Foundation of Mathematical Olympiad

Euclidiad 2025-2026 Working Committee

Contents

1 Introduction

Welcome! This handout is put together by the **Euclidiad 2025–2026 Working Committee**.

1.1 About us

Euclidiad was founded by 4 IMO alumni with a shared vision, and beyond the founding members, the **Euclidiad 2025–2026 Working Committee** was formed to carry this mission forward. Our current committee consists of 6 members, most of whom were contestants for the IMO 2025. Together, we aim to create resources, organize activities, and provide guidance that can help beginners take their first steps in proof-based mathematics while also supporting advanced students preparing for competitions.

1.2 Purpose of the handout series

Our goal is simple: to make it easier for new students to enter the world of proof-based mathematics and olympiads.

We know that starting out can feel overwhelming; resources are scattered and advanced while problems can look intimidating, starting down from decrypting the phrasing of each problem to understanding what it's asking us to do.

That's why we've gathered a collection of notes, examples, and practice problems in one place and publish our handout series. Our aim is to give you a smoother path into rigorous problem solving and to enjoy problem solving. This is just the start of the series, with many more handouts planned.

The problems and discussions here are written with Myanmar Open Mathematics Competition (MOMC) and Team Selection Test (TST) style problems in mind, but they're also great practice if you're preparing for other math olympiad contests. Most importantly, we hope this handout will make learning maths more approachable and encourage you to keep exploring, practicing, and enjoying the challenge.

2 Algebra

2.1 A.M-G.M Inequality (Cauchy Induction)

The A.M-G.M inequality is probably the most famous inequality for any high school Olympiads, defined over all positive reals. We will see in later events why this is the building block of most problems and theorems.

The Inequality.

For all $\mathbb{R}^+$ $a_1, a_2, \ldots, a_n$,

$$\frac{(a_1 + a_2 + \cdots + a_n)}{n} \geq (a_1 a_2 \ldots a_2)^{1/n}$$

Equality occurs iff $a_1 = a_2 = \cdots = a_n$.

Proof.

A nice way to prove this is by using *Cauchy Induction*, which follows:
For $n = 2$,

$$(a_1 - a_2)^2 \geq 0$$

due to the $\mathbb{R}^+$ condition, leading to

$$a_1 + a_2 \geq 2a_1 a_2$$

Assume statement is true for some $n = k$ then we claim it is also true for $n = 2k$, for $k \in \mathbb{N}$.

Proof.

$$a_1 + a_2 + \cdots + a_k \geq k(a_1 a_2 \ldots a_k)^{\frac{1}{k}}$$

by initial assumption.

$$\implies a_1 + a_2 + \cdots + a_{2k} \geq k(a_1 a_2 \ldots a_k)^{\frac{1}{k}} + k(a_{k+1} a_{k+2} \ldots a_{2k})^{\frac{1}{k}} \geq 2k(a_1 a_2 \ldots a_{2k})^{\frac{1}{2k}}$$

by exploiting the $n = 2$ base case. Hence, by *P.M.I*, statement is true for all even n.

We now attempt to prove that if statement is true for k then it is also true for $k - 1$, using the previous fact. Just like earlier, assume true for k and thus:

$$a_1 + a_2 + a_3 + \cdots + a_k \geq k(a_1 a_2 a_3 \ldots a_k)^{1/k}$$

Since a_n can be any $\mathbb{R}^+$, we can set $a_k = \frac{a_1 + a_2 + \cdots + a_{k-1}}{k-1}$ WLOG. For $n = k - 1$:

$$\frac{a_1 + a_2 + \cdots + a_k}{k} \geq (a_1 a_2 \ldots a_k)^{\frac{1}{k}}$$

$$\frac{a_1 + a_2 + \cdots + a_{k-1} + \frac{a_1 + a_2 + \cdots + a_{k-1}}{k-1}}{k} \geq (a_1 a_2 \ldots a_k)^{\frac{1}{k}}$$

$$\frac{\frac{(k-1)(a_1 + a_2 + \ldots a_{k-1})}{k-1} + \frac{a_1 + a_2 + \cdots + a_{k-1}}{k-1}}{k} \geq (a_1 a_2 \ldots a_k)^{\frac{1}{k}}$$

$$\frac{k(a_1 + a_2 + \ldots a_{k-1})}{k(k-1)} \geq (a_1 a_2 \ldots a_k)^{\frac{1}{k}}$$

$$(\frac{a_1 + a_2 + \cdots + a_{k-1}}{k-1})^k \geq a_1 a_2 \cdots a_{k-1} \times (\frac{a_1 + a_2 + \cdots + a_{k-1}}{k-1})$$

We cancel common terms on both sides and then take the $k-1$ root to achieve the result:

$$(\frac{a_1 + a_2 \cdots + a_{k-1}}{k-1})^{k-1} \geq a_1 a_2 \cdots a_{k-1}$$

$$a_1 + a_2 + \cdots + a_{k-1} \geq (k-1)(a_1 a_2 \cdots a_{k-1})^{\frac{1}{k-1}}$$

Hence, if statement is true for $n = k$ then $n = k-1$ is true. Since any even n satisfies the conditions, we have just proved the general A.M-G.M inequality.

Example: Prove that $\frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2} \geq a^2 + b^2 + c^2$ for $a, b, c \in \mathbb{R}^+$.

Solution:

$$\frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2} \geq 2\sqrt{\frac{a^2b^4c^2}{a^2c^2}} = 2b^2$$

And we repeat this 'pairwise':

$$\frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2} \geq 2c^2$$

$$\frac{c^2a^2}{b^2} + \frac{a^2b^2}{c^2} \geq 2a^2$$

Adding the 3 inequalities above gives the desired result. By A.M-G.M inequality, equality occurs iff $a = b = c$. Extra care is advised when dealing with the equality.

Example: Find the maximum value of $2 - a - \frac{1}{2a}$ for all positive real numbers a.

Solution: We rewrite it as $2 - (a + \frac{1}{2a})$. We know that $\frac{1}{2a}$ is positive so we use AM-GM to get:

$$\frac{a + \frac{1}{2a}}{2} \geq \sqrt{a \cdot \frac{1}{2a}} = \frac{\sqrt{2}}{2}$$

$$a + \frac{1}{2a} \geq \sqrt{2}$$

Now we can manipulate this to get: $2 - (a + \frac{1}{2a}) \leq 2 - \sqrt{2}$ where equality occurs if and only if $a = \frac{1}{2a}$. Therefore, the maximum value is $2 - \sqrt{2}$.

2.2 Cauchy-Schwarz Inequality

Our inequality goes:

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + \cdots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2$$

for $a_n, b_n \in \mathbb{R}$.

Equality holds *iff* $\frac{a_n}{b_n} = k$ where k is a non-zero real constant.

(Secret Tip: Cauchy-Schwarz Inequality can be helpful when there are ugly terms we want to cancel out by multiplying a_i and b_i from LHS so that we get $a_i b_i$ on RHS, where the ugly term we want gone is probably gone. This will be demonstrated in the next example.)

Titu's Lemma

Tip

Titu's Lemma is just another form of the Cauchy-Schwarz Inequality that comes in handy you see series of fractions instead of two separate groups of a_n and b_n.

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n}$$

Equality condition remains consistent with the original theorem.

Example: Given then $a, b, c \in \mathbb{R}^+$ and $abc = 8$ prove that $\dfrac{ab+4}{a+2} + \dfrac{bc+4}{b+2} + \dfrac{ca+4}{c+2} \geq 6$.
Solution: Note that the given expression is the same as

$$\frac{ab + \frac{abc}{2}}{a + \frac{abc}{4}} + \cdots + \frac{ca + \frac{abc}{2}}{c + \frac{abc}{4}}$$

The motivation is just trying to make use of the $abc = 8$ condition in the problem statement. We can cancel out the a, b, c in the new fractions:

$$\frac{2b(2+c)}{bc+4} + \frac{2c(2+a)}{ac+4} + \frac{2a(2+b)}{ab+4}$$

Using the Cauchy-Schwarz Inequality, we cancel out the annoying $ab + 4$ in the problem statement:

$$\left(\frac{ab+4}{a+2} + \frac{bc+4}{b+2} + \frac{ca+4}{c+2}\right)\left(\frac{2a(2+b)}{ab+4} + \frac{2b(2+c)}{bc+4} + \frac{2c(2+a)}{ca+4}\right) \geq \left(\sqrt{\frac{2a(2+b)}{a+2}} + \cdots + \sqrt{\frac{2c(2+a)}{c+2}}\right)^2$$

Using A.M-G.M Inequality:

$$\left(\sqrt{\frac{2a(2+b)}{a+2}} + \cdots + \sqrt{\frac{2c(2+a)}{c+2}}\right)^2 \geq \left(3\left(\sqrt{8abc}\right)^{\frac{1}{3}}\right)^2 = 36$$

$$\therefore \left(\frac{ab+4}{a+2} + \frac{bc+4}{b+2} + \frac{ca+4}{c+2}\right)\left(\frac{2a(2+b)}{ab+4} + \frac{2b(2+c)}{bc+4} + \frac{2c(2+a)}{ca+4}\right) \geq 36$$

$$\left(\frac{ab+4}{a+2}+\frac{bc+4}{b+2}+\frac{ca+4}{c+2}\right)^2 \geq 36$$

$$\left(\frac{ab+4}{a+2}+\frac{bc+4}{b+2}+\frac{ca+4}{c+2}\right) \geq 6$$

Due to the $\mathbb{R}^+$ condition. Equality occurs *iff* $a=b=c=2$.

Practice Problems

Problem 1. Russia 1992: For positive reals $x, y > 1$ prove that

$$\frac{x^2}{y-1}+\frac{y^2}{x-1} \geq 8$$

Problem 2. (Nesbitt's Inequality): For $a, b, c \in \mathbb{R}^+$, prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}$$

Problem 3: Try to make sense of the equality case of the AM-GM general inequality.

Problem 4. Austria 2013 Junior: For $a, b \in \mathbb{R}$ with $0 \leq a, b \leq 1$ prove that

$$\frac{a}{b+1}+\frac{b}{a+1} \leq 1$$

Problem 5. Hong Kong 2019: For $a, b, c \in \mathbb{R}^+$ with $ab+bc+ca \geq 1$, prove that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \geq \frac{\sqrt{3}}{abc}$$

Problem 6 Poland 1998 First Round: For $a, b, c, d \in \mathbb{R}$ prove that

$$(a+b+c+d)^2 \leq 3(a^2+b^2+c^2+d^2)+6ab$$

Problem 7. Japan 2010: Prove that

$$\frac{1+xy+xz}{(1+y+z)^2}+\frac{1+yz+yx}{(1+z+x)^2}+\frac{1+zx+zy}{(1+x+y)^2} \geq 1$$

for $x, y, z \in \mathbb{R}^+$.

2.3 Functional Equations

2.3.1 Things to know before solving FEs

For beginners, Functional Equation (FE) is a well-known topic in Algebra, which can be found in most exams and contests. It is like playing a game since we can insert all the given values of x well within the given range (like real numbers, rational numbers, integers, etc.) and find all functions which satisfy the equation. Let's now see how FEs work together!

2.3.2 Guide

Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$2f(x) - 5f(y) = 2x - 5y,$$

$\forall x, y \in \mathbb{R}$.

Notation. $\forall x, y$ means "for all" which means the given equation is satisfied for all real numbers x and y simultaneously. This means we can substitute any pair of real numbers into (x, y). For example, we can make the following substitutions:

$$\begin{aligned}
&P(x, y):\ 2f(x) - 5f(y) = 2x - 5y \\
&P(x, 0):\ 2f(x) - 5f(0) = 2x \\
&P(x, 1):\ 2f(x) - 5f(1) = 2x - 5
\end{aligned}$$

Most FE in olympiads ask you to solve over any pair of (x, y), so any substitution is free to use as long as they're well within the range of domain. (Range is very important.) In this problem, we're dealing with a function $f\colon \mathbb{R} \to \mathbb{R}$, but some other problems might ask you to deal with, for example, $f\colon \mathbb{N} \to \mathbb{N}$. You can't substitute anything outside of natural numbers for x and y in this case.

The most fundamental skill in solving FE problems is to spot a smart substitution (or) assertion to kill the problem. In this case, substituting (x, x) kills the problem.

$$\begin{aligned}
P(x, x) : 2f(x) - 5f(x) &= 2x - 5x \\
-3f(x) &= -3x \\
f(x) &= x
\end{aligned}$$

$\therefore f(x) = x\ \forall x$ satisfies the problem statement.

2.3.3 FE with one variable

When dealing with systems of equations, it's critical that we do nice assertions to produce equations that are suitable for us to solve. Let's see an example of a problem which can be solved like linear equations. **Example 1.2.2.1:** Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x) + xf(1 - x) = x$$

$\forall x \in \mathbb{R}$.

Solution:

$$\begin{aligned}
P(x):&\ f(x) + xf(1 - x) = x \\
P(1 - x):&\ f(1 - x) + (1 - x)f(x) = 1 - x \\
&\ xf(1 - x) + x(1 - x)f(x) = x(1 - x)
\end{aligned}$$

so now we have 2 linear eqs with 2 unknowns, $f(x)$ and $f(1-x)$. Of course, we can easily solve it just by subtracting the last eq from the first equation. It will be left as an exercise for readers (you will get $f(x) = $ something).

Example 1.2.2.2: Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(\frac{1}{x}) + \frac{1}{x}f(-x) = x$$

$\forall x \in \mathbb{R}$ and $x \neq 0$.

Example 1.2.2.3: Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(\frac{1}{x}) + 2f(x) = 3x$$

$\forall x \in \mathbb{R}$ and $x \neq 0$.

2.3.4 Standard FE

Before solving standard FE, let me share the most common tricks to kill easy problems. Firstly, we need to guess the answer of the problem (basically f(x) = x, f(x) = mx + c, f(x) = constant, f(x) = x^n). Mostly, in basic problems, we try to get the value of f(0) or f(1) since they are strong enough to get more observations. Let's see an example problem.
Example 1.2.4.1: Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x-y) = f(x) + f(y) - 2xy$$

$\forall x, y \in \mathbb{R}$.

Solution: First of all, we guess the answer. Of course, the ans is very obvious, f(x) = x^2.

$$\begin{aligned} P(x,y):&\ f(x-y) = f(x) + f(y) - 2xy \\ P(0,0):&\ f(0) = 2f(0) \\ &\ f(0) = 0 \\ P(x,x):&\ 0 = 2f(x) - 2x^2 \\ &\ f(x) = x^2 \end{aligned}$$

Let's now try more problems.

Exercise 1.2.4.2: Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) + f(x)f(y) = x^2y^2 + 2xy$$

$\forall x, y \in \mathbb{R}$.

Exercise 1.2.4.3: Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(xy + 2x + 2y - 1) = f(x)f(y) + f(y) + x - 2$$

$\forall x, y \in \mathbb{R}$.

Exercise 1.2.4.4: Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$xf(x) - yf(y) = (x-y)f(x+y)$$

$\forall x, y \in \mathbb{R}$.

Exercise 1.2.4.5: Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)) + f(f(y)) = 2y + f(x - y)$$

$\forall x, y \in \mathbb{R}$.

Exercise 1.2.4.6: Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$(x + y)(f(x) - f(y)) = (x - y)f(x + y)$$

$\forall x, y \in \mathbb{R}$.

Exercise 1.2.4.7: Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(y) - y^2) = (y + 1)f(x - y)$$

$\forall x, y \in \mathbb{R}$.

3 Combinatorics

This section will be an introduction to common themes that one will find in the combinatorics Olympiad.

3.1 Pigeonhole Principle

Definition. Suppose there are N pigeons and M holes such that $N > M$. If these pigeons are placed into these holes, then at least one hole must contain two or more pigeons.

This is really useful in problems for proving that there exists... or deducing the minimum and maximum value of a problem.

Example: If Jake has an infinite number of socks of 10 different colors in a drawer, what is the minimum number of socks he can draw from the drawer to ensure that he has a pair of the same color?

Solution: Note that Jake can draw ten socks of different colours. However, if he draws 11 socks, the pigeonhole principle guarantees that there must exist a pair of same colour (Where 11 socks are "pigeons" and 10 colours are "holes"). Therefore, the minimum number of socks is **10**.

Exercise 2.1.1: Let $S = \{1, 2, 3, \ldots, 2n\}$. Show that if we choose $n + 1$ numbers from S, then there exist two numbers such that one is a multiple of the other.

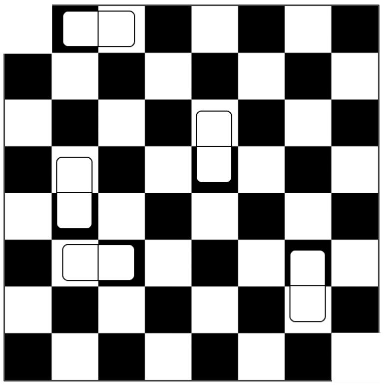
Exercise 2.1.2: Prove that whenever there are six people in the room, there exist three people that either know each other or do not know each other.

3.2 Invariant

Definition. It is the property of the situation that remains the same in each operation.

Example: Suppose a standard 8×8 chessboard has two diagonally opposite corners removed. Is it possible to place dominoes of size 2×1 to cover all of these squares?

Solution: In this board, we have 62 squares to cover, it will be boring and tedious to show whether it is possible or not for each dominoes arrangement, so let us find an invariant.



You will notice that each domino placed on the board takes up exactly 1 black square and 1 white square which is an invariant. But we only have 30 white and 32 black squares so we will be left with 2 black squares. Therefore, it is impossible to cover all squares with dominoes.

Useful Tips

You can see how useful an invariant is and they are usually found in game-style problems. Here are the common variations:

- Playing with boards (it is useful when you colour them in such a way some properties are constant)
- Parity (Even/Odd)
- Showing something is impossible
- Sum/Product of numbers
- Remainder of an expression when divided by a number

Be used to writing algorithms, i.e. a series of steps, to show something is possible.

For problems about finding a minimum or maximum and "find all X such that ...", they are basically two part (bi-directional) problems: one is showing how that answer can be achieved and proving the bound or they are the only possible answers.

Don't be surprised if one direction is obvious. Sometimes, the other direction can be the entirety of the problem.

Exercise 2.2.1: The positive integers 1 through 10 are written on a blackboard. At any given point, Evan can erase any three numbers a, b, and c and replace them with $\sqrt{a^2+b^2+c^2}$. What is the greatest number that can appear on the board at any given point?

Exercise 2.2.2: In a class, all 99 students in class are facing forward. At the end of each minute during class, 10 students turn around (if they are facing forward, they turn to face backwards; if they are facing backward, they turn to face forward). Prove that the students will never all face backward at the same time.

3.3 Practice Problems

We implore you to start with small cases to play around with it and try to question about these configurations. Don't worry if they are difficult because there is a lot of writing in combinatorial problems.

Problem 1. (IMO 1972/P1) Let X be a subset of $\{1, 2, 3, \ldots, 99\}$ of size 10. Prove that there exist disjoint nonempty sets of Y and Z of X such that the sum of the elements in Y is equal to that of Z.

Problem 2. Consider the set: $A = \{1, 2, ..., 100\}$. Let S be the subset of A with 11 different elements. Prove that there exists distinct x, y from S such that $0 < |\sqrt{x} - \sqrt{y}| < 1$.

Problem 3. Numbers $\frac{49}{1}, \frac{49}{2}, ..., \frac{49}{97}$ are written on a blackboard. For each move, we remove two numbers a, b and write $2ab - a - b + 1$. We make many moves until one number is left on the board. Find all the possible values of that number.

Problem 4. Show that given any 9 points inside a square of side 1 we can always find 3 which form a triangle with area less than $\frac{1}{8}$.

Problem 5. Consider a grid 5×5 with a light bulb in each square. When a bulb is chosen and toggled, other 4 bulbs adjacent to it are toggled (from on to off and off to on). Initally, all the bulbs are off. We say that a bulb is *good* if there exists a series of toggles such that only that bulb is on. Find all possible positions of *good* bulbs.

Problem 6. Each two-digit number is coloured in one of k colours. What is the minimum value of k such that there exist three numbers a, b, c with different colors with a and b having the same unit digit and b and c having the same tens digit?

Problem 7. Suppose that there is only one positive real number r written on the board. We do the following operation: choose a number n on the board, erase that number, and write a pair of positive real numbers a, b satisfying $2r^2 = ab$ on the board.
Prove that, after $k^2 - 1$ operations, there exists a number m on the board such that $m < kr$.

Problem 8. Bob wrote the numbers $1, 2, \ldots, 2025$ on a blackboard. He picks any two numbers x, y, erases them, and writes the number $|3x - 5y|$ on the board. This process continues until only one number is left. Prove that the remaining number is odd.

4 Geometry

In this section, you are going to study solvable angles which is extremely helpful for angle chasing (an essential tool in geometry).

4.1 Solvable Angles

Definition. Solvable Angle is an angle which we can write in terms of the main angles ($\angle A$, $\angle B$, $\angle C$ of original $\triangle ABC$).

They are extremely useful since we can leave solvable angles alone until we are handling them. But ! you shouldn't try to convert everything into solvable angles since this normally doesn't work for every angle of the problem !. Let's see some examples of Solvable Angles.

Warm-up 3.1.1: Let O be the circumcentre of $\triangle ABC$. Show $\angle OAB$, $\angle OBC$, $\angle OAC$ in terms of main angles of the triangle($\angle A$, $\angle B$, $\angle C$).

Warm-up 3.1.2: Let H be the orthocentre of $\triangle ABC$ and D, E, F be the foots of perpendicular from A, B, C to BC, AC and AB. Show $\angle BAH$, $\angle CAH$, $\angle ABH$, $\angle CBH$, $\angle ACH$, $\angle BCH$ in terms of main angles of the triangle($\angle A$, $\angle B$, $\angle C$).

- Do you notice that some angles have same value? (Why?) Let's dive deeper to orthocentre configurations.

4.2 Cyclic Quadrilaterals

Some elementary ways to show that a quadrilateral is cyclic

- (1) By showing that the butterfly angles are equal.
- (2) By showing that the sum of a pair of opposite angles is 180.
- (3) By showing that an exterior angle is equal to the interior opposite angle.

Exercise 3.2.1: Let H be the orthocentre of $\triangle ABC$ and D, E, F be the foots of perpendicular from A, B, C to BC, AC and AB. Then, find six cyclic quadrilaterals with their diameters and centres of circumcircles.

Exercise 3.2.1: Two circles intersect at A and B. A point P is taken on one circle so that PA and PB cut the other at Q and R respectively. The tangents at Q and R meet the tangent at P in S and T respectively. Prove that $ST//QR$. and that $PBQS$ is cyclic.

4.3 Angle Chasing with solvable angles

We have gained enough knowledge for solvable angles and cyclic quadrilaterals. Let's combine the previous results we got!
In the following problems, let H be the orthocentre of $\triangle ABC$ and D,E,F be the foots of perpendicular from A,B,C to BC, AC and AB and Let O be the circumcentre of $\triangle ABC$ and M be the midpoint of BC

Exercise 3.3.1: Prove that $\angle OAC = \angle BAH = \angle BCH = \angle FEH$

Exercise 3.3.2: Prove that H is incentre of $\triangle DEF$.
"This fact is sometimes also referred to as "Orthic-triangle".

Exercise 3.3.3: Prove that OA is perpendicular to EF

Exercise 3.3.4: Prove that the tangents to the circumcircle of $\triangle AEF$ at points E and F intersect at M.

After finishing all of these problems, you would have known the tremendous power of solvable angles, making the problem a lot more clear. Let's move to a well-known lemma then.

4.4 Orthocenter Lemma

Let H be the Orthocenter of $\triangle ABC$, as shown in the figure. Let X be the reflection of H over BC and Y the reflection over the midpoint of BC.

(i) Show that X lies on (ABC).

(ii) Show that $BHCY$ is a parallelogram.

(b) Show that AY is a diameter of (ABC).

4.5 Every configuration about orthocentre

We will list a set of configurations which can be assumed as exercises and proved by just pure angle chasing using solvable angles. *Let H be the orthocentre of $\triangle ABC$ and D,E,F be the foots of perpendicular from A,B,C to BC, AC and AB and Let O be the circumcentre of $\triangle ABC$, K be circumcentre of $\triangle AEF$ and M be the midpoint of BC.*

Exercise 3.5.1: Prove that $KOMH$ and $AOMK$ are parallelograms.

Exercise 3.5.2: If MH intersects arc BC containing point A at S, prove that SHEA is cyclic.

Exercise 3.5.3: If MA intersects (AEF) at N, prove that $BHNC$ is cyclic.

Exercise 3.5.4: If MH intersects arc BC containing point A at S, prove that $\triangle SEF$ similar to $\triangle SCB$

Exercise 3.5.5: If MA intersects (AEF) at N, prove that $\angle CBN = \angle BAN$ and $\angle BCN = \angle CAN$

Exercise 3.5.6: If MA intersects (AEF) at N, prove that $NB/NC = AB/AC$.

4.6 Practice Problems

I highly encourage you to solve the practice problems since I strongly believe that you have all gained enough knowledge to solve all of them, and I personally put a lot of effort into this handout. But it might be challenging for some beginners, so don't be sad if you cannot solve yet. Lastly, don't give up and try to spend more time if you don't get. You can ask me directly or also ask in our page's chatbox if you want hints.

Exercise 3.6.1: (BAMO 2013) Let H be the orthocenter of an acute triangle ABC. Consider the circumcenters of triangles ABH, BCH, and CAH. Prove that they are the vertices of a triangle that is congruent to ABC.

Exercise 3.6.2: (My Fav Geo Problem) Let ABC be a triangle with circumcircle w. Let L_b and L_c be two lines through the points B and C, respectively, such that L_b // L_c. The second intersections of L_b and L_c with w are D and E, respectively ($D \in$ arc AB, $E \in$ arc AC). Let DA intersect L_c at F and let EA intersect L_b at G. If O, $O1$ and $O2$ are circumcenters of $\triangle ABC$, $\triangle ADG$ and $\triangle AEF$, respectively, and P is the circumcenter of $\triangle OO1O2$, prove that L_b // OP // L_c.

Exercise 3.6.3: (IGO 2014, Junior) In a right triangle ABC we have $\angle A = 90$ and angle $\angle C = 30$. Denote by w the circle passing through A which is tangent to BC at the midpoint. Assume that w intersects AC and the circumcircle of $\triangle ABC$ at N and M, respectively. Prove that MN perpendicular to BC.

Exercise 3.6.4: (JBMO Shortlist 2015) Let t be the tangent at the vertex C to the circumcircle of triangle ABC. A line p parallel to t intersects BC and AC at points D and E, respectively. Prove that the points A, B, D and E are concyclic.

Exercise 3.6.5: Two circles intersect at A and B. One of their common tangents touches the circles at P and Q. Let A' be the reflection of A across the line PQ. Prove that $A'PBQ$ is a cyclic quadrilateral.

Exercise 3.6.6: (IGO 2016, Intermediate) Let two circles $C1$ and $C2$ intersect in points A and B. The tangent to $C1$ at A intersects $C2$ in P and the line PB intersects $C1$ for the second time in Q (suppose that Q is outside $C2$). The tangent to $C2$ from Q intersects $C1$ and $C2$ in C and D, respectively. (The points A and D lie on different sides of the line PQ.) Show that AD is the bisector of angle CAP.

Exercise 3.6.7: (IGO 2018, Intermediate) Let $w1$ and $w2$ be two circles with centers $O1$ and $O2$, respectively. These two circles intersect at points A and B. Line $O1B$ intersects $w2$ for the second time at point C, and line $O2A$ intersects $w1$ for the second time at point D. Let X be the second intersection of AC and $w1$ and let Y be the second intersection of BD and $w2$. Prove that $CX = DY$.

Exercise 3.6.8: (IGO 2018, Advanced) In acute triangle ABC, $\angle A = 45$, points O and H are the circumcenter and the orthocenter, respectively. The foot of the altitude from B is D. Point X is the midpoint of are ADH of the circumcircle of triangle ADH. Prove that $DX = DO$.

Exercise 3.6.9: (EGMO 2012) Let ABC be a triangle with circumcenter 0 . The points D, E, F lie in the interiors of the sides BC, CA, AB respectively, such that DE is perpendicular to CO and DF is perpendicular to BO. Let K be the circumcenter of triangle AFE. Prove that the lines DK and BC are perpendicular.

Exercise 3.6.10: (JBMO Shortlist 2014) Let ABC be a triangle such that AB is not equal to AC. Let M be the midpoint of BC and H be the orthocenter of triangle ABC. Let D be the midpoint of AH and O the circumcenter of triangle HBC. Prove that $DAMO$ is a parallelogram.

Exercise 3.6.11: Let ABC be a triangle and let H and O be its orthocenter and circumcenter, respectively. Let K be the midpoint of AH. The perpendicular to OK through K intersects AB and AC at P and Q, respectively. Prove that $OP = OQ$.

Exercise 3.6.12: (JBMO Shortlist 2010) In a triangle ABC, let $\angle ACB = 90°$. Let F be the foot of the altitude from C. Circle w touches the line segment FB at point P, the altitude CF at point Q and the circumcircle of triangle ABC at point R. Prove that points A, Q, R are collinear and $AP = AC$.

Exercise 3.6.13: (USAMO 1990) Let ABC be an acute-angled triangle. The circle with diameter AB intersects altitude CE and its extension at points M and N, and the circle with diameter AC intersects altitude BD and its extension at points P and Q. Prove that the points M, N, P, Q lie on a common circle.

Exercise 3.6.14: (APMO 2015) Let ABC be a triangle, and let D be a point on the side BC. A line through D intersects side AB at X and ray AC at Y. The circumcircle of triangle BXD intersects the circumcircle w of triangle ABC again at point Z distinct from point B. The lines ZD and ZY intersect w again at V and W respectively. Prove that $AB = VW$.

5 Number Theory

This section is an introduction to the notations and ideas you will face when you start Number Theory.

5.1 Prime Numbers

Definition. A prime number is a positive integer whose only divisors are 1 and itself.

We are interested in prime numbers in number theory because they serve as the fundamental *building blocks* of all integers. Every whole number can be decomposed into primes, just as molecules are built from atoms. Looking further onward, many useful theorems like Fermat's Little Theorem and Wilson's Theorem give us beautiful results with prime numbers.

5.2 Fundamental Theorem of Arithmetic

The "building block" idea can be formalized as follows:

$$\text{Every } n \in \mathbb{N} \text{ can be uniquely expressed as } n = \prod_i p_i^{a_i} = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r},$$

where the p_i are prime numbers and the a_i are nonnegative integers.

Exercise 4.2.1. A positive integer $n = \prod_i p_i^{a_i}$, where p_i are prime numbers and the a_i are positive integers, is a perfect square. Prove that any a_i is an even number. Can this be generalised to k-th power?

5.3 Greatest Common Divisor and Least Common Multiple (GCD, LCM)

Definition. The *greatest common divisor* (GCD) of two or more positive integers is the largest integer that divides all of them. The *least common multiple* (LCM) of two or more positive integers is the smallest positive integer that is divisible by all of them.
Suppose

$$a = \prod_i p_i^{a_i}, \qquad b = \prod_i p_i^{b_i}.$$

Then,

$$\gcd(a, b) = \prod_i p_i^{\min(a_i, b_i)}, \qquad \operatorname{lcm}(a, b) = \prod_i p_i^{\max(a_i, b_i)}.$$

From this, an important identity holds:

$$\gcd(a, b) \cdot \operatorname{lcm}(a, b) = a \cdot b.$$

Warm up. Try proving the above identity using the fundamental theorem of arithmetic.

Useful Trick

Let $a, b \in \mathbb{N}$ and define $g = \gcd(a, b)$. Then we can write

$$a = ga_1, \qquad b = gb_1,$$

where $\gcd(a_1, b_1) = 1$. This often simplifies divisibility arguments.

Exercise 4.3.1. Prove that $\gcd(a + b, b) = \gcd(a, b)$.

Exercise 4.3.2. Let a, b, c be positive integers. Prove that a divides bc if and only if

$$\frac{a}{\gcd(a, b)} \mid c.$$

Proof-Writing Note

When a statement is written as "P if and only if Q" (often abbreviated as $P \iff Q$), it means two things:

$$P \implies Q \quad \text{and} \quad Q \implies P.$$

You must prove in two directions.

Exercise 4.3.3 (All-Russian MO 1995 Regional (R4)). Let m, n be positive integers such that

$$\gcd(m,n) + \operatorname{lcm}(m,n) = m + n$$

Show that one of the two numbers is divisible by the other.

5.3.1 Euclid's Division Algorithm

Lemma. Let a, b be integers. We can write $a = bq + r$ for integers q, r where $0 \leq r \leq b$. Then the lemma states that

$$gcd(a,b) = gcd(r,b).$$

Euclid's Division Algorithm is to just keep on reducing $\gcd(a,b)$ to $\gcd(b,r)$ repeatedly until one number will divide the other. For example,

$$\gcd(250,150) = \gcd(150,100) = \gcd(100,50) = 50$$

We usually prefer GCD over LCM because of Euclid's Division Algorithm. We can easily change LCM to GCD, fortunately.

Exercise 4.3.4 (IMO 1959). Prove that for any natural number n, the fraction $\dfrac{21n+4}{14n+3}$ is irreducible. (Tip: In an irreducible fraction, the numerator and denominator are relatively prime.)

5.4 Factors

We often define divisors of a positive integer n as d_1, d_2, ..., d_k where $1 = d_1 < d_2 < \cdots < d_k = n$. For instance, if $n = 30$, then $d_1 = 1, d_2 = 2, d_3 = 3, d_4 = 5, d_5 = 6, d_6 = 10, d_7 = 15, d_8 = 30$. Understanding how each divisors connect to each other will make it much easier to tackle harder number theory problems. Here are a few general truths, I advice you to read through all of them and think about the proof behind each statement to get more familiar with them:

- d_2 will always be a prime number. This also implies $d_2 \geq 2$.
- d_3 will either be a prime number unique from d_2 or $d_3 = d_2^2$.
- $d_i \cdot d_{k+1-i} = n$. This implies $d_{k-1} \leq \dfrac{n}{2}$. (For example, it's easier to see d_{k-1} as $\dfrac{n}{d_2}$.)
- $d_i \leq \sqrt{n}$ for $1 \leq i \leq \left\lceil \dfrac{k}{2} \right\rceil$
- $d_i \geq \sqrt{n}$ for $\left\lfloor \dfrac{k+2}{2} \right\rfloor \leq i \leq k$
- k is odd for perfect square numbers while even for non-perfect square numbers.
- For a perfect square number $n = m^2$, $k \leq 2m - 1$.

Exercise 4.4.1. (Belarusian National Olympiad 2022) Positive integers a and b satisfy the equality $a + d(a) = b^2 + 2$ where $d(n)$ denotes the number of divisors of n. Prove that $a + b$ is even.

5.4.1 Number of Factors

Suppose the prime factorization of n is

$$n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}.$$

Then the number of positive divisors of n, denoted $d(n)$, is given by

$$d(n) = (a_1 + 1)(a_2 + 1) \cdots (a_r + 1).$$

5.4.2 Sum of factors

With the same prime factorization

$$n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r},$$

the sum of all positive divisors of n, denoted $\sigma(n)$, is

$$\sigma(n) = \prod_{i=1}^{r} \left(1 + p_i + p_i^2 + \cdots + p_i^{a_i}\right).$$

5.5 Practice Problems

Problems are arranged in no particular order. Try not to think much of the source while problem solving to maximize mental strength.

Problem 1. Let a and b be positive integers such that $a \mid b^2,\ b^2 \mid a^3,\ a^3 \mid b^4,\ b^4 \mid a^5, \ldots$. Prove that $a = b$.

Problem 2 (USAMO 1972). The symbols $(a, b, \ldots, g)$ and $[a, b, \ldots, g]$ denote the greatest common divisor and least common multiple, respectively, of the positive integers $a, b, \ldots, g$. For example, $(3, 6, 18) = 3$ and $[6, 15] = 30$. Prove that

$$\frac{[a, b, c]^2}{[a, b][b, c][c, a]} = \frac{(a, b, c)^2}{(a, b)(b, c)(c, a)}.$$

Problem 3 (Romania Mathematical Olympiad). Let a, b be positive integers such that there exists a prime p with the property

$$\operatorname{lcm}(a, a + p) = \operatorname{lcm}(b, b + p)$$

Prove that $a = b$.

Problem 4 (IMO 2023). Determine all composite integers $n > 1$ that satisfy the following property: if $d_1, d_2, \ldots, d_k$ are all the positive divisors of n with $1 = d_1 < d_2 < \cdots < d_k = n$, then d_i divides $d_{i+1} + d_{i+2}$ for every $1 \le i \le k - 2$.

Problem 5 (IMO 2002). Let $n \ge 2$ be a positive integer, with divisors $1 = d_1 < d_2 < \ldots < d_k = n$. Prove that $d_1 d_2 + d_2 d_3 + \ldots + d_{k-1} d_k$ is always less than n^2, and determine when it is a divisor of n^2.

Problem 6. Find all integers x and y such that

$$\frac{1}{\gcd(x, y)} + \frac{3}{xy} + \frac{y}{\operatorname{lcm}(x, y)} = y.$$

Problem 7 (Myanmar TST 2024). Find all integers a, b and c such that

$$\begin{aligned} \operatorname{lcm}(a, b) &= 5040, \\ \operatorname{lcm}(b, c) &= 2024, \\ \gcd(a, c) &= 4 \end{aligned}$$

Problem 8. Find all pairs of positive integers (a, b) such that

$$\gcd(a, b) + \operatorname{lcm}(a, b) + a + b = 2026$$

Problem 9 (JBMO 2025). Determine all numbers of the form

$$20252025 \cdots 2025$$

(consisting of one or more consecutive blocks of 2025) that are perfect squares of positive integers. (Note that $2025 = 45 \times 45$.)

6 Proofwriting

In my opinion, this is the most important part of Math Olympiads, especially the higher the level of the tournament. It's necessary to get all the good habits in right now as it would be hard to neatly write proofs later on when you're used to prior methods.

6.1 Differences between School Math, and Olympiad Math

Example: Find all real solutions for x when $\sqrt{5x+5} = 5x+3$

What you might do if you were faced with this question in your normal school settings would be something along the lines of:

$$(\sqrt{5x+5})^2 = (5x+3)^2$$

$$5x+5 = 25x^2+30x+9$$

$$25x^2+25x+4 = 0$$

$$(5x+4)(5x+1) = 0$$

$$5x+4 = 0, 5x+1 = 0$$

$$x = -\frac{4}{5}, -\frac{1}{5}$$

Reject $x = -\frac{4}{5}$ as then L.H.S < 0 which is not possible. Thus,

$$x = -\frac{1}{5}$$

This is indeed correct, and in school, you would get full marks for it. However, this won't cut it in olympiad. This might be harder to see for rudimentary problems like this, but for an IMO problem, if you do not write well, it would be extremely easy to lose marks.
The same question for olympiads would be answered along the lines of:

Claim: We have $x = -1/5$.

Proof. We shall prove that the claimed answer satisfies the equation.
When $x = -1/5$, the $L.H.S$ is equal to $\sqrt{5(-1/5)+5} = \sqrt{-1+5} = \sqrt{4} = 2$,
while the $R.H.S$ is equal to $5(-1/5)+3 = -1+3 = 2$.
Therefore, the two sides are equal to each other, which shows that $x = -1/5$ is a solution to the equation.

Now we shall prove that $x = -1/5$ is the only solution. Suppose that x is any real number satisfying $\sqrt{5x+5} = 5x+3$. Then,

$$\begin{aligned}
& \sqrt{5x+5} = 5x+3 \\
\implies & (\sqrt{5x+5})^2 = (5x+3)^2 \\
\implies & 5x+5 = 25x^2+30x+9 \\
\implies & 25x^2+25x+4 = 0 \\
\implies & (5x+4)(5x+1) = 0 \\
\implies & 5x+4 = 0 \quad \text{or} \quad 5x+1 = 0 \\
\implies & x = -4/5 \quad \text{or} \quad x = -1/5.
\end{aligned}$$

Thus, $x = -4/5$ or $x = -1/5$. When $x = -4/5$, L.H.S. is not equal to R.H.S. Therefore, $x = -1/5$ is the only solution. $\square$

You may have figured out that this is not that different from the first solution, and indeed, the first solution is part of this. However, in the second solution, it's more clear, and the examiner knows exactly what you're trying to achieve with each step.

The first part proved '**Existence**' , where you showed that your claimed answer does work. The second part proved '**Uniqueness**', where we proved that the answer is the only one(s) that work. These are, in essence, the two main parts of a solution. This is the format that you should remember, not just for NT, but for other topics as well.

State what you're trying to prove first. As Evan Chen once said, "Emphasize the point where you cross the ocean". Whatever progress you make, without stating exactly what the progress is, it might be hard to get marks. Always state explicitly, do not count on the examiner finding out your implicit answers.

6.2 Exercises

Try writing the following in as much detail as possible. These are by no means hard questions, and you should be able to get them easily. However, this is a practice more for proof-writing so keep that in mind. If you have answered the questions in the exercises above, try answering them again with proof-writing in mind. Remember, proof-writing is just to help you communicate your maths to others. It means nothing if your maths is wrong to begin with, and that's why redoing questions you've already answered correctly is a great exercise.

Exercise 5.1: Find the maximum value of $x^2 + 2x + 10$ for all real numbers x.

Exercise 5.2: What is the maximum area of a triangle whose vertices belong to a circle with radius 1?

Exercise 5.3: Let $f : R \to R$ be a function such that $f(x) + 2f(1 - x) = 5 - x$ for all real numbers x. Prove that $f(x) = x + 1$ for all real numbers x.

Exercise 5.4: Let A,B,C be three points on a line in that order. Show that there exists a point P such that $\angle APB = \angle BPC = 60°$.

7 References

7.1 Algebra

1. Functional Equations by Pang-Cheng Wu
2. U Naing Zaw Lu's 2024 MSM December Training

7.2 Combinatorics

1. Evan Chen's OTIS "Entry Combinatorics" Unit

7.3 Geometry

1. "Euclidean Geometry in Mathematical Olympiads" by Evan Chen
2. "Geometry Configurations" of Canada 2020 Winter Camp by Victor Rong

7.4 Number Theory

1. Aditya Khurmi's "Modern Olympiad Number Theory"

7.5 Proof-Writing

1. Evan Chen's handout, "Remarks on English"
2. U Hein Thant Aung's 2025 IMO Team Myanmar training camp lectures