On Computing the Maximum Time-Delay Bound for Stability of Linear Neutral Systems

Qing-Long Han, Xinghuo Yu, and Keqin Gu

***Abstract*—This note is concerned with the stability problem of linear delay-differential systems of neutral type. A discretized Lyapunov functional approach is developed. The resulting stability criterion is formulated in the form of a linear matrix inequality. For nominal systems, the analytical results can be approached with fine discretization. Numerical examples show significant improvement over approaches in the literature.**

***Index Terms*—Linear matrix inequality (LMI), Lyapunov functional, stability, time-delay, uncertainty.**

I. Introduction

During the last two decades, the problem of stability of delay-differential systems of neutral type has received considerable attention, see for example, [7] and the references therein. The practical examples of neutral delay-differential systems include the distributed networks containing lossless transmission lines [2], and population ecology [14]. Some earlier results are based on matrix measure and matrix norm [13] or a simple Lyapunov functional [16], [17]. The resulting criteria are independent of delay. Although these criteria are easy to use, they are often overly conservative for practical applications.

Delay-dependent stability results, which take the delay into account, are usually less conservative than the delay-independent stability ones. A model transformation technique is often used to transform the point-wise delay system to a distributed delay system, and delay-dependent stability criteria are obtained; see, for example, [8], [9], and [15]. The model transformation may introduce additional dynamics, i.e., additional poles that are not present in the original system, and one of these additional poles may cross the imaginary axis before any of the poles of the original system do as the delay increases from zero [6]. Moreover, there are no obvious ways to obtain less conservative results even if one is willing to commit more computational effort to the problem.

For a linear system of retarded type with a constant time-delay, it has been proven that the existence of a more general quadratic form Lyapunov–Krasovskii functional is necessary and sufficient for the stability of an uncertainty-free time-delay system [12]. A discretized Lyapunov functional approach has been proposed to enable one to write the stability criterion in a linear matrix inequality (LMI) form [4]. The criteria have shown significant improvements over the existing results even under very coarse discretization. The results in [4] have been extended to the cases where the uncertainty is norm-bounded in [10].

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In this note, we develop the discretized Lyapunov functional approach to study the stability of linear neutral systems. A stability criterion is formulated in the form of LMIs. Numerical examples are presented to illustrate the effectiveness of the approach.

Notation: For a symmetric matrix W, "$W > 0$" denotes that W is a positive–definite matrix. Similarly, "$\geq$," "$<$," and "$\leq$" denote positive semidefiniteness, negative definiteness, and negative semidefiniteness. Use $\dot{W}$ to denote the derivative of W with respect to time t while $\dot{W}(\alpha)$ denotes the derivative of W with respect to the argument and evaluated at α. $\lambda_i(C)$ is the ith eigenvalue of matrix C. Use I to stand for the identity matrix of appropriate dimensions.

II. Problem Statement

Consider the following time-delay system:

$$\dot{x}(t) - C\dot{x}(t-r) = A(t)x(t) + B(t)x(t-r) \tag{1}$$

with initial condition

$$x(t) = \varphi(t), \quad \dot{x}(t) = \dot{\varphi}(t) \qquad \forall t \in [-r, 0] \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state, $r > 0$ is a constant time delay, $\varphi(t)$ is the initial condition, $C \in \mathbb{R}^{n\times n}$ is a constant matrix, and $A(t) \in \mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^{n\times n}$ are uncertain matrices, which are unkown and possibly time varying, but are known to be bounded by some compact set Ω, i.e.,

$$(A(t)\ B(t)) \in \Omega \subset \mathbb{R}^{n\times 2n}, \text{ for all } t \in (0, \infty). \tag{3}$$

Define $\mathcal{C}$ as the set of continuous $\mathbb{R}^n$ valued function on the interval $[-r, 0]$, and let $x_t \in \mathcal{C}$ be a segment of system trajectory defined as

$$x_t(\theta) = x(t+\theta), \qquad -r \leq \theta \leq 0. \tag{4}$$

In this note, we will attempt to formulate some practically computable criteria for robust stability of uncertain system described by (1)–(3).

Define the operator

$$\mathcal{D}\phi = \phi(0) - C\phi(-r). \tag{5}$$

Throughout this note, we assume the following.

A1) All the eigenvalues of matrix C are inside the *open* unit circle, i.e. $|\lambda_i(C)| < 1$ $(i = 1, 2, \ldots, n)$.

Choose a Lyapunov–Krasovskii functional $V(t, \phi)$ of a quadratic form

$$\begin{aligned} &V(t,\phi) : \mathbb{R}\times\ \mathcal{C} \mapsto \mathbb{R} \\ &V(t,\phi) = \frac{1}{2}(\mathcal{D}\phi)^T P(\mathcal{D}\phi) + (\mathcal{D}\phi)^T \int_{-r}^{0} Q(\xi)\phi(\xi)d\xi \\ &\qquad + \frac{1}{2}\int_{-r}^{0}\left[\int_{-r}^{0}\phi^T(\xi)R(\xi,\eta)\phi(\eta)d\eta\right]d\xi \\ &\qquad + \frac{1}{2}\int_{-r}^{0} x^T(t+\xi)S(\xi)x(t+\xi)d\xi \end{aligned} \tag{6}$$

where

$$\begin{aligned} &P \in \mathbb{R}^{n\times n} \quad P = P^T \quad Q : [-r, 0] \to \mathbb{R}^{n\times n} \\ &S : [-r, 0] \to \mathbb{R}^{n\times n} \quad S^T(\xi) = S(\xi) \\ &R : [-r, 0]\times[-r, 0] \to \mathbb{R}^{n\times n} \quad R(\eta,\xi) = R^T(\xi,\eta) \end{aligned}$$

and Q, R and S are Lipschitz matrix functions with piecewise continuous derivatives.

Remark 1: When $C = 0$ and there is no uncertainty in system matrices A and B, system (1) reduces to the following system of *retarded* type:

$$\dot{x}(t) = Ax(t) + Bx(t - r)$$

which is a special case of [5, eq. (7.20)]. It is pointed out in [5] that the existence of the Lyapunov–Krasovskii functional $V(t, \phi)$ with $C = 0$ in (5) and (6) is a *necessary* and *sufficient* condition for the stability of the system of *retarded* type without uncertainty while Lyapunov–Krasovskii functionals employed in some existing results such as [3], [8], [9], [11], and [18] are only *sufficient* conditions for the stability of the same case. Correspondingly, the Lyapunov–Krasovskii functional $V(t, \phi)$ (6) is more effective than those in [3], [8], [9], [11], and [18] for checking the stability of (1).

For asymptotic stability of (1)–(3), we have the following result.

Theorem 1: Under A1), the system (1)–(3) is asymptotically stable if there exist $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and a quadratic Lyapunov–Krasovskii functional V of the form (6) which staifies

$$\varepsilon_1(\mathcal{D}\phi)^T(\mathcal{D}\phi) \leq V(t, \phi) \tag{7}$$

and its derivative along the solution of (1) satisfies

$$\dot{V}(\phi) \leq -\varepsilon_2 \phi^T(0)\phi(0) \tag{8}$$

for any $\phi \in \mathcal{C}$, where $\dot{V}(t, \phi) \triangleq (d/dt)V(t, x_t)|_{x_t = \phi}$.

Proof: For the quadratic functional $V(t, \phi)$, there always exists a sufficiently large $\varepsilon_3 > 0$ such that $V(t, \phi) \leq \varepsilon_3 \max_{-r \leq \theta \leq 0} \|\phi\|$. Note that A1 guarantees that the operator $\mathcal{D}$ is stable. By [7, Ch. 9, Th. 8.1], system (1) and (2) is asymptotically stable. Q.E.D.

Choose Q, R and S to be continuous piecewise linear [4], i.e.,

$$\begin{aligned}
Q^i(\alpha) &\triangleq Q(\delta_{i-1} + \alpha h) = (1 - \alpha)Q_{i-1} + \alpha Q_i \\
S^i(\alpha) &\triangleq S(\delta_{i-1} + \alpha h) = (1 - \alpha)S_{i-1} + \alpha S_i \\
R(\delta_{i-1} + \alpha h, \delta_{j-1} + \eta h) &= R^{ij}(\alpha, \eta) \\
&\triangleq \begin{cases} (1 - \alpha)R_{i-1,j-1} + \eta R_{ij} \\ \quad + (\alpha - \eta)R_{i,j-1}, \ \alpha \geq \eta \\ (1 - \eta)R_{i-1,j-1} + \alpha R_{ij} \\ \quad + (\eta - \alpha)R_{i-1,j}, \ \alpha < \eta \end{cases}
\end{aligned} \tag{9}$$

for $0 \leq \alpha \leq 1$, $0 \leq \eta \leq 1$, where

$$\delta_i = -r_m + ih, \qquad i = 0, 1, 2, \ldots, N, \quad h = \frac{r}{N}.$$

i.e., N is the number of divisions of the interval $[-r, 0]$, and h is the length of each division. It is convenient to write

$$\phi^i(\alpha) = \phi(\delta_{i-1} + \alpha h).$$

III. Main Results

With the choice of piecewise linear functions, the Lyapunov–Krasovskii functional condition (7) can be written in the form of a linear matrix inequality. Using similar argument to the proof of [4, Prop. 3] yields the following result.

Proposition 1: For piecewise linear Q, S, and R, as described by (9), there exists an $\varepsilon_1 > 0$ such that the Lyapunov–Krasovskii functional satisfies (7) if

$$\tilde{S} > 0 \tag{10}$$

and

$$\begin{bmatrix} P & \tilde{Q} \\ \tilde{Q}^T & \frac{1}{h}\hat{S} + \tilde{R} \end{bmatrix} > 0 \tag{11}$$

where

$$\begin{aligned}
\tilde{S} &= diag(S_0 \ S_1 \ \cdots \ S_{N-1} \ S_N) \\
\tilde{R} &= \begin{bmatrix} R_{00} & R_{01} & \cdots & R_{0N} \\ R_{10} & R_{11} & \cdots & R_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{N0} & R_{N1} & \cdots & R_{NN} \end{bmatrix} \\
\tilde{Q} &= (Q_0, \ Q_1, \ \ldots, \ Q_N).
\end{aligned}$$

Similar to (7), the Lyapunov–Krasovskii derivative condition (8) can also be written in the form of an LMI. We have the following proposition.

Proposition 2: For piecewise linear Q, S, and R, as described by (9), (8) is satisfied for some $\varepsilon_2 > 0$, and arbitrary $\phi \in \mathcal{C}$ if

$$\Xi(t) = \begin{bmatrix} G_{11}(t) & -G_{12}(t) & \Gamma_1^s & \Gamma_1^a \\ -G_{12}^T(t) & G_{22}(t) & \Gamma_2^s & \Gamma_2^a \\ \Gamma_1^{sT} & \Gamma_2^{sT} & \frac{1}{h}S_d + R_d & 0 \\ \Gamma_1^{aT} & \Gamma_2^{aT} & 0 & \frac{3}{h}S_d \end{bmatrix} > 0 \tag{12}$$

for all $(A(t) \ B(t)) \in \Omega$, where

$$\begin{aligned}
G_{11}(t) &= -PA(t) - A^T(t)P - S_N - Q_N - Q_N^T \\
G_{12}(t) &= PB(t) - A^T(t)PC - Q_N^T C - Q_0 \\
G_{22}(t) &= C^T PB(t) + B^T(t)PC - C^T Q_0 - Q_0^T C + S_0 \\
S_d &= diag(S_{d1} \ S_{d2} \ \ldots \ S_{dN}) \\
S_{di} &= \frac{1}{h}(S_i - S_{i-1}) \\
R_d &= \begin{bmatrix} R_{d11} & R_{d12} & \cdots & R_{d1N} \\ R_{d21} & R_{d22} & \cdots & R_{d2N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{dN1} & R_{dN2} & \cdots & R_{dNN} \end{bmatrix} \\
R_{dij} &= \frac{1}{h}(R_{ij} - R_{i-1,j-1}), \qquad i, j = 1, 2, \ldots, N \\
\Gamma_J^s &= \left[\Gamma_{j1}^s \ \Gamma_{j2}^s \ \cdots \ \Gamma_{jN}^s\right], \qquad j = 1, 2 \\
\Gamma_{1i}^s &= -\frac{1}{2}A^T(t)(Q_i + Q_{i-1}) + \frac{1}{h}(Q_i - Q_{i-1}) \\
&\quad - \frac{1}{2}\left(R_{i,N}^T + R_{i-1,N}^T\right) \\
\Gamma_{2i}^s &= -\frac{1}{2}B^T(t)(Q_i + Q_{i-1}) - \frac{1}{h}C^T(Q_i - Q_{i-1}) \\
&\quad + \frac{1}{2}\left(R_{i,0}^T + R_{i-1,0}^T\right) \\
\Gamma_j^a &= \left[\Gamma_{j1}^a \ \Gamma_{j2}^a \ \cdots \ \Gamma_{jN}^a\right], \qquad j = 1, 2 \\
\Gamma_{1i}^a &= \frac{1}{2}A^T(t)(Q_i - Q_{i-1}) + \frac{1}{2}\left(R_{i,N}^T - R_{i-1,N}^T\right) \\
\Gamma_{2i}^a &= \frac{1}{2}B^T(t)(Q_i - Q_{i-1}) - \frac{1}{2}\left(R_{i,0}^T + R_{i-1,0}^T\right) \\
&\quad i = 1, 2, \ldots, N.
\end{aligned}$$

Proof: Using (1), one obtains

$$\begin{aligned}\dot{V}(t,\phi) &= (\mathcal{D}\phi)^T P\left[A(t)\phi(0)+B(t)\phi(-r)\right]\\ &\quad+\left[A(t)\phi(0)+B(t)\phi(-r)\right]^T\int_{-r}^{0}Q(\xi)\phi(\xi)d\xi\\ &\quad+(\mathcal{D}\phi)^T\int_{-r}^{0}Q(\xi)\dot{\phi}(\xi)d\xi\\ &\quad+\int_{-r}^{0}d\xi\int_{-r}^{0}\phi^T(\xi)R(\xi,\eta)\dot{\phi}(\eta)d\eta\\ &\quad+\int_{-r}^{0}\phi^T(\xi)S(\xi)\dot{\phi}(\xi)d\xi. \end{aligned}\tag{13}$$

Integrating by parts yields

$$\begin{aligned}\dot{V}(t,\phi) =& -\frac{1}{2}\phi^T(0)\Big[-PA(t)-A^T(t)P-Q(0)\\ &\qquad -Q^T(0)-S(0)\Big]\phi(0)+\phi^T(0)\\ &\times\Big[PB(t)-A^T(t)PC-Q^T(0)C-Q(-r)\Big]\\ &\times\phi(-r)-\frac{1}{2}\phi^T(-r)\\ &\times\Big[C^TPB(t)+B^T(t)PC+S(-r)\\ &\qquad -C^TQ(-r)-Q^T(-r)C\Big]\phi(-r)\\ &+\phi^T(0)\int_{-r}^{0}\Big[A^T(t)Q(\xi)-\dot{Q}(\xi)+R^T(\xi,0)\Big]\phi(\xi)d\xi\\ &+\phi^T(-r)\int_{-r}^{0}\Big[B^T(t)Q(\xi)+C^T\dot{Q}(\xi)-R^T(\xi,-r)\Big]\\ &\times\phi(\xi)d\xi\\ &-\frac{1}{2}\int_{-r}^{0}d\xi\int_{-r}^{0}\phi^T(\xi)\left(\frac{\partial R(\xi,\eta)}{\partial\xi}+\frac{\partial R(\xi,\eta)}{\partial\eta}\right)\phi(\eta)d\eta\\ &-\frac{1}{2}\int_{-r}^{0}\phi^T(\xi)\dot{S}(\xi)\phi(\xi)d\xi. \end{aligned}\tag{14}$$

With the piecewise linear Q, S, and R chosen, (14) can be written as

$$\begin{aligned}\dot{V}(t,\phi) =& -\frac{1}{2}\phi^T(0)G_{11}(t)\phi(0)+\phi^T(0)G_{12}(t)\phi(-r)\\ &-\frac{1}{2}\phi^T(-r)G_{22}(t)\phi(-r)\\ &-\frac{h}{2}\sum_{i=1}^{N}\int_0^1\phi^{iT}(\alpha)S_{di}\phi^i(\alpha)d\alpha\\ &-\frac{h^2}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\int_0^1\phi^i(\alpha)d\alpha\right)^T R_{dij}\left(\int_0^1\phi^j(\alpha)d\alpha\right)\\ &-h\phi^T(0)\sum_{i=1}^{N}\int_0^1\left[\Gamma_{1i}^s+(1-2\alpha)\Gamma_{1i}^a\right]\phi^i(\alpha)d\alpha\\ &-h\phi^T(-r)\sum_{i=1}^{N}\int_0^1\left[\Gamma_{2i}^s+(1-2\alpha)\Gamma_{2i}^a\right]\\ &\times\phi^i(\alpha)d\alpha. \end{aligned}\tag{15}$$

Rewrite (15) as shown in (16) at the bottom of the page, where $\tilde{\phi}(\alpha)=[\tilde{\phi}^{1T}(\alpha)\ \tilde{\phi}^{2T}(\alpha)\ \ldots\ \tilde{\phi}^{NT}(\alpha)]^T$.

Introducing the following notation:

$$G(t)=\begin{bmatrix}G_{11}(t) & -G_{12}(t)\\ -G_{12}^T(t) & G_{22}(t)\end{bmatrix}\quad \phi_{0r}=\begin{pmatrix}\phi(0)\\ \phi(-r)\end{pmatrix}$$
$$\Gamma^s=\begin{pmatrix}\Gamma_1^s\\ \Gamma_2^s\end{pmatrix},\quad \Gamma^a=\begin{pmatrix}\Gamma_1^a\\ \Gamma_2^a\end{pmatrix}.$$

Equation (16) can be written as

$$\begin{aligned}\dot{V}(t,\phi) =& -\frac{1}{2}\phi_{0r}^T G(t)\phi_{0r}\\ &-h\phi_{0r}^T\int_0^1\left[\Gamma^s+(1-2\alpha)\Gamma^a\right]\tilde{\phi}(\alpha)d\alpha\\ &-\frac{1}{2}h^2\int_0^1\tilde{\phi}^T(\alpha)\frac{1}{h}S_d\tilde{\phi}(\alpha)d\alpha\\ &-\frac{1}{2}h^2\left(\int_0^1\tilde{\phi}^T(\alpha)d\alpha\right)R_d\left(\int_0^1\tilde{\phi}(\alpha)d\alpha\right). \end{aligned}\tag{17}$$

$$\begin{aligned}\dot{V}(t,\phi) = -\frac{1}{2}\int_0^1\left[\phi^T(0)\ \phi^T(-r)\ h\tilde{\phi}^T(\alpha)\right]&\begin{bmatrix}G_{11}(t) & -G_{12}(t) & \Gamma_1^s+(1-2\alpha)\Gamma_1^a\\ -G_{12}^T(t) & G_{22}(t) & \Gamma_2^s+(1-2\alpha)\Gamma_2^a\\ \Gamma_1^{sT}+(1-2\alpha)\Gamma_1^{aT} & \Gamma_2^{sT}+(1-2\alpha)\Gamma_2^{aT} & \frac{1}{h}S_d\end{bmatrix}\\ &\times\begin{pmatrix}\phi(0)\\ \phi(-r)\\ h\tilde{\phi}(\alpha)\end{pmatrix}d\alpha-\frac{1}{2}h^2\left(\int_0^1\tilde{\phi}^T(\alpha)d\alpha\right)R_d\left(\int_0^1\tilde{\phi}(\alpha)d\alpha\right). \end{aligned}\tag{16}$$

Rewrite (17) as

$$\begin{aligned}\dot{V}(t,\phi) = & -\frac{1}{2}\int_0^1 \left(\phi_{0r}^T\left[\Gamma^s + (1-2\alpha)\Gamma^a\right]h\tilde{\phi}^T(\alpha)\right)\\ & \times \begin{bmatrix} W & I \\ I & \frac{1}{h}S_d\end{bmatrix}\begin{pmatrix}\left[\Gamma^s + (1-2\alpha)\Gamma^a\right]^T\phi_{0r} \\ h\tilde{\phi}(\alpha)\end{pmatrix}d\alpha \\ & + \frac{1}{2}\phi_{0r}^T\left[\Gamma^s W \Gamma^{sT} + \frac{1}{3}\Gamma^a W \Gamma^{aT}\right]\phi_{0r} \\ & - \frac{1}{2}\phi_{0r}^T G(t)\phi_{0r} - \frac{1}{2}h^2\left(\int_0^1 \tilde{\phi}^T(\alpha)d\alpha\right) \\ & \times R_d\left(\int_0^1 \tilde{\phi}(\alpha)d\alpha\right)\end{aligned} \tag{18}$$

for arbitrary W. If W is chosen to satisfy

$$\begin{bmatrix} W & I \\ I & \frac{1}{h}S_d\end{bmatrix} > 0 \tag{19}$$

then use [4, Lemma 1] in the first term on the right-hand side of (18) to obtain

$$\begin{aligned}\dot{V}(t,\phi) \le & -\frac{1}{2}\left(\int_0^1 \phi_{0r}^T\left[\Gamma^s + (1-2\alpha)\Gamma^a\right]d\alpha \quad \int_0^1 h\tilde{\phi}^T(\alpha)d\alpha\right)\\ & \times \begin{bmatrix} W & I \\ I & \frac{1}{h}S_d\end{bmatrix}\begin{pmatrix}\int_0^1\left[\Gamma^s + (1-2\alpha)\Gamma^a\right]^T\phi_{0r}d\alpha \\ \int_0^1 h\tilde{\phi}(\alpha)d\alpha\end{pmatrix} \\ & + \frac{1}{2}\phi_{0r}^T\left[\Gamma^s W \Gamma^{sT} + \frac{1}{3}\Gamma^a W \Gamma^{aT}\right]\phi_{0r} \\ & - \frac{1}{2}\phi_{0r}^T G(t)\phi_{0r} - \frac{1}{2}h^2\left(\int_0^1 \tilde{\phi}^T(\alpha)d\alpha\right) \\ & \times R_d\left(\int_0^1 \tilde{\phi}(\alpha)d\alpha\right) \\ = & -\frac{1}{2}\left(\phi_{0r}^T \quad \int_0^1 h\tilde{\phi}^T(\alpha)d\alpha\right)\begin{bmatrix}\Gamma^s W \Gamma^{sT} & \Gamma^s \\ \Gamma^{sT} & \frac{1}{h}S_d\end{bmatrix} \\ & \times \begin{pmatrix}\phi_{0r} \\ \int_0^1 h\tilde{\phi}(\alpha)d\alpha\end{pmatrix} - \frac{1}{2}\left(\phi_{0r}^T \quad \int_0^1 h\tilde{\phi}^T(\alpha)d\alpha\right) \\ & \times \begin{bmatrix} G(t) - \Gamma^s W\Gamma^{sT} - \frac{1}{3}\Gamma^a W \Gamma^{aT} & 0 \\ 0 & 0\end{bmatrix} \\ & \times \begin{pmatrix}\phi_{0r} \\ \int_0^1 h\tilde{\phi}(\alpha)d\alpha\end{pmatrix} - \frac{1}{2}\left(\phi_{0r}^T \quad \int_0^1 h\tilde{\phi}^T(\alpha)d\alpha\right) \\ & \times \begin{bmatrix} 0 & 0 \\ 0 & R_d\end{bmatrix}\begin{pmatrix}\phi_{0r} \\ \int_0^1 h\tilde{\phi}(\alpha)d\alpha\end{pmatrix} \\ = & -\frac{1}{2}\left(\phi_{0r}^T \quad h\int_0^1 \tilde{\phi}^T(\alpha)d\alpha\right) \\ & \times \begin{bmatrix} G(t) - \frac{1}{3}\Gamma^a W\Gamma^{aT} & \Gamma^s \\ \Gamma^{sT} & \frac{1}{h}S_d + R_d\end{bmatrix} \\ & \times \begin{pmatrix}\phi_{0r} \\ h\int_0^1 \tilde{\phi}(\alpha)d\alpha\end{pmatrix}.\end{aligned} \tag{20}$$

Therefore, to satisfy (8), it is sufficient for (19) and

$$\begin{bmatrix} G(t) - \frac{1}{3}\Gamma^a W\Gamma^{aT} & \Gamma^s \\ \Gamma^{sT} & \frac{1}{h}S_d + R_d\end{bmatrix} > 0 \tag{21}$$

to be satisfied. Use [4, Prop. 2] to eliminate matrix W in (19) and (21) to obtain (12). Q.E.D.

From this discussion, we prove the following stability criterion.

Proposition 3: Under A1), the system (1)–(3) is asymptotically stable if there exist real matrices $P = P^T$, Q_i, S_i, $(i = 0, 1, 2, \ldots, N)$, and $R_{ij}(i, j = 0, 1, 2, \ldots, N)$ such that $S_0 > 0$, (11), and (12) hold for all $(A(t)\ B(t)) \in \Omega$.

Proof: By Theorem 1, one can see that the system (1)–(3) is asymptotically stable if (10), (11) and (12) hold. Note that (10) is implied by $S_0 > 0$ and (12). Therefore, the conclusion of Proposition 3 is true. Q.E.D.

For nominal systems, matrices $A(t)$ and $B(t)$ in (12) are replaced with the nominal matrices A and B. For polytopic uncertainty, it is clearly sufficient that (12) only needs to be satisfied at all the vertices. For norm-bounded uncertainty, similar to [10], (12) needs to be reformulated in order to be reduced to a finite number of LMIs.

IV. EXAMPLES

To illustrate the effectiveness of the approach, two numerical examples are presented.

Example 1: Consider the system

$$\dot{x}(t) - c\dot{x}(t-r) = -bx(t-r) \tag{22}$$

where b and c are scalars, $|c| < 1$, $b > 0$. In [15], it was concluded that the system is asymptotically stable for

$$r < r_{\max}^{\text{Niculescu}} = \frac{1-|c|}{b}. \tag{23}$$

The exact stability limit was analytically calculated as [15]

$$r_{\max}^{\text{analytical}} = \frac{\sqrt{1-c^2}}{b}\arctan\sqrt{\frac{1}{c^2}-1}. \tag{24}$$

Let $c = 0.2$ and $b = 1$, applying the discretized Lyapunov functional approach, the resulting stability limits obtained for different N are listed in Table I, along with the analytical limit $r_{\max}^{\text{analytical}}$ calculated using (24) and the estimated limit $r_{\max}^{\text{Niculescu}}$ obtained using (23).

It is clear that the stability limit obtained by the discretized Lyapunov functional approach is less conservative than the results in [15] and $r_{\max}^N$ converges to the analytical solution $r_{\max}^{\text{analytical}}$ as N increases.

Example 2: Consider the nominal system of neutral type

$$\dot{x}(t) - C\dot{x}(t-r) = Ax(t) + Bx(t-r) \tag{25}$$

where

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9\end{bmatrix}\ B = \begin{bmatrix} -1 & 0 \\ -1 & -1\end{bmatrix}\ C = \begin{bmatrix} c & 0 \\ 0 & c\end{bmatrix}\ |c| < 1.$$

The exact stability limit can be analytically calculated as follows.

TABLE I
BOUND $r_{\max}$ CALCULATED USING THE METHODS IN [15] AND THIS NOTE

$r_{\max}^{analytical}$	$r_{\max}^{Niculescu}$	$r_{\max}^{N=1}$	$r_{\max}^{N=2}$	$r_{\max}^{N=3}$
1.3418	0.8	1.3407	1.3417	1.3418

TABLE II
BOUND $r_{\max}$ CALCULATED USING THE METHODS IN [3], [9], [11], [18] AND THIS NOTE FOR $c = 0.5$

$r_{\max}^{YueHan}$	$r_{\max}^{He}$	$r_{\max}^{Han}$	$r_{\max}^{Fridman}$
3.69	3.67	3.62	1.14
$r_{\max}^{analytical}$	$r_{\max}^{N=1}$	$r_{\max}^{N=2}$	$r_{\max}^{N=3}$
4.7388	4.6850	4.7357	4.7381

TABLE III
BOUND $r_{\max}$ CALCULATED USING THE METHODS IN [3], [9], [11], [18] AND THIS NOTE FOR VARIOUS c

c	0	0.10	0.30
[3]	4.47	3.49	2.06
[9]	4.35	4.33	4.10
[11]	4.47	4.35	4.13
[18]	4.47	4.42	4.17
This note	6.17	6.03	5.54
c	0.50	0.70	0.90
[3]	1.14	0.54	0.13
[9]	3.62	2.73	0.99
[11]	3.67	2.87	1.41
[18]	3.69	2.87	1.41
This note	4.73	3.50	1.57

i) For $|c| < 1$ and $c \neq 0$

$$r_{\max}^{\text{analytical}} = \frac{1}{\omega} \arccos\left(\frac{c\omega^2 - 0.9}{1 + c^2\omega^2}\right) \tag{26}$$

where

$$\omega = \sqrt{\frac{-1 + 1.19c^2 + \sqrt{1 - 1.62c^2 + 0.6561c^4}}{2(c^2 - c^4)}}.$$

ii) For $c = 0$, $r_{\max}^{\text{analytical}} = 6.172\,58$.

Let $c = 0.5$, the maximum time-delay for asymptotic stability as judged by the criteria in [3], [9], [11], and [18], and the discretized Lyapunov functional approach, are estimated in Table II, along with the analytical limit $r_{\max}^{\text{analytical}}$ calculated using (26).

It is seen again that the stability limit obtained by the discretized Lyapunov functional approach is less conservative than the results in [3], [9], [11], [18], and $r_{\max}^{N}$ converges to the analytical solution $r_{\max}^{\text{analytical}}$ as N increases.

We now consider the effect of parameter c on the maximum time-delay for stability $r_{\max}$. Table III gives the $r_{\max}$ by the criteria in [3], [9], [11], [18], and this note for $N = 3$. It is clear that the new criterion here significantly improve the estimate of stability limit over the results in [3], [9], [11], and [18].

V. CONCLUSION

The stability problem of linear delay-differential systems of neutral type has been investigated. The discretized Lyapunov functional approach has been developed. Stability criteria have been obtained. Numberical examples show that the results derived by these new criteria significantly improve the estimate of stability limit over the existing results in the literature.

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A High-Gain Scaling Technique for Adaptive Output Feedback Control of Feedforward Systems

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***Abstract*—In this note, we propose an adaptive output feedback control design technique for feedforward systems based on our recent results on dynamic high-gain scaling techniques for controller design for strict-feedback systems. The system is allowed to contain uncertain functions of all the states and the input as long as the uncertainties satisfy certain bounds. Unknown parameters are allowed in the bounds assumed on the uncertain functions. If the uncertain functions involve the input, then the output-dependent functions in the bounds on the uncertain functions need to be polynomially bounded. It is also shown that if the uncertain functions can be bounded by a function independent of the input, then the polynomial boundedness requirement can be relaxed. The designed controllers have a very simple structure being essentially a linear feedback with state-dependent dynamic gains and do not involve any saturations or recursive computations. The observer utilized to estimate the unmeasured states is similar to a Luenberger observer with dynamic observer gains. The Lyapunov functions are quadratic in the state estimates, the observer errors, and the parameter estimation error. The stability analysis is based on our recent results on uniform solvability of coupled state-dependent Lyapunov equations. The controller design provides strong robustness properties both with respect to uncertain parameters in the system model and additive disturbances. This robustness is the key to the output feedback controller design. Global asymptotic results are obtained.**

***Index Terms*—Adaptive output feedback, feedforward form, high gain, nonlinear systems.**

I. Introduction

We consider the class of systems given by

$$\begin{aligned}\dot{x}_1 &= \phi_{(1,2)}(y)x_2 + \phi_1(t,y,x_3,\ldots,x_n,u)\\ \dot{x}_2 &= \phi_{(2,3)}(y)x_3 + \phi_2(t,y,x_4,\ldots,x_n,u)\\ &\vdots\\ \dot{x}_{n-2} &= \phi_{(n-2,n-1)}(y)x_{n-1} + \phi_{n-2}(t,y,x_n,u)\\ \dot{x}_{n-1} &= \phi_{(n-1,n)}(y)x_n + \phi_{n-1}(t,y,u)\\ \dot{x}_n &= \mu(y)u\\ y &= [x_1,x_n]^T \qquad (1)\end{aligned}$$

where $x=[x_1,\ldots,x_n]^T\in\mathcal{R}^n$ is the state of the system, $u\in\mathcal{R}$ the input, and $y\in\mathcal{R}$ the measured output. $\mu:\mathcal{R}^2\to\mathcal{R}$ and $\phi_{(i,i+1)}:\mathcal{R}^2\to\mathcal{R}, i=1,\ldots,n-1$, are known continuous functions of y. $\phi_i:\mathcal{R}^{n+3-i}\to\mathcal{R}, i=1,\ldots,n-1$, are uncertain time-varying functions.[1] It is assumed that sufficient conditions (e.g., local Lipschitz property) on ϕ_i needed for local existence and uniqueness of solutions of (1) are satisfied.

Available controller design techniques for feedforward systems in the literature include saturation-based designs [1]–[4] and forwarding [5], [6]. Nested saturation designs rely on the use of small inputs and require the ϕ_i functions to involve only quadratic or higher powers in their arguments. Since the saturation levels are restricted to be sufficiently small, the scheme is sensitive to additive disturbances. Forwarding is a recursive passivation scheme which proceeds by adding one integrator at a time through the design of cross terms. However, forwarding is computationally complicated and the cross terms often need to be approximated numerically. Adaptive state feedback stabilization of feedforward systems was considered in [7]. A combination of forwarding and nested saturation was proposed in [8] to obtain weaker growth conditions. An adaptive state-feedback scaling-based design with the scaling governed by a switching logic was considered in [9] and [10]. However, due to lack of robustness to additive disturbances in the aforementioned designs, the extension to the output feedback case has not been reported.

In [11]–[13] and [23], we developed a technique for dynamic high-gain scaling based output feedback control for strict-feedback systems [14] of the form

$$\begin{aligned}\dot{x}_i &= \phi_{(i,i+1)}(y)x_{i+1} + \phi_i(t,x_1,\ldots,x_i),\\ &\qquad\qquad i=1,\ldots,n-1\\ \dot{x}_n &= \mu(y)u + \phi_n(t,x_1,\ldots,x_n)\\ y &= x_1 \qquad (2)\end{aligned}$$

with $\phi_{(i,i+1)}$ being known functions of output and ϕ_i being uncertain functions. The observer and controller designs are based on a high-gain scaling applied to the nominal system

$$\begin{aligned}\dot{x}_i &= \phi_{(i,i+1)}(y)x_{i+1}, \qquad i=1,\ldots,n-1\\ \dot{x}_n &= \mu(y)u \qquad (3)\end{aligned}$$

and the dynamics of the high-gain parameter are designed to compensate for the uncertain terms ϕ_i. The high-gain scaling essentially achieves an approximation of the system as the chain of nonlinear integrators (3). It was noted that a complexity of bounds on ϕ_i does not result in greater complexity of the observer and controller structures and the Lyapunov function but is rather handled through the dynamics of the high-gain parameter. The controller design in [12] can be interpreted as a dual of the observer in [15]. The dual high-gain observer/controller design in [12] was based on results on uniform solvability of coupled state-dependent Lyapunov equations [11], [16]. Previous high-gain results either utilized a constant high-gain parameter to obtain semiglobal results [17], [18] or utilized a high-gain parameter $r=\int_0^t y^2(\pi)d\pi$ to obtain global results for minimum-phase systems with relative-degree one [19], [20] or linear systems with appended stable nonlinear zero dynamics and input-matched nonlinearities [21]. The designs in [15], [11], and [12] utilized a high-gain parameter with dynamics given by a scalar Riccati equation driven by y guaranteeing boundedness of r if y remains bounded (which is not guaranteed by the dynamics $\dot{r}=y^2$).

In this note, we consider the extension of the dynamic high-gain scaling technique for feedforward systems. Motivated by the observation that the nominal form of the feedforward system (1) neglecting unknown functions ϕ_i is of the form (3), the controller and observer designs are carried out analogous to [11], [12] using the nominal system (3). The high-gain parameter dynamics are then designed to compensate for the unknown functions ϕ_i. The scaling order and the form of the dynamics of the high-gain parameter are crucial differences between the designs for strict-feedback and feedforward systems. The unknown functions $\phi_i, i=1,\ldots,n-1$, are allowed to be functions of all the states and the input as long as they satisfy certain bounds. Unknown parameters are allowed in the bounds on the unknown functions. The structure of the bounds essentially requires that ϕ_i must be

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[1] $\phi_i, i=1,\ldots,n-1$, can depend on all the states and the input. However, ϕ_i are shown in (1) to depend only on subsets of the state to emphasize the state dependence of the bounds to be introduced.