# Solving Dynamic Euler-Bernoulli Beam Eq (PART B)

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 v}{\partial x}\right) = -\rho A\frac{\partial^2 v}{\partial t^2}$$

$I = I(x)$ $\qquad$ $A = A(x)$

If material and cross section are uniform

$$EI\frac{\partial^4 v}{\partial x^4} = -\rho A\left(\frac{\partial^2 v}{\partial t^2}\right)$$

They are because of simplification explained on page 2

There are 4 boundary conditions because the order of the dynamic euler-bernoulli beam equation in x is 4

$L^* = \frac{L}{N}$

BC: $v(0,t)=0$ $\qquad$ $v''(L^*,t) = 0$

$v'(0,t)=0$ $\qquad$ $v'''(L^*,t) = 0$

There are 2 initial conditions because the order of the dynamic euler-bernoulli beam equation in t is 2

Initial conditions

$v(x,0) = v_0(x)$ $\qquad$ $\dot{v}(x,0) = 0$

initial shape of beam $\qquad$ initial velocity

Explanation for why 1st segment influences the fundamental frequency more than other segments

The first segment has a much larger influence on fundamental frequency compared to other segments because it has the largest cross-sectional area causing it to have the largest second moment of area. Another reason we can make this approximation is because we only have insert number of segments N we have. This means we will have a reasonable estimate. If we had more segments, this approximation would be less accurate and not be considered reasonable. We are only making this estimation so we can get a reasonable estimate for the fundamental frequency. Lastly,

## seperation of variables solution (PART C & D)

Looking for seperable equation of form

$$v(x,t) = X(x)\,T(t) \leftarrow \text{Ansatz}$$

to plug into the governing equation

$$EI\frac{\partial^4 v}{\partial x^4} = -\rho A\left(\frac{\partial^2 v}{\partial t^2}\right)$$ ** found earlier

Substituting the seperable equation into the governing equation, we obtain

$$X''''(x)\,T(t) = -\frac{1}{K^2}X(x)\,\ddot{T}(t).$$

Dividing both sides of the equation by

$$X(x)\,T(t),$$

we obtain

$$\frac{X''''(x)}{X(x)} = -\frac{1}{K^2}\frac{\ddot{T}(t)}{T(t)}.$$

where the left side of the equation is only a function of x and the right side of the equation is only a function of T.

We made a tacit assumption that

$$X \neq 0 \text{ and } T(t) \neq 0$$

in order to prevent getting a trivial solution

$$v(x,t)=0.$$

we are not interested in the trivial solution because that would mean that the beam doesn't move.

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Looking back the seperable solution

$$\frac{X''''(x)}{X(x)} = -\frac{1}{k^2}\frac{\ddot{T}(t)}{T(t)}$$

what is this eq called?

seperable solution?

we know that both sides of this equation must be equal to a constant

$$\frac{X''''(x)}{X(x)} = -\frac{1}{k^2}\frac{T(t)}{T(t)} = \text{constant}$$

first solve for this

which one is it? red or blue

sign of the seperation constant or

To determine the admissable values of the seperation constant, recall the boundary conditions,

then solve for this

$$v(0,t)=0$$

$$v'(0,t)=0$$

$$v''(L^*,t)=0$$

$$v'''(L^*,t)=0.$$

Looking at each boundary condition individually,

$$v(0,t)=0 \rightarrow X(0)T(t)=0$$

$$X(0)=0 \quad *$$

or

$$T(t)=0$$

$$v'(0,t)=0 \rightarrow X'(0)T(t)=0$$

$$X'(0)=0 \quad *$$

or

$$T(t)=0$$

$$v''(L,t)=0 \rightarrow X''(L)T(t)=0$$

$$X''(L^*)=0 \quad *$$

or

$$T(t)=0$$

$$v'''(L^*,t)=0 \rightarrow X'''(L)T(t)=0$$

$$X'''(L^*)=0 \quad *$$

or

$$T(t)=0$$

Looking at each initial condition individually

$$\dot{u}(x,0) = 0 \longrightarrow X(x)\dot{T}(0) = 0$$

$$X(x) = 0$$

or

$$\dot{T}(0) = 0 \quad *$$

$$u(x,0) = u_0(x)$$

Admissable values of seperation constant

$$\frac{X''''(x)}{X(x)} = -\frac{1}{c^2}\frac{\ddot{T}(t)}{T(t)} = \text{constant}$$

case 1: const $= 0$ ✗

case 2: const $< 0$ ✗

case 3 : const $> 0$ ✓

In order to determine the sign of the constant, we must examine the boundary conditions and initial conditions.

$$\frac{X''''(x)}{X(x)} = \text{constant} \quad \text{seperable solution}$$

Examining case 1 (constant = 0), we first must isolate $X''''(x)$ from the seperable solution

Multiplying both sides of the seperable solution by $X(x)$,

$$X(x)\left(\frac{X''''(x)}{X(x)}\right) = (0)\, X(x)$$

we obtain a function of x to the 4th order

$$X''''(x) = 0. \quad \text{eq}$$

Integrating both sides of eq

$$\int X''''(x) = \int 0$$

we get,

$$X'''(x) = c_1. \quad \text{eq } *$$

Recall boundary condition

$$X'''(L^*) = 0.$$

Replacing x with $L^*$, our constant will be

$$c_1 = 0.$$

Integrating both sides of eq *,

$$\int X'''(x) = \int 0$$

gives us

$$X''(x) = c_2. \quad \text{eq } **$$

Recall boundary condition

$$X''(L^*) = 0$$

Replacing $x$ with $L^*$ we get,

$$c_2 = 0.$$

Integrating both sides of eq **,

$$\int X''(x) = \int 0$$

gives us

$$X'(x) = c_3. \qquad \text{eq ***}$$

Recall boundary condition

$$X'(0) = 0$$

Replacing $x$ with $0$ we get,

$$c_3 = 0.$$

Integrating both sides of eq ***,

$$\int X'(x) = \int 0$$

gives us

$$X(x) = c_4. \qquad \text{eq ****}$$

Recall boundary condition

$$X(0) = 0$$

Replacing $x$ with $0$ we get,

$$c_4 = 0.$$

This shows that

$$X(x) = 0,$$

which would be a trivial solution which we do not want.

Now that we proved that the constant cannot be 0, we will examine case 2:

constant < 0.

First we must isolate

$$X''''(x).$$

Multiplying both sides of the seperable solution by $X(x)$,

$$X(x)\left(\frac{X''''(x)}{X(x)}\right)=(-\beta^4)\,X(x)$$

we obtain a function of x to the $4^{th}$ order

$$X''''(x) = X(x)(-\beta^4) \quad \text{eq}$$

Substitute

$$X = e^{\lambda x}$$

we are left with

$$\lambda^4 e^{\lambda x} = e^{\lambda x}(-\beta^4) \quad \text{eq}*$$

because

$$e^{\lambda x} \neq 0$$

we can cancel out on both sides of eq*

leaving us with

$$\lambda^4 = -\beta^4 \quad \text{eq}**$$

Taking the square root of eq**

$$\lambda^2 = \pm\sqrt{-1}\,\beta^2 = \pm i\beta^2 \quad \text{eq}***$$

Taking another square root of eq***

$$\lambda = \pm\sqrt{\pm i}\,\beta$$

Gives us 4 solutions for $\lambda$

$$\lambda = \pm\sqrt{\pm i}\,\beta$$

where

$$\lambda_1 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$$
$$\lambda_2 = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$$
$$\lambda_3 = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$$
$$\lambda_4 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$$

The equation for $X$ is

$$X = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + c_3 e^{\lambda_3 x} + c_4 e^{\lambda_4 x} \quad \text{eq ****}$$

Recall boundary condition

$$X(0) = 0.$$

Replacing x with 0 we get,

$$c_1 + c_2 + c_3 + c_4 = 0 \quad \text{eq 1}$$

Taking the derivative of eq ****

$$X' = (c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + c_3 e^{\lambda_3 x} + c_4 e^{\lambda_4 x})',$$

we obtain

$$X' = (c_1 \lambda_1 e^{\lambda_1 x} + c_2 \lambda_2 e^{\lambda_2 x} + c_3 \lambda_3 e^{\lambda_3 x} + c_4 \lambda_4 e^{\lambda_4 x}). \quad \text{eq 2}$$

Recall boundary condition

$$X'(0)=0.$$

Replacing x with 0 we get,

$$c_1\lambda_1+c_2\lambda_2+c_3\lambda_3+c_4\lambda_4=0 \quad \text{eq 3}$$

Taking the derivative of eq 2

$$X''=(c_1\lambda_1e^{\lambda_1x}+c_2\lambda_2e^{\lambda_2x}+c_3\lambda_3e^{\lambda_3x}+c_4\lambda_4e^{\lambda_4x})'$$

we obtain

$$X''=(c_1\lambda_1^2e^{\lambda_1x}+c_2\lambda_2^2e^{\lambda_2x}+c_3\lambda_3^2e^{\lambda_3x}+c_4\lambda_4^2e^{\lambda_4x}). \quad \text{eq 4}$$

Recall boundary condition

$$X''(L^*)=0.$$

Replacing x with L we get,

$$(c_1\lambda_1^2e^{\lambda_1L^*}+c_2\lambda_2^2e^{\lambda_2L^*}+c_3\lambda_3^2e^{\lambda_3L^*}+c_4\lambda_4^2e^{\lambda_4L^*})=0. \quad \text{eq 5}$$

Taking the derivative of eq 4

$$X''' = \left(c_1\lambda_1^2 e^{\lambda_1 x} + c_2\lambda_2^2 e^{\lambda_2 x} + c_3\lambda_3^2 e^{\lambda_3 x} + c_4\lambda_4^2 e^{\lambda_4 x}\right)'$$

we obtain

$$X''' = \left(c_1\lambda_1^3 e^{\lambda_1 x} + c_2\lambda_2^3 e^{\lambda_2 x} + c_3\lambda_3^3 e^{\lambda_3 x} + c_4\lambda_4^3 e^{\lambda_4 x}\right). \quad \text{eq 6}$$

Recall boundary condition

$$X'''(L^*) = 0.$$

Replacing x with L we get,

$$\left(c_1\lambda_1^3 e^{\lambda_1 L^*} + c_2\lambda_2^3 e^{\lambda_2 L^*} + c_3\lambda_3^3 e^{\lambda_3 L^*} + c_4\lambda_4^3 e^{\lambda_4 L^*}\right) = 0. \quad \text{eq 7}$$

$$c_1 D e^{Dx} + c_2 D e^{Dx} = -c_3 F e^{Fx} - c_4 F e^{Fx}$$

$$D e^{Dx}(c_1 - c_2) = -F e^{Fx}(c_3 - c_4)$$

$$(c_1\lambda_1^3 e^{\lambda_1 L^*} + c_2\lambda_2^3 e^{\lambda_2 L^*} + c_3\lambda_3^3 e^{\lambda_3 L^*} + c_4\lambda_4^3 e^{\lambda_4 L^*})$$

$$=$$

$$(c_1\lambda_1^2 e^{\lambda_1 L^*} + c_2\lambda_2^2 e^{\lambda_2 L^*} + c_3\lambda_3^2 e^{\lambda_3 L^*} + c_4\lambda_4^2 e^{\lambda_4 L^*})$$

$$\Downarrow$$

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$$

$$\lambda_1 + \lambda_2 = -\lambda_3 - \lambda_4$$

$$X = c_1 + c_2 + c_3 + c_4 = 0$$

This shows that

$$X = 0,$$

which again is a trivial solution that we do not want.

This means the only possible sign for the constant to be is positive.

Now if the constant is positive, the seperable solution is

$$\frac{X''''(x)}{X(x)} = -\frac{1}{k^2}\frac{\ddot{T}(t)}{T(t)} = \lambda$$

where

$$\lambda = \beta^4.$$

Looking at the equation for spatial dependence, we have

$$\frac{X''''(x)}{X(x)} = \beta^4.$$

multiplying $X(x)$ to both sides and subtracting the right side, we obtain the equation

$$X''''(x) - \beta^4 X(x) = 0 \quad \text{eq 45 6}$$

Now looking at the equation for time dependence,

$$-\frac{1}{\kappa^2}\frac{\ddot{T}(t)}{T(t)} = \beta^4 \quad \text{eq....}$$

Rearranging eq.... we obtain

$$\ddot{T}(t) + \kappa^2\beta^4 T(t) = 0. \quad \text{eq...}$$

eq... is equivalent to the equation for the simple harmonic oscillator

$$\ddot{x} + w^2 x = 0.$$

where

$$w = K\beta^2.$$

The general solution for this equation is

$$T(t) = E\cos(K\beta^2 t) + F\sin(K\beta^2 t),$$

where E and F are constants.

Enforcing the initial conditions on the general solution,

$$\dot{T}(t) = -K\beta^2 E\sin(K\beta^2 t) + K\beta^2 F\cos(K\beta^2 t)$$

Evaluating $\dot{T}(t)$ at $t=0$,

$$\dot{T}(0) = K\beta^2 F = 0.$$

we know that $\beta$ can't be 0 because the seperation constant is positive. on top of that, we know $K \neq 0$ because we are dividing by $K$ in the equation for time dependence. It is impossible to divide by 0, thus we can conclude that

$$F = 0.$$

Due to the fact that $F=0$, the time dependence function simplifies to

$$T(t) = E\cos(KB^2 t).$$

and the 1st derivative of the time dependence function simplifies to

$$\dot{T}(t) = -KB^2 E\sin(KB^2 t).$$

Now, we must prove that the general solution for $X$ is

$$X(x) = A^* \cos(Bx) + B^* \sin(Bx) + C^* \cosh(Bx) + D^* \sinh(Bx) \quad \text{eq 123}$$

To prove this, we need to plug eq 123 back into eq 456.

First, we need the fourth derivative of eq123 in order to substitute it back into eq456.

The first derivative is

$$X'(x)=\beta(-A^*\sin(\beta x)+B^*\cos(\beta x)+C^*\sinh(\beta x)+D^*\cosh(\beta x)).$$

The second derivative is

$$X''(x)=\beta^2(-A^*\cos(\beta x)-B^*\sin(\beta x)+C^*\cosh(\beta x)+D^*\sinh(\beta x)).$$

The third derivative is

$$X'''(x)=\beta^3(A^*\sin(\beta x)-B^*\cos(\beta x)+C^*\sinh(\beta x)+D^*\cosh(\beta x)).$$

Finally, the fourth derivative is

$$X''''(x)=\beta^4(A^*\cos(\beta x)+B^*\sin(\beta x)+C^*\cosh(\beta x)+D^*\sinh(\beta x)).$$

Now lets substitute back into eq456.

$$\beta^4(A^*\cos(\beta x)+B^*\sin(\beta x)+C^*\cosh(\beta x)+D^*\sinh(\beta x))-$$
$$\beta^4(A^*\cos(\beta x)+B^*\sin(\beta x)+C^*\cosh(\beta x)+D^*\sinh(\beta x))=0$$

After substituting, we have proven that eq 123 is the general solution for $X$.

Now that we have the correct general solution, we will enforce our boundary conditions to determine the admissable values of the seperation constant.

First let's recall our boundary equations

$$X(0)=0$$
$$X'(0)=0$$
$$X''(L^*)=0$$
$$X'''(L^*)=0$$

and the general solution along with its first three derivatives

$$X(x)=A^*\cos(\beta x)+B^*\sin(\beta x)+C^*\cosh(\beta x)+D^*\sinh(\beta x)$$
$$X'(x)=\beta\left(-A^*\sin(\beta x)+B^*\cos(\beta x)+C^*\sinh(\beta x)+D^*\cosh(\beta x)\right)$$
$$X''(x)=\beta^2\left(-A^*\cos(\beta x)-B^*\sin(\beta x)+C^*\cosh(\beta x)+D^*\sinh(\beta x)\right)$$
$$X'''(x)=\beta^3\left(A^*\sin(\beta x)-B^*\cos(\beta x)+C^*\sinh(\beta x)+D^*\cosh(\beta x)\right).$$

Enforcing our first boundary condition

$$X(0)=0,$$

we obtain

$$A^* + C^* = 0. \qquad \text{eq 79}$$

Enforcing our second boundary condition

$$X'(0)=0,$$

we obtain

$$B^* + D^* = 0. \qquad \text{eq 80}$$

Enforcing our third boundary condition

$$X''(L^*)=0,$$

we obtain

$$(-A^* \cos(\beta L^*) - B^* \sin(\beta L^*) + C^* \cosh(\beta L^*) + D^* \sinh(\beta L^*)) = 0. \qquad \text{eq 81}$$

Enforcing our last boundary condition

$$X'''(L)=0,$$

we obtain

$$(A^* \sin(\beta L^*) - B^* \cos(\beta L^*) + C^* \sinh(\beta L^*) + D^* \cosh(\beta L^*)) = 0. \quad \text{eq } 82$$

Note: we can divide by $\beta$ to simplify our last three equations.

Solving for $C^*$ and $D^*$ in eq 79 and eq 80, we obtain

$$C^* = -A^* \quad \text{eq } 83$$

and

$$D^* = -B^*, \quad \text{eq } 84$$

respectively.

We can now eliminate $C^*$ and $D^*$ from eq 81 and eq 82.

Substituting eq 83 and eq 84 into eq 81, we obtain

$(-A^*\cos(\beta L^*) - B^*\sin(\beta L^*) - A^*\cosh(\beta L^*) - B^*\sinh(\beta L^*)) = 0.$ eq 85

simplifying eq 85, we obtain

$A^*(\cos(\beta L^*) + \cosh(\beta L^*)) = -B^*(\sin(\beta L^*) + \sinh(\beta L^*))$ eq 86

Solving for $A^*$ in eq 86, we obtain

$$A^* = -B^* \frac{(\sin(\beta L^*) + \sinh(\beta L^*))}{(\cos(\beta L^*) + \cosh(\beta L^*))}. \quad \text{eq 87}$$

Substituting eq 83 and eq 84 into eq 82, we obtain

$(A^*\sin(\beta L^*) - B^*\cos(\beta L^*) - A^*\sinh(\beta L^*) - B^*\cosh(\beta L^*)) = 0.$ eq 88

using eq 87 we can get eq 88 in terms of B

$$\left(-B^*\sin(\beta L^*)\frac{(\sin(\beta L^*) + \sinh(\beta L^*))}{(\cos(\beta L^*) + \cosh(\beta L^*))} - B^*\cos(\beta L^*) + B^*\sinh(\beta L^*)\frac{(\sin(\beta L^*) + \sinh(\beta L^*))}{(\cos(\beta L^*) + \cosh(\beta L^*))} - B^*\cosh(\beta L^*)\right) = 0. \quad \text{eq 89}$$

Combining like terms from eq 89, we obtain

$$B^*(\sinh(\beta L^*) - \sin(\beta L^*))\frac{(\sin(\beta L^*) + \sinh(\beta L^*))}{(\cos(\beta L^*) + \cosh(\beta L^*))} = B^*(\cos(\beta L^*) + \cosh(\beta L^*)). \quad \text{eq 90}$$

simplifying eq 90, we obtain

$$(\sin(\beta L^*)+\sinh(\beta L^*))(\sinh(\beta L^*)-\sin(\beta L^*))=(\cos(\beta L^*)+\cosh(\beta L^*))(\cos(\beta L^*)+\cosh(\beta L^*)) \quad \text{eq 91}$$

simplifying eq 91 further, we obtain

$$(-\sin^2(\beta L^*)+\sinh^2(\beta L^*))=(\cos^2(\beta L^*)+2\cos(\beta L^*)\cosh(\beta L^*)+\cosh^2(\beta L^*)) \quad \text{eq 92}$$

simplifying eq 92 further, we obtain

$$-(\cosh^2(\beta L^*)-\sinh^2(\beta L^*))=\sin^2(\beta L^*)+\cos^2(\beta L^*)+2\cos(\beta L^*)\cosh(\beta L^*) \quad \text{eq 93}$$

simplifying eq 93 once again, we obtain

$$-2 = 2\cos(\beta L^*)\cosh(\beta L^*) \quad \text{eq 94}$$

After one final simplification, we are left with

$$\cos(\beta L^*)\cosh(\beta L^*)=-1 \quad \text{eq 95}$$

which shows $\beta$ satisfies EQ 7 IN CHECKPOINT by enforcing the boundary conditions on eq 123.