MATH 2164 Test 2 Review

Your Name: ____________________

Solutions to be posted July 27, 2022

1. Given that A and B are two 5×5 matrices, $\det(A) = -1$, and $\det(B) = 2$, find the following:

   $\det(A^4 B^2) =$ ____________

   $\det(-2A) =$ ____________

   $\det((AB)^{-1}) =$ ____________

   $\det(2A + B) =$ ____________

2. Let A be a 4×4 matrix whose determinant is 13 and let
   A_1 be the matrix obtained from A by the row operation $-13R_1 + R_4 \to R_4$.
   A_2 be the matrix obtained from A by the row operation $R_1 \leftrightarrow R_4$
   A_3 be the matrix obtained from A by the row operation $5R_2 \to R_2$
   A_4 be the matrix obtained from A by the following four row operations consecutively: the first operation is $R_1 \leftrightarrow R_3$, followed by the operation $4R_2 + R_3 \to R_3$, then the operation $-2R_4 \to R_4$, and finally the operation $R_2 \leftrightarrow R_4$.
   then $\det(A_1) =$ ____________, $\det(A_2) =$ ____________, $\det(A_3) =$ ____________,

   $\det(A_4) =$ ____________

3. Select all statements below which are true for all invertible $n \times n$ matrices A and B.
   - ◯ $\det(A - B) = \det(A) - \det(B)$.
   - ◯ kB is invertible for any scalar k that is not zero.
   - ◯ $(AB)^{-1} = B^{-1}A^{-1}$.
   - ◯ $\det(A^{-1}B^2) = \frac{(\det(B))^2}{\det(A)}$.
   - ◯ A^k is invertible for any positive integer k and $(A^k)^{-1} = (A^{-1})^k$.
   - ◯ $\det(5B) = 5\det(B)$.
   - ◯ $(A + B)^2 = A^2 + 2AB + B^2$.
   - ◯ The equation $A\mathbf{x} = \mathbf{b}$ always has a unique solution for any vector $\mathbf{b}$.
   - ◯ A may have an echelon form with fewer than n pivots.
   - ◯ $A + B$ is always invertible and $(A + B)^{-1} = A^{-1} + B^{-1}$

4. Find the inverse of the matrix $\begin{bmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix}$ Write your answer in the blank space below.

5. Given that $\mathbf{v_1} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$, and $\mathbf{v_3} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, choose the correct statements.

- ◯ The span of $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{v_3}$ has dimension 2.
- ◯ Any vector in $\mathbb{R}^3$ can be written as a linear combination of $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{v_3}$.
- ◯ $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{v_3}$ are linearly independent.
- ◯ The span of $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{v_3}$ is $\mathbb{R}^3$.
- ◯ $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{v_3}$ are linearly dependent.

6. Let $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \\ 3 \\ -2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ -3 \\ 5 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -5 \\ 3 \\ -27 \\ 39 \end{bmatrix}$. Determine if $\mathbf{b}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$. If it is, find the coefficients a_1 and a_2 such that $a_1\mathbf{u} + a_2\mathbf{v} = \mathbf{b}$. If not, simply write "None" in the spaces provided.

$a_1 =$ ______________

$a_2 =$ ______________

7. Use the row reduction method to find the dimension and a basis for the vector space $W = \text{Span}(\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3})$, where $u_1 = 1 - x - 2x^2 + x^3$, $u_2 = 3 - 3x - 7x^2 + 2x^3$, and $u_3 = -4 + 4x + 9x^2 - 3x^3$

- ◯ $\dim(W) = 3$ and a basis is $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$.
- ◯ $\dim(W) = 2$ and a basis is $\{\mathbf{u_1}, \mathbf{u_2}\}$.
- ◯ $\dim(W) = 1$ and a basis is $\{\mathbf{u_3}\}$.
- ◯ $\dim(W) = 1$ and a basis is $\{\mathbf{u_1}\}$.
- ◯ $\dim(W) = 2$ and a basis is $\{\mathbf{u_1}, \mathbf{u_3}\}$.

8. Solve the linear equation system

$$\begin{cases} 3x_1 - 4x_2 = a \\ 7x_1 - 9x_2 = b \end{cases}$$

using Cramer's rule. Write your answers in terms of a and b.

9. Determine the value of k so that the vectors $\begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} k & 1 \\ 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} 2 & -2 \\ 0 & 3 \end{bmatrix}$ are linearly dependent.

$k =$ _______________

10. Let V be the vector space of all 3×3 matrices with real number entries. Let H_1 be the subset of V that contains all 3×3 triangular matrices with at most one nonzero entry and H_2 be the subset of V that contains all 3×3 matrices whose traces are integers. Choose all statements that are correct.

- ○ H_1 is a subspace of V.
- ○ H_2 is a subspace of V.
- ○ H_1 is closed under scalar multiplication.
- ○ H_2 is closed under scalar multiplication.
- ○ H_1 is closed under matrix addition.
- ○ H_2 is closed under matrix addition.

11. Select all of the following statements that are true.

- ○ If $V = \text{Span}\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}\}$, then $\dim(V) = 4$.
- ○ If $V = \text{Span}\{\mathbf{u_1}, \mathbf{u_2}\}$, and $\mathbf{u_1}$ and $\mathbf{u_2}$ are linearly independent, then $\dim(V) = 2$.
- ○ If $\mathbf{u_1}, \mathbf{u_2}$, $\mathbf{u_3}$, and $\mathbf{u_4}$ are linearly independent and $V = \text{Span}\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}\}$, then $\dim(V) < 4$.
- ○ $\mathbb{R}^5$ cannot be spanned by less than five vectors.

12. If W is a subspace of the vector space V and we know that $\dim(V) = 4$, and we also know that $W \neq V$, what are the possible dimensions of W?

- ○ 1 or 3
- ○ 2
- ○ 1, 2, or 3
- ○ 0, 1, 2, 3, or 4
- ○ 0 or 3

13. Find the LU factorization of the matrix $\begin{bmatrix} 6 & 9 \\ 4 & 5 \end{bmatrix}$. Write your answer in the space below.

14. Use the given LU factorization $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -4 & 3 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -4 & -3 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ to solve the equation $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \begin{bmatrix} 1 \\ 7 \\ 0 \\ 3 \end{bmatrix}$.

15. Write the standard matrix if the linear transformation T where $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the vertical x_2 axis and then rotates points (about the origin) through $\pi/2$ radians (counterclockwise).

16. If an $n \times n$ matrix A can be row reduced to I_n, what can you say about A? Select all that are true.

- ◯ The columns of A are linearly independent.
- ◯ The columns of A are linearly dependent.
- ◯ The determinant of A is zero.
- ◯ The determinant of A cannot be determined by this information.
- ◯ The transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(x) = Ax$ is one-to-one.
- ◯ The transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(x) = Ax$ is onto.

17. Assume that A_{11} is invertible and that the following matrices are partitioned comfortably for multiplication. Find X and Y such that. $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}$ where $S = A_{22} - A_{21}A_{11}^{-1}A_{21}$ is called the Schur complement of A_{11}.

18. Compute the following determinant by combining the methods of row reduction and cofactor expansion. $\begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 11 & 4 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{vmatrix}$