Problem 1

A line in the plane is called *sunny* if it is not parallel to any of the x–axis, the y–axis, or the line $x + y = 0$.
Let $n \geq 3$ be a given integer. Determine all nonnegative integers k such that there exist n distinct lines in the plane satisfying both of the following:

- for all positive integers a and b with $a + b \leq n + 1$, the point (a, b) lies on at least one of the lines; and
- exactly k of the n lines are sunny.

Solution

Answer: $k = 0, 1, 3$ for all $n \in \mathbb{N} \geq 3$.

Part A: $k = 0, 1, 3$ works

Proof. For $n = 3$, consider the following 3 lines for each k:

$k = 0$	$k = 1$	$k = 3$
$x = 3$	$x + y = 4$	$x = y$
$x = 2$	$x + y = 3$	$x + 2y = 5$
$x = 1$	$x = y$	$2x + y = 5$

For $n > 3$, consider the following n lines for each k:

$k = 0$	$k = 1$	$k = 3$
$x + y = n + 1$	$x + y = n + 1$	$x + y = n + 1$
$x + y = n$	$x + y = n$	$x + y = n$
.	.	.
.	.	.
.	.	.
$x + y = 5$	$x + y = 5$	$x + y = 5$
$x = 3$	$x + y = 4$	$x = y$
$x = 2$	$x + y = 3$	$x + 2y = 5$
$x = 1$	$x = y$	$2x + y = 5$

Part B: only $k = 0, 1, 3$ works

Proof.
Claim 1. For $n > 3$, the n lines must include at least one of $x = 1$, $y = 1$, and $x + y = n + 1$.
For the sake of contradiction, assume that neither of $x = 1$, $y = 1$, and $x + y = n + 1$ are one of the n lines. This means that

- all n points on $x = 1$ have n unique lines passing through them.
- all n points on $y = 1$ have n unique lines passing through them.

- all n points on $x + y = n + 1$ have n unique lines passing through them.

We have $3n - 3$ points in total on line $x = 1$, $y = 1$, and $x + y = n + 1$ — call these points *border points*. However, each line can only cover at most 2 *border points*. Hence,

$$2n \geq 3n - 3$$

We are left with $n \leq 3$, so contradiction! Claim 1 is proven.

Claim 2. All k which works for all n is the same as k which works for $n = 3$. *Proof.* Suppose that only $k = 0, 1, 3$ works for $n = 3$. Assume that for a constant $c \geq 3$, only $k = 0, 1, 3$ work. Then, for $n = c + 1$:

Case 1: $x = 1$ is one of the $c + 1$ lines

We are then left with the exact arrangement and configurations as $n = c$. Specifically, the points not on $x = 1$ are the points on the $n = c$ case translated by 1 unit to the right, and there are c lines left to use. $\therefore$ The answer of k is exactly the same as $n = c$ case.

Case 2: $y = 1$ is one of the $c + 1$ lines

We are then left with the exact arrangement and configurations as $n = c$. Specifically, the points not on $y = 1$ are the points on the $n = c$ case translated by 1 unit upwards, and there are c lines left to use. $\therefore$ The answer of k is exactly the same as $n = c$ case.

Case 3: $x + y = c + 2$ is one of the $c + 1$ lines

We are then left with the exact arrangement and configurations as $n = c$. $\therefore$ The answer of k is exactly the same as $n = c$ case.
$\therefore$ By Induction, Claim 2 is proven!

Claim 3. $k \neq 2$ for $n = 3$.
Proof. Assume $k = 2$ is possible. Then:
If the one and only non-sunny line was either $x = 1$ or $y = 1$ or $x + y = 4$, it is obvious that both of the other 2 lines cannot be sunny lines. Hence, the only non-sunny line can't be either of them, which means

- all 3 points on $x = 1$ have 3 unique lines passing through them.
- all 3 points on $y = 1$ have 3 unique lines passing through them.
- all 3 points on $x + y = 4$ have 3 unique lines passing through them.
- all 3 lines can pass through at most 2 points.

But, there are exactly 6 points needed to be passed through, so each line must pass through exactly 2 points. In order to avoid having $x = 1$ or $y = 1$ or $x + y = 4$ as one of the lines, (1,3) must share a line with (2,1); (1,1) must share a line with (2,2). Note that both of the previous lines are sunny. Then, the two points left must also share the last line, but that line — specifically, $x + 2y = 5$ — is also sunny. $\therefore$ Contradiction, as this makes $k = 3$! Claim 3 is done!

Solution Summary

- $k = 0, 1, 3$, and constructions to get these answers.
- For $n \geq 3$, one of the n lines must be at least one of the *borderlines*, specifically $x = 1$, $y = 1$, and $x + y = n + 1$.
- The above condition is used to prove that k is constant for all n.
- $k \neq 2$ for $n = 3$.

Problem 4

A *proper divisor* of a positive integer N is a positive divisor of N other than N itself.

The infinite sequence $a_1, a_2, \ldots$ consists of positive integers, each of which has at least three proper divisors. For each $n \geq 1$, the integer a_{n+1} is the sum of the three largest proper divisors of a_n.

Determine all possible values of a_1.

Solution

Answer: All positive integers of the form

$$2^x \cdot 3^y \cdot Z$$

where:

- x is an odd number,
- $x, y \in \mathbb{N}$ with $y \geq \dfrac{x+1}{2}$,
- $Z \in \mathbb{N}$ such that $2 \nmid Z$, $3 \nmid Z$, and $5 \nmid Z$.

Part A: Proof that no other integers are possible

Claim 1. For any $n \in \mathbb{N}$, if a_n is odd, then a_{n+1} is odd.
Proof. Suppose a_n is odd. Then the three largest proper divisors of a_n — say d_1, d_2, d_3 — must also be odd, since all proper divisors of an odd number are odd.

Thus, we can write:

$$a_{n+1} = d_1 + d_2 + d_3$$

which is a sum of three odd numbers. Therefore, a_{n+1} is odd.

Note that Claim 1 also implies that if $2 \nmid a_n$, then $2 \nmid a_i$ for all $i \geq n$, assuming that the sequence goes on infinitely.

Claim 2. a_n is even for all $n \in \mathbb{N}$.

Proof. Assume for contradiction that a_n is odd. Then by Claim 1, all terms of the sequence after a_n are odd:

$$a_n, a_{n+1}, a_{n+2}, \ldots \text{ are all odd.}$$

Since a_n has at least three proper divisors and is odd, its three largest proper divisors are at most $\frac{a_n}{3}$, $\frac{a_n}{5}$, and $\frac{a_n}{7}$. Thus:

$$a_{n+1} = d_1 + d_2 + d_3 \leq \frac{a_n}{3} + \frac{a_n}{5} + \frac{a_n}{7} = a_n \left(\frac{1}{3} + \frac{1}{5} + \frac{1}{7} \right) < a_n.$$

Therefore, $a_{n+1} < a_n$. Applying the same logic repeatedly, we get a strictly decreasing sequence:

$$a_n > a_{n+1} > a_{n+2} > \dots$$

This shows that this sequence will not go on infinitely.

Hence, a_n cannot be odd.

Case 1.1: $v_2(a_n) = 1$

Then the largest proper divisor is $d_1 = \frac{a_n}{2}$, which is odd.

Since $a_{n+1} = d_1 + d_2 + d_3$ and Claim 2 implies a_{n+1} must be even, d_2 and d_3 cannot both be odd or both be even — they must have different parity.

Let p be the smallest odd prime dividing a_n (i.e., $p > 2$). Then the only choice for the next two largest proper divisors is:

$$d_2 = \frac{a_n}{p}, \quad d_3 = \frac{a_n}{2p}$$

This gives:

$$a_{n+1} = d_1 + d_2 + d_3 = \frac{a_n}{2} + \frac{a_n}{p} + \frac{a_n}{2p} = a_n \left(\frac{1}{2} + \frac{1}{p} + \frac{1}{2p} \right)$$

Case 1.2: $v_2(a_n) > 1$

4 must be a divisor of a_1.

$$a_{n+1} = d_1 + d_2 + d_3 = \frac{a_n}{2} + \frac{a_n}{4} + \frac{a_n}{m}$$

where m is either 3 or the smallest divisor of a_n after 4.

Claim 3. If $3 \nmid a_n$, then $3 \nmid a_{n+1}$.

Proof. Suppose $3 \nmid a_n$.

Case 1.1: $v_2(a_n) = 1$

$$a_{n+1} = a_n(\frac{1}{2} + \frac{1}{p} + \frac{1}{2p}) = a_n(\frac{p+3}{2p})$$

$3 \nmid a_n$ leads to $3 \nmid p + 3$

$\therefore 3 \nmid a_{n+1}$

Case 1.2: $v_2(a_n) > 1$

$$a_{n+1} = a_n(\frac{1}{2} + \frac{1}{4} + \frac{1}{m}) = a_n(\frac{3m+4}{4m})$$

$3 \nmid 3m + 4$

$\therefore$ Both cases lead to $3 \nmid a_{n+1}$. Claim 3 is proven.

Note that Claim 3 also implies that if $3 \nmid a_n$, then $3 \nmid a_i$ for all $i \geq n$, assuming that the sequence goes on infinitely.

Claim 4. $3 \mid a_n$ for all $n \in \mathbb{N}$

Proof. Suppose $3 \nmid a_n$ for the sake of contradiction. Then,

$$a_{n+1} \leq a_n(\frac{1}{2} + \frac{1}{4} + \frac{1}{5}) < a_n$$

Using Claim 3, we get

$$a_n > a_{n+1} > a_{n+2} > \dots$$

This shows that the sequence cannot go on infinitely. $\therefore$ By contradiction, Claim 4 is proven. Using Claim 2 and 4, we can deduce that $6 \mid a_n$.

Case 2.1: $v_2(a_1) = 1$

$$a_2 = a_1(\frac{1}{2} + \frac{1}{p} + \frac{1}{2p}) = a_1(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}) = a_1$$

(Note that due to the parity check above after Claim 2, d_3 can only be an odd number, which is why d_3 cannot be $\frac{a_1}{5}$.)
It is then obvious, by induction, that

$$a_1 = a_2 = a_3 = \dots = a_n = \dots$$

The sequence will go on infinitely: this solution is a piece of the claimed answer! Specifically, it is when $x = 1$ and $y \geq 1$ and $5 \nmid a_n$!

Case 2.2: $v_2(a_1) > 1$

$$a_2 = a_1(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}) = \frac{13}{12}a_n$$

Let $v_2(a_1) = x$ and $v_3(a_1) = y$.

Claim 5. $y \geq \frac{x}{2}$

Proof. Suppose, for contradiction, that $y < \frac{x}{2}$. Then for all $2 \leq i \leq y + 1$,

$$a_i = a_1 \cdot (\frac{13}{12})^{i-1}$$

(This is because for all $1 \leq j \leq y$, $v_2(a_j) = x - 2(j-1) \geq 2$, $v_3(a_j) = y - (j-1) \geq 1$: the next term gives $\frac{13}{12}$ times the previous term.)
However,
$v_3(a_{y+1}) = 0$, meaning it goes against Claim 4. $\therefore$ Contradiction! Claim 5 is proven.

Claim 6. x is odd.

Proof. Suppose, for contradiction, that x is even. Then for all $2 \leq i \leq \frac{x+2}{2}$,

$$a_i = a_1 \cdot (\frac{13}{12})^{i-1}$$

(Note that using Claim 5, we can guarantee that $3 \mid a_j$ for all $1 \leq j \leq \frac{x}{2}$, which is why the next term gives $\frac{13}{12}$ times the previous term.)
However,
$v_2(a_{\frac{x+2}{2}}) = 0$, meaning it goes against Claim 2. $\therefore$ Contradiction! Claim 6 is proven.

Since x is odd, Claim 5 is basically $y \geq \frac{x+1}{2}$. For all $2 \leq i \leq \frac{x-1}{2}$,

$$a_i = a_1 \cdot (\frac{13}{12})^{i-1}$$

However, $v_2(a_{\frac{x-1}{2}}) = 1$. This goes back to Case 1.1 with $n = \frac{x-1}{2}$ where we can confirm that $5 \nmid a_1$.

$\therefore$ Part A is done!

Part B: Verification

Suppose a_1 fits the claimed the answer.
If $x = 1$,

$$a_2 = \frac{a_1}{2} + \frac{a_1}{3} + \frac{a_1}{6} = a_1$$

$\therefore$ By induction, it is obvious that the sequence is constant and will go on infinitely.

If $x > 1$, for all $2 \leq i \leq \frac{x+1}{2}$:

$$a_i = a_1 \cdot (\frac{13}{12})^{i-1}$$

$$v_2(a_i) = x - 2(i-1)$$

$$v_3(a_i) = y - (i-1)$$

Then, $v_2(a_{\frac{x+1}{2}}) = x - 2(\frac{x+1}{2} - 1) = 1$.

$$a_{\frac{x+3}{2}} = a_{\frac{x+1}{2}}(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}) = a_{\frac{x+1}{2}}$$

It is then obvious that

$$a_{\frac{x+3}{2}} = a_{\frac{x+5}{2}} = \cdots = a_n = \ldots$$

which will go on infinitely. □

Solution Summary

- Proving that $\frac{a_n}{2}$ is one of the three largest proper divisors for all $n \in \mathbb{N}$. (Claim 1, 2)
- Proving that $\frac{a_n}{3}$ is one of the three largest proper divisors for all $n \in \mathbb{N}$. (Claim 3, 4)
- Necessary bonding with powers of 2 and 3. (Claim 5, 6)