Algebra 2.

- Functional Equation.

We will begin at 8:05 PM MMT.

BEFORE WE GET TO THE PROBLEMS,

FE is ...

One or more unknown function in an equation.

Two Parts to prove:

1. Uniqueness
2. Existence.

$\mathbb{R}$: real number

$\mathbb{Q}$: rational number.

$\mathbb{Z}$: integer.

$\mathbb{N}$: natural number.

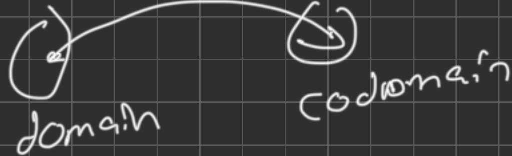
(1) Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that

$$x^2 f(x) + f(1-x) = 2x - x^4$$

for all real numbers x.

Hint: Substitute $(1-x)$ into x.

1.5 Similar Problem



Find all functions $f: \mathbb{R}_{>0} \to \mathbb{R}$ such that

$$\boxed{2f(x) + f\left(\frac{1}{x}\right) = 3x.}$$

$P(x)$:

$f(x) = ?$

$\frac{1}{\frac{1}{x}} = x$ [illegible] for all x. $\mathbb{R}_{>0}$

$P\left(\frac{1}{x}\right)$: $\boxed{2f\left(\frac{1}{x}\right) + f(x) = \frac{3}{x}}$

$f\left(\frac{1}{x}\right) + 2f(x) = 3x.$

$f(x) = ?$

$f(x) = a$

$f\left(\frac{1}{x}\right) = b$

(1): $2a + b = 3x$

(2): $a + 2b = \frac{3}{x}$

$a = ?$

$\times 2 \Rightarrow 4a + 2b = 6x$

$3a = 6x - \frac{3}{x}$

$a = 2x - \frac{1}{x}$

$\therefore f(x) = \boxed{2x - \frac{1}{x}}$. uniqueness.

Existence: LHS $= 2f(x) + f\left(\frac{1}{x}\right)$

$= 4x - \cancel{\frac{2}{x}} + \cancel{\frac{2}{x}} - x$

$= 3x =$ RHS.

A few tips

- Check $f: \boxed{} \to \boxed{}$
- Guess the answer first.
- Try substituting
  - ~~(0, 0)~~
  - (1, 1)
  - (x, x)
  - (x, −x)
  - (y, x) ← $\boxed{\text{symmetry}}$

  in (x, y).
- check for injective, surjective

8:26

② Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$P(x,y)$: $f(x+y) - f(x-y) = f(x)f(y)$, $\forall x, y \in \mathbb{R}$.

$f(x) - 0 = f(x)f(-y)$

$P(0,0) \to f(0) - f(0) = f(0)^2$

$0 = f(0)^2$

$\boxed{f(0) = 0}$

$P(0,x)$: $f(x) - f(-x) = f(0)f(x) = 0$.

$\boxed{f(x) - f(-x) = 0}$

$\boxed{f(x) = f(-x)}$

$P(x,-y)$: $f(x-y) - f(x+y) = f(x)f(-y) = f(x)f(y)$

$f(x-y) - f(x+y) = f(x)f(y) = f(x+y) - f(x-y)$ $f(x)=0$

$f(x-y) - f(x+y) = f(x+y) - f(x-y)$

$\boxed{f(x+y) = f(x-y)}$

8:37 : $P(\frac{x}{2},\frac{x}{2})$: $f(x) = f(0)$.

$f(x) = 0$

LHS $= 0 - 0$
$= 0$
$=$ RHS.

Existence ✓

Uniqueness ✓

Ans: $f(x) = 0$ $\forall x$

SYMMETRY $ax+b$.

(3) Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that

$f(x,y)$: $f(x+y) = \boxed{x + f(y)}$ $\forall x, y \in \mathbb{R}$.

$y + f(x)$ $\to f(x) = ?$

$f(x) = x$.

$f(x) = x+1$ $x+c$

$f(x) = x$ ✓

$f(x) = x+1$ ✓

$f(y,x)$: $f(x+y) = y + f(x)$ $f(x) = x+c$ ✓

cx $cx+d$

$f(x)$

$x + f(y) = y + f(x)$.

$(x,0)$: $x + f(0) = f(x)$. $\leftarrow$ uniqueness

$\boxed{f(x) = x + c.}$ $c \in \mathbb{R}$

Existence:

$LHS = f(x+y) = x + \boxed{y+c} = f(y)$

$= x + f(y)$

$= RHS$.

$f(x) = x + c$ $\forall x \in \mathbb{R}$

Made with Goodnotes

$f(x) = x + c.$

④ Find all functions $f: \mathbb{R} \to \mathbb{R}$ satisfying

$f(x) = x+1$

$$f(xy + 2x + 2y - 1) = f(x)f(y) + f(y) + x - 2$$

$\forall x, y \in \mathbb{R}.$

$f(yx + 2y + 2x - 1)$

$f(y)f(x) + f(x) + y - 2$

$P(y, x): f(xy + 2x + 2y - 1) = f(y)f(x) + f(x) + y - 2$

$$\cancel{f(x)f(y)} + f(y) + x \cancel{- 2} = \cancel{f(x)f(y)} + f(x) + y \cancel{- 2}$$

$$f(y) + x = f(x) + \cancel{y}$$

$$x + f(0) = f(x)$$

constant.

$$f(x) = \boxed{x + c} \quad \text{uniqueness.}$$

Existence: $LHS = \cancel{xy} + 2x + 2y - 1 + c$

$RHS = (x+c)(y+c) + (y+c) + x - 2.$

$= \cancel{xy} + cx + cy + c^2 + y + c + x - 2.$

$2x = cx + x \implies c = 1.$

$1 + c = c^2 + c - 2$

$c = 1$

$2y = cy + y \Rightarrow c = 1$ $f(x) = x+1$

⑤ Find all functions $f: \mathbb{R}^+ \to \mathbb{R}^+$ s.t.

$$x^2(f(x)+f(y)) = (x+y) f(yf(x))$$

$\forall x, y \in \mathbb{R}^+$

$(f(x) = \frac{1}{x})$

$P(1,1): f(1)+f(1) = 2f(f(1))$

$\cancel{2}f(1) = \cancel{2}f(f(1))$

$f(1) = f(f(1))$

$f(?)$

$x, y \Rightarrow f(?)$

$f(1)$

$P(f(1),1): f(1)^2 (f(1)+f(1)) = (f(1)+1)f(1)$

$2f(1)^3 = f(1)^2 + f(1)$

$2f(1)^3 - f(1)^2 - f(1) = 0$

$2f(1)^3 - f(1)^2 - f(1) - 1 = 0$

$\cancel{f(1)}(2f(1)^2 - f(1) - 1)$

$\Downarrow$

$f(1) = 1$

$(1,x): (f(1)+f(x)) = (x+1)f(x)$

$f(x) + 1 = (x+1) f(x)$

$1 = x f(x)$

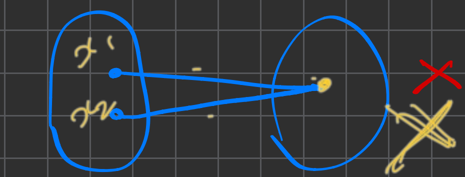
$f(x) = \frac{1}{x}$

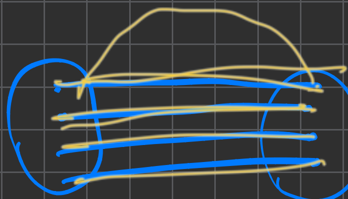
$$x^2\left(\frac{1}{x}+\frac{1}{y}\right) \neq (x+y)\frac{x}{y}$$

$x + \frac{x^2}{y}$, $\cancel{x + \frac{x}{y}}$ $\quad \cancel{\frac{x^2+xy}{y}}$

Injective (one to one)

if $f(x) = f(y)$, then $x=y$.





$\frac{x^2}{y} + x$

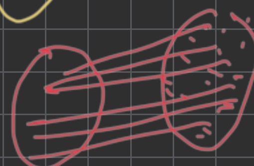
$f(x) = \frac{1}{x}$ $\forall x \in \mathbb{R}^+$

a, b.

Let $\boxed{f(a) = f(b)}$

$\Rightarrow \boxed{a=b}$

$f(\bigcirc) = f(\bigcirc)$



Surjective

there exists $f(x)=y$ $\forall y$.

Bijective

= both injective & surjective

$f(x) = x$

$f: \mathbb{R} \to \mathbb{R}$

$f(x) =$

$f(x) = 0$

$f: \mathbb{R} \to \mathbb{R}$

$f(x) = -3$

variable

$f(x) = y$

⑤ Let $f:\mathbb{Q}\to\mathbb{Q}$ be any function s.t.

$$f(x+f(y+f(z)))=\boxed{y}+f(x+z)$$

$\forall x,y\in\mathbb{Q}$. Show that f is bijective.

$f(\quad)=y.$

$x, z=0.$

$f(0+f(y+f(0)))=y+f(0).$

f is surjective.

$y=y-f(0):$

$f(0+f(y))=y$

f is $\boxed{\text{surjective.}}$

$y=\boxed{y-f(x+z)}$

$\boxed{f(\quad)}=y$ ← variable.

injective ?

assume $\boxed{f(a)=f(b)}$, then $\boxed{a=b.}$

injective ✓

$(x, a, z): f(x+f(a+f(z)))=a+f(x+z)$

$(x, b, z): f(x+f(b+f(z)))=b+f(x+z)$

equal. $\begin{cases} f(x+f(a))=a+\boxed{f(x+z)} \\ f(x+f(b))=b+\boxed{f(x+z)} \end{cases}$

$\Rightarrow \boxed{a=b.}$ ✓

$\boxed{f(z)=0}$