Divisibility

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Symbol

The symbol of divisibility is the vertical bar, " | ". The notation "$a \mid b$" is used to express that *a divides b* (or) *b is divisible by a*. Conversely, " $a \nmid b$ " means *a does not divide b.*

Some divisibility tests

2	Even numbers
3	Sum of all digits is divisible by 3
9	Sum of all digits is divisible by 9
4	The last two digits are divisible by 4
25	The last two digits are divisible by 25
8	The last three digits are divisible by 8
125	The last three digits are divisible by 125
11	The difference between the sum of digits of odd places and sum of digits of even places is divisible by 11

Exercise 1A

1.Given that *A2018B* is a 6-digit number which is divisible by *72*, find the value of *A*.

Solution: N is divisible by 72 ⇔ N is divisible by both 8 and 9.

A2018B is divisible by 8 ➔ 18B is divisible by 8 ➔ B = 4.

A20184 is divisible by 9 ➔ $9 \mid (A+2+0+1+8+4)$ ➔ A = 3.

Ans: A = 3

2. Find the sum of all four-digit natural numbers of the form *4AB5* which are divisible by *225.*

Solution: N is divisible by 225 ⇔ N is divisible by both 9 and 25.

4AB5 is divisible by 25 ➔ 25 | B5 ➔ B = 2 or 7.

Case 1 : B = 2.

4A25 is divisible by 9 ➔ 9 | (4 + A + 2 + 5) ➔ A = 7

Case 2 : B = 7

4A75 is divisible by 9 ➔ 9 | (4 + A + 7 + 5) ➔ A = 2

Sum of possible numbers = 4725 + 4275 = 9000

Ans : 9000

3. [ASSIGNMENT] (Modified IMO 1960 P1) There is a 3-digit number N such that N is divisible by 11 and $\frac{N}{11}$ is equivalent to the sum of the squares of the digits of N. If the sum of the first digit and the last digit does not exceed 11, find the possible values of N.

Some divisibility rules

- $x \mid x$
- $1 \mid x$, $x \mid 0$
- $x \mid y$ **and** $y \mid z$ ➔ $x \mid z$
- $z \mid x$ **and** $y \mid x$ ➔ $lcm(z , y) \mid x$
- $x \mid y$ ➔ $|x| \leq |y|$ (unless y = 0)
- $z \mid x , y$ ➔ $z \mid ax + by$ for any integers *a and b*

<u>Exercise 1B</u>

1. How many positive perfect squares less than 10^6 are multiples of 24?

[Solution provided at the official solution of MOMC Scholarship Test]

2. Prove that for any positive odd integer *n*, $n^3 - n$ is divisible by *8*.

<u>Solution</u>: $n^3 - n = (n - 1)\ n\ (n + 1)$

n is given to be an odd number. Let $n = 2m + 1$ where m is any positive integer.

Therefore, $n - 1 = 2m$ and $n + 1 = 2m + 2$.

$n^3 - n = (n - 1)\, n\, (n + 1) = (2m)\,(2m+1)\,(2m+2) = 4(m)\,(2m+1)\,(m+1) = \frac{8\,(m)\,(2m+1)\,(m+1)}{2}$.

Note that one of m and $m + 1$ must be an even integer. Thus, $\frac{m(m+1)}{2}$ is an integer.

Therefore, $n^3 - n$ must be divisible by 8.

3. Show that, for any positive integer n ≥ 1, *6 | n (n + 1) (2n + 1).*

Solution: N is divisible by 6 ⇔ N is divisible by both 2 and 3.

Part 1: We will prove that $2 \mid n\,(n + 1)\,(2n + 1)$.

Exactly one of n and $n+1$ must be an even number ➔ $2 \mid n\,(n + 1)$ ➔ $2 \mid n\,(n + 1)\,(2n + 1)$.

Part 2: We will prove that $3 \mid n\,(n + 1)\,(2n + 1)$.

Remainder of n will be either 0, 1, or 2 when divided by 3.

This means $n = 3m$ or $n = 3m + 1$ or $n = 3m + 2$ for some positive integer m.

Case 1: n = 3m

$3 \mid n$ ➔ $3 \mid n\,(n + 1)\,(2n + 1)$.

Case 2: n = 3m + 2

$3 \mid (3m + 3)$ ➔ $3 \mid (n + 1)$ ➔ $3 \mid n\,(n + 1)\,(2n + 1)$.

Case 3: n = 3m + 1

$2n + 1 = 6m + 3 = 3\,(2m + 1)$

Thus, $3 \mid (2n + 1)$ ➔ $3 \mid n\,(n + 1)\,(2n + 1)$.

In both part 1 and part 2, we can see that $n\,(n + 1)\,(2n + 1)$ *is always divisible by 2 or 3.* Therefore, $6 \mid n\,(n + 1)\,(2n + 1)$.

Euclidian and Division Algorithm

For positive integers numbers *a* and *b*, division algorithm is used to determine a quotient and remainder q, r, such that $a = bq + r$.

Then $\gcd(a, b) = \gcd(b, r)$. This idea is used repetitively in the Euclidian Algorithm to find $\gcd(a, b)$.

$$a = bq_1 + r_1$$

$$b = r_1q_2 + r_2$$

$$r_1 = r_2\, q_3 + r_3$$

$$\dots$$

$$r_{n-2} = r_{n-1}\, q_n + r_n$$

$$r_{n-1} = r_n\, q_{n+1}$$

$$\gcd(a, b) = \gcd(b, r_1) = \gcd(r_1, r_2) = \dots = \gcd(r_{n-2}, r_{n-1}) = \gcd(r_{n-1}, r_n) = r_n$$

Exercise 1C

1. Show that $\gcd(4n + 3, 2n) \in \{1, 3\}$.

Solution: $\gcd(4n + 3, 2n) = \gcd(2n + 3, 2n) = \gcd(2n, 3)$.

Note that n is either divisible by 3 or not divisible by 3.

If $3 \mid n$ ➔ $\gcd(2n, 3) = 3$.

Otherwise, $\gcd(2n, 3) = 1$.

Therefore, $\gcd(4n + 3, 2n) \in \{1, 3\}$.

2. (IMO 1959 P1) Prove that this fraction is irreducible for every natural number *n*.

$$\frac{21n+4}{14n+3}$$

Solution: A fraction is said to be irreducible when its numerator and denominator have a greatest common divisor of 1.

$\gcd(21n+4, 14n+3) = \gcd(14n+3, 7n+1) = \gcd(7n+1, 7n+2) = \gcd(1, 7n+1) = 1$

Therefore, the fraction is irreducible.