14.1 Homework 1, 5, 9, 13, 17, 21, 25, 29, 33, 37, 41, 45, 49

(1) a. $f(-15, 40) = -27$

$w = f(T, v)$ At $-15°$ and with wind blowing at 40km/h, the actual temperature feels like $-27°$ w/o wind

b. $f(-20, v) = -30$ $v = 20$

At $-20°$ and with wind blowing at 20km/h, the actual temperature feels like $-30°$ w/o wind

c. $f(T, 20) = -49$ $T = -35$

At $-35°$ and with wind blowing at 20km/h the actual temperature feels like $-49°$ without wind

d. $w = f(-5, v)$ Means that when the temperature is $-5°$, the wind blowing at speed v affects the actual temperature without wind. As v increases in value, w decreases thereby lowering the actual temperature.

e. $w = f(T, 50)$ When the wind blows at 50 km/h, the temperature T affects the actual temperature without wind. As T decreases, w decreases thereby lowering the actual percieved temperature.

⑤ $s = f(w,h) = 0.1091\, w^{0.425} h^{0.725}$

a. $f(160,70) = 0.1091\,(160)^{0.425}(70)^{0.725} = 20.52 \text{ lb-in}^{2}$

At 160 lbs and 70 inches tall, the surface area of this particular person is 20.5 square feet

b. $F(165,72) = 0.1091(165)^{0.425}(72)^{0.725} = 21.22 \text{ ft}^2$

At 165 lbs and 72 inches tall, my surface area is aproximately 21.2 square feet.

⑨ $g(x,y) = \cos(x+2y)$

a. $g(2,-1) = \cos(2-2) = \cos(0) = \boxed{1}$

b. $D = (x,y) \mid x+2y \geq 0$

$y \geq -\frac{1}{2}x$ $\quad \boxed{D = \mathbb{R}^2}$

c. $\boxed{[-1,1]}$ = range

$\cos(x) \in [-1,1]$

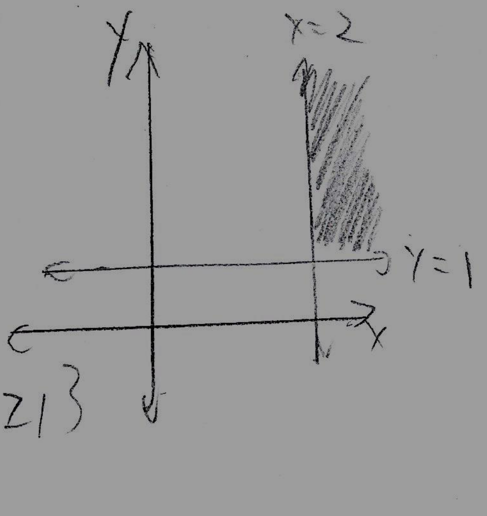
⑬ $f(x,y) = \sqrt{x-2} + \sqrt{y-1}$ $\quad \boxed{\begin{array}{c} x \geq 2 \\ y \geq 1 \end{array}}$

$z = \sqrt{x-2} + \sqrt{y-1}$

$x \not< 2$

$y \not< 1$

$\{(x,y) \mid x \geq 2, y \geq 1\}$



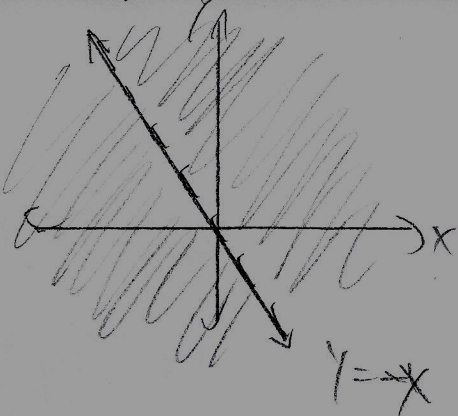
(17) $g(x,y) = \frac{x-y}{x+y}$ $\{(x,y) \mid x \neq -y\}$

$\boxed{\{(x,y) \mid y \neq -x\}}$

$x+y=0$

$x=-1$

$y=-x$

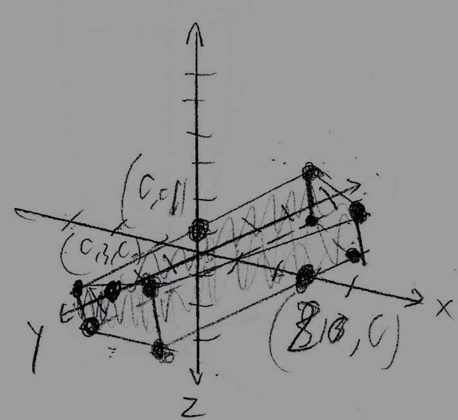


(21.) $f(x,y,z) = \sqrt{4-x^2} + \sqrt{9-y^2} + \sqrt{1-z^2}$

$\boxed{\{(x,y,z) \mid x \in [-2,2] \wedge y \in [-3,3] \wedge z \in [-1,1]\}}$

edges $(-2,-3,-1)$ $(-2,-3,1)$ $(2,-3,-1)$ $(2,-3,1)$

$(-2,3,-1)$ $(-2,3,1)$ $(2,3,-1)$ $(2,3,1)$



(25) $f(x,y) = 10-4x-5y$ plane

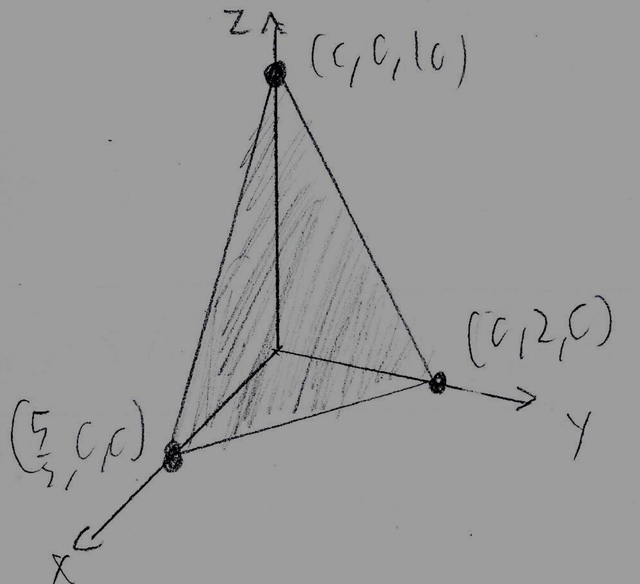
$z = 10-4x-5y$

$4x+5y+z=10$

$(0,0,10)$

$(0,2,0)$

$(\frac{5}{2},0,0)$



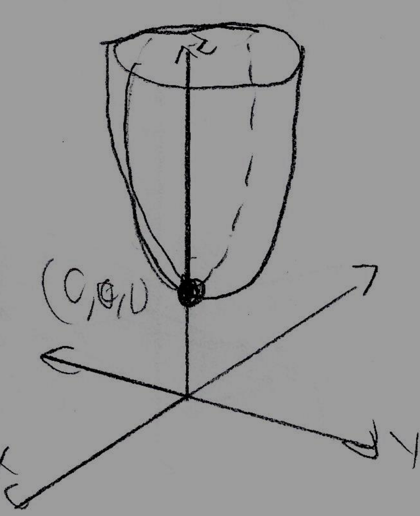
(29) $f(x,y) = x^2+4y^2+1$

$z = x^2+4y^2+1$

$-x^2-4y^2+z=1$

$z = x^2+4y^2+1$

$(0,0,1)$



$k=x^2$

$z=k+4y^2+1$

$-4y^2+z-1=k$

parabola

$k=x^2$

$z=x^2+k+1$

parabola

$k=z$

$k=x^2+4y^2+1$

elipsis

(33.) $f(-3,3) \approx 56$ as it is halfway between ring 50 and ring 60

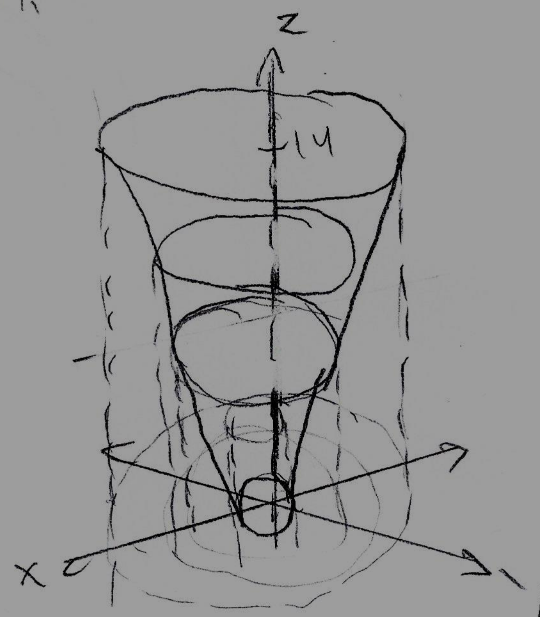
$f(3,-2) \approx 36$ as it is sligtly more than halfway between ring 40 and ring 30

The shape of this graph appears to be an ellip toid of sorts given each ring is an elipsis.

(37.) Given that in point A there are a lot of contours surrounding A with minimal distance, it can be inferred that point A is steep. Point B is nearly flat as it is farther away from the center with maximum distance between contours.

(41.) $f(x,y) = x^2 + y^2 + k^2$

- Thin center
- widest at 14
- Shape looks like upside down funnel with circular xy crossections

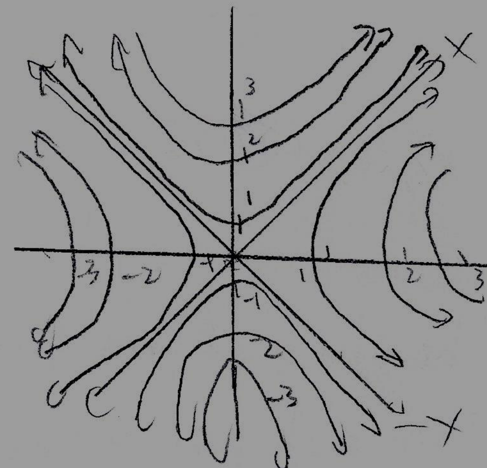


45.) $f(x,y) = x^2 - y^2$

- hyperbola

aysmpltes $y = x$ $y = -x$

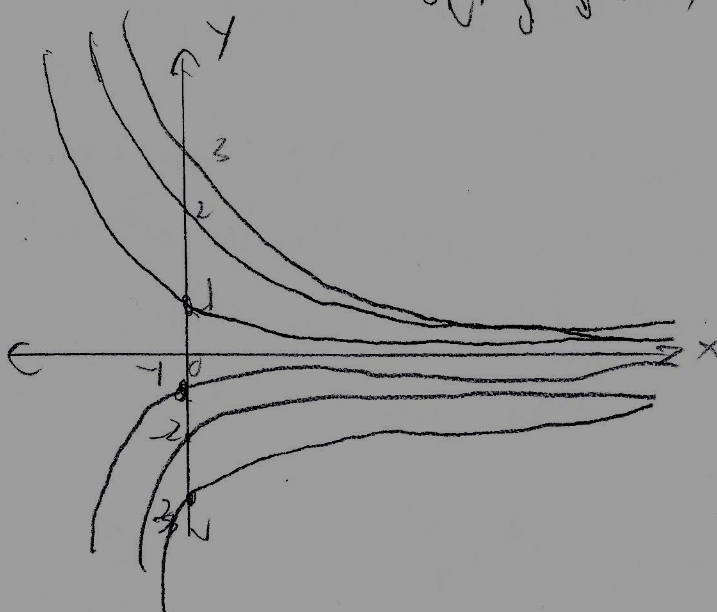
$x^2 - y^2 = k$



49.) $f(x,y) = ye^x$

$k = ye^x$

$y = ke^{-x}$



14.2

5, 7, 9, 11, 13, 15, 17, 19, 21

25, 29, 33, 37

5. $\lim_{(x,y)\to(3,2)} (x^2y^3 - 4y^2) = 9(8) - 4(4)$

$72 - 16 = \boxed{56}$

7. $\lim_{(x,y)\to(\pi,\frac{\pi}{2})} y\sin(x-y) = \frac{\pi}{2}\sin(\pi - \frac{\pi}{2}) = \frac{\pi}{2}\sin(\frac{\pi}{2}) = \boxed{\frac{\pi}{2}}$

9. $\lim_{(x,y)\to(0,0)} \frac{x^4 - 4y^2}{x^2 + 2y^2}$

$\lim_{x\to 0} \frac{0 - 4y^2}{0 + 2y^2} = -2$

$\lim_{y\to 0} \frac{x^4 - 0}{x^2 - 0} = x^2$

$\boxed{DNE}$

11. $\lim_{(x,y)\to(0,0)} \frac{y^2 \sin^2 x}{x^4 + y^4}$ $\lim_{(x,y)\to(0,0)} \frac{y^2}{x^4+y^4} \cdot \lim_{(x,y)\to(0,0)} \sin^2 x$

$\lim_{(x,y)\to(x,0)} = 0$ $\lim_{(x,y)\to(0,y)} = \frac{1}{1+y^2}$ $\cdot\, 0$

$\boxed{DNE}$

13. $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \frac{0}{0}$

$\lim_{x\to 0} \frac{0}{\sqrt{y^2}} = 0$

$\lim_{y\to 0} \frac{0}{\sqrt{x^2}} = 0$

$\boxed{0}$

15. $\lim_{(x,y)\to(0,0)} \frac{xy^2\cos y}{x^2+y^2} = \frac{0}{0}$ $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} \cdot \lim_{(x,y)\to(0,0)} \cos y$

$DNE \cdot 1 = \boxed{DNE}$

(17) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

$\lim\limits_{x\to 0} \dfrac{x^2+m^2x^2}{\sqrt{x^2+m^2x^2+1}-1} = \lim\limits_{x\to 0} \dfrac{\cancel{(x^2+m^2x^2)}\left(\sqrt{x^2+m^2x^2+1}+1\right)}{\cancel{x^2+m^2x^2}}$

$\lim\limits_{x\to 0} \sqrt{x^2+m^2x^2+1}+1 = \sqrt{1}+1 = 2$

$\lim\limits_{y\to 0} \dfrac{m^2y^2+y^2}{\sqrt{m^2y^2+y^2+1}-1} = \dfrac{(m^2y^2+y^2)\left(\sqrt{m^2y^2+y^2+1}+1\right)}{m^2y^2+y^2}$

$\lim\limits_{y\to 0} \sqrt{m^2y^2+y^2+1}+1 = \sqrt{1}+1 = 2 \quad \boxed{2}$

(19) $\lim\limits_{(x,y,z)\to(\pi,0,1/3)} e^{y^2}\tan(xz) = e^0 \tan\left(\frac{\pi}{3}\right) = \tan\left(\frac{\pi}{3}\right) = \boxed{\sqrt{3}}$

(21) $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xy+yz^2+xz^2}{x^2+y^2+z^2}$

$\lim\limits_{(x,0,0)} = \dfrac{x(0)+0}{x^2+0} = 0$

$\lim\limits_{(0,y,0)} = \dfrac{0(y)+0+0}{y^2} = 0$

$\lim\limits_{(0,0,z)} = \dfrac{0+0+0}{z^2} = 0$

$f(x,x,x) = \dfrac{x^2+x^3+x^3}{x^2+x^2+x^2}$

$\dfrac{x^2+2x^3}{3x^2} = \dfrac{1+2x}{3}$

$\boxed{\text{DNE}}$

(25) $g(f(x,y)) = \underbrace{(2x+3y-6)^2}_{\mathbb{R}} + \sqrt{\underbrace{(2x+3y)-6}}$

$2x+3y \geq 6$

$\{(x,y) \mid 2x+3y \geq 6\}$

(29) $F(x,y) = \dfrac{xy}{1+e^{x-y}}$

$$F(x,y) = \begin{cases} \dfrac{xy}{1+e^{x-y}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{1+e^{x-y}} = \frac{0}{1+e^0} = \frac{0}{2} = 0 = F(0,0)$$

continuous on $(0,0)$ $\rightarrow$ $\boxed{\text{continuous on } \mathbb{R}^2}$

(33) $G(x,y) = \sqrt{x} + \sqrt{1-x^2-y^2}$

$\sqrt{x}$ $D = x \in [0,\infty)$

$\sqrt{1-x^2-y^2}$ $D = (x^2+y^2) \leq 1$

$\sqrt{1-(x^2+y^2)}$

$$\boxed{\{(x,y) \mid x^2+y^2 \leq 1 \wedge x \geq 0\}}$$

(37) $f(x,y) = \begin{cases} \dfrac{x^2y^3}{2x^3+y^2} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$

$$f(0,0) = \frac{0 \cdot 0}{0}$$

$$f(x,0) = \left[\frac{0}{2x^3}\right) = 0$$

$$f(0,y) = y$$

$0 \neq y$ DNE

$$\boxed{\{(x,y) \mid (x,y) \neq (0,0)\}}$$

14.3 homework 3, 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, 47, 51, 55, 59, 63, 67, 71

(3) a. $f_T(-15, 30) \approx 1.3$ for a temp of $-15°C$ and speed 30 km/hr the wind chill index rises by $1.3°C$ for each degree temp increase.

$f_v(-15, 30) \approx -0.15$. for a temp of $-15°C$ and speed 30 km/hr w.c.i decreases each km/hr speed.

b. $\frac{\partial W}{\partial T}$ = Positive $\frac{\partial W}{\partial V}$ = negative

C. Approaches 0.

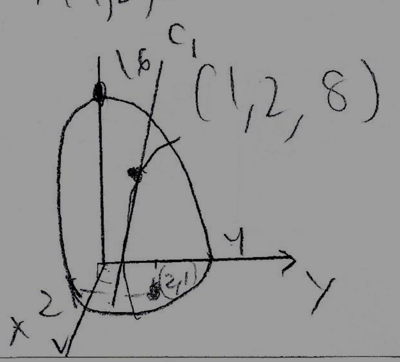
(7) a. $f_{xx}(-1, 2)$: Positive

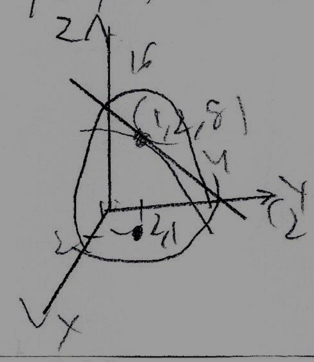
b. $f_{yy}(-1, 2)$: Negative

(11) $f(x, y) = 16 - 4x^2 - y^2$

$\frac{\partial f}{\partial x} = -8x$ $\frac{\partial f}{\partial x} = -2y$

$f_x(1, 2) = -8$ $f_y(1, 2) = -4$





(15) $f(x,y) = x^4 + 5xy^3$

$$\boxed{\frac{\partial f}{\partial x} = 4x^3 + 5y^3 \qquad \frac{\partial f}{\partial y} = 15y^2x}$$

(19) $z = \ln(x+t^2)$

$$\boxed{\frac{\partial z}{\partial x} = \frac{1}{x+t^2} \qquad \frac{\partial z}{\partial t} = \frac{1}{x+t^2}(2t)}$$

(23) $f(x,y) = \dfrac{ax+by}{cx+dy}$

$$\frac{\partial f}{\partial x} = \frac{a(cx+dy) - c(ax+by)}{(cx+dy)^2} = \boxed{\frac{(ad-bc)y}{(cx+dy)^2}}$$

$$\frac{\partial f}{\partial y} = \frac{b(cx+dy) - d(ax+by)}{(cx+dy)^2} = \boxed{\frac{(bc-ad)y}{(cx+dy)^2}}$$

(27) $R(p,q) = \tan^{-1}(pq^2)$

$$\boxed{\frac{\partial R}{\partial p} = \frac{1}{1+p^2q^4}(q^2) \qquad \frac{\partial R}{\partial q} = \frac{1}{1+p^2q^4}(2qp)}$$

(31) $f(x,y,z) = x^3yz^2 + 2yz$

$$\boxed{\frac{\partial f}{\partial x} = 3x^2yz^2} \qquad \boxed{\frac{\partial f}{\partial y} = x^3z^2 + 2z} \qquad \boxed{\frac{\partial f}{\partial z} = 2x^3yz + 2y}$$

(35.) $p = \sqrt{t^4 + u^2 \cos v}$

$$\frac{\partial p}{\partial t} = \frac{1}{2\sqrt{t^4 + u^2\cos v}}(4t^3) = \boxed{\frac{2t^3}{\sqrt{t^4+u^2\cos v}} = \frac{\partial p}{\partial t}}$$

$$\frac{\partial p}{\partial u} = \frac{1}{2\sqrt{t^4+u^2\cos v}}(2u\cos v) = \boxed{\frac{u\cos v}{\sqrt{t^4+u^2\cos v}} = \frac{\partial p}{\partial u}}$$

$$\frac{\partial p}{\partial v} = \frac{1}{2\sqrt{t^4+u^2\cos v}}(-u^2\sin v) = \boxed{\frac{-u^2\sin v}{2\sqrt{t^4+u^2\cos v}} = \frac{\partial p}{\partial v}}$$

(39.) $u = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$

$$\frac{\partial u}{\partial x_1} = \frac{2x_1}{2\sqrt{x_1^2 + \dots x_n^2}} \qquad \frac{\partial u}{\partial x_n} = \frac{2x_n}{2\sqrt{x_1^2+x_2^2+\dots x_n^2}}$$

$$\boxed{\frac{\partial u}{\partial x_i} = \frac{x_i}{\sqrt{x_1^2+x_2^2+\dots x_n^2}}}$$

(43.) $f(x,y,z) = \ln \frac{1-\sqrt{x^2+y^2+z^2}}{1+\sqrt{x^2+y^2+z^2}}$ $\quad f_y(1,2,2)$

$$f_y(x,y,z) = \frac{1+\sqrt{x^2+y^2+z^2}}{1-\sqrt{x^2+y^2+z^2}}\left(\frac{\left(1+\sqrt{x^2+y^2+z^2}\right)\left(\frac{-y}{\sqrt{x^2+y^2+z^2}}\right) - \left(1-\sqrt{x^2+y^2+z^2}\right)\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)}{\left(1+\sqrt{x^2+y^2+z^2}\right)^2}\right)$$

$$f_y(1,2,2) = \left(\frac{1+\sqrt{9}}{1-\sqrt{9}}\right)\left(\frac{(1+3)\left(-\frac{2}{3}\right) - (1-3)\left(\frac{2}{3}\right)}{(4)^2}\right) = \left(\frac{4}{-2}\right)\left(\frac{\left(-\frac{8}{3}+\frac{4}{3}\right)}{16}\right)$$

$$= \frac{-\frac{4}{3}}{-8} = \boxed{\frac{1}{6}}$$

(47) $x^2+2y^2+3z^2=1$

$2x+6z\frac{\partial z}{\partial x}=0$ $\qquad$ $4y+6z\frac{\partial z}{\partial y}=0$

$\boxed{\frac{\partial z}{\partial x}=\frac{-x}{3z}}$ $\qquad$ $\boxed{\frac{\partial z}{\partial y}=\frac{-2y}{3z}}$

(51) a. $z=f(x)+g(y)$

$\boxed{\frac{\partial z}{\partial x}=f'(x) \qquad \frac{\partial z}{\partial y}=g'(y)}$

b. $z=f(x+y)$

$\boxed{\frac{\partial z}{\partial x}=f'(x+y) \qquad \frac{\partial z}{\partial y}=f'(x+y)}$

(55) $z=\frac{y}{2x+3y}$ $\qquad$ $\frac{\partial z}{\partial x}=\frac{-2y}{(2x+3y)^2}$ $\qquad$ $\frac{\partial z}{\partial y}=\frac{(2x+3y)-3y}{(2x+3y)^2}=\frac{2x}{(2x+3y)^2}$

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{2y(2(2x+3y)(2))}{(2x+3y)^4}=\frac{8y}{(2x+3y)^3}$ $\qquad$ $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{2(2x+3y)^2-(2x)(4(2x+3y))}{(2x+3y)^4}$ $=6y-4x$

$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{-2(2x+3y)^2+2y(6(2x+3y))}{(2x+3y)^4}=\frac{6y-4x}{(2x+3y)^3}$ $\qquad$ $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{-2x(6(2x+3y))}{(2x+3y)^4}=\frac{-12x}{(2x+3y)^3}$

$$\boxed{\frac{\partial^2 z}{\partial x^2}=\frac{8y}{(2x+3y)^3} \qquad \frac{\partial^2 z}{\partial y^2}=\frac{-12x}{(2x+3y)^3} \qquad \frac{\partial^2 z}{\partial y\partial x}=\frac{\partial^2 z}{\partial x\partial y}=\frac{6y-4x}{(2x+3y)^3}}$$

59) $u = x^4 y^3 - y^4$

$$\frac{\partial u}{\partial x} = 4x^3 y^3 \qquad \frac{\partial u}{\partial y} = 3y^2 x^4 - 4y^3$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 12x^3y^2 \qquad \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 12x^3y^2$$

$$\boxed{u_{xy} = u_{yx} \therefore QED}$$

63) $f(x,y) = x^4y^2 - x^3y$

$$\frac{\partial f}{\partial x} = 4x^3y^2 - 3x^2y$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 12x^2y^2 - 6xy \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 8x^3y - 3x^2$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\right) = 24xy^2 - 6y \qquad \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right) = 24x^2y - 6x$$

$$\boxed{\begin{aligned} f_{xxx} &= 24xy^2 - 6y \\ f_{xyx} &= 24x^2y - 6x \end{aligned}}$$

(67) $w = \sqrt{u+v^2}$

$$\frac{\partial w}{\partial v} = \frac{-v}{\sqrt{u+v^2}}$$

$$\frac{\partial}{\partial u}\left(\frac{\partial w}{\partial v}\right) = \frac{v\left(\frac{1}{2\sqrt{u+v^2}}\right)}{u+v^2} = \frac{v}{2}(u+v^2)^{-1/2}$$

$$\frac{\partial}{\partial u}\left(\frac{\partial}{\partial u}\left(\frac{\partial w}{\partial v}\right)\right) = \frac{(u+v^2)\left(-\frac{v}{4}(u+v^2)^{-3/2}\right) - \frac{v}{2}(u+v^2)^{-1/2}}{(u+v^2)^2}$$

$$= \boxed{\frac{3}{4}v(u+v^2)^{-5/2}}$$

(71) $f(x,y,z) = xy^2z^3 + \sin^{-1}(x\sqrt{z})$ f_{xyz}

$$\frac{\partial f}{\partial x} = y^2z^3 + \frac{\sqrt{z}}{\sqrt{1-x^2z}}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 2yz^3$$

$$\frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right) = \boxed{6yz^2}$$

14.4 homework 1, 5, 11, 15, 19, 23, 25, 27, 29, 31, 33, 35.

(1.) $z = 2x^2 + y^2 - 5y \quad (1, 2, -4)$

$\frac{\partial z}{\partial x} = 4x \qquad \frac{\partial z}{\partial y} = 2y - 5$

$\frac{\partial z}{\partial x}\Big|_{x=1} = 4 \qquad \frac{\partial z}{\partial y}\Big|_{y=2} = 4 - 5 = -1$

$z + 4 = 4(x-1) - 1(y-2)$

$\boxed{z = 4x - y - 6}$

(5.) $z = x\sin(x+y) \quad (-1, 1, 0)$

$\frac{\partial z}{\partial x} = \sin(x+y) + x\cos(x+y) \qquad \frac{\partial z}{\partial y} = x\cos(x+y)$

$\frac{\partial z}{\partial x}\Big|_{(-1,1)} = 0 - 1 = -1 \qquad \frac{\partial z}{\partial y}\Big|_{(-1,1)} = -1$

$z = -1(x+1) - 1(y-1) \quad \boxed{x + y + z = 0}$

(11.) $f(x,y) = 1 + x\ln(xy-5) \quad (2, 3)$

$f(2,3) = 1 + 2(\ln(1)) = 1$

$L(x,y) = f(2,3) + f_x(2,3)(x-2) + f_y(2,3)(y-3)$

$\frac{\partial f}{\partial x} = \ln(xy-5) + \frac{xy}{xy-5} \overset{x=2,\ y=3}{=} 6 \qquad \frac{\partial f}{\partial y} = \frac{x^2}{xy-5} \overset{x=2,\ y=3}{=} 4$

$L(x,y) = 1 + 6(x-2) + 4(y-3)$

$1 + 6x - 12 + 4y - 12 = \boxed{6x + 4y - 23}$

15) $f(x,y) = 4\tan^{-1}(xy) \quad (1,1) \quad f(1,1) = \pi$

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$$\frac{\partial f}{\partial x} = \frac{4y}{1+x^2y^2} \qquad \frac{\partial f}{\partial y} = \frac{4x}{1+x^2y^2}$$

$y=1$
$x=1$

$$\frac{4}{2} = 2 \qquad = \frac{4}{2} = 2$$

$L(x,y) = \pi + 2(x-1) + 2(y-1) = \boxed{2x+2y+\pi-4}$

19) $f(2,5) = 6 \quad f_x(2,5) = 1 \quad f_y(2,5) = -1 \quad f(2.2, 4.9) = ?$

$L(x,y) = 6 + (x-2) - (y-5)$

$L(2.2, 4.9) = 6 + (0.2) - (-0.1) = 6 + 0.3 = \boxed{6.3}$

23) $(94, 80) \quad (95, 78)$

$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$f(94,80) = f(94,80) + f_T(94,80)(T-94) + f_H(94,80)(H-80) \qquad h=2$

$g(T) = f(T,80) \qquad g'(94) = f'(94,80) = \lim_{h \to 0} \frac{f(a+h,80) - f(a,80)}{h} = \frac{f(96+80) - f(94,80)}{2} = 4$

$G'(80) = \lim_{h \to 0} \frac{(94, 80-5) - f(94,80)}{-5} = 1$

$\boxed{L(T,H) = 4T + H - 329} \qquad f(95,78) = 4 \cdot 95 + 78 - 329 = \boxed{129}$

25. $z = e^{-2x} \cos 2\pi t$

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial t}\,dt$$

$$\frac{\partial z}{\partial x} = -2e^{-2x}\cos 2\pi t \qquad \frac{\partial z}{\partial t} = -2\pi e^{-2x}\sin 2\pi t$$

$$\boxed{dz = -2e^{-2x}\cos 2\pi t\,dx - 2\pi e^{-2x}\sin 2\pi t\,dt}$$

27. $m = p^5 q^3$

$$\frac{\partial m}{\partial p} = 5p^4q^3 \qquad \frac{\partial m}{\partial q} = 3q^2p^5$$

$$\boxed{dm = 5p^4q^3\,dp + 3q^2p^5\,dq}$$

29. $R = \alpha\beta^2\cos\gamma$

$$\frac{\partial R}{\partial \alpha} = \beta^2\cos\gamma \qquad \frac{\partial R}{\partial \beta} = 2\alpha\beta\cos\gamma \qquad \frac{\partial R}{\partial \gamma} = -\alpha\beta^2\sin\gamma$$

$$\boxed{dR = \beta^2\cos\gamma\,d\alpha + 2\alpha\beta\cos\gamma\,d\beta - \alpha\beta^2\sin\gamma\,d\gamma}$$

(31) $z = 5x^2 + y^2$

$\frac{\partial z}{\partial x} = 10x \qquad \frac{\partial z}{\partial y} = 2y$

$\Delta z = 5(1.05)^2 + (2.1)^2 - 5 - 4$
$= \boxed{0.9225}$

$dz = 10x\,dx + 2y\,dy$

$dz = 10x(x - x_0) + 2y(y - y_0)$
$= 10(1)(0.05) + 2(2)(0.1)$
$= \boxed{0.9}$

(33) $(30, 24) \qquad (29.9, 23.9)$

$A = LW$

$\frac{\partial A}{\partial L} = W \qquad \frac{\partial A}{\partial W} = L$

$dA = W(30 - 29.9) + L(0.1)$
$dA = W(0.1) + L(0.1)$
$dA = 0.1(54) = \boxed{5.4\text{cm}^2}$

(35) $V = \pi r^2 h$
$dr = 0.04\text{cm}$

$dh = 0.08$

$\frac{\partial f}{\partial r} = 2\pi r h \qquad \frac{\partial f}{\partial h} = \pi r^2$

$dV = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial h}dh$
$= 2\pi r h(0.04) + \pi r^2(0.08)$
$= 96\pi(0.04) + 16\pi(0.08) \approx \boxed{16\text{cm}^3}$

14.5 homework 1, 5, 9, 13, 17, 21, 25, 29, 33, 37, 41, 45, 49, 53

(1) $z = xy^3 - x^2y$ $\quad x = t^2+1 \quad y = t^2 - 1$

$\frac{\partial z}{\partial x} = y^3 - 2xy \quad \frac{dx}{dt} = 2t \quad \frac{dy}{dt} = 2t$

$\frac{\partial z}{\partial y} = 3y^2x - x^2$

$$\boxed{\frac{dz}{dt} = (y^3-2xy)(2t) + (3y^2x - x^2)(2t)}$$

(5) $w = xe^{y/z} \quad x = t^2 \quad y = 1-t \quad z = 1+2t$

$\frac{\partial w}{\partial x} = e^{y/z} \quad \frac{\partial w}{\partial y} = \frac{x}{z}e^{y/z} \quad \frac{dx}{dt} = 2t \quad \frac{dy}{dt} = -1 \quad \frac{dz}{dt} = 2$

$\frac{\partial w}{\partial z} = \frac{xy}{z}e^{y/z}\left(\frac{1}{z}\right)$ $\boxed{\frac{dw}{dt} = e^{y/z}(2t) - \frac{x}{z}e^{y/z} - 2e^{y/z}\left(\frac{xy}{z^2}\right)}$

(9) $z = \ln(3x+2y) \quad x = s\sin t \quad y = t\cos s$

$\frac{\partial z}{\partial x} = \frac{3}{3x+2y} \quad \frac{\partial x}{\partial s} = \sin t \quad \frac{\partial y}{\partial s} = -t\sin s$

$\frac{\partial x}{\partial t} = s\cos t \quad \frac{\partial y}{\partial t} = \cos s$

$\frac{\partial z}{\partial y} = \frac{2}{3x+2y}$

$$\frac{\partial z}{\partial s} = \left(\frac{3}{3x+2y}\right)(\sin t) + \left(\frac{-2}{3x+2y}\right)(t\sin s) = \boxed{\frac{3\sin t - 2t\sin s}{3x+2y}}$$

$$\frac{\partial z}{\partial t} = \left(\frac{3}{3x+2y}\right)(s\cos t) + \left(\frac{2}{3x+2y}\right)(\cos s) = \boxed{\frac{3s\cos t + 2\cos s}{3x+2y}}$$

(13) $p(t) = f(g(t), h(t))$ $\quad g(2)=4 \quad g'(2)=-3$

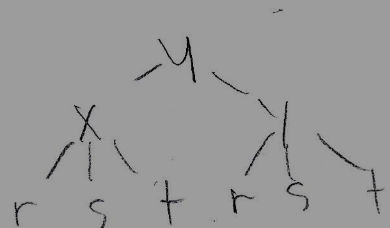
$h(2)=5 \quad h'(2)=6$

$f_x(4,5)=2 \quad f_y(4,5)=8$

$$p'(2) = f_x(4,5)(g'(2)) + f_y(4,5)(h'(2))$$

$$2(-3) + 8(6) = \boxed{42}$$

(17) $u = f(x,y) \quad x = x(r,s,t) \quad y = y(r,s,t)$



$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t}$$

(21) $z = x^4 + x^2 y \quad x = s + 2t - u \quad y = stu^2 \quad (4,2,1)$

$x = 4+4-1 = 7$

$y = 4(2)(1) = 8$

$\frac{\partial z}{\partial x} = 4x^3 + 2xy \quad \frac{\partial x}{\partial s} = 1 \quad \frac{\partial x}{\partial t} = 2 \quad \frac{\partial y}{\partial s} = tu^2 \quad \frac{\partial y}{\partial t} = su^2$

$\frac{\partial z}{\partial y} = x^2 \quad \frac{\partial x}{\partial u} = -1 \quad \frac{\partial y}{\partial u} = 2stu$

$\frac{\partial z}{\partial s} = (4x^3 + 2xy) + x^2(tu^2) \qquad (4(7^3) + 2(56)) + 49(2) = \boxed{1582}$

$\frac{\partial z}{\partial t} = (4x^3 + 2xy)(2) + (x^2)(su^2) \qquad 2(4(7^3) + 2(56)) + 49(4) = \boxed{3164}$

$\frac{\partial z}{\partial u} = (4x^3 + 2xy)(-1) + (x^2)(2stu) \qquad -(4(7^3) + 2(56)) + 49(16) = \boxed{-700}$

(25.) $N = \dfrac{p+q}{p+r}$ $\quad p = u + vw \quad q = v + uw \quad r = w + uv$

$u = 2$
$v = 3$
$w = 4$
$p = 14$
$q = 11$
$r = 10$

$\dfrac{\partial N}{\partial p} = \dfrac{r-q}{(p+r)^2}$ $\quad N_p = \dfrac{-1}{(24)^2}$

$\dfrac{\partial N}{\partial q} = \dfrac{1}{p+r}$ $\quad N_q = \dfrac{1}{24}$

$\dfrac{\partial N}{\partial r} = \dfrac{-(p+q)}{(p+r)^2}$ $\quad N_r = \dfrac{-25}{(24)^2}$

$\dfrac{\partial p}{\partial u} = 1 \quad \dfrac{\partial p}{\partial v} = w \quad \dfrac{\partial p}{\partial w} = v$

$\dfrac{\partial q}{\partial u} = w \quad \dfrac{\partial q}{\partial v} = 1 \quad \dfrac{\partial q}{\partial w} = u$

$\dfrac{\partial r}{\partial u} = v \quad \dfrac{\partial r}{\partial v} = u \quad \dfrac{\partial r}{\partial w} = 1$

$$\frac{\partial N}{\partial u} = \frac{\partial N}{\partial p}(1) + \frac{\partial N}{\partial q}(w) + \frac{\partial N}{\partial r}(v) = \frac{-1}{24^2} + \frac{4}{24} - \frac{75}{(24)^2} = \boxed{\frac{5}{144}}$$

$$\frac{\partial N}{\partial v} = \frac{\partial N}{\partial p}(w) + \frac{\partial N}{\partial q}(1) + \frac{\partial N}{\partial r}(u) = \frac{-4}{(24)^2} + \frac{1}{24} - \frac{50}{(24)^2} = \boxed{\frac{-5}{96}}$$

$$\frac{\partial N}{\partial w} = \frac{\partial N}{\partial p}(v) + \frac{\partial N}{\partial q}(u) + \frac{\partial N}{\partial r}(1) = \frac{-3}{(24)^2} + \frac{2}{24} + \frac{-25}{(24)^2} = \boxed{\frac{5}{144}}$$

29. $\tan^{-1}(x^2 y) = x + xy^2$

$F(x,y) = \tan^{-1}(x^2 y) - x - xy^2 = 0$

$F_x = \frac{1}{1+x^4y^2}(2xy) - 1 - y^2 = \frac{2xy}{1+x^4y^2} - 1 - y^2$

$= \frac{2xy - (1+y^2)(1+x^4y^2)}{(1+x^4y^2)}$

$F_y = \frac{x^2}{1+x^4y^2} - 2xy = \frac{x^2 - 2xy(1+x^4y^2)}{1+x^4y^2}$

$\frac{dy}{dx} = -\frac{F_x}{F_y} = \boxed{\frac{(1+y^2)(1+x^4y^2) - 2xy}{x^2 - 2xy(1+x^4y^2)}}$

33. $e^z = xyz$

$e^z \frac{\partial z}{\partial x} = yz + xy\frac{\partial z}{\partial x}$

$\boxed{\frac{\partial z}{\partial x} = \frac{yz}{e^z - xy}}$

$e^z \frac{\partial z}{\partial y} = xz + xy\frac{\partial z}{\partial y}$

$\boxed{\frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}}$

37. $D \sim 7$ $T \sim 12.5$ $C(T,D) = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + 0.016D$

$\frac{dC}{dt} = \frac{\partial C}{\partial T}\frac{dT}{dt} + \frac{\partial C}{\partial D}\frac{dD}{dt} = (4.6 - 0.11T + 0.00087T^2)\cdot T_t + 0.016D_t$

$\frac{dC}{dt}(20) = (4.6 - 0.11\cdot T(20) + 0.00087(T(20))^2)\cdot T_t(20) + 0.016D_t(20)$

$\frac{dT}{dt}(20) \approx \frac{T(25) - T(20)}{25-20} = \frac{12-12.5}{5} = -0.1$ $\frac{dD}{dt}(20) = \frac{D(24)-D(20)}{24-20} = \frac{3}{4} = 0.75$

$\frac{dC}{dt}(20) = (4.6 - 0.11\cdot T(20) + 0.00087\, T(20)^2)\cdot T_t(20) + 0.016\cdot D_t(20) =$

-0.325 m/s $\approx \boxed{-0.33 \text{ m/s per min}}$

(41) $V = 8.31 \frac{T}{P}$

$P(t_0) = 20\ kPa \quad T(t_0) = 320\ K$

$\frac{dP(t_0)}{dt} = 0.05\ kPa/s \quad \frac{dT(t_0)}{dt} = 0.15$

$$\frac{dV(t_0)}{dt} = \frac{\partial V(P(t_0), T(t_0))}{\partial P} \cdot \frac{\partial P(t_0)}{dt} + \frac{\partial V(P,T)}{\partial T} \cdot \frac{dT(t_0)}{dt} =$$

$$-8.31 \frac{T(t_0)}{P^2(t_0)} \cdot \frac{dP(t_0)}{dt} + 8.31\left(\frac{1}{P(t_0)}\right) \cdot \frac{dT(t_0)}{dt} =$$

$$8.31\left(-\frac{320}{400} \cdot 0.05 + \frac{1}{20} \cdot 0.15\right) = \boxed{-0.27\ L/s}$$

(45) $z = f(x, y) \quad x = r\cos\theta \quad y = r\sin\theta$

(a) $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$

$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$

$z = f(x,y) \quad x = r\cos\theta \quad y = r\sin\theta$

$\frac{\partial x}{\partial r} = \cos\theta \quad \frac{\partial y}{\partial r} = \sin\theta$

$$\boxed{\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta}$$

$\frac{\partial x}{\partial \theta} = -r\sin\theta \quad \frac{\partial y}{\partial \theta} = r\cos\theta$

$$\boxed{\frac{\partial z}{\partial \theta} = -\frac{\partial z}{\partial x} r\sin\theta + \frac{\partial z}{\partial y} r\cos\theta}$$

(b) $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$

$z_x^2 + z_y^2 = z_r^2 + \frac{1}{r^2} z_\theta^2$

$z_x^2 + z_y^2 = (z_x\cos\theta + z_y\sin\theta)^2 + z_\theta^2/r^2$

$= z_x^2\cos^2\theta + z_x z_y\cos\theta\sin\theta + z_y^2\sin^2\theta + \frac{z_\theta^2}{r^2}$

$z_\theta = \frac{\partial z}{\partial x}\frac{dx}{d\theta} + \frac{\partial z}{\partial y}\frac{dy}{d\theta}$

$\frac{(z_x - r\sin\theta + z_y r\cos\theta)^2}{r^2} = z_x^2\sin^2\theta - z_x z_y \sin\theta\cos\theta + z_y^2\cos^2\theta$

$= z_x^2\cos^2\theta + z_x z_y\sin\theta\cos\theta + z_y^2\sin^2\theta + z_x^2\sin^2\theta - z_x z_y\sin\theta\cos\theta + z_y^2\cos^2\theta$

$= z_x^2(1) + z_y^2(1)$

$\boxed{z_x^2 + z_y^2 = z_x^2 + z_y^2}$ Q.E.D.

49. $u = x + at \quad v = x - at \qquad z = f(u) + g(v)$

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x}\frac{df}{du} + \frac{\partial v}{\partial x}\cdot\frac{dg}{dv} = \frac{df}{du} + \frac{dg}{dv}$$

$$\frac{\partial z}{\partial t} = \frac{df}{du}\frac{\partial u}{\partial t} + \frac{dg}{dv}\frac{\partial v}{\partial t} = a\frac{df}{du} - a\frac{dg}{dv}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{df}{du} + \frac{dg}{dv}\right) = \frac{d^2 f}{du^2} + \frac{d^2 g}{dv^2}$$

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t}\left(a\left(\frac{df}{du}\right) - a\frac{dg}{dv}\right) = a^2\left(\frac{d^2 f}{du^2} + \frac{d^2 g}{dv^2}\right)$$

$$\frac{\partial^2 z}{\partial t^2} = a^2\frac{\partial^2 z}{\partial x^2} \quad \therefore \text{QED}$$

53) $z=f(x,y)$ $x=r\cos\theta$ $y=r\sin\theta$

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 z}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}+\frac{1}{r}\frac{\partial z}{\partial r}$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x}=\cos\theta\frac{\partial z}{\partial r}-\frac{\sin\theta}{r}\frac{\partial z}{\partial \theta}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial y}=\sin\theta\frac{\partial z}{\partial r}+\frac{\cos\theta}{r}\frac{\partial z}{\partial \theta}$$

$$\frac{\partial^2 z}{\partial x^2}=\cos^2\theta\frac{\partial^2 z}{\partial r^2}-\frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 z}{\partial r\,\partial \theta}+\frac{\sin^2\theta}{r^2}\frac{\partial^2 z}{\partial \theta^2}+\frac{\sin^2\theta}{r}\frac{\partial z}{\partial r}+$$

$$\left.\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial z}{\partial \theta}\right)$$

$$\frac{\partial^2 z}{\partial y^2}=\sin^2\theta\frac{\partial^2 z}{\partial r^2}+\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 z}{\partial r\,\partial \theta}+\frac{\cos^2\theta}{r^2}\frac{\partial^2 z}{\partial \theta^2}+\frac{\cos^2\theta}{r}\frac{\partial z}{\partial r}$$

$$-\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial z}{\partial \theta}$$

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=(\sin^2\theta+\cos^2\theta)\frac{\partial^2 z}{\partial r^2}+0\,\frac{\partial^2 z}{\partial r\,\partial \theta}+\frac{(\sin^2\theta+\cos^2\theta)}{r^2}\frac{\partial^2 z}{\partial \theta^2}+\frac{\sin^2\theta+\cos^2\theta}{r}\frac{\partial z}{\partial r}$$

$$+0\left(\frac{\partial z}{\partial \theta}\right)$$

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 z}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}+\frac{1}{r}\frac{\partial z}{\partial r}\quad \therefore \text{QED}$$

14.6 homework

5, 9, 13, 17, 21, 25, 29, 33, 37, 41, 45, 49, 53, 57

(5) $f(x,y) = y\cos(xy)$ $(0,1)$ $\theta = \frac{\pi}{4}$

$\frac{\partial f}{\partial x} = -y^2\sin(xy)$ $\frac{\partial f}{\partial y} = \cos(xy) - xy\sin(xy)$

$\nabla f = \langle -y^2\sin(xy), \cos(xy) - xy\sin(xy) \rangle$

$\nabla f(0,1) = \langle 0, 1 \rangle$ $|\nabla f(0,1)| = \sqrt{1} = 1$

$D_u f(0,1) = (1)\cos\left(\frac{\pi}{4}\right) = \boxed{\frac{\sqrt{2}}{2}}$

(9) $f(x,y,z) = x^2yz - xyz^3$ $P(2,-1,1)$ $u = \langle 0, \frac{4}{5}, \frac{-3}{5} \rangle$

$\frac{\partial f}{\partial x} = 2xyz - yz^3$ $\frac{\partial f}{\partial y} = x^2z - xz^3$ $\frac{\partial f}{\partial z} = x^2y - 3xyz^2$

$\nabla f(x,y,z) = \langle 2xyz - yz^3, x^2z - xz^3, x^2y - 3xyz^2 \rangle$

$\nabla f(2,-1,1) = \langle 2(2)(-1) - (-1)(1), 4(1) - 2(1), 4(-1) + 3(2) \rangle = \langle -3, 2, 2 \rangle$

$D_u f(2,-2,1) = \langle -3, 2, 2 \rangle \cdot \langle 0, \frac{4}{5}, \frac{-3}{5} \rangle = 0 + \frac{8}{5} - \frac{6}{5} = \boxed{\frac{2}{5}}$

(13) $g(s,t) = s\sqrt{t}$, $(2,4)$, $v = \langle 2, -1 \rangle$ $|v| = \sqrt{5}$

$\frac{\partial g}{\partial s} = \sqrt{t}$ $\frac{\partial g}{\partial t} = \frac{s}{2\sqrt{t}}$ $\nabla g(s,t) = \langle \sqrt{t}, \frac{s}{2\sqrt{t}} \rangle$ $\nabla g(2,4) = \langle 2, \frac{1}{2} \rangle$

$\frac{1}{|v|} D_v g(2,4) = \frac{\langle 2, \frac{1}{2} \rangle \cdot \langle 2, -1 \rangle}{|v|} = \frac{4 - \frac{1}{2}}{|v|} = \frac{3\frac{1}{2}}{|v|} = \frac{7}{2}\left(\frac{1}{|v|}\right) = \boxed{\frac{7}{2\sqrt{5}}}$

(17.) $h(r,s,t) = \ln(3r+6s+9t)$, $(1,1,1)$, $v = \langle 4,12,6 \rangle$ $|v| = \sqrt{16+144+36} = \sqrt{196} = 14$

$$\frac{\partial h}{\partial r} = \frac{3}{3r+6s+9t} \quad \frac{\partial h}{\partial s} = \frac{6}{3r+6s+9t} \quad \frac{\partial h}{\partial t} = \frac{9}{3r+6s+9t}$$

$$\nabla h(r,s,t) = \left\langle \frac{3}{3r+6s+9t}, \frac{6}{3r+6s+9t}, \frac{9}{3r+6s+9t} \right\rangle$$

$$\nabla h(1,1,1) = \left\langle \frac{3}{18}, \frac{6}{18}, \frac{9}{18} \right\rangle = \left\langle \frac{1}{6}, \frac{1}{3}, \frac{1}{2} \right\rangle$$

$$\frac{1}{|v|} D_u h(1,1,1) = \frac{1}{14}\left(\left\langle \frac{1}{6}, \frac{1}{3}, \frac{1}{2} \right\rangle \cdot \langle 4,12,6 \rangle\right) = \frac{1}{14}\left(\frac{2}{3}+4+3\right) = \frac{1}{14}\left(\frac{23}{3}\right) = \boxed{\frac{23}{42}}$$

(21.) $f(x,y) = 4y\sqrt{x}$ $(4,1)$

$$\frac{\partial f}{\partial y} = 4\sqrt{x} \quad \frac{\partial f}{\partial x} = \frac{2y}{\sqrt{x}} \quad \nabla f(x,y) = \left\langle \frac{2y}{\sqrt{x}}, 4\sqrt{x} \right\rangle$$

$$\nabla f(4,1) = \boxed{\langle 1, 8 \rangle}$$

$$|\nabla f(4,1)| = \sqrt{1+64} = \boxed{\sqrt{65}}$$

(25.) $f(x,y,z) = x\dfrac{1}{y+z}$ $(8,1,3)$

$$\frac{\partial f}{\partial x} = \frac{1}{y+z} \quad \frac{\partial f}{\partial y} = \frac{-x}{(y+z)^2} \quad \frac{\partial f}{\partial z} = \frac{-x}{(y+z)^2}$$

$$\nabla f(x,y,z) = \left\langle \frac{1}{y+z}, \frac{-x}{(y+z)^2}, \frac{-x}{(y+z)^2} \right\rangle \quad \nabla f(8,1,3) = \left\langle \frac{1}{4}, \frac{-1}{2}, \frac{-1}{2} \right\rangle = \boxed{\langle 1,-2,-2 \rangle}$$

$$|\nabla f(8,1,3)| = \sqrt{\frac{1}{16}+\frac{1}{4}+\frac{1}{4}}$$

$$= \sqrt{\frac{1}{16}+\frac{1}{2}} = \sqrt{\frac{9}{16}} = \boxed{\frac{3}{4}}$$

(29) $f(x,y)=x^2+y^2-2x-4y$ $\quad i+j=\langle 1,1\rangle$

$\nabla f(x,y)=\langle 2x-2,\,2y-4\rangle = \langle 1,1\rangle$

$2x-2=1 \quad 2y-4=1 \qquad \left(\frac{3}{2},\frac{5}{2}\right)$

$2x-2=2y-4$

$2x+2=2y$

$y=x+1 \qquad \boxed{\text{All points on the line } y=x+1}$

(33) $V(x,y,z)=5x^2-3xy+xyz \quad P(3,4,5) \quad v=\langle 1,1,-1\rangle \quad |v|=\sqrt{3}$

$\frac{\partial V}{\partial x}=10x-3y+yz \quad \frac{\partial V}{\partial y}=-3x+xz \quad \frac{\partial V}{\partial z}=xy$

$\nabla V(x,y,z)=\langle 10x-3y+yz,\,-3x+xz,\,xy\rangle$

$\nabla V(3,4,5)=\langle 38,6,12\rangle \quad D_v V(3,4,5)=\langle 38,6,12\rangle\cdot\langle 1,1,-1\rangle=$

$38+6-12=32$

a. $\frac{D_v V(3,4,5)}{|v|}=\boxed{\frac{32}{\sqrt{3}}}$

b. $\boxed{\langle 38,6,12\rangle}$

c. $|\langle 38,6,12\rangle|=\sqrt{38^2+36+144}=\sqrt{38^2+180}=\sqrt{1624}=\boxed{2\sqrt{406}}$

37) a) $\nabla(au+bv) = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j\right)(au+bv)$

$= a\left(\frac{\partial}{\partial x} u i + \frac{\partial}{\partial y} u j\right) + b\left(\frac{\partial}{\partial x} v i + \frac{\partial}{\partial x} v j\right)$

$= \boxed{a\nabla u + b\nabla v}$

b.) $\nabla(uv) = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j\right) uv = \frac{\partial uv}{\partial x} i + \frac{\partial uv}{\partial y} j$

$\frac{\partial uv}{\partial y} = v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y}$ $\qquad \frac{\partial uv}{\partial x} = v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x}$

$\nabla uv = \left(v\frac{\partial u}{\partial x} + \frac{u\,\partial v}{\partial x}\right)i + \left(v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y}\right)j = v\left(\frac{\partial u}{\partial x} i + \frac{\partial u}{\partial y} j\right) +$

$u\left(\frac{\partial v}{\partial x} i + \frac{\partial v}{\partial y} j\right) =$

$\boxed{v\nabla u + u\nabla v}$

c.) $\nabla\left(\frac{u}{v}\right) = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j\right)\left(\frac{u}{v}\right) = \frac{\partial}{\partial x}\left(\frac{u}{v}\right)i + \frac{\partial}{\partial y}\left(\frac{u}{v}\right)j$

$= \frac{v\frac{\partial}{\partial x} u_i - u\frac{\partial}{\partial x} v_i}{v^2} + \frac{v\frac{\partial}{\partial y} u_j - u\frac{\partial}{\partial y} v_j}{v^2}$

$= \frac{v\left[\frac{\partial}{\partial x} u i + \frac{\partial}{\partial y} u j\right] - u\left[\frac{\partial}{\partial x} v i + \frac{\partial}{\partial y} v j\right]}{v^2} = \boxed{\frac{v\nabla u - u\nabla v}{v^2}}$

d) $\nabla u^n = \langle \frac{\partial}{\partial x} u^n, \frac{\partial}{\partial y} u^n \rangle = \langle n u^{n-1}\frac{\partial u}{\partial x}, n u^{n-1}\frac{\partial u}{\partial y} \rangle$

$= n u^{n-1} \langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \rangle$

$= \boxed{n u^{n-1} \nabla u}$

(41) $2(x-2)^2 + (y-1)^2 + (z-3)^2 = 10 \qquad (3,3,5)$

$F(x,y,z) = 2(x-2)^2 + (y-1)^2 + (z-3)^2$

$F_x = 4(x-2) \qquad F_y = 2(y-1) \qquad F_z = 2(z-3)$

$F_x(3,3,5) = 4 \qquad F_y(3,3,5) = 4 \qquad F_z = 2(2) = 4$

$4(x-3) + 4(y-3) + 4(z-5) = 0$

$\boxed{x+y+z-11=0}$

$\frac{(x-3)}{4} = \frac{(y-3)}{4} = \frac{(z-5)}{4} = \boxed{(x-3) = (y-3) = (z-5)}$

(45) $x+y+z = e^{xyz} \qquad (0,0,1)$

$x+y+z - e^{xyz} = 0 \qquad F(x,y,z) = x+y+z-e^{xyz}$

$F_x = 1 - yze^{xyz} \qquad F_y = 1 - xze^{xyz} \qquad F_z = 1 - xye^{xyz}$

$F_x(0,0,1) = 1 \qquad F_y(0,0,1) = 1 \qquad F_z(0,0,1) = 1$

$(x-0) + (y-0) + (z-1) = 0$

$\boxed{x+y+z-1=0}$

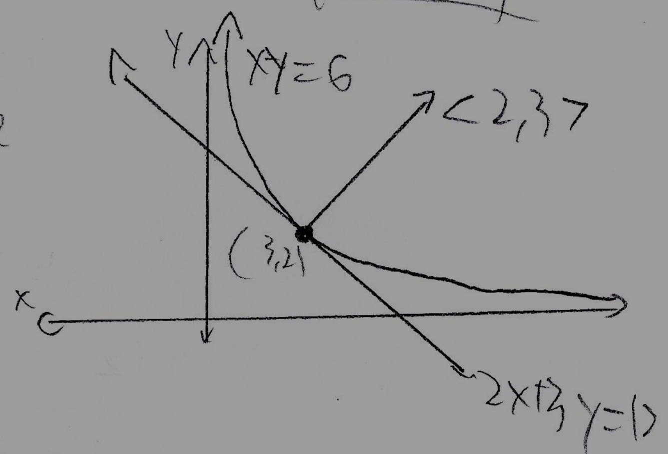
$\boxed{x = y = (z-1)}$

(49a) $f(x,y) = xy$ $\quad \nabla f(3,2)$ tangent to $f(x,y) = 6$ at $(3,2)$

$f_x = y \quad f_y = x \quad \nabla f(x,y) = \langle y, x \rangle \quad \nabla f(3,2) = \boxed{\langle 2,3 \rangle}$

$2(x-3) + 3(y-2) = 0$ = tangent line

$\boxed{2x + 3y = 12}$



(53) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z}{c} = 0 \quad (x_0, y_0, z_0)$

$\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} = \dfrac{z_0}{c}$

$F_x = \frac{1}{a^2} 2x \quad F_y = \frac{1}{b^2} 2y \quad F_z = \frac{-1}{c}$

$F_x(x_0,y_0,z_0) = \dfrac{2x_0}{a^2} \quad F_y(x_0,y_0,z_0) = \dfrac{2y_0}{b^2} \quad F_z(x_0,y_0,z_0) = \dfrac{-1}{c}$

$\dfrac{2x_0}{a^2}(x-x_0) + \dfrac{2y_0}{b^2}(y-y_0) - \dfrac{1}{c}(z-z_0) = 0$

$\dfrac{2xx_0}{a^2} + \dfrac{2yy_0}{b^2} - \left(\dfrac{2x_0^{\,}x_0}{a^2} + \dfrac{2y_0y_0}{b^2}\right) - \dfrac{z}{c} + \dfrac{z_0}{c} = 0$

$\dfrac{2xx_0}{a^2} + \dfrac{2yy_0}{b^2} - \dfrac{2z_0}{c} - \dfrac{z}{c} + \dfrac{z_0}{c} = 0$

$\dfrac{2yy_0}{b^2} + \dfrac{2xx_0}{a^2} = \dfrac{z}{c} + \dfrac{z_0}{c}$

$\boxed{\dfrac{2xx_0}{a^2} + \dfrac{2yy_0}{b^2} = \dfrac{z+z_0}{c}}$ $\quad \therefore$ QED

57. $x^2+y^2=z^2$ $\quad P(x_0,y_0,z_0)$ $\quad x_0^2+y_0^2-z_0^2=0$

$x^2+y^2-z^2=0$

$F_x=2x$ $\quad F_y=2y$ $\quad F_z=-2z$ $\quad \nabla F(x,y,z)=\langle 2x,2y,-2z\rangle$

$\nabla F(0,0,0)=\langle 0,0,0\rangle$

$F_x(x_0^2,y_0^2,z_0^2)=2x_0$ $\quad F_y(x_0,y_0,z_0)=2y_0$ $\quad F_z=-2z_0$

$2x_0(x-x_0)+2y_0(y-y_0)-2z_0(z-z_0)=0$

$x_0x-x_0^2+y_0y-y_0^2-z_0z+z_0^2=0 \Rightarrow x_0x+y_0y-z_0z=x_0^2+y_0^2-z_0^2$

$x_0x+y_0y-z_0z=0$

if $x=0, y=0, z=0$ then $0=0$ which satifies the statement that every tangent plane to the cone contains $(0,0,0)$

$\therefore$ QEP

14.7 homework 1, 5, 9, 13, 17, 21, 31, 33, 35, 37, 41, 45, 49, 53.

1. a. $f_{xx}(1,1)=4$ $f_{xy}(1,1)=1$ $f_{yy}(1,1)=2$

$D(1,1)=4(2)-[1]^2=5$ $D(1,1)>0$ $f_{xx}(1,1)>0$ $\boxed{f(1,1)=\text{local min}}$

b. $f_{xx}(1,1)=4$ $f_{xx}(1,1)=3$ $f_{yy}(1,1)=2$

$D(1,1)=4(2)-3^2=-1$ $D<0$ $f(1,1)\neq$ local min or max

$\boxed{\text{Saddle point at } f(1,1)}$

5. $f(x,y)=x^2+xy+y^2+y$

$f_x(x,y)=2x+y$ $f_y(x,y)=x+2y+1$

$f_{xx}(x,y)=2$ $f_{xy}(x,y)=1$ $f_{yy}(x,y)=2$

$D(x,y)=4-1=3$

$0=2x+y$ $0=x+2y+1$

$y=-2x$ $0=x-4x+1$

$x=\frac{1}{3}$ $y=-\frac{2}{3}$

$\boxed{f(\frac{1}{3},-\frac{2}{3})=-\frac{1}{3} = \text{local minimum}}$

(9) $f(x,y) = x^2 + y^4 + 2xy$

$f_x(x,y) = 2x + 2y$ $\quad$ $f_y = 4y^3 + 2x$

$f_{xx}(x,y) = 2$ $\quad$ $f_{xy}(x,y) = 2$ $\quad$ $f_{yy} = 12y^2$

$D = 24y^2 - 4$ $\quad$ $2x + 2y = 4y^3 + \cancel{2x}$ $\quad$ $y = 0$ $\quad$ $(0,0)$

$D(0) = -4 < 0 \rightarrow$ saddle point $\quad$ $y = 2y^3$

$D\left(-\frac{\sqrt{2}}{2}\right) = 12 - 4 = 8$ min $\quad$ $\frac{1}{2} = y^2$

$D\left(\frac{\sqrt{2}}{2}\right) = 12 - 4 = 8$ min $\quad$ $y = -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$

$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ $\quad$ $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$f\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = -\frac{1}{4}$

$f\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = -\frac{1}{4}$

$\boxed{\text{Saddlepoint at } (0,0) \quad \text{minima at } f\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = f\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = -\frac{1}{4}}$

(13) $f(x,y) = x^4 - 2x^2 + y^3 - 3y$

$f_x(x,y) = 4x^3 - 4x$ $\quad$ $f_y(x,y) = 3y^2 - 3$ $\quad$ $0 = 3y^2 - 3$ $\quad$ $y = \pm 1$

$f_{xx}(x,y) = 12x^2 - 4$ $\quad$ $f_{xy} = 0$ $\quad$ $f_{yy}(x,y) = 6y$ $\quad$ $0 = x(x+1)(x-1)$

$x = 0 \quad x = -1 \quad x = 1$

$D = (12x^2 - 4)(6y)$

$(0,-1) \quad (0,1)$

$(-1,-1) \quad (-1,1)$

$(1,-1) \quad (1,1)$

$D(0,-1) = (-4)(-6) = 24$ max $\quad$ $f(0,-1) = 2$

$D(0,1) = (-4)(6) = -24$ saddle $\quad$ $f(0,1) = -2$

$D(-1,-1) = (8)(-6) = -24$ saddle $\quad$ $f(-1,-1) = 1$

$D(-1,1) = (8)(6) = 24$ min $\quad$ $f(-1,1) = -3$

$D(1,-1) = (8)(-6) = -24$ saddle $\quad$ $f(1,-1) = 1$

$D(1,1) = (8)(6) = 24$ min $\quad$ $f(1,1) = -3$

$$\boxed{\begin{array}{l} \text{max } f(0,-1) = 2 \\ \text{saddle } (0,1) \\ \text{saddle } (-1,-1) \\ \text{min } f(-1,1) = -3 \\ \text{saddle } (1,-1) \\ \text{min } f(1,1) = -3 \end{array}}$$

(17) $f(x,y) = xy + e^{-xy}$

$f_x(x,y) = y - ye^{-xy}$ $\quad$ $f_y(x,y) = x - xe^{-xy}$

$f_{xx}(x,y) = y^2 e^{-xy}$ $\quad$ $f_{xy}(x,y) = 1 - e^{-xy} + xye^{-xy}$ $\quad$ $f_{yy}(x,y) = x^2 e^{-xy}$

$0 = y - ye^{-xy}$ $\quad$ $0 = x - xe^{-xy}$ $\quad$ $1 = e^{-xy}$ $\quad$ $x = 0 \,||\, y = 0$

$y = ye^{-xy}$ $\quad$ $x = xe^{-xy}$

$D = f_{xx} f_{yy} - (f_{xy})^2 = (y^2 e^{-xy})(x^2 e^{-xy}) - (xye^{-xy} - e^{-xy} + 1)^2$

$x = 0$ $\quad$ $(y^2 e^0)(0) - (0 - e^0 + 1)^2 = 0$ $\quad$ $D = 0$ when $x \,||\, y = 0$

$f(0,0) = xy + e^{-xy} = 1$ $\quad$ $x = y = \pm 1$ $\quad$ $f(x,y) = xy + e^{-xy} = 1 + e^{-1}$

$x = 1$ $\quad y = -1$ $\quad f(x,y)$ $\quad f(x,y) = xy + e^{-xy} = 1 + e$

$x = -1$ $\quad y = 1$ $\quad f(x,y) = xy + e^{-xy} = 1 + e$

Local minimum on $f(x,y) = 1$ at all points along x and y axes

(21) $f(x,y) = x^2 + 4y^2 - 4xy + 2$

$f_x(x,y) = 2x - 4y$ $\quad$ $f_y(x,y) = 8y - 4x$ $\quad$ $0 = 2x - 4y$

$x = 2y$

$f_{xx}(x,y) = 2$ $\quad$ $f_{xy}(x,y) = -4$ $\quad$ $f_{yy}(x,y) = 8$ $\quad$ $8y = -4x$

$y = \frac{1}{2}x$

$2x - 4y = 0 \Rightarrow 0$ $\quad -4x + 8y = 0$

$-4x + 8y = 0$ $\quad -4x + 8y = 0$

$0 + 0 = 0$

$0 = 0$ $\quad$ Infinite solutions $\rightarrow$ infinite critical points

$D = f_{xx} \cdot f_{yy} - (f_{xy})^2 = 2(8) - (-4)^2 = 16 - 16 = 0$ $\quad$ $D = 0$ No info

$f_{xx} = 2 > 0 \rightarrow$ local minimum

Because $D = 0$ and because $f_x = f_y$ has infinite soln. There is local min at all points

(31) $f(x,y)=x^2+y^2-2x$

$f_x(x,y)=2x-2$ $f_y(x,y)=2y$

$f_{xx}(x,y)=2$ $f_{xy}(x,y)=0$ $f_{yy}(x,y)=2$

$D=4$ $0=2x-2$ $0=2y$

$x=1$ $y=0$

$(1,0)$ $f(1,0)=1-2=-1$ min

$f(2,0)=4-4=0$

$f(0,2)=0+4=4$

$f(0,-2)=0+4=4$

$\boxed{\text{Absolute min} = f(1,0)=-1 \quad \text{Absolute max} = f(0,-2)=f(0,2)=4}$

(33) $f(x,y)=x^2+y^2+x^2y+4$

$f_x(x,y)=2x+2xy$ $f_y(x,y)=2y+x^2$ $D=(2+2y)(2)-(2x)^2$

$f_{xx}(x,y)=2+2y$ $f_{xy}(x,y)=2x$ $f_{yy}(x,y)=2$

$0=2x(1+y)$ $f_y(x,y)=2y+x^2$ $(0,0)$

$x=0$ $y=-1$ $x=0$ $y=0$

$x=-1$ $x=1$ $y=-1$ $y=1$

$f(-1,y)=1+y^2+y+4$ $f(1,y)=y^2+y+5$ $f(x,-1)=x^2+1-x^2+4$ $f(x,1)=x^2+1+x^2+4$

y^2+y+5 $=5$ $=2x^2+5$

$(1,-\frac{1}{2})$

$f'(-1,y)=2y+1$ $f'(x,1)=4x$

$y=-\frac{1}{2}$ $x=0$ $(0,1)$

$(-1,-\frac{1}{2})$

critical points: $(0,0)$, $(-1,-\frac{1}{2})$, $(1,-\frac{1}{2})$, $(0,1)$

Corner values: $(1,1)$, $(-1,1)$, $(1,-1)$, $(-1,-1)$

$f(0,0)=4$ $D(0,0)=4-0=4$ min

$f(0,1)=5$ $f(1,-1)=1+1-1+4=5$ $D(1,1)=8-4=4$ max

$f(-1,-\frac{1}{2})=1+\frac{1}{4}-\frac{1}{2}+4=4.75$ $f(1,1)=1+1+1+4=7$ $f(-1,-1)=5$ $D(-1,1)=8-4=4$ max

$f(1,-\frac{1}{2})=1+\frac{1}{4}$ $f(-1,1)=1+1+1+4=7$

$\boxed{\text{abs max } f(\pm 1,1)=7 \quad \text{abs min } f(0,0)=4}$

35) $f(x,y) = x^2+2y^2-2x-4y+1$ $D = \{(x,y) \mid x \in [0,2] \wedge y \in [0,3]\}$

$f_x = 2x-2$ $f_y = 4y-4$

$f_{xx} = 2$ $f_{xy} = 0$ $f_{yy} = 4$ $D = 8 = f_{xx}f_{yy} - 0 = 2(4)$

$0 = 2x-2$ $0 = 4y-4$
$x = 1$ $y = 1$

critical point $(1,1)$
cornerpoints: $(0,0), (0,3), (2,0), (2,3)$
$(1,0), (2,1), (1,3)$
$(0,1)$

$f(x,0) = x^2-2x+1$
$f'(x,0) = 2x-2$
$0 = 2x-2$ $(1,0)$
$x = 1$

$f(2,y) = 4+2y^2-4-4y+1$
$f'(2,y) = 4y-4$
$y = 1$ $(2,1)$

$f(x,3) = x^2+18-2x-12+1$
$f'(x,3) = 2x-2$
$x = 1$ $(1,3)$

$f(0,y) = 2y^2-4y+1$
$f'(0,y) = 4y-4$ $(0,1)$
$y = 1$

$f(1,1) = 1+2-2-4+1 = -2$
$f(0,0) = 1$
$f(0,3) = 0+18+0-12+1 = 7$

$f(2,0) = 4-4+1 = 1$
$f(2,3) = 4+18-4-12+1 = 7$
$f(1,0) = 1-2+1 = 0$

$f(2,1) = 4+2-4-4+1 = -1$
$f(1,3) = 1+18-2-12+1 = 6$
$f(0,1) = 2-4+1 = -1$

$\boxed{\begin{array}{l} \text{minimum} = f(1,1) = -2 \\ \text{maximum} = f(0,3) = f(2,3) = 7 \end{array}}$

$D(1,1) = 8 > 0$
$D(0,3) = 8 > 0$
$D(2,3) = 8 > 0$

(37) $f(x,y) = 2x^3 + y^4$ $\quad D = \{(x,y) \mid x^2+y^2 \le 1\}$ $\quad x \in [-1,1]$, $y \in [-1,1]$

$f_x = 6x^2 \qquad f_y = 4y^3$

$f_{xx} = 12x \quad f_{xy} = 0 \quad f_{yy} = 12y^2 \qquad D = 144xy^2$

$0 = 6x^2 \qquad 0 = 4y^3 \qquad$ critical point: $(0,0)$

$x = 0 \qquad y = 0$

$f(x,-1) = 2x^3+1 \quad f'(x,-1) = 6x^2 \quad x=0 \quad (0,-1) \quad f(0,0) = 0$

$f(x,1) = 2x^3+1 \quad f'(x,1) = 6x^2 \quad x=0 \quad (0,1) \quad f(0,-1) = 1$

$f(-1,y) = -2+y^4 \quad f'(-1,y) = 4y^3 \quad y=0 \quad (-1,0) \quad f(0,1) = 1$

$f(1,y) = 2+y^4 \quad f'(1,y) = 4y^3 \quad y=0 \quad (1,0) \quad f(-1,0) = -2$

$f(1,0) = 2$

$D(-1,0) = 0 \qquad D(1,0) = 0$

$$\boxed{\begin{array}{l}\text{Minimum } f(-1,0) = -2 \\ \text{Maximum } f(1,0) = 2\end{array}}$$

(41) $(2,0,-3) \quad x+y+z=1 \quad z = 1-x-y \qquad \left|\frac{1(2)+0-3-1}{\sqrt{1^2+1^2+1^2}}\right| = \frac{2}{\sqrt{3}}$

$d = \sqrt{(x-2)^2+y^2+(z+3)^2} = \sqrt{(x-2)^2+y^2+(4-x-y)^2}$

$d_x(x,y) = \dfrac{2x+y-6}{\sqrt{(x-2)^2+y^2+(z+3)^2}} \qquad d_y(x,y) = \dfrac{x+2y-4}{\sqrt{(x-2)^2+y^2+(z+3)^2}}$

$0 = 2x+y-6 \qquad 0 = x+2y-4$

$2x+y = 6 \Rightarrow \quad 2x+y = 6$

$x+2y = 4 \qquad -2x-4y = -8$

$-3y = -2$

$2x + \frac{2}{3} = \frac{18}{3} \qquad y = \frac{2}{3}$

$x = \frac{8}{3} \qquad \left(\frac{8}{3}, \frac{2}{3}\right)$

$d = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2}$

$= \sqrt{\frac{8}{9} + \frac{4}{9}} = \sqrt{\frac{12}{9}} = \sqrt{\frac{4}{3}}$

$= \boxed{\frac{2}{\sqrt{3}} = \text{shortest dist.}}$

45) $a+b+c=100$ $c=100-a-b$

$P(a,b,c)=abc$ $P(a,b)=ab(100-a-b)$

$P(a,b)=100ab-a^2b-ab^2$

$\frac{\partial P}{\partial b}=100a-a^2-2ab$ $\frac{\partial P}{\partial a}=100b-2ab-b^2$

$P_{aa}(a,b)=-2b$ $P_{bb}(a,b)=-2a$ $P_{ab}(a,b)=100-2a-2b$

$0=(100-a-2b)a$ $0=(100-2a-b)b$

$0=b$

$b=100-2a$

$f_b=100a-a^2-2ab=0$

$b=0$ $f_b=100a-a^2=0$

$a=0$ $a=100$

$b=100-2a$ $f_b=3a^2-100a=0$ $a=0$ $a=\frac{100}{3}$ $b=\frac{100}{3}$

Critical points: $(0,0)$, $(100,0)$, $(0,100)$, $\left(\frac{100}{3},\frac{100}{3}\right)$

$D(0,0)=-2(0)\times-2(0)-[100-0-0]^2=-10000$

$D(0,100)=-2(0)\times-2(100)-[100-2(100)-0]^2=-10000$

$D(100,0)=-10000$

$D\left(\frac{100}{3},\frac{100}{3}\right)=-2\left(\frac{100}{3}\right)\times\frac{-200}{3}-\left[100-\frac{200}{3}-\frac{200}{3}\right]^2=\frac{200}{3}>0$

$f_{xx}=\frac{-200}{3}<0$

max

$\frac{100}{3}+\frac{100}{3}+c=100$

$c=\frac{100}{3}$

$\boxed{a=\frac{100}{3}\quad b=\frac{100}{3}\quad c=\frac{100}{3}}$

49. $x+2y+3z=6 \quad x=6-2y-3z$

$V=xyz = (6-2y-3z)yz = 6yz-2y^2z-3yz^2$

$V_y = 6z-4yz-3z^2 \qquad V_z = 6y-2y^2-6yz$

$V_{yy}=-4z \quad V_{yz}=6-4y-6z \quad V_{zz}=-6y$

$0=z(6-4y-3z) \qquad\qquad 0=y(6-2y-6z)$

$z=0 \qquad z=2-\frac{4}{3}y \qquad\qquad y=0 \quad y=3-3z$

$z=0 \quad 6y-2y^2=0 \quad 2y(3-y)=0$

$y=0 \quad y=3$

$z=2-\frac{4}{3}y \qquad (0,1) \quad (3,0)$

$6y-2y^2-12y+8y^2=0 \qquad -6y+6y^2=0$

$y(1-y)=0 \qquad (1,\frac{2}{3})$

Critical pts: $(0,1),(3,0)\,(0,2)\,(1,\frac{2}{3})$

$y=0 \quad y=1$

$(0,2) \quad z=-\frac{4}{3}(1)+2=\frac{2}{3}$

$D(0,1) = -4(0)\cdot-6(1)-[6-4(0)-6]^2=0$

$D(3,0) = 0-[6-12-0]^2=-36$

$D(0,2) = 0-[6-12]^2=-36$

$D(1,\frac{2}{3}) = 24(\frac{2}{3})-[6-4-4]^2 = 16-4=12 \checkmark >0 \quad y=1 \quad z=\frac{2}{3}$

$x=6-2(1)-3(\frac{2}{3}) = 6-2-2=2$

$$\boxed{\begin{array}{c} x=2 \quad y=1 \quad z=\frac{2}{3} \\ V=2(1)(\frac{2}{3})=\frac{4}{3}\end{array}}$$

53) $xyz = 32000$ $S = xy + 2yz + 2xz$

$z = \frac{32000}{xy}$ $S = xy + \frac{64000}{x} + \frac{64000}{y}$

$S_x = y - \frac{64000}{x^2}$ $S_y = x - \frac{64000}{y^2}$

$S_{xx} = 2(64000)x^{-3}$ $S_{xy} = 1$ $S_{yy} = 2(64000)y^{-3}$

$0 = y - \frac{64000}{x^2}$ $x = 64000y^{-2}$

$y = 64000x^{-2}$

$x = 64000(64000x^{-2})^{-2}$

$x = 64000^{-1}(x^4)$

$64000 = x^3$ $y = 64000(40)^{-2}$

$x = 40$ $y = 20$

$y = 64000(64000y^{-2})^{-2}$

$y = 64000^{-1}y^4$

$64000 = y^3$ $x = 20$

$y = 40$

critical points: $(20,40)$, $(40,20)$

$D(20,40) = 2^2(64000)^2(20)^{-3}(40)^{-3} - 1^2 = \frac{4(64000)^2}{(8)(64)10^6} - 1$

$D(40,20) = 2^2(64000)^2(20)^{-3}(40)^{-3} - 1^2 = \frac{4(64)^2(1000)^2}{8(64)10^6} - 1 > 0$

max

$x = 40$ $y = 20$

$(40)(20)z = 32000 \Rightarrow 800z = 32000$

$z = 40$

$x = 40$ $y = 20$
$z = 40$
base: 40cm x 40cm hight = 20 cm

14.8 homework 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25

(1) The red constraint curve $g(x,y)=8$ is roughly between the 30 and 60 rings. The curve $g(x)$ intersects the curve $f(x,y)=60$ on the y axis, so the max is roughly equivalent to 59. Because $g(x,y)$ intersects negative y axis at 30, the min is 30.

(3) $f(x,y)=x^2-y^2$ $\quad x^2+y^2=1=g(x,y)$ $\quad x\in[-1,1]$, $y\in[-1,1]$ $\quad 1=x^2-y^2$ $\quad -1=x^2-y^2$

$f_x=\lambda g_x$ $\quad f_y=\lambda g_y$ $\quad x^2+y^2=1$ $\quad x=\pm 1$, $y=0$ $\quad x=0$, $y=\pm 1$

$2x=\lambda 2x$ $\quad -2y=\lambda 2y$

$\lambda=1$ $\quad \lambda=-1$

$\boxed{\begin{array}{l}\text{maximum } f(\pm 1,0)=1\\ \text{minimum } f(0,\pm 1)=-1\end{array}}$

(5) $f(x,y)=xy$ $\quad 4x^2+y^2=8=g(x,y)$

$f_x=\lambda g_x$ $\quad f_y=\lambda g_y$ $\quad y=8x\lambda$ $\quad y=\pm 2x$

$y=\lambda 8x$ $\quad x=\lambda 2y$ $\quad =\pm 2x$ $\quad y=\pm 2(\pm 1)=\pm 2$

$x=x^2 16x$ $\quad 4x^2+y^2=8$ $\quad (1,\pm 2)$

$x-x^2 16x=0$ $\quad 4x^2+4x^2=8$ $\quad (-1,\pm 2)$

$x(1-16x^2)=0$ $\quad 8x^2=8$

$x=0$ $\quad x=\sqrt{\frac{1}{16}}=\pm\frac{1}{4}$ $\quad x=\pm 1$

$f(0,y)=y^2=8$ $\quad y=\pm\sqrt{8}$ $\quad (0,\pm\sqrt{8})$

$f(0,-\sqrt{8})=0$
$f(0,\sqrt{8})=0$
$f(-1,-2)=2$
$f(-1,2)=-2$
$f(1,-2)=-2$
$f(1,2)=2$

$\boxed{\begin{array}{l}\max: f(1,2)=f(-1,-2)=2\\ \min: f(-1,2)=f(1,-2)=-2\end{array}}$

(7) $f(x,y,z) = 2x + 2y + z \qquad x^2+y^2+z^2 = 9$

$\langle 2, 2, 1 \rangle = \lambda \langle 2x, 2y, 2z \rangle$

$2 = \lambda 2x \qquad 2 = \lambda 2y \qquad 1 = \lambda 2z \qquad x^2 + x^2 + \frac{x^2}{4} = 9$

$\lambda = \frac{1}{x} \qquad 2 = \frac{2y}{x} \qquad 1 = \frac{2z}{x} \qquad \frac{9}{4}x^2 = 9$

$2x = 2y \qquad x = 2z \qquad x^2 = 4$

$x = y \qquad z = \frac{1}{2}x \qquad x = \pm 2$

Points: $(-2,-2,-1)$, $(-2,-2,1)$, $(2,-2,-1)$, $(2,-2,1)$ $\quad y = x = \pm 2$

$(-2,2,-1)$, $(-2,2,1)$, $(2,2,-1)$, $(2,2,1)$ $\quad z = \frac{1}{2}x = \pm 1$

$f(-2,-2,-1) = -9 \qquad f(-2,-2,1) = -8 \qquad f(2,-2,-1) = -1 \qquad f(2,-2,1) = 1$

$f(-2,2,-1) = -1 \qquad f(-2,2,1) = 1 \qquad f(2,2,-1) = 8 \qquad f(2,2,1) = 9$

$$\boxed{\text{minimum: } f(-2,-2,-1) = -9 \qquad \text{maximum: } f(2,2,1) = 9}$$

(9) $f(x,y,z) = xy^2z \qquad x^2+y^2+z^2 = 4$

$\langle y^2z, 2xyz, xy^2 \rangle = \lambda \langle 2x, 2y, 2z \rangle$

$y^2z = \lambda 2x \qquad 2xyz = 2y\lambda \qquad xy^2 = \lambda 2z$

$\frac{y^2z}{2x} = \lambda \qquad 2xyz = \frac{2y^3z}{2x} \qquad xy^2 = \frac{2z^2y^2}{2x}$

$2x^2 = y^2 \qquad z^2 = x^2$

$x^2 + 2x^2 + x^2 = 4$

$4x^2 = 4$

$x = \pm 1$

$y^2 = 2(\pm 1)^2 \quad y = \pm\sqrt{2}$

$z = \sqrt{(\pm 1)^2} = \pm 1$

Points: $(-1,-\sqrt{2},-1)$, $(-1,-\sqrt{2},1)$, $(-1,\sqrt{2},-1)$, $(-1,\sqrt{2},1)$, $(1,-\sqrt{2},-1)$, $(1,-\sqrt{2},1)$, $(1,\sqrt{2},-1)$, $(1,\sqrt{2},1)$

$f(-1,-\sqrt{2},-1) = 2 \qquad f(-1,\sqrt{2},1) = -2 \qquad f(1,\sqrt{2},-1) = -2$

$f(-1,-\sqrt{2},1) = -2 \qquad f(1,-\sqrt{2},-1) = -2 \qquad f(1,\sqrt{2},1) = 2$

$f(-1,\sqrt{2},-1) = 2 \qquad f(1,-\sqrt{2},1) = 2$

$$\boxed{\begin{aligned} &\text{max } f(1,\pm\sqrt{2},1) = 2 = f(-1,\pm\sqrt{2},-1) = 2 \\ &\text{min: } f(1,\pm\sqrt{2},-1) = -2 = f(-1,\pm\sqrt{2},1) = -2 \end{aligned}}$$

(11.) $f(x,y,z) = x^2+y^2+z^2$ $\quad x^4+y^4+z^4=1$ $\quad 3x^4=1$

$\langle 2x, 2y, 2z\rangle = \lambda\langle 4x^3, 4y^3, 4z^3\rangle$ $\quad x = \sqrt[4]{\frac{1}{3}}$

$2x = \lambda 4x^3$ $\quad 2y = \lambda 4y^3$ $\quad 2z = \lambda 4z^3$ $\quad y = \sqrt[4]{\frac{1}{3}}$

$\lambda = \frac{1}{2x^2}$ $\quad \lambda = \frac{1}{2y^2}$ $\quad \lambda = \frac{1}{2z^2}$ $\quad z = \sqrt[4]{\frac{1}{3}}$

$f\left(\sqrt[4]{\frac{1}{3}}, \sqrt[4]{\frac{1}{3}}, \sqrt[4]{\frac{1}{3}}\right) = 3\left(\sqrt[4]{\frac{1}{3}}\right)^2$

$\frac{1}{x^2} = \frac{1}{y^2} = \frac{1}{z^2}$

$= 3\sqrt{\frac{1}{3}} = \sqrt{3}$

$\frac{1}{x} = \frac{1}{y} = \frac{1}{z}$ $\quad$ If $x=0 \wedge y=0$ $z=\pm 1$

$x=y=z$ $\quad f(0,0,\pm 1) = 0^2+0^2+(\pm 1)^2 = 1$

$\boxed{\text{Maximum! } f\left(\sqrt[4]{\frac{1}{3}}, \sqrt[4]{\frac{1}{3}}, \sqrt[4]{\frac{1}{3}}\right) = \sqrt{3}}$

minimum
$f(0,0,\pm 1) = 1$
$f(0,\pm 1,0) = 1$
$f(\pm 1,0,0) = 1$

(13.) $f(x,y,z,t) = x+y+z+t$ $\quad x^2+y^2+z^2+t^2=1$

$\langle 1,1,1,1\rangle = \lambda\langle 2x, 2y, 2z, 2t\rangle$ $\quad 4x^2=1$

$1 = \lambda 2x$ $\quad 1 = \lambda 2y$ $\quad 1 = \lambda 2z$ $\quad 1 = \lambda 2t$ $\quad x^2 = \frac{1}{4}$

$\lambda = \frac{1}{2x}$ $\quad y = x$ $\quad z = x$ $\quad t = x$ $\quad x = \pm\frac{1}{2}$ $\quad y = \pm\frac{1}{2}$ $\quad z = \pm\frac{1}{2}$

$t = \pm\frac{1}{2}$

$f\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) = -2$ $\quad f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = 2$

$f\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) = f\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) = f\left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) = f\left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) = -1$

$f\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = f\left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) = f\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) = f\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) =$
$= f\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) = f\left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) = 0$

$f\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) = f\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) = f\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = 1$

Minimum! $f\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) = -2$ $\quad$ maximum $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = 2$

(15) $f(x,y)=x^2+y^2 \quad xy=1$

$\langle 2x, 2y \rangle = \lambda \langle y, x \rangle$

$2x = \lambda y \qquad 2y = \lambda x$

$\lambda = \frac{2x}{y} \qquad 2y = \frac{2x^2}{y}$

$2x^2 = 2y^2 \quad y = \pm x$

$x^2 = y^2$

$x^2=1 \quad x=\pm 1 \quad y=\pm 1$

$-x^2=1 \quad (1,1) \quad (-1,-1)$

$f(1,1)=2$

$f(-1,-1)=2$

No max $f(x,y)=x^2+y^2$
$xy=1$ if $x=x$ $y=\frac{1}{x}$
and if $x<-1 \wedge x>1$,
Then new max will be
achieved. $f(1,1)=f(-1,-1)$
$=2$ is minimum
with constraints

(17) $f(x,y,z) = x+y+z \quad x^2+z^2=2 \quad x+y=1$

$\langle 1,1,1 \rangle = \lambda \langle 2x, 0, 2z \rangle + \mu \langle 1,1,0 \rangle$

$1 = \lambda 2x + \mu \qquad 1 = \mu \qquad 1 = \lambda 2z \qquad x^2+z^2=2 \qquad x+y=1$

$0 = \lambda 2x$

$x=0$

$z = \pm\sqrt{2} \quad y=1$

$f(0,1,\sqrt{2}) = 1+\sqrt{2}$

$f(0,1,-\sqrt{2}) = 1-\sqrt{2}$

Maximum: $f(0,1,\sqrt{2}) = 1+\sqrt{2}$
Minimum: $f(0,1,-\sqrt{2}) = 1-\sqrt{2}$

(19.) $f(x,y,z) = yz + xy \quad xy = 1 \quad y^2+z^2=1$

$\langle y, x+z, y\rangle = \lambda\langle y, x, 0\rangle + \mu\langle 0, 2y, 2z\rangle$

$y = \lambda y \quad x+z = \lambda x + 2\mu y \quad y = 2\mu z \qquad xy = 1 \quad y^2+z^2=1$

$\lambda = 1 \quad x+z = x + 2\mu y \qquad y = \pm z$

$z = 2\mu(2\mu z) \qquad 2z^2 = 1$

$0 = (4\mu^2 - 1)z \qquad z = \pm\frac{\sqrt{2}}{2}$

$z = 0 \quad \mu = \pm\frac{1}{2} \qquad y = \pm\frac{\sqrt{2}}{2}$

$x = \pm\sqrt{2}$

Points: $(-\sqrt{2}, -\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}) \quad (\sqrt{2}, \frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2})$

$f(-\sqrt{2}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) = \frac{1}{2} + 1$

$f(-\sqrt{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = -\frac{1}{2} + 1$

$f(\sqrt{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) = -\frac{1}{2} + 1$

$f(\sqrt{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = \frac{1}{2} + 1$

Maximum $f(-\sqrt{2}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) = \frac{3}{2}$

$= f(\sqrt{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = \frac{3}{2}$

Minimum $f(-\sqrt{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = f(\sqrt{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) = \frac{1}{2}$

21. $f(x,y) = x^2 + y^2 + 4x - 4y \quad x^2 + y^2 \le 9$ $\quad x \in [-3,3]$, $y \in [-3,3]$

$\langle 2x+4, 2y-4 \rangle = \lambda \langle 2x, 2y \rangle$

$2x+4 = \lambda 2x \qquad 2y-4 = \lambda 2y \qquad x^2+y^2 \le 9$

$\lambda = 1 + \frac{2}{x} \qquad 2y-4 = 2y + \frac{4y}{x} \qquad 2x^2 \le 9$

$2x+4=0 \quad x=-2$
$2y-4=0 \quad y=2 \quad (-2,2) \qquad -4x = 4y \qquad x^2 \le \frac{9}{2}$

$y = -x \qquad x \le \frac{3}{\sqrt{2}} \quad x \le \frac{-3}{\sqrt{2}}$

$f\left(\frac{3}{\sqrt{2}}, \frac{-3}{\sqrt{2}}\right) = 9 + \frac{24}{\sqrt{2}} = 9 + 12\sqrt{2} \qquad \left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) \quad \left(\frac{-3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$

$f\left(\frac{-3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = 9 - \frac{24}{\sqrt{2}}$

$f(-2,2) = 4+4-8-8 = -8$

$$\boxed{\begin{array}{l} \text{Maximum } f\left(\frac{3}{\sqrt{2}}, \frac{-3}{\sqrt{2}}\right) = 9 + 12\sqrt{2} \\ \text{Minimum } f(-2,2) = -8 \end{array}}$$

23. $f(x,y) = e^{-xy} \quad x^2 + 4y^2 \le 1$

$\langle -ye^{-xy}, -xe^{-xy} \rangle = \lambda \langle 2x, 8y \rangle \qquad 8y^2 \le 1$

$y \le \pm\sqrt{\frac{1}{8}} = \pm \frac{1}{2\sqrt{2}}$

$-ye^{-xy} = \lambda 2x \qquad -xe^{-xy} = \lambda 8y$

$\lambda = \frac{-ye^{-xy}}{2x} \qquad -xe^{-xy} = -4y^2 \frac{e^{-xy}}{x}$

Points: $\left(\frac{-1}{\sqrt{2}}, \frac{\pm 1}{2\sqrt{2}}\right)$, $\left(\frac{-1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, \frac{-1}{2\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$

$f\left(\frac{-1}{\sqrt{2}}, \frac{-1}{2\sqrt{2}}\right) = e^{-(-1/\sqrt{2})(-1/8)} = e^{-1/4} \qquad -x^2 e^{-xy} = -4y^2 e^{-xy}$

$f\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) = e^{-1/4} \qquad x^2 = 4y^2$

$f\left(-\frac{1}{\sqrt{2}}, \frac{1}{8}\right) = e^{\frac{1}{\sqrt{2}}\left(\frac{1}{8}\right)} = e^{1/4} \qquad x^2 = \frac{1}{2}$

$f\left(\frac{1}{\sqrt{2}}, \frac{-1}{8}\right) = e^{1/4} \qquad x = \pm \frac{1}{\sqrt{2}}$

$$\boxed{\text{minimum: } f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2\sqrt{2}}\right) = e^{-1/4} \quad \text{maximum } f\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{2\sqrt{2}}\right) = e^{1/4}}$$

(25.) $f(x,y) = x$ $\quad g(x,y) = y^2 + x^4 - x^3 = 0$

$\langle 1, 0 \rangle = \lambda \langle 4x^3 - 3x^2, 2y \rangle$

$1 = \lambda(4x^3 - 3x^2)$ $\quad 0 = \lambda 2y$ $\quad y^2 + x^4 - x^3 = 0$

$\lambda \neq 0$ $\quad y = 0$

$x^4 - x^3 = 0$

$x^3(x-1) = 0$

$\cancel{x = 0}, x = 1$

a. The only extreme value is $f(1,0) = 1$ $\quad x = 1$

b. $\nabla f = \lambda \nabla g$ $\quad g(x,y) = 0$

$1 = \lambda(4x^3 - 3x^2)$ $\quad 0 = 2\lambda y$ $\quad x^4 - x^3 + y^2 = 0$

Since $\nabla f(x,y) = (1,0)$ and $\nabla g(x,y) = (4x^3 - 3x^2, 2y)$ at $(x,y) = (0,0)$ we get

$$(1,0) = \lambda(0,0)$$

Such that λ doesn't exist. $f(0,0) = 0$ is the minimum, but lagrange condition isn't satisfied

c. Lagrange Multiplier method requires that $\nabla g \neq 0$ everywhere on the constraint curve, but $g(0,0) = 0$. (the point $(0,0)$ belongs to the constraint curve). The Lagrange multiplier isn't satisfied because $\nabla g(0,0) = 0$ on curve $g(x,y) = 0$.