15. $r = \cos 3\theta$ $\quad \cos 3\theta = 0$

$r^2 = \cos^2 3\theta$ $\quad 3\theta = \cos^{-1}(0) = \frac{\pi}{2}, -\frac{\pi}{2}$

$\theta = -\frac{\pi}{6}, \frac{\pi}{6}$

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_0^{\cos 3\theta} r \, dr \, d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2} \cos^2 3\theta \, d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}\left(\frac{1+\cos 6\theta}{2}\right) d\theta$$

$$= \frac{1}{4} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 1 + \cos 6\theta \, d\theta = \frac{1}{4}\left(\theta + \frac{1}{6}\sin 6\theta\right)\Big]_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = \frac{2}{24}\pi = \boxed{\frac{\pi}{12}}$$

19. $z = x^2 + y^2$ $\quad x^2+y^2 \le 25$ $\quad r \in [0,5]$ $\quad$ full disk: $\theta \in [0, 2\pi]$

$$\int_0^{2\pi} \int_0^5 r^3 \, dr \, d\theta = \int_0^{2\pi} \frac{1}{4} r^4 \Big]_0^5 d\theta = 2\left(\frac{625}{4}\pi\right) = \boxed{\frac{625\pi}{2}}$$

23. sphere radius a: $\quad V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi a^3$

$x^2+y^2+z^2 = a^2$

$r \in [0,a]$

$z = \pm\sqrt{a^2-x^2-y^2}$

θ of circular crossections! ~~top half of sphere~~

$\theta \in [0, \pi]$

$z = \pm\sqrt{a^2-r^2}$

$z = 2\sqrt{a^2-r^2}$

$$\int_0^{2\pi} \int_0^a 2\sqrt{a^2-r^2} \, r \, dr \, d\theta$$

$v = a^2 - r^2$

$dv = -2r\,dr$

$-dv = 2r\,dr$

$$-\int_0^{2\pi} \int_{a^2}^0 \sqrt{v} \, dv \, d\theta = \int_0^{2\pi} \left[\frac{2}{3} v^{3/2}\right]_0^{a^2} d\theta = \int_0^{2\pi} \frac{2}{3} a^3 \, d\theta$$

$$= \frac{2}{3} a^3 \theta \Big]_0^{2\pi} = \boxed{\frac{4}{3}\pi a^3}$$