QSMT Mini-camp

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Algebra

Contents

1 Introduction

This lecture file serves as an introduction to Olympiad Algebra, tailored to the difficulty level of the Quadri-School Math Tournament (QSMT). It is highly recommended to cover this lecture before QSMT, academic year 2025-2026.

1.1 About QSMT

QSMT was founded in the year 2023-2024 by the IMO-2024 Myanmar Team to recruit math enthusiasts from selected schools for the journey toward IMO and provide a fee-waiver to the most talented students from each school. This tradition has been carried forward and has now become an annual contest.

QSMT, academic year 2025-2026, include the following the events to prepare students for the Myanmar Open Mathematical Competition (MOMC):

- Mini-camp (online)
- QSMT exam (on campus and online)
- Main-camp (online)

1.2 About us

QSMT academic 2025-2026 is organized by 5 members from the IMO-2025 Myanmar Team. Meet the team members:

- Aung Nyan Zaw, currently studying IB diploma at St. Clare's, Oxford, United Kingdom;
- Hsu Wutt Yee Lin, currently studying IAL at Hinthar International School;
- Nyan Phone Win, currently studying IAS at Ascend International College;
- Swan Htet Naing, Stumble Guys God;
- Swan Htet Nay Khaing, alumnus of IIP International School, currently studying IAS in Bangkok, Thailand;

each of whom are going to be the lecturers of the mini-camp (pre-QSMT) and the main camp (post-QSMT).

2 Solving Equations

Skills required to solve QSMT Algebra

- Solve linear and quadratic equations accurately
- Form equations according to the problem context, which are appropriate to solve
- Know which equations could be solved through known givens and observations, and which can't

2.1 Examples

Question 1 (QSMT 2024-2025). Adults made up $\frac{5}{12}$ of people at a concert. After a bus carrying 50 more people arrived, adults made up $\frac{11}{25}$ of the people at the concert. Find the minimum number of adults who could have been at the concert after the bus arrived.

Answer: $\boxed{154}$
Solution:
Let the number of people before the bus be P,
and the number of additional adults from the bus be x.

Then,

the number of adults before the bus $= \dfrac{5P}{12}$,
the number of people after the bus $= P + 50$,
and the number of adults after the bus $= \dfrac{11(P+50)}{25}$.

We can then form the following equation:

$$\begin{aligned} x &= \frac{11(P+50)}{25} - \frac{5P}{12} \\ &= \frac{7P + 6600}{300} \\ &= \frac{7P}{300} + 22 \end{aligned}$$

Since x is a positive integer, we can see that P must certainly be a multiple of 300.

Since we are interested in the minimum number, take $P = 300$ and we reach the conclusion that the number of adults who were at the concert after the bus is

$$\frac{11(P+50)}{25} = \boxed{154}.$$

Question 2 (Well known). Let a, b be positive integers such that

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{101}$$

Find all pairs of such (a, b).

Answer: $\boxed{(202, 202), (102, 10302), (10302, 102)}$
Solution:

$$\begin{aligned}
\frac{1}{a} + \frac{1}{b} &= \frac{1}{101} \\
\frac{a+b}{ab} &= \frac{1}{101} \\
ab - 101(a+b) &= 0 \\
ab - 101(a+b) + 101^2 &= 101^2 \\
(a-101)(b-101) &= 101^2 \qquad \text{(Simon's Favorite Factoring Trick)}
\end{aligned}$$

Taking advantage of a, b being positive integers, we can rewrite 101^2 as a product of two numbers.

$$\begin{aligned}
101^2 &= 1 \times 101^2 \\
&= 101 \times 101 \\
&= 101^2 \times 1
\end{aligned}$$

By equating each of the terms with $(a - 101)$ and $(b - 101)$, we get the answer.

Remark: The main point of this example was Simon's Favorite Factoring Trick. Next time you're facing a term $xy + Ax + By + C$, you can turn it into $(x + B)(y + A)$ which might be easier to handle.

Domain of a, b is also extremely important. If a, b was over all integers, there would be more solutions.

2.2 Practice Question

Problems are arranged in no particular order. It is recommended that you try to solve these problems before QSMT. Don't worry if you get stuck; you can request hints by emailing me at `cholerossi198@gmail.com`.

Exercise 1 (QSMT 2024-2025). Let x, y, and z be real numbers which satisfy the following system of equations:

$$xy + 4z = 60$$
$$yz + 4x = 60$$
$$zx + 4y = 60$$

Let S be the set of possible values of x. Find the sum of the squares of the elements of S.

Exercise 2. Let x be a positive integer such that $x \geq 1$ and $x + \dfrac{1}{x} = 4$. Find the value of
(i) $x^2 + \frac{1}{x^2}$
(ii) $x^3 + \frac{1}{x^3}$
(iii) $x - \frac{1}{x}$

Exercise 3. Find all real numbers x such that $4^x + 6^x = 9^x$.

Exericse 4. Let a, b be integers such that

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{101}$$

Find all pairs of such (a, b). (Note that a, b can also be negative integers.)

Exercise 5 (Germany 2008). Solve over real numbers:

$$(x + y)(x^2 - y^2) = 675$$
$$(x - y)(x^2 + y^2) = 3$$

3 Polynomials

Skills required to solve QSMT Algebra

- Properties of roots of quadratic equations
- Know the properties of a polynomial
- Apply Vieta's Formula well

3.1 General Polynomial form

An unknown polynomial is generally considered expressed as

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0.$$

All polynomials are continuous, which means their graphs can be drawn in a single, unbroken stroke without lifting the pencil. For example, functions like $\dfrac{7x-2}{3x-2}$ which has vertical and horizontal asymptotes are not considered polynomials.

A root, α, of $P(x)$ is defined as the value which satisfies the equation $P(\alpha) = 0$.

3.2 Quadratic equations

Quadratic equations are the simpest polynomials in my opinion. Allow me to present my case.
For a quadratic equation $ax^2 + bx + c = 0$ and its roots α and β, the following are true:

- $(x - \alpha)(x - \beta) = 0$.
- $\alpha + \beta = -\frac{b}{a}$.
- $\alpha\beta = \frac{c}{a}$.
- if $b^2 - 4ac = 0$, $\alpha = \beta$ and they are real.
- if $b^2 - 4ac > 0$, α, β are real.
- if $b^2 - 4ac < 0$, α, β are not real.
- $\alpha, \beta = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

3.3 Vieta's formula

Vieta's formulas relate the coefficients of a polynomial to sums and products of its roots. Consider a polynomial P(x):

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0, \quad a_n \neq 0$$

with roots $r_1, r_2, \ldots, r_n$ (not necessarily distinct). Then

$$
\begin{aligned}
r_1 + r_2 + \cdots + r_n &= -\frac{a_{n-1}}{a_n}, \\
\sum_{1 \le i < j \le n} r_i r_j &= \frac{a_{n-2}}{a_n}, \\
\sum_{1 \le i < j < k \le n} r_i r_j r_k &= -\frac{a_{n-3}}{a_n}, \\
&\vdots \\
r_1 r_2 \dots r_n &= (-1)^n \frac{a_0}{a_n}.
\end{aligned}
$$

This may look intimidating but let's try untangling it with a smaller n in the example.

3.4 Examples

Question 1. Consider the polynomial

$$x^3 - 15x^2 + 59x - 45$$

with roots α, β, θ, which are pairwise distinct. Find the values of:

(i) $\alpha + \beta + \theta$
(ii) $\alpha\beta + \beta\theta + \theta\alpha$
(iii) $\alpha\beta\theta$

(Hint: The given statement says that $(x-\alpha)(x-\beta)(x-\theta) = x^3 - 15x^2 + 59x - 45$.)

Answer: $\boxed{\text{(i)}45, \text{(ii)}59, \text{(iii)}15}$

Solution:

Approach 1:

$$
\begin{aligned}
(x-\alpha)(x-\beta)(x-\theta) &= x^3 - 15x^2 + 59x - 45 \\
x^3 - (\alpha+\beta+\theta)x^2 + (\alpha\beta + \beta\theta + \theta\alpha)x - \alpha\beta\theta &= x^3 - 15x^2 + 59x - 45
\end{aligned}
$$

By equating each of the coefficient from two sides, we get the desired answers.

Approach 2: directly from Vieta's formula, we have

$$
\begin{aligned}
\alpha + \beta + \theta &= -\frac{-45}{1} = 45 \\
\alpha\beta + \beta\theta + \theta\alpha &= \frac{59}{1} = 59 \\
\alpha\beta\theta &= -\frac{-15}{1} = 15
\end{aligned}
$$

Remark: Both approaches are provided so that you might get a sense of where Vieta's formula come from.

Question 2 (QSMT 2024-2025). What is the sum of all possible values of k for which the polynomials $x^2 - 3x + 2$ and $x^2 - 5x + k$ have a root in common?

Answer: $\boxed{10}$

Solution:
Roots of $x^2 - 3x + 2$ are 1 and 2. Let the roots of $x^2 - 5x + k$ be α and β. Then,

$$\begin{aligned} \alpha + \beta &= 5 \\ \alpha\beta &= k \end{aligned}$$

Case 1: The common root is 1. Let $\alpha = 1$. Then,

$$\beta = 5 - 1 = 4$$

$$k = 1 \times 4 = 4$$

Case 2: The common root is 2. Let $\alpha = 2$. Then,

$$\beta = 5 - 2 = 3$$

$$k = 2 \times 3 = 6$$

From both cases, the sum of possible values of $k = 4 + 6 = \boxed{10}$.

3.5 Practice Problems

Problems are arranged in no particular order. It is recommended that you try to solve these problems before QSMT. Don't worry if you get stuck; you can request hints by emailing me at `cholerossi198@gmail.com`.

Exercise 1 (Vieta's formula practice). Evaluate.

1. The polynomial
$$x^3 - 6x^2 + 11x - 6$$
has roots α, β, γ. Without solving the polynomial explicitly, find the value of
$$(\alpha + 1)(\beta + 1)(\gamma + 1).$$

2. Let α, β, γ be the roots of
$$x^3 - 3x^2 + 4x - 2.$$
Find the value of $\alpha^2 + \beta^2 + \gamma^2$.

3. The roots of $x^3 - 7x^2 + 14x - 8$ are α, β, γ. Compute
$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}.$$

4. Three students solved a polynomial problem and found the roots to be the ages (in years) of three siblings. The polynomial was
$$x^3 - 12x^2 + 47x - 60 = 0.$$
Determine the following the products of their age one year ago.

5. Let α, β be the roots of
$$x^2 - 7x + 11 = 0.$$
Without finding α, β, evaluate $\alpha^3 + \beta^3$.

Exercise 2 (2020 Myanmar TST nerfed). Three real numbers x, y, z are such that $y + z = 2 - x$ and $y^2 + z^2 = 4 - x^2$.

(i) Prove that $(y - z)^2 = -3x^2 + 4x + 4$
(ii) Show that $-\frac{2}{3} \leq x \leq 2$.

Exercise 3. Three consecutive terms in a sequence of real numbers are given by
$$k, 1 + 2k, 3 + 3k$$
where k is a constant. Show that the sequence is not a geometric sequence.

Exercise 4. Find all possible value of integers a, b, c such that $ab - c = 3$ and $abc = 18$.

4 Inequalities

Skills required to solve QSMT Algebra

- Form inequalities according to the problem context
- Solve linear and quadratic inequalities
- Apply the AM-GM inequality well

4.1 AM-GM inequality

The Arithmetic Mean (AM) - Geometric Mean (GM) inequality states that for any positive real numbers a and b,

$$\frac{a+b}{2} \geq \sqrt{ab}$$

The statement is derived from the inequality

$$(\sqrt{a}-\sqrt{b})^2 \geq 0$$

which is always true. From here, we can see that the equality case occurs when $\sqrt{a}=\sqrt{b}$, thus $a=b$.

A few useful applications from this inequality:
(i) $a^2+b^2 \geq 2ab$
(i) $a+\frac{1}{a} \geq 2$

Gernerally, the AM-GM inequality states that

$$\frac{a_1+a_2+\cdots+a_n}{n} \geq (a_1a_2\cdots a_n)^{\frac{1}{n}}$$

Equality occurs when $a_1=a_2=\cdots=a_n$.

4.2 Examples

Question 1. Find the maximum value of $2-a-\frac{1}{2a}$ for all positive real numbers a.
Answer: $\boxed{2-\sqrt{2}}$

Solution:
In order to achieve the maximum value of $2-a-\frac{1}{2a}$, we should get the minimum value of $a+\frac{1}{2a}$. By AM-GM inequality, we know that

$$a+\frac{1}{2a} \geq 2\times\sqrt{\frac{1}{2}} = \sqrt{2}$$

With the minimum value of $a+\frac{1}{2a}$ being $\sqrt{2}$, the maximum value of $2-a-\frac{1}{2a} = \boxed{2-\sqrt{2}}$.

Question 2. Let positive real numbers x, y, and z be such that $xy + yz + zx = 3$. Find the minimum value of
(i) $x^2 + y^2 + z^2$
(ii) $x + y + z$, by expanding $(x + y + z)^2$.
State when the case of equality occurs.

Answer: $\boxed{\text{(i)}3, \text{(ii)}3}$

Solution:
(i) By AM-GM inequality,

$$\begin{aligned}
x^2 + y^2 &\geq 2xy \\
y^2 + z^2 &\geq 2yz \\
z^2 + x^2 &\geq 2zx
\end{aligned}$$

Adding them together, we get

$$\begin{aligned}
2(x^2 + y^2 + z^2) &\geq 2(xy + yz + zx) \\
(x^2 + y^2 + z^2) &\geq (xy + yz + zx) \\
&= 3
\end{aligned}$$

Equality occurs when $x = y = z$ (according to AM-GM inequality). Therefore, $x = y = z = 1$.

(ii)

$$\begin{aligned}
(x + y + z)^2 &= x^2 + y^2 + z^2 + 2xy + 2yz + 2zx \\
&\geq 3(xy + yz + zx) \\
&= 9 \\
\therefore x + y + z &\geq \sqrt{9} = 3
\end{aligned}$$

Equality occurs when $x = y = z = 1$.

Remark: Make note of that inequality $\boxed{x^2 + y^2 + z^2 \geq xy + yz + zx}$ as it is one of the most very well-known results of AM-GM inequality.

4.3 Practice Problems

Problems are arranged in no particular order. It is recommended that you try to solve these problems before QSMT. Don't worry if you get stuck; you can request hints by emailing me at `cholerossi198@gmail.com`.

Exercise 1 (Warm up). Form inequalities according to the problem context.

1. A tank can hold 500 liters of water. If it is filled at a rate of $4x + 5y$ liters per minute, where x and y are the number of large and small pipes opened, form the inequality ensuring the tank doesn't overflow in 20 minutes.

2. A town's population grows at a rate of $r\%$ per year compoundly. If the current population is P, and after 10 years it must not exceed $2P$, form an inequality involving r.

3. In a race, the number of people who placed behind Alice was more than half of the number of people before her. If she placed at 19^{th}, form an inequality for the amount of people who participated in the race.

Exercise 2 (Own). Alice and Bob are playing a game. Alice has four positive real numbers in mind. Alice tells Bob that

- One of the numbers is 3.
- The sum of cubes of the four numberes is exactly 27 greater than the product of all four numbers.
- The sum of the highest and second highest number is 10.

Help Bob find the four positive real numbers.

Exercise 3. Bob practices Algebra for a hours and Geometry for g hours. His coach requires that

$$ag \geq 100$$

However, Bob wants the total practice hours to be as little as possible. How many hours of Algebra should he practice?

Exercise 4 (Hongkong TST). Find all real triples (a, b, c) satisfying

$$(2^{2a} + 1)(2^{2b} + 2)(2^{2c} + 8) = 2^{a+b+c+5}.$$