Machine Learning: Chenhao Tan

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LECTURE 8

Slides adapted from Jordan Boyd-Graber, Chris Ketelsen

Logistics

- HW2 available on Github, due in 11 days
- First social time at the end of this lecture

Learning objectives

- Understand some standard feature engineering techniques
- Understand probabilistic classification
- Understand logistic regression

Outline

Feature engineering techniques

Probabilistic classification

Logistic regression

- Logistic Regression Example

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What about *redundant* features ϕ_j and $\phi_{j'}$ such that $\phi_j \approx \phi_{j'}$?

Technique: Feature Pruning

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Generalization: if a feature has variance (in D) **lower** than some threshhold value, remove it.

$$\text{sample\_mean}(\phi; D) = \frac{1}{N}\sum_{n=1}^{N}\phi(x_n) \qquad \text{(call it "}\bar{\phi}\text{")}$$

$$\text{sample\_variance}(\phi; D) = \frac{1}{N-1}\sum_{n=1}^{N}\left(\phi(x_n) - \bar{\phi}\right)^2 \qquad \text{(call it "Var}(\phi)\text{")}$$

Technique: Feature Normalization

Center a feature:

$$\phi(x) \rightarrow \phi(x) - \bar{\phi}$$

(This was a required step for principal components analysis!)

Scale a feature. Two choices:

$$\phi(x) \rightarrow \frac{\phi(x)}{\sqrt{\text{Var}(\phi)}} \qquad \text{"variance scaling"}$$

$$\phi(x) \rightarrow \frac{\phi(x)}{\max_n |\phi(x_n)|} \qquad \text{"absolute scaling"}$$

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Remember that you’ll need to normalize test data before you test!

Technique: Example Normalization

We have been talking about normalizing columns.

We can also normalize rows. l_2 normalization is commonly used for bag of words.

$$\boldsymbol{x} = \frac{\boldsymbol{x}}{||\boldsymbol{x}||_2} = \frac{\boldsymbol{x}}{\sqrt{\sum_j \boldsymbol{x}[j]^2}}$$

Techniques: Creating New Features

1. Consider two binary features, ϕ_j and $\phi_{j'}$. A new *conjunction* feature can be defined by:

$$\phi_{j \wedge j'}(x) = \phi_j(x) \wedge \phi_{j'}(x)$$

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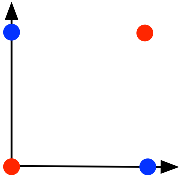

The classic "xor" problem: these points are *not* linearly separable.

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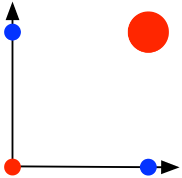

Define $\boldsymbol{x}[3] = \boldsymbol{x}[1] \wedge \boldsymbol{x}[2]$.

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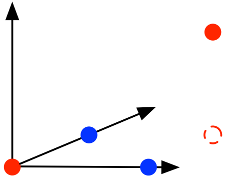

Rotating the view.

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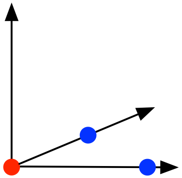

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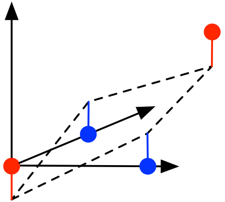

$$2 \cdot \boldsymbol{x}[1] + 2 \cdot \boldsymbol{x}[2] - 4 \cdot \boldsymbol{x}[3] - 1 = 0$$

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Generalization: take the *product* of two features.

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2. Even more generally, we can create conjunctions (or products) using as many features as we'd like.
   This is one view of what decision trees are doing!
   - Every leaf's path (from root) is a conjunction feature.
   - Why not build decision trees, extract the features and toss them into the perceptron?

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2. Even more generally, we can create conjunctions (or products) using as many features as we'd like.
   This is one view of what decision trees are doing!
   - Every leaf's path (from root) is a conjunction feature.
   - Why not build decision trees, extract the features and toss them into the perceptron?
3. Transformations on features can be useful. For example,

$$\phi(x) \rightarrow \text{sign}(\phi(x)) \cdot \log(1 + |\phi(x)|)$$

Techniques: Creating New Features

Remember that adding features does not always bring benefits.

You could be just bring irrelevant, redundant, or features that make linearly separable datasets not linearly separable.

A more realistic but easy example

Given the following data about the locations of two cities, predict whether it is possible to drive between these two cities.

City 1 lat.	City 1 long.	City 2 lat.	City 2 long.	drivable
123.24	46.71	121.33	47.34	Yes
123.24	56.91	121.33	55.23	Yes
123.24	46.71	121.33	55.34	No
123.24	46.71	130.99	47.34	No

Feature engineering

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- Normalization
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In practice, feature engineering requires a deep understanding of the problem.

Outline

Recap

K-nearest neighbor

- Find $\mathcal{N}_K(\boldsymbol{x})$: the set of K training examples nearest to $\boldsymbol{x}$
- Predict $\hat{y}$ to be majority label in $\mathcal{N}_K(\boldsymbol{x})$
- Admits a probabilistic interpretation of class given data: $p(y = c \mid \boldsymbol{x})$

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Perceptron

- Learn weights $\boldsymbol{w}$ and b via the perceptron algorithm
- Predict $\hat{y}$ via $\hat{y} = \text{sign}(\boldsymbol{w} \cdot \boldsymbol{x} + b)$
- Has no probabilistic interpretation

Probabilistic Models

- hypothesis function $h : X \to Y$.

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  In this special case, we define h based on estimating a probabilistic model $P(X, Y)$.

Probabilistic Classification

Input: $S_{\text{train}} = \{(\boldsymbol{x}_i, y_i)\}_{i=1}^N$ training examples

$$y_i \in \{c_1, c_2, \ldots, c_J\}$$

Goal: $h : X \to Y$
For each class c_j, estimate

$$P(y = c_j \mid \boldsymbol{x}, S_{\text{train}})$$

Assign to $\boldsymbol{x}$ the class with the highest probability

$$\hat{y} = h(\boldsymbol{x}) = \arg\max_c P(y = c \mid \boldsymbol{x}, S_{\text{train}})$$

Outline

Feature engineering techniques

Probabilistic classification

Logistic regression

Logistic Regression Example

What are we talking about?

- Probabilistic classification: $p(y|x)$
- Classification uses: ad placement, spam detection
- Building block of other machine learning methods

Logistic Regression: Definition

- Weight vector β_i
- Observations X_i
- "Bias" β_0 (like intercept in linear regression)

$$P(Y = 0|X) = \frac{1}{1 + \exp\left[\beta_0 + \sum_i \beta_i X_i\right]} \tag{1}$$

$$P(Y = 1|X) = \frac{\exp\left[\beta_0 + \sum_i \beta_i X_i\right]}{1 + \exp\left[\beta_0 + \sum_i \beta_i X_i\right]} \tag{2}$$

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What is the decision boundary?

Logistic Regression: Definition

- Weight vector β_i
- Observations X_i
- For shorthand, we'll say that

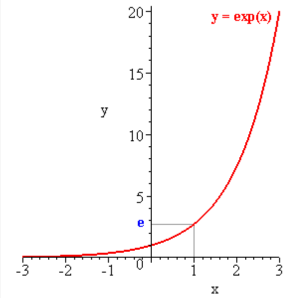
$$P(Y=1|X) = \sigma((\beta_0 + \sum_i \beta_i X_i)) \tag{3}$$

$$P(Y=0|X) = 1 - \sigma((\beta_0 + \sum_i \beta_i X_i)) \tag{4}$$

- Where $\sigma(z) = \frac{1}{1+exp[-z]}$

What's this "exp" doing?

Exponential function



- $\exp[x]$ is shorthand for e^x
- e is a special number, about 2.71828
  - e^x is the limit of compound interest formula as compounds become infinitely small
  - It's the function whose derivative is itself
- The "logistic" function is $\sigma(z) = \frac{1}{1+e^{-z}}$
- Looks like an "S"
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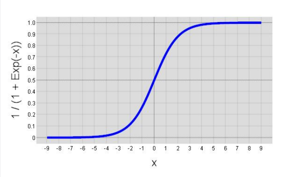

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- Always between 0 and 1.
  - Allows us to model probabilities
  - Different from **linear** regression

Logistic Regression Example

feature	coefficient	weight
bias	β_0	0.1
"viagra"	β_1	2.0
"mother"	β_2	-1.0
"work"	β_3	-0.5
"nigeria"	β_4	3.0

- What does $Y = 1$ mean?

Example 1: Empty Document?

$X = \{\}$

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- $P(Y = 0) = \frac{1}{1+\exp[0.1]} = 0.48$
- $P(Y = 1) = \frac{\exp[0.1]}{1+\exp[0.1]} = 0.52$
- Bias β_0 encodes the prior probability of a class

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- Include bias, and sum the other weights

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- $P(Y=1) = \frac{\exp\left[0.1-1.0+3.0\right]}{1+\exp\left[0.1-1.0+3.0\right]} = 0.89$
- Include bias, and sum the other weights

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$X = \{\text{Mother}, \text{Work}, \text{Viagra}, \text{Mother}\}$

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- Multiply feature presence by weight

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Example 3

$X = \{$Mother, Work, Viagra, Mother$\}$

- $P(Y=0) = \frac{1}{1+\exp\left[0.1-1.0-0.5+2.0-1.0\right]} = 0.60$
- $P(Y=1) = \frac{\exp\left[0.1-1.0-0.5+2.0-1.0\right]}{1+\exp\left[0.1-1.0-0.5+2.0-1.0\right]} = 0.40$
- Multiply feature presence by weight

How is Logistic Regression Used?

- Given a set of weights $\vec{\beta}$, we know how to compute the conditional likelihood $P(y|\beta, x)$
- Find the set of weights $\vec{\beta}$ that maximize the conditional likelihood on training data (next week)
- **Intuition**: higher weights mean that this feature implies that this feature is a good feature for the positive class