1$.*

Proof. From (5.3) and Remark B.9 we see that $\mathcal{F}(L(\mathbb{F}_r))$ is independent of r. Let $x, y > 1$, and let $t = \left(\frac{y-1}{x-1}\right)^{\frac{1}{2}}$. Then we have $x = (y-1)t^{-2} + 1$, and hence $L(\mathbb{F}_y)_t \simeq L(\mathbb{F}_x)$. It follows that $L(\mathbb{F}_x) \simeq L(\mathbb{F}_y)$ if and only if $t \in \mathcal{F}(L(\mathbb{F}_y))$, and so this happens if and only if $t \in \mathcal{F}(L(\mathbb{F}_r))$ for every r.

Suppose (ii) fails to hold. Then there is some $r > 1$ and $t \neq 1$ such that $L(\mathbb{F}_r)_t \simeq L(\mathbb{F}_r)$. Using (5.3) this means that there is some $x \neq y$ such that $L(\mathbb{F}_x) \simeq L(\mathbb{F}_y)$. Then by (5.2)

$$L(\mathbb{F}_{x+a}) \simeq L(\mathbb{F}_{y+a}), \quad a > 1,$$

and it follows that $\mathcal{F}(L(\mathbb{F}_r))$ contains an open interval. Since $\mathcal{F}(L(\mathbb{F}_r))$ is a multiplicative subgroup of $]0, \infty[$, it follows that $\mathcal{F}(L(\mathbb{F}_r)) = \mathbb{R}_+$. $\square$

We have the following immediate strengthening, which is the first version of our main theorem of this thesis.

Theorem 5.9. *One (and only one) of the following holds*

(i) *For every finite $r, s > 1$ we have $L(\mathbb{F}_r) \simeq L(\mathbb{F}_s)$, and the fundamental group of $L(\mathbb{F}_r)$ is $\mathbb{R}_+$.*

(ii) *For every finite $r, s > 1$ we have $L(\mathbb{F}_r) \not\simeq L(\mathbb{F}_s)$, and the fundamental group of $L(\mathbb{F}_r)$ is $\{1\}$.*

Proof. Suppose (ii) fails to hold. Then there is some $r > 1$ such that $\mathcal{F}(L(\mathbb{F}_r)) \neq \{1\}$ or there is $r \neq s$ such that $L(\mathbb{F}_s) \simeq L(\mathbb{F}_r)$. In the first case we see from Theorem 5.8 that $L(\mathbb{F}_r) = \mathbb{R}_+$ for every $r > 1$, and it follows from (5.3) that $L(\mathbb{F}_r) \simeq L(\mathbb{F}_s)$ for all r, s. So (i) holds.

If $L(\mathbb{F}_s) \simeq L(\mathbb{F}_r)$ for some $r \neq s$, let $t = \left(\frac{r-1}{s-1}\right)^{\frac{1}{2}}$. Then $t \neq 1$ and $t \in \mathcal{F}(L(\mathbb{F}_r))$, and we are back in the first case. $\square$

The result does not relate $L(\mathbb{F}_r)$ with r finite to the free group factor $L(\mathbb{F}_\infty)$. Rădulescu managed to include $L(\mathbb{F}_\infty)$ in the above result, and we shall also do so, but first we need some preparations.

Lemma 5.10. *Let $(\mathcal{M}, \tau)$ be a W^*-probability space, where τ is a faithful trace, and suppose $\mathcal{M}$ contains a copy R of the hyperfinite II_1 factor which is free of an infinite semicircular family $(x_s)_{s\in S}$. Partition $S = S' \cup S''$ with S' infinite, let $\sigma, \nu \in S'$ be distinct, and take a projection $p \in R$ with trace $1/n$. Let $\mathcal{A} = W^*(px_\sigma p, px_\nu p) \subseteq p\mathcal{M}p$. Suppose $(z_t)_{t\in T}$ is an at most countable semicircular family in $(\mathcal{A}, n\tau)$. Then there is a semicircular family $(a_t)_{t\in T}$ in $\mathcal{M}$, free of $R \cup (x_s)_{s\in S''}$, such that $z_t = pa_tp$ for each $t \in T$.*

Proof. Imbed p in a system of matrix units $(e_{ij})_{i,j=1}^n$ in R such that $p = e_{11}$. Let $f_s = px_sp$ for $s \in S'$. Then by Proposition 3.20 the family $(f_s)_{s\in S'}$ is a semicircular family in $p\mathcal{M}p$ free of $p(R \cup (x_s)_{s\in S''})p$. With $\mathcal{A} = W^*(px_\sigma p, px_\nu p)$ and $V = S' \setminus \{\sigma, \nu\}$, we see that $(f_s)_{s\in V}$ is free of $\mathcal{A}$ and hence of $(z_t)_{t\in T}$.

Let $\alpha : \{1, \ldots, n\}^2 \times T \to V$ be an injective map. This is possible, since T is at most countable, and V is infinite. With $g_{\alpha(i,j,t)} = (f_{\alpha(i,j,t)} + if_{\alpha(j,i,t)})/\sqrt{2}$ put

$$a_t = pz_tp + \sum_{2\le i\le n} e_{i1}f_{\alpha(i,j,t)}e_{1i} + \sum_{1\le i<j\le n} e_{i1}g_{\alpha(i,j,t)}e_{1j} + e_{j1}(g_{\alpha(i,j,t)})^* e_{1i}.$$

Then we see from Theorem 3.19, that $(a_t)_{t\in T}$ is a semicircular family free of $R \cup (x_s)_{s\in S''}$. □

We have now reached the final version of the main theorem of this thesis.

Theorem 5.11. *One (and only one) of the following holds*

(I) *For $r \in \,]1, \infty]$ the factors $L(\mathbb{F}_r)$ are all isomorphic, and the fundamental group of $L(\mathbb{F}_r)$ is $\mathbb{R}_+$.*

(II) *For $r \in \,]1, \infty]$ the factors $L(\mathbb{F}_r)$ are mutually non-isomorphic, and the fundamental group of $L(\mathbb{F}_r)$ is $\{1\}$, when $r < \infty$, while $\mathcal{F}(L(\mathbb{F}_\infty)) = \mathbb{R}_+$.*

Proof. By Theorem 5.9, and since $\mathcal{F}(L(\mathbb{F}_\infty)) = \mathbb{R}_+$, it suffices to prove that if all $L(\mathbb{F}_r)$ are isomorphic for finite r, then $L(\mathbb{F}_r) \simeq L(\mathbb{F}_\infty)$ for some (equivalently, all) $r < \infty$.

Let $(X_s)_{s\in S}$ be an infinite semicircular family free of R in some W*-probability space $(\mathcal{M}, \tau)$, where τ is a faithful trace. Let

$$\{\sigma_1, \sigma_2, \ldots\} \cup \{\nu_1, \nu_2, \cdots\}$$

be an infinite subset of distinct indices in S with infinite complement. Fix some finite $r > 1$ and infinitely many projections $(p_i)_{i=1}^\infty$ such that $\tau(p_i) = 1/n_i$ for some $n_i \in \mathbb{N}$, and such that

$$r = 1 + 2\sum_{i=1}^{\infty} \tau(p_i)^2.$$

By definition $L(\mathbb{F}_r)$ is

$$L(\mathbb{F}_r) = W^*(R \cup \{p_i X_{\sigma_i} p_i, p_i X_{\nu_i} p_i \mid i \in \mathbb{N}\}).$$

Let S be partitioned as $S = \bigcup_{i \geq 1} S_i'$ with infinite sets, and let S_i be finite subsets of S_i' such that $\sigma_i, \nu_i \in S_i$. Let N_i be the cardinality of S_i.

Let $\mathcal{A}_i = W^*(p_i X_{\sigma_i} p_i, p_i X_{\nu_i} p_i)$ inside $p_i \mathcal{M} p_i$. Since we have assumed that all $L(\mathbb{F}_s)$ are isomorphic for finite s, and $\mathcal{A}_i \simeq L(\mathbb{F}_2)$, there is a semicircular family $(Z_s)_{s \in S_i}$ in $\mathcal{A}_i$ that generates $\mathcal{A}_i$.

Claim. *There is a semicircular family $(T_s)_{s \in \bigcup_i S_i}$ in $\mathcal{M}$ free of R such that $Z_s = p_i T_s p_i$ for all $s \in S_i$, $i \in \mathbb{N}$.*

Proof. By Lemma 5.10 we see that there is a semicircular family $(T_s)_{s \in S_1}$ such that $p_1 T_s p_1 = Z_s$ for $s \in S_1$ and such that $(T_s)_{s \in S_1}$ is free of $R \cup \{X_s \mid s \in S_i', i = 2, 3, \ldots\}$. Inductively, we may assume that up to a given $n \in \mathbb{N}$ we have constructed the family $(T_s)_{s \in \bigcup_{i=1}^n S_i}$ such that

1. $(T_s)_{s \in \bigcup_{i=1}^n S_i}$ is a semicircular family,
2. $(T_s)_{s \in \bigcup_{i=1}^n S_i}$ is free of $R \cup \{X_s \mid s \in S_i', i = n+1, n+2, \ldots\}$,
3. $p_i T_s p_i = Z_s$ for $s \in S_i$, $i = 1, \ldots, n$.

Now we apply Lemma 5.10 with

$$\begin{array}{rcl} \{x_s\}_{s \in S'} & = & \{X_s\}_{s \in S_{n+1}'}, \\ \{x_s\}_{s \in S''} & = & (T_s)_{s \in \bigcup_{i=1}^n S_i} \cup \{X_s \mid s \in S_i', i = n+2, \ldots\}, \\ p = p_{n+1}, & & x_\sigma = X_{\sigma_{n+1}}, \quad x_\nu = X_{\nu_{n+1}}, \quad (z_t)_{t \in T} = (Z_s)_{s \in S_{n+1}}. \end{array}$$

As the family $(T_s)_{s \in S_{n+1}}$ we take the family $(a_t)_{t \in T}$ provided by the lemma. Now the claim follows easily by induction. ■

From the claim we see that

$$\begin{aligned} L(\mathbb{F}_r) &= W^*(R \cup \{p_i X_{\sigma_i} p_i, p_i X_{\nu_i} p_i \mid i \in \mathbb{N}\}) \\ &= W^*(R \cup \{Z_s \mid s \in S_i,\ i \in \mathbb{N}\}) \\ &= W^*(R \cup \{p_i T_s p_i \mid s \in S_i,\ i \in \mathbb{N}\}) = L(\mathbb{F}_t), \end{aligned}$$

where

$$t = 1 + \sum_{s \in S_i, i \in \mathbb{N}} \tau(p_i)^2 = 1 + \sum_{i=1}^{\infty} N_i \tau(p_i)^2 = 1 + \sum_{i=1}^{\infty} \frac{N_i}{n_i^2}.$$

Choosing N_i large enough (for instance $N_i \geq n_i^2$) we obtain $t = \infty$. This concludes the proof and also this thesis. $\square$

APPENDIX A

Direct limits

When arguing that the interpolated free group factors are well-defined in the way we have defined them (see Definition 5.1) we needed some limit results about von Neumann algebras. We have put the needed result, Prosition A.3, here for reference.

Proposition A.1 (C*-version)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras. Let $(\mathcal{A}_i)_{i\in I}$ be a directed system of C*-subalgebras of $\mathcal{A}$ whose union is dense in $\mathcal{A}$, and let $(\mathcal{B}_i)_{i\in I}$ be a directed system of C*-subalgebras of $\mathcal{B}$ whose union is dense in $\mathcal{B}$. Suppose that for each $i \in I$ there is an isomorphism $\varphi_i : \mathcal{A}_i \to \mathcal{B}_i$, and that these isomorphisms are compatible in the sense that if $i \leq j$, then the restriction of φ_j to $\mathcal{A}_i$ agrees with φ_i. Then $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.*

Sketch of proof. Define $\varphi : \bigcup \mathcal{A}_i \to \mathcal{B}$ by $\varphi(a) = \varphi_i(a)$, if $a \in \mathcal{A}_i$. This is well-defined and isometric, hence extends to an isomorphism $\mathcal{A} \simeq \mathcal{B}$. □

For the extension in the above proof one uses the fact that $\mathcal{B}$ is complete in the norm topology. When we pass to von Neumann algebras, we no longer have completeness, not even a metric for that matter, in the strong operator topology. Hence the direct limit proposition becomes a bit more technical, but is still very useful. But first of all we need a lemma that ensures some sort of completeness.

Lemma A.2. *Let $(\mathcal{M}, \varphi)$ be a von Neumann algebra with a faithful, normal state. Then $||a||_2 = \varphi(a^*a)^{1/2}$ defines a norm on $\mathcal{M}$ (called the 2-norm). On bounded sets of $\mathcal{M}$ the topology induced by the 2-norm coincides with the strong operator topology, and the closed unit ball $(\mathcal{M})_1$ is complete with respect to the 2-norm.*

Proof. Using the GNS construction we may assume that $\mathcal{M} \subseteq B(H)$ has a cyclic, separating vector $\xi \in H$ such that $\varphi(a) = \langle a\xi, \xi\rangle$. Then a simple calculation shows that $||b||_\varphi = ||b\xi||$ for any $b \in \mathcal{M}$. It is then clear that $||\ ||_2$ is a norm and that strong operator convergence implies 2-norm convergence.

Suppose conversely, that $(a_\lambda)_{\lambda\in\Lambda}$ is a bounded net that converges to a in 2-norm. For any $b \in \mathcal{M}'$ we find

$$||(a - a_\lambda)b\xi|| = ||b(a - a_\lambda)\xi|| \leq ||b||\ ||(a - a_\lambda)\xi|| = ||b||\ ||a - a_\lambda||_2 \to 0.$$

Since ξ is separating for $\mathcal{M}$, it is cyclic for $\mathcal{M}'$, and this proves that $a_\lambda \to a$ strongly.

To see that $(\mathcal{M})_1$ is complete under $||\ ||_2$, let $(a_n)_{n=1}^\infty$ be a Cauchy sequence in $(\mathcal{M})_1$. Then $||(a_n - a_m)\xi|| \to 0$ as $m, n \to \infty$. Hence for any $b \in \mathcal{M}'$

$$||(a_n - a_m)b\xi|| \to 0$$

so that $(a_n b\xi)_{n=1}^\infty$ is a Cauchy sequence in H. Define $a : \mathcal{M}'\xi \to H$ as $a(b\xi) = \lim_{n\to\infty} a_n b\xi$. Since ξ is separating for $\mathcal{M}'$, this is well-defined. Notice also that

$$||a(b\xi)|| = \lim_{n\to\infty} ||a_n b\xi|| \leq \sup_{n\in\mathbb{N}} ||a_n||\ ||b\xi|| \leq ||b\xi||, \quad b \in \mathcal{M}' \tag{A.1}$$

so that a extends to a bounded operator $a \in B(H)$. We claim that $a \in (\mathcal{M})_1$ and that $a_n \to a$ with respect to $||\ ||_2$. For the first let $b, c \in \mathcal{M}'$. Then

$$abc\xi = \lim a_n bc\xi = b \lim a_n c\xi = bac\xi,$$

and by cyclicity of ξ this proves $ab = ba$, so that $a \in \mathcal{M}'' = \mathcal{M}$. It is clear that

$$||a - a_n||_2 = ||(a - a_n)\xi|| = ||\lim_m a_m\xi - a_n\xi|| = \lim_m ||a_m - a_n||_2 \to 0, \quad \text{as } n \to \infty.$$

So the sequence $(a_n)_{n=1}^\infty$ converges to a with respect to $||\ ||_2$. To complete the proof, observe that $||a|| \leq 1$ by (A.1). ☐

Proposition A.3 (W*-version)**.** *Let $(\mathcal{A}, \psi)$ and $(\mathcal{B}, \tau)$ be von Neumann algebras with faithful, normal states. Let $(\mathcal{A}_i)_{i\in I}$ be a directed system of von Neumann subalgebras of $\mathcal{A}$ whose union is weakly dense in $\mathcal{A}$, and let $(\mathcal{B}_i)_{i\in I}$ be a directed system of von Neumann subalgebras of $\mathcal{B}$ whose union is weakly dense in $\mathcal{B}$. Suppose that for each $i \in I$ there is a state-preserving isomorphism $\varphi_i : \mathcal{A}_i \to \mathcal{B}_i$, and that these isomorphisms are compatible in the sense that if $i \leq j$, then the restriction of φ_j to $\mathcal{A}_i$ agrees with φ_i. Then $\mathcal{A}$ and $\mathcal{B}$ are isomorphic with a state-preserving isormorphism.*

Proof. Let $\mathcal{A}_0$ be the norm closure of $\bigcup \mathcal{A}_i$, and let $\mathcal{B}_0$ be the norm closure of $\bigcup \mathcal{B}_i$. By the previous proposition there is an isomorphism $\varphi : \mathcal{A}_0 \to \mathcal{B}_0$, and since $\tau \circ \varphi = \psi$ on a norm dense subalgebra, it follows by continuity that $\tau \circ \varphi = \psi$ on all of $\mathcal{A}_0$.

Let $a \in (\mathcal{A})_1$. By Kaplansky's Density Theorem and Lemma A.2 we see that a is the limit with repect to $||\ ||_2$ of a bounded sequence $(a_n)_{n\in\mathbb{N}}$ in $(\mathcal{A}_0)_1$. Hence (a_n) is a Cauchy sequence, and $(\varphi(a_n))$ is a Cauchy sequence in $(\mathcal{B})_1$ (with respect to $||\ ||_2$ on $\mathcal{B}$). Denote the limit by $\varphi(a)$. We claim that the definition of $\varphi(a)$ is independent of the choice of sequence converging to a and that with this definition (extrapolated linearly to the rest of $\mathcal{A}$) φ is a state-preserving isomorphism of $\mathcal{A}$ onto $\mathcal{B}$. The proof of these claims is left to the reader. $\square$

APPENDIX B

The fundamental group of a II_1 factor

B.1 Definition

If there are any topology geeks around, they should be warned. The fundamental group of a II_1 factor described here has nothing to do with the fundamental groups encountered in algebraic topology.

In all of the following $\mathcal{A}$ will denote a II_1 factor with trace τ. It is well-known that the image of the projections under τ is the interval $[0,1]$. Also, for any two projections $p, q \in \mathcal{A}$ we have $p \sim q$ if and only if $\tau(p) = \tau(q)$, and in particular τ is unique.

Since $p \sim q$ implies that $p\mathcal{A}p \simeq q\mathcal{A}q$, the following definition makes sense.

Definition B.1. Let $\mathcal{A}$ be a II_1 factor, and let $t \in \,]0,1]$. Then $\mathcal{A}_t$ denotes any II_1 factor isomorphic to $p\mathcal{A}p$, where p is a projection in $\mathcal{A}$ with $\tau(p) = t$.

We will always simply think of $\mathcal{A}_t$ as the von Neumann algebra $p\mathcal{A}p$ instead of a class of von Neumann algebras, and we write $\mathcal{A}_t = p\mathcal{A}p$.

Lemma B.2. *Let $s, t \in \,]0,1]$. Then $(\mathcal{A}_s)_t = \mathcal{A}_{st}$.*

Proof. Choose a projection p in $\mathcal{A}$ such that $\tau(p) = s$. Then $p\mathcal{A}p$ is again a II_1 factor, and τ' given by $\tau'(a) = \frac{\tau(a)}{\tau(p)}$ is a trace on $p\mathcal{A}p$ (and hence the only trace on $p\mathcal{A}p$). Choose a projection $q \in p\mathcal{A}p$ with $\tau'(q) = t$. Then

$$(\mathcal{A}_s)_t = q(p\mathcal{A}p)q = q\mathcal{A}q,$$

and since $\tau(q) = \tau(p)\tau'(q) = st$, we have proved $(\mathcal{A}_s)_t = \mathcal{A}_{st}$. □

Lemma B.3. *Let $t \in \,]0,1]$ and $n \in \mathbb{N}$ be given. Then $M_n(\mathcal{A}_t) = M_n(\mathcal{A})_t$.*

Proof. With p a projection in $\mathcal{A}$ of trace t we deduce by straightforward calculation that

$$M_n(p\mathcal{A}p) = PM_n(\mathcal{A})P,$$

where P is the diagonal matrix

$$P = \begin{pmatrix} p & & \\ & \ddots & \\ & & p \end{pmatrix} \in M_n(\mathcal{A}).$$

Since $M_n(\mathcal{A})$ is a II_1 factor with trace τ_n given by

$$\tau_n \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \frac{1}{n}\sum_{i=1}^{n} \tau(a_{ii}),$$

we see that $\tau_n(P) = \tau(p) = t$, and hence

$$M_n(\mathcal{A}_t) = PM_n(\mathcal{A})P = M_n(\mathcal{A})_t.$$

□

Corollary B.4. *With $n \in \mathbb{N}$ we have $M_n(\mathcal{A}_{1/n}) = M_n(\mathcal{A})_{1/n} = \mathcal{A}$.*

Proof. The first equality follows from the lemma, and the second is obvious. □

Lemma B.5. *Let $t > 0$, and let $m, n \in \mathbb{N}$ be such that $m \geq t$, $n \geq t$. Then*

$$M_m(\mathcal{A})_{t/m} = M_n(\mathcal{A})_{t/n}.$$

Proof. From the previous corollary and Lemma B.2 we see that

$$M_m(\mathcal{A})_{t/m} = \Big(M_n(M_m(\mathcal{A}))_{1/n}\Big)_{t/m} = M_n(M_m(\mathcal{A}))_{\frac{t}{mn}}.$$

Similarly,

$$M_n(\mathcal{A})_{t/n} = M_m(M_n(\mathcal{A}))_{\frac{t}{mn}}.$$

This proves the lemma, since $M_m(M_n(\mathcal{A})) \simeq M_{mn}(\mathcal{A}) \simeq M_n(M_m(\mathcal{A}))$. □

With Lemma B.5 well in place the following definition makes sense.

Definition B.6. Let $\mathcal{A}$ be a II_1 factor, and let $t > 0$ be given. Then we let $\mathcal{A}_t$ be $(M_n(\mathcal{A}))_{t/n}$, where n is an integer such that $n \geq t$.

Of course, this new definition of $\mathcal{A}_t$ for $0 < t \leq 1$ agrees with the old definition; simply take $n = 1$. An immediate consequence of Lemma B.2 and Lemma B.3 is the following proposition.

Proposition B.7. *Let $s, t > 0$. Then $(\mathcal{A}_s)_t = \mathcal{A}_{st}$.*

It may or may not happen that $\mathcal{A} \simeq \mathcal{A}_t$ for some $t > 0$. The set of $t > 0$, where $\mathcal{A}_t = \mathcal{A}$ is of particular interest.

Definition B.8. Let $\mathcal{A}$ be a II_1 factor. The *fundamental group* of $\mathcal{A}$ is the set of real numbers $t > 0$ such that $\mathcal{A} \simeq \mathcal{A}_t$. We denote it by $\mathcal{F}(\mathcal{A})$.

It is a consequence of Proposition B.7, that $\mathcal{F}(\mathcal{A})$ is a subgroup of $\mathbb{R}_+$.

Remark B.9. It follows from Proposition B.7 that the fundamental groups of $\mathcal{A}$, $p\mathcal{A}p$ and $M_n(\mathcal{A})$ are all the same. Here $p \in \mathcal{A}$ is any non-zero projection. And more generally, we get that the fundamental groups of $\mathcal{A}$ and $\mathcal{A}_t$ are the same for any $t \in \mathbb{R}_+$.

B.2 The fundamental group of the hyperfinite II_1 factor

The fundamental group was introduced by Murray and von Neumann [9] as an invariant of von Neumann algebras with the hope to be able to distinguish between various von Neumann algebras. As it turned out, calculating fundamental groups of II_1 factors is very difficult. For a long time the only known fundamental group was that of the hyperfinite II_1 factor. And as mentioned in the introduction the fundamental groups of the free group factors $L(\mathbb{F}_n)$ are not yet known.

We will give a short argument of the fact that $\mathcal{F}(R) = \mathbb{R}_+$ build on the uniqueness of a hyperfinite factor of type II_1 and another characterization of hyperfiniteness. For a proof of the following theorem as well as the uniqueness of the hyperfinite II_1 factor we refer to [9].

In the following theorem $|| \, ||_2$ denotes the norm induced by the unique, faithful trace on a II_1 factor, see also Lemma A.2. Note the following inequalities

$$||ab||_2 \leq ||a|| \, ||b||_2 \qquad \text{and} \qquad ||ab||_2 \leq ||a||_2 \, ||b||.$$

Theorem B.10. *Let $\mathcal{A}$ be a II_1 factor on a separable Hilbert space. Then $\mathcal{A}$ is hyperfinite if and only if the following condition is true. To each $\varepsilon > 0$ and each finite set $\{a_1, \dots, a_n\}$ of operators in $\mathcal{A}$ there is a finite dimensional C^*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ and operators $b_1, \dots, b_n$ in $\mathcal{B}$ such that $||a_i - b_i||_2 < \varepsilon$ for all $i = 1, \dots, n$.*

Theorem B.11. *The fundamental group of the hyperfinite II_1 factor is $\mathbb{R}_+$.*

Proof. Since the fundamental group is a subgroup of $\mathbb{R}_+$, it will suffice to prove that it contains $]0, 1[$. So we need only show that pRp is hyperfinite for any non-zero projection p in the hyperfinite II_1 factor R. And for this it will suffice to see that the condition of the previous theorem is satisfied for the II_1 factor $\mathcal{A} = pRp$. Note that if we let $|| \, ||_2$ denote the 2-norm on R, then the 2-norm on $\mathcal{A}$ is $||p||_2^{-1} || \, ||_2$.

Let $\varepsilon > 0$ be arbitrary, and let $a_1, \dots, a_n \in pRp$ be given. Find $C > 0$ such that $||a_i|| \leq C$ for each i. Choose a subprojection q of p with rational trace such that $||p - q||_2 < \varepsilon/(2C||p||_2)$.

There is a generating, increasing sequence $(\mathcal{M}_i)_{i=1}^{\infty}$ of finite type I subfactors of R that contains projections of every rational trace. In particular, there is a projection $\widetilde{q}$ in some $\mathcal{M}_j$ with the same trace as q, and so there is a unitary $u \in R$ such that $u\widetilde{q}u^* = q$. By replacing $(\mathcal{M}_i)_{i=1}^{\infty}$ with $(u\mathcal{M}_i u^*)_{i=1}^{\infty}$, we may assume that $q \in \mathcal{M}_j$.

Then by Kaplansky's Density Theorem and Lemma A.2 there are operators $c_1, \ldots, c_n$ in some $\mathcal{M}_k$ for a $k > j$ such that $||a_i - c_i||_2 < \varepsilon/(2||p||_2)$ for each $i = 1, \ldots, n$. Let $\mathcal{B} = \mathcal{M}_k$, and let $b_i = qc_iq$. Then $b_i \in \mathcal{B}$, the C*-algebra $\mathcal{B}$ is finite dimensional, and

$$a_i - b_i = pa_ip - qc_iq = qa_iq - qc_iq + (p-q)a_i(p-q).$$

And so,

$$||a_i - b_i||_2 \leq ||q(a_i - c_i)q||_2 + ||(p-q)a_i(p-q)||_2 \leq ||a_i - c_i||_2 + C||p-q||_2 < ||p||_2\varepsilon.$$

We have now shown that the condition in the previous theorem is satisfied, and this completes the proof that $\mathcal{F}(R) = \mathbb{R}_+$. □

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