Machine Learning: Chenhao Tan

University of Colorado Boulder

LECTURE 10

Slides adapted from Jordan Boyd-Graber, Chris Ketelsen

Logistics

- HW2 available on Github, due in 7 days

Learning objectives

- Understand stochastic gradient descent

Outline

Stochastic Gradient Descent

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Stochastic Gradient Descent

Review of Wednesday's lecture

Objective function:

$$\begin{aligned}\mathscr{L} &= -\sum_i \log P(y^{(i)} \mid \boldsymbol{x}^{(i)}, \beta)) + \frac{1}{2}\lambda \sum_j \beta_j^2 \\ &= \sum_i -y^{(i)} \left(\beta_0 + \sum_j \beta_i \boldsymbol{x}_j^{(i)}\right) + \log \left[1 + \exp\left(\beta_0 + \sum_j \beta_i \boldsymbol{x}_j^{(i)}\right)\right] + \frac{1}{2}\lambda \sum_j \beta_j^2\end{aligned}$$

Gradient descent:

$$\beta_j^{l+1} = \beta_j^l - \eta \frac{\partial \mathscr{L}}{\partial \beta_j}$$

Gradient:

$$\frac{\partial \mathscr{L}}{\partial \beta_j} = \sum_i [-(y_i - \pi_i)x_j] + \lambda \beta_j$$

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- Our datasets are big (to fit into memory)
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- Average over all observations

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- Average over all observations
- What if we compute an update just from one observation?

Getting to Union Station

Pretend it's a pre-smartphone world and you want to get to Union Station

Stochastic Gradient for Regularized Regression

$$\mathscr{L} = -\log p(y \,|\, \boldsymbol{x}; \beta) + \frac{1}{2}\lambda \sum_j \beta_j^2 \tag{2}$$

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Taking the derivative (with respect to example x_i)

$$\frac{\partial \mathscr{L}}{\partial \beta_j} = -(y_i - \pi_i)x_{ij} + \lambda \beta_j \tag{3}$$

Stochastic Gradient for Logistic Regression

Given a **single observation** x_i chosen at random from the dataset,

$$\beta_j \leftarrow \beta_j' - \eta \left(\lambda \beta_j' - x_{ij} \left[y_i - \pi_i \right] \right) \tag{4}$$

Example Documents

$$\beta_j = \beta_j + \eta(y_i - \pi_i)x_{ij}$$
$$\vec{\beta} = \langle \beta_{bias} = 0, \beta_A = 0, \beta_B = 0, \beta_C = 0, \beta_D = 0 \rangle$$

$y_1 = 1$	$y_2 = 0$
A A A A B B B C	B C C C D D D D

(Assume step size $\eta = 1.0$.)

You first see the positive example. First, compute π_1

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Algorithm

1. Initialize a vector β to be all zeros
2. For $t = 1, \dots, T$
   - For each example $\boldsymbol{x}_i, y_i$ and feature j:
     - Compute $\pi_i \equiv \text{Pr}(y_i = 1 \mid \boldsymbol{x}_i)$
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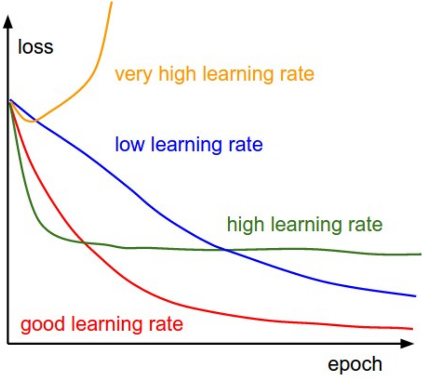
How to decide η?

Choosing learning rate



http://cs231n.github.io/neural-networks-3/

Learning rate decay

- Decay after each epoch (e.g., $\frac{\eta_0}{t^2}$, $\eta_0 e^{-kt}$)
- Decay after each example (e.g., $\frac{\eta_0}{1+kn}$)

Decay schedule can be seen as a hyperparameter too.

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Advanced stochastic gradient descent:
`http://ruder.io/optimizing-gradient-descent/`