Machine Learning: Chenhao Tan

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LECTURE 20

Slides adapted from Jordan Boyd-Graber, Chris Ketelsen

Logistics

- Homework 4 is due on Sunday!
- Prelim next Friday

Learning objects

- Learn about Adaboost
- Understand the math behind boosting

Overview

Recap of Boosting

Adaboost

The underlying math

Outline

Recap of Boosting

Adaboost

The underlying math

Boosting intuition

Boosting is an ensemble method, but with a different twist.
Idea:

- Build a sequence of dumb models
- Modify training data along the way to focus on difficult to classify examples
- Predict based on weighted majority vote of all the models

Challenges:

- What do we mean by dumb?
- How do we promote difficult examples?
- Which models get more say in vote?

Boosting intuition

What do we mean by dumb?
Each model in our sequence will be a weak learner

$$\mathrm{err} = \frac{1}{m}\sum_{i=1}^{m} I(y_i \neq h(\mathbf{x}_i)) = \frac{1}{2} - \gamma, \gamma > 0$$

Most common weak learner in Boosting is a decision stump - a decision tree with a single split

Boosting intuition

How do we promote difficult examples?
After each iteration, we'll increase the importance of training examples that we got wrong on the previous iteration and decrease the importance of examples that we got right on the previous iteration
Each example will carry around a weight w_i that will play into the decision stump and the error estimation
Weights are normalized so they act like a probability distribution

$$\sum_{i=1}^{m} w_i = 1$$

Boosting intuition

Which models get more say in vote?
The models that performed better on training data get more say in the vote
For our sequence of weak learners: $h_1(\mathbf{x}), h_2(\mathbf{x}), \ldots, h_K(\mathbf{x})$
Boosted classifier defined by

$$H(\mathbf{x}) = \text{sign}\left[\sum_{k=1}^{K} \alpha_k h_k(\mathbf{x})\right]$$

Weight α_k is measure of accuracy of h_k on training data

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Adaboost

- Training set $\{(\mathbf{x}_i, y_i)\}_{i=1}^{m}$
- Feature vector $\mathbf{x} \in \mathbb{R}^d$
- Labels are $y_i \in \{-1, 1\}$

Adaboost

1 . Initialize data weights to $w_i = \frac{1}{m}$, $i = 1, \ldots, m$

2 . For $k = 1$ to K:

   (a) Fit classifier $h_k(\mathbf{x})$ to training data with weights w_i

   (b) Compute weighted error $\mathrm{err}_k = \frac{\sum_{i=1}^{m} w_i I(y_i \neq h_k(\mathbf{x}_i))}{\sum_{i=1}^{m} w_i}$

   (c) Compute $\alpha_k = \frac{1}{2} \log((1 - \mathrm{err}_k)/\mathrm{err}_k)$

   (d) Set $w_i \leftarrow \frac{w_i}{Z_k} \cdot \exp[-\alpha_k y_i h_k(\mathbf{x}_i)]$, $i = 1, \ldots, m$

3 . Output $H(\mathbf{x}) = \mathrm{sign}\left[\sum_{k=1}^{K} \alpha_k h_k(\mathbf{x})\right]$

Adaboost

1 . Initialize data weights to $w_i = \frac{1}{m}, i = 1, \ldots, m$
Weights are initialized to uniform distribution. Every training example counts equally on first iteration.

Adaboost

2a. Fit classifier $h_k(\mathbf{x})$ to training data with weights w_i
Decide split based on info gain with weighted entropy:

$$H(D) = -p \log_2 p - (1-p)\log_2(1-p)$$

$$I(D,a) = H(D) - \sum_{v \in vals(a)} \frac{|\{x \in D | x_a = v\}|}{|D|} H(\{x \in D | x_a = v\})$$

$$p = \frac{\sum_{i=1}^{m} w_i \cdot I(y_i = 1)}{\sum_{i=1}^{m} w_i \cdot I(y_i = \pm 1)}$$

Adaboost

2b. Compute weighted error

$$\mathrm{err}_k = \frac{\sum_{i=1}^{m} w_i I(y_i \neq h_k(\mathbf{x}_i))}{\sum_{i=1}^{m} w_i}$$

Adaboost

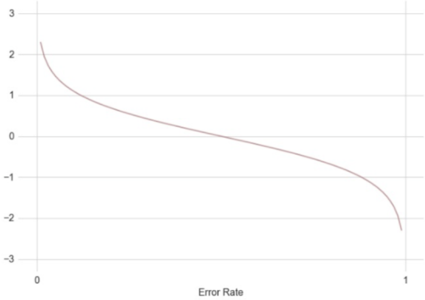
2b. Compute weighted error

$$\text{err}_k = \frac{\sum_{i=1}^{m} w_i I(y_i \neq h_k(\mathbf{x}_i))}{\sum_{i=1}^{m} w_i}$$

Still gives error rate in $[0,\ 1]$

Adaboost

2c. Compute $\alpha_k = \frac{1}{2} \log((1 - \mathrm{err}_k)/\mathrm{err}_k)$
Models with small err get promoted (exponentially)
Models with large err get demoted (exponentially)



Adaboost

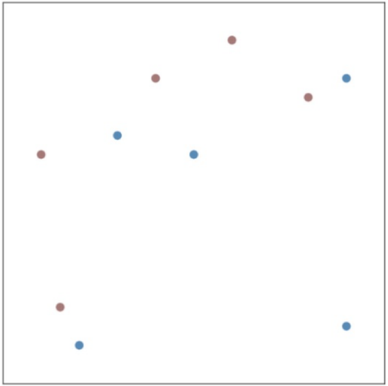
2d. Set $w_i \leftarrow \frac{w_i}{Z_k} \cdot \exp[-\alpha_k y_i h_k(\mathbf{x}_i)]$, $i = 1, \ldots, m$
If example was misclassified weight goes up
If example was classified correctly weight goes down
How big of a jump depends on accuracy of model
Z_k is just a normalizing constant to ensure that the w's are a distribution

Adaboost

3. Output $H(\mathbf{x}) = \text{sign}\left[\sum_{k=1}^{K} \alpha_k h_k(\mathbf{x})\right]$
Sum up weighted votes from each model
Classify $y = 1$ if positive and $y = -1$ if negative

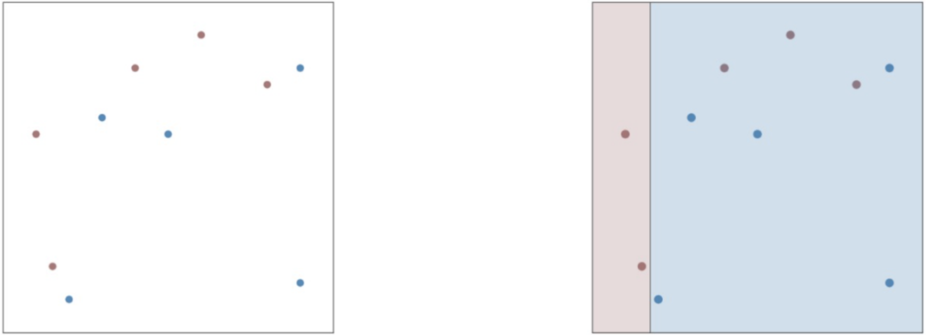
Adaboost example

Suppose we have the following training data



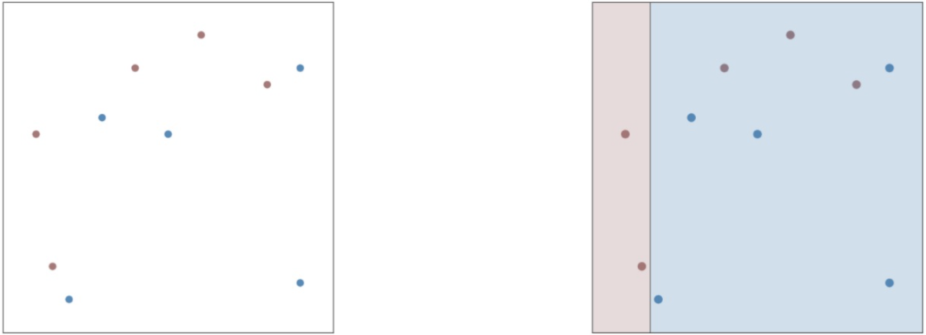
Adaboost example

First decision stump



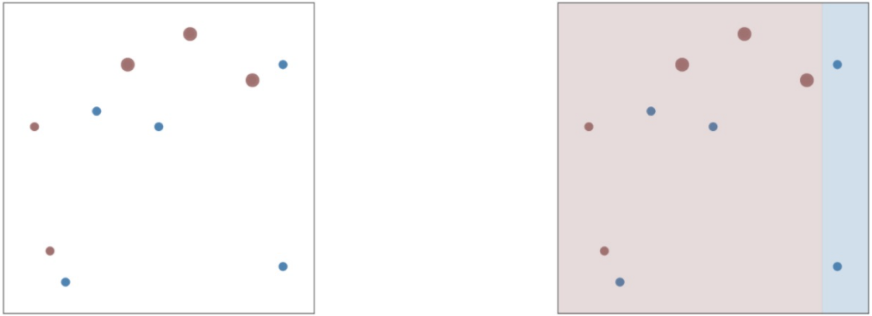
Adaboost example

First decision stump , $err_1 = 0.3, \alpha_1 = 0.42$



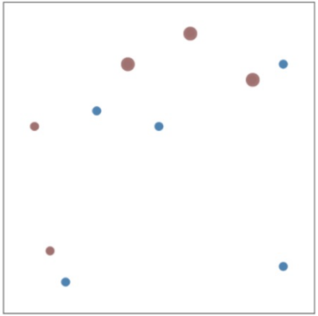
Adaboost example

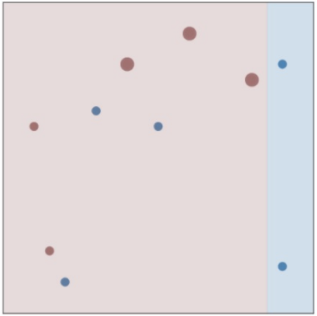
Second decision stump



Adaboost example

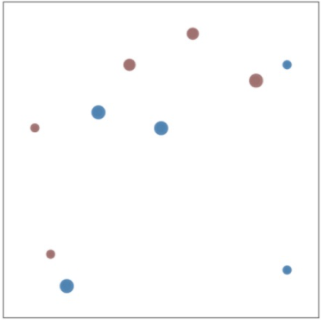
Second decision stump , $err_2 = 0.21, \alpha_1 = 0.65$

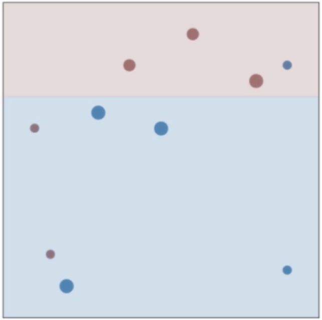




Adaboost example

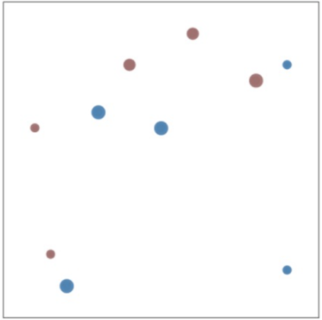
Third decision stump

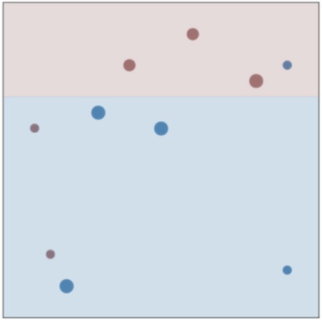




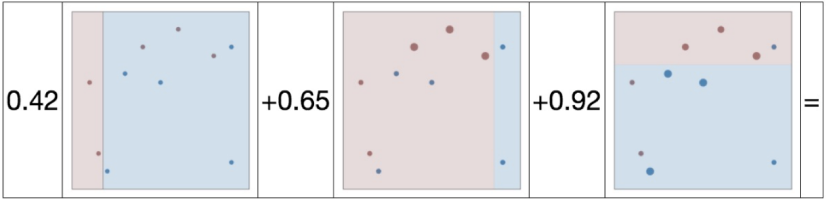
Adaboost example

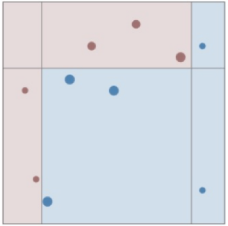
Third decision stump , $err_2 = 0.14, \alpha_1 = 0.92$





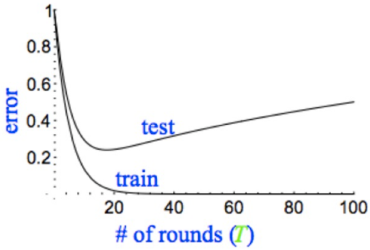
Adaboost example





Generalization performance

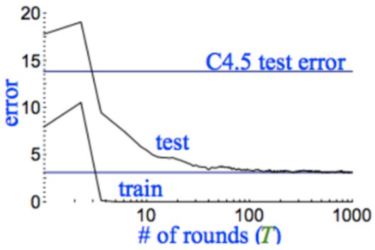
Recall the standard experiment of measuring test and training error vs. model complexity



Once overfitting begins, test error goes up

Generalization performance

Boosting has remarkably uncommon effect



Happens much slower with boosting

Adaboost

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2 . For $k = 1$ to K:

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   (b) Compute weighted error $\mathrm{err}_k = \frac{\sum_{i=1}^m w_i I(y_i \neq h_k(\mathbf{x}_i))}{\sum_{i=1}^m w_i}$

   (c) Compute $\alpha_k = \frac{1}{2}\log((1 - \mathrm{err}_k)/\mathrm{err}_k)$

   (d) Set $w_i \leftarrow \frac{w_i}{Z_k} \cdot \exp[-\alpha_k y_i h_k(\mathbf{x}_i)]$, $i = 1, \ldots, m$

3 . Output $H(\mathbf{x}) = \mathrm{sign}\left[\sum_{k=1}^K \alpha_k h_k(\mathbf{x})\right]$

Quiz

Which of the following statement is false?

A. Adaboost can return the same weak classifier at different K.

B. Adaboost can improve the generalization performance with many rounds.

C. Every misclassified data point has the same multiplicative factor in the update.

D. α_k always grows as k grows.

Outline

Recap of Boosting

Adaboost

The underlying math

The underlying math

So far this looks like a reasonable thing that just worked out
But is there math behind it?

The underlying math

Yep! It is minimization of a loss function, like always
Formulate the problem as finding a classifier $H(\mathbf{x})$ such that

$$H^* = \arg\min_H \sum_{i=1}^{m} L(y_i, H(\mathbf{x}_i))$$

where here we take the loss function L to be the following exponential function

$$L = \exp[-y\, H(\mathbf{x})]$$

Notice if $y \neq H(\mathbf{x})$ we get a positive exponent and a large loss.

The underlying math

Since we're doing this in an iterative way, we're going to assume a form of $H(\mathbf{x})$ that is amenable to iterative improvement. Specifically

$$H(\mathbf{x}) = \sum_{k=1}^{K} \alpha_k \phi_k(\mathbf{x})$$

So the problem becomes to choose the optimal weights, α, and optimal basis functions ϕ_k.
For everything I will assume that we've already computed a good H_{k-1} and attempt to build a better H_k.

The underlying math

At step k we have

$$L_k = \sum_{i=1}^{m} \exp\left[-y_i\left(H_{k-1}(\mathbf{x}_i) + \alpha\phi(\mathbf{x_i})\right)\right]$$

Taking the things we know out of the exponent gives

$$L_k = \sum_{i=1}^{m} w_{i,k} \exp\left[-\alpha y_i \phi(\mathbf{x_i})\right], \quad w_{i,k} = \exp\left[-y_i\, H_{k-1}(\mathbf{x}_i)\right]$$

Want to choose good α and ϕ to reduce loss

The underlying math

Can rewrite as

$$L_k = e^{\alpha} \sum_{y_i \neq \phi(\mathbf{x}_i)} w_{i,k} + e^{-\alpha} \sum_{y_i = \phi(\mathbf{x}_i)} w_{i,k}$$

Add zero in a fancy way

$$L_k = (e^{\alpha} - e^{-\alpha}) \sum_{i=1}^{m} w_{i,k} I(y_i \neq \phi(\mathbf{x}_i)) + e^{-\alpha} \sum_{i=1}^{m} w_{i,k}$$

The underlying math

Choose ϕ and α separately.
A good ϕ would be one that minimizes weighted misclassifications

$$h_k = \arg\min_{\phi} w_{i,k} I(y_i \neq \phi(\mathbf{x}_i))$$

That's what our weak learner is for

The underlying math

Plugging that into our Loss function gives

$$L_k = (e^{\alpha} - e^{-\alpha}) \sum_{i=1}^{m} w_{i,k} I(y_i \neq h_k(\mathbf{x}_i)) + e^{-\alpha} \sum_{i=1}^{m} w_{i,k}$$

Now we want to minimize w.r.t. α. Take derivative, set equal to zero

$$0 = \frac{dL_k}{d\alpha} = (e^{\alpha} + e^{-\alpha}) \sum_{i=1}^{m} w_{i,k} I(y_i \neq h_k(\mathbf{x}_i)) - e^{-\alpha} \sum_{i=1}^{m} w_{i,k}$$

which gives $e^{2\alpha} = \frac{\sum_i w_{i,k}}{\sum_i w_{i,k} I(y_i \neq h_k(\mathbf{x}))} - 1 = \frac{1}{\mathrm{err}_k} - 1 = \frac{1-\mathrm{err}_k}{\mathrm{err}_k}$

And finally $\alpha_k = \frac{1}{2} \log \left(\frac{1-\mathrm{err}_k}{\mathrm{err}_k} \right)$

The underlying math

What about the weight update?
Remember we got the weights by pulling the already computed function out of L.
For the new weights, we have

$$w_{i,k+1} = \exp[-y_i H_k(\mathbf{x}_i)] = \exp[-y_i H_{k-1}(\mathbf{x}_i) - \alpha_k y_i h_k(\mathbf{x}_i)] = w_{i,k} \exp[-\alpha_k y_i h_k(\mathbf{x}_i)]$$

which gives the update

$$w_i \leftarrow w_i \exp[-\alpha_k y_i h_k(\mathbf{x}_i)]$$

And finally, $H_K(\mathbf{x}) = \text{sign}\left[\sum_{k=1}^{K} \alpha_k h_k(\mathbf{x})\right]$
This is exactly Adaboost