HW6

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1 APPM 5370 Homework 6

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In [1]:
```
import numpy as np
from scipy.integrate import odeint
import numpy.random as rnd
import matplotlib.pyplot as plt
%matplotlib inline
# plt.style.use('dark_background')
import math
```

In [2]:
```
horiz_scaling = 3 # scale plots to fill page, so must adjust the phase plane plots as 
height = 5

plt.rcParams['figure.figsize'] = [height*horiz_scaling, height]
```

1.1.1 Problem 4: Training the NOT AND Function With a Two Layer Network

The network will have inputs x_1 and x_2 with a single hidden layer containing two neurons:

$$y_1 = f(w_{11}x_1 + w_{12}x_2 + \theta_1)$$

$$y_2 = f(w_{21}x_1 + w_{22}x_2 + \theta_2)$$

$$z = f(J_1 y_1 + J_2 y_2 + \eta)$$

The loss function is given as:

$$E = (z - z_{\text{targ}})^2$$

We will use a sigmoid activation for $f(x)$. The partial derivatives for each parameter can be found using the chain rule:

$$\frac{\partial E}{\partial w_{11}} = \frac{\partial E}{\partial z}\frac{\partial z}{\partial y_1}\frac{\partial y_1}{\partial w_{11}} = 2(z - z_{targ}) \cdot f'(J_1 y_1 + J_2 y_2 + \eta) \cdot J_1 \cdot f'(w_{11}x_1 + w_{12}x_2 + \theta_1) \cdot x_1$$

$$\frac{\partial E}{\partial w_{12}} = \frac{\partial E}{\partial z}\frac{\partial z}{\partial y_1}\frac{\partial y_1}{\partial w_{12}} = 2(z - z_{targ}) \cdot f'(J_1 y_1 + J_2 y_2 + \eta) \cdot J_1 \cdot f'(w_{11}x_1 + w_{12}x_2 + \theta_1) \cdot x_2$$

$$\frac{\partial E}{\partial w_{21}} = \frac{\partial E}{\partial z}\frac{\partial z}{\partial y_2}\frac{\partial y_2}{\partial w_{21}} = 2(z - z_{targ}) \cdot f'(J_1 y_1 + J_2 y_2 + \eta) \cdot J_2 \cdot f'(w_{21}x_1 + w_{22}x_2 + \theta_1) \cdot x_1$$

$$\frac{\partial E}{\partial w_{22}} = \frac{\partial E}{\partial z}\frac{\partial z}{\partial y_2}\frac{\partial y_2}{\partial w_{22}} = 2(z - z_{targ}) \cdot f'(J_1 y_1 + J_2 y_2 + \eta) \cdot J_2 \cdot f'(w_{21}x_1 + w_{22}x_2 + \theta_1) \cdot x_2$$

$$\frac{\partial E}{\partial J_1} = \frac{\partial E}{\partial z}\frac{\partial z}{\partial J_1} = 2(z - z_{targ}) \cdot f'(J_1 y_1 + J_2 y_2 + \eta) \cdot y_1$$

$$\frac{\partial E}{\partial J_2} = \frac{\partial E}{\partial z}\frac{\partial z}{\partial J_2} = 2(z - z_{targ}) \cdot f'(J_1 y_1 + J_2 y_2 + \eta) \cdot y_2$$

$$\frac{\partial E}{\partial \theta_1} = \frac{\partial E}{\partial z}\frac{\partial z}{\partial y_1}\frac{\partial y_1}{\partial \theta_1} = 2(z - z_{targ}) \cdot f'(J_1 y_1 + J_2 y_2 + \eta) \cdot J_1 \cdot f'(w_{11}x_1 + w_{12}x_2 + \theta_1)$$

$$\frac{\partial E}{\partial \theta_2} = \frac{\partial E}{\partial z}\frac{\partial z}{\partial y_2}\frac{\partial y_2}{\partial \theta_2} = 2(z - z_{targ}) \cdot f'(J_1 y_1 + J_2 y_2 + \eta) \cdot J_1 \cdot f'(w_{21}x_1 + w_{22}x_2 + \theta_2)$$

$$\frac{\partial E}{\partial \theta_3} = \frac{\partial E}{\partial z}\frac{\partial z}{\partial \theta_3} = 2(z - z_{targ})f'(J_1 y_1 + J_2 y_2 + \theta_3)$$

In [3]:
```
def f(x):
    return 1/(1+np.exp(-x))
```

In [4]:
```
def fp(x):
    return np.exp(-x)/(1+np.exp(-x))**2
```

For the "soft" NOT AND function, the following inputs and targets are used:

$$x_1 = x_2 = 0 \rightarrow z_{\text{targ}} = 0.95$$

$$x_1 = x_2 = 1 \rightarrow z_{\text{targ}} = 0.05$$

$$x_1 = 1, x_2 = 0 \rightarrow z_{\text{targ}} = 0.95$$

$$x_1 = 0, x_2 = 1 \rightarrow z_{\text{targ}} = 0.95$$

Part A We will use a learning rate of 0.5, 10000 iterations, and an error threshold of 0.05

In [11]:
```
def z_target(x1, x2):
    if x1 == 1 and x2 == 1:
        return 0.05
    return 0.95
```

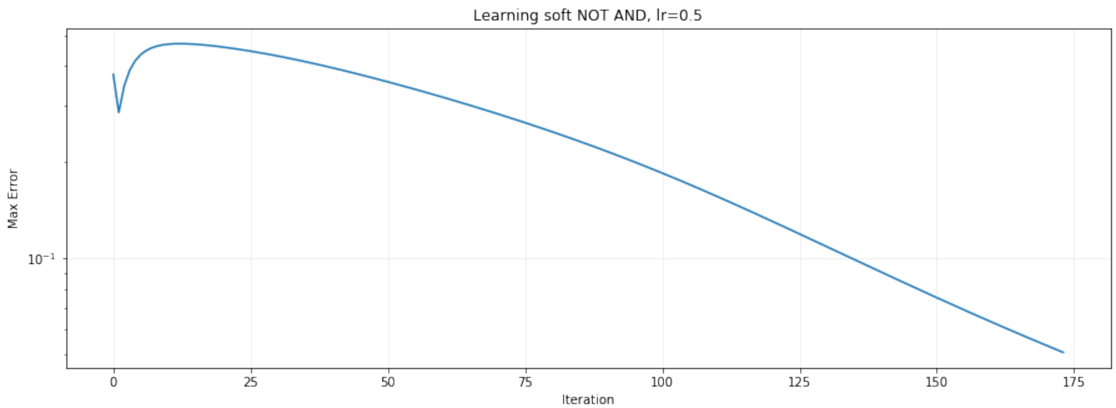
In [12]:
```
def train_net(r, errthresh, maxiter):
    # randomize weights and bias
    w11 = rnd.uniform(-1,1,1)
    w12 = rnd.uniform(-1,1,1)
    w21 = rnd.uniform(-1,1,1)
    w22 = rnd.uniform(-1,1,1)
```

```
J1 = rnd.uniform(-1,1,1)
J2 = rnd.uniform(-1,1,1)
theta1 = rnd.uniform(-1,1,1)
theta2 = rnd.uniform(-1,1,1)
eta = rnd.uniform(-1,1,1)

inputs = [[0,0], [0,1], [1,0], [1,1]]

errvec = []
for i in range(maxiter):
    maxerr = -np.infty
    for x1, x2 in inputs:
        y1 = f(w11*x1 + w12*x2 + theta1)
        y2 = f(w21*x1 + w22*x2 + theta2)
        z = f(J1*y1 + J2*y2 + eta)

        zt = z_target(x1, x2)
        err = (z - zt)**2
        maxerr = max(err, maxerr)

        # backpropagate based on observed error
        w11_new = w11 - r * 2*(z - zt) * fp(J1*y1 + J2*y2 + eta) * J1\
            * fp(w11*x1 + w12*x2 + theta1) * x1

        w12_new = w12 - r * 2*(z - zt) * fp(J1*y1 + J2*y2 + eta) * J1\
            * fp(w11*x1 + w12*x2 + theta1) * x2

        w21_new = w21 - r * 2*(z - zt) * fp(J1*y1 + J2*y2 + eta) * J2\
            * fp(w21*x1 + w22*x2 + theta2) * x1

        w22_new = w22 - r * 2*(z - zt) * fp(J1*y1 + J2*y2 + eta) * J2\
            * fp(w21*x1 + w22*x2 + theta2) * x2

        J1_new = J1 - r * 2*(z - zt) * fp(J1*y1 + J2*y2 + eta) * y1\

        J2_new = J2 - r * 2*(z - zt) * fp(J1*y1 + J2*y2 + eta) * y2

        theta1_new = theta1 - r * 2*(z - zt) * fp(J1*y1 + J2*y2 + eta)\
            * J1 * fp(w11*x1 + w12*x2 + theta1)

        theta2_new = theta2 - r * 2*(z - zt) * fp(J1*y1 + J2*y2 + eta)\
            * J2 * fp(w21*x1 + w22*x2 + theta2)

        eta_new = eta - r * 2*(z - zt) * fp(J1*y1 + J2*y2 + eta)

        # update the weights and biases
        w11 = w11_new
        w12 = w12_new
```

```
                w21 = w21_new
                w22 = w22_new
                J1 = J1_new
                J2 = J2_new
                theta1 = theta1_new
                theta2 = theta2_new
                eta = eta_new


            if maxerr < errthresh:
                break
            errvec.append(maxerr)

        return errvec
```

In [13]:
```
def plot_err(errvec, lr):
    fig = plt.figure()
    plt.grid(True, alpha=0.2)
    plt.semilogy(np.arange(len(errvec)), errvec)
    plt.title('Learning soft NOT AND, lr=%s'%lr)
    plt.xlabel('Iteration')
    plt.ylabel('Max Error')
```

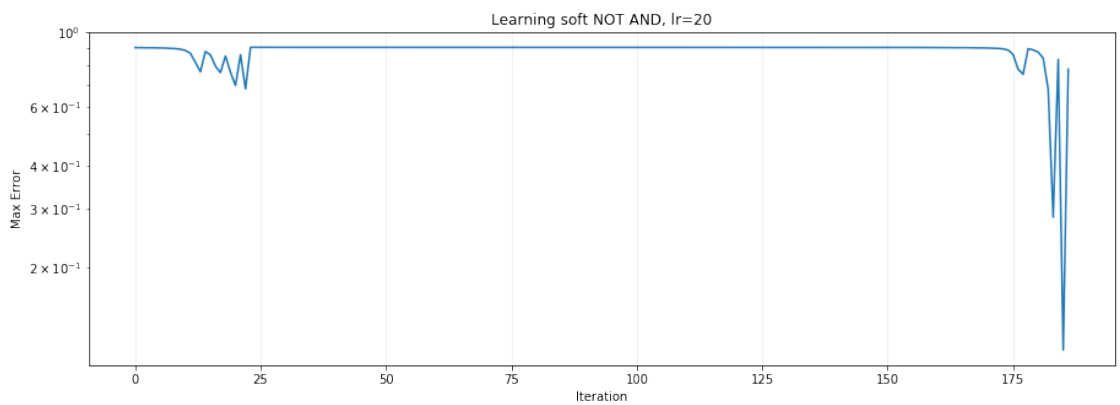
In [14]:
```
# set hyperparams
r = 0.5
errthresh = 0.05
maxiter = 10000

# train and plot
err = train_net(r, errthresh, maxiter)
plot_err(err, r)
```

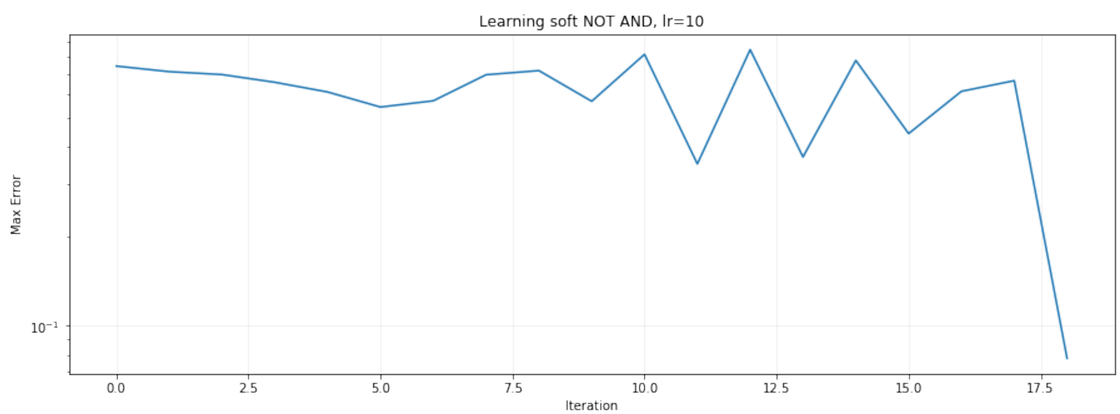


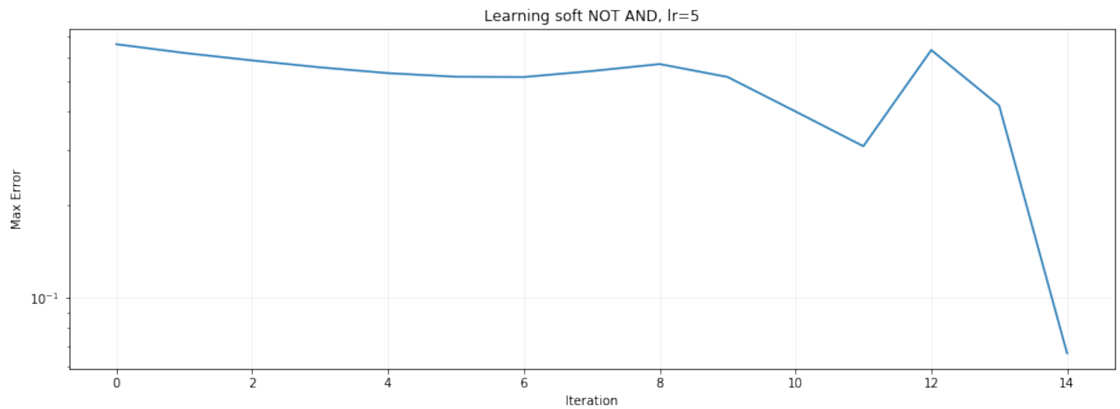
The error doesn't always decrease, but after many trials has a very similar curve each time. The initial increase is an artifact of overshooting or missing the direction to descend in parameter space.

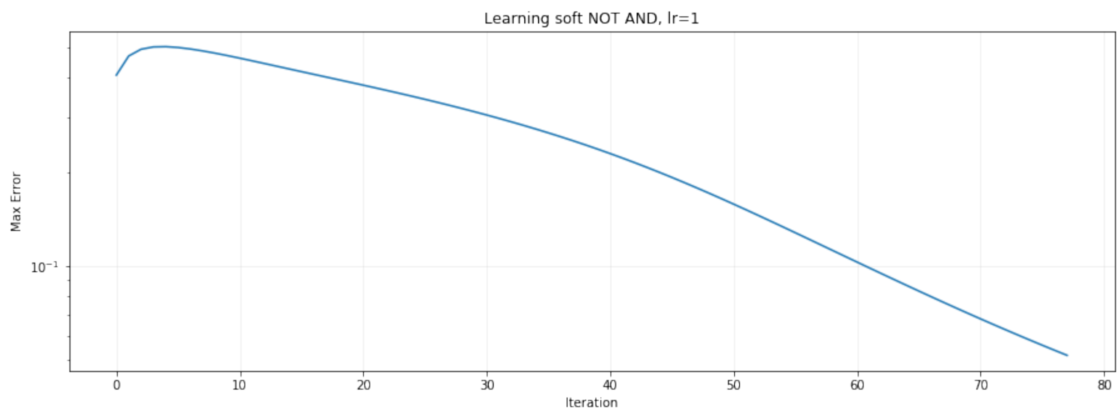
Part B: Tuning Learning Rate Hyperparameter

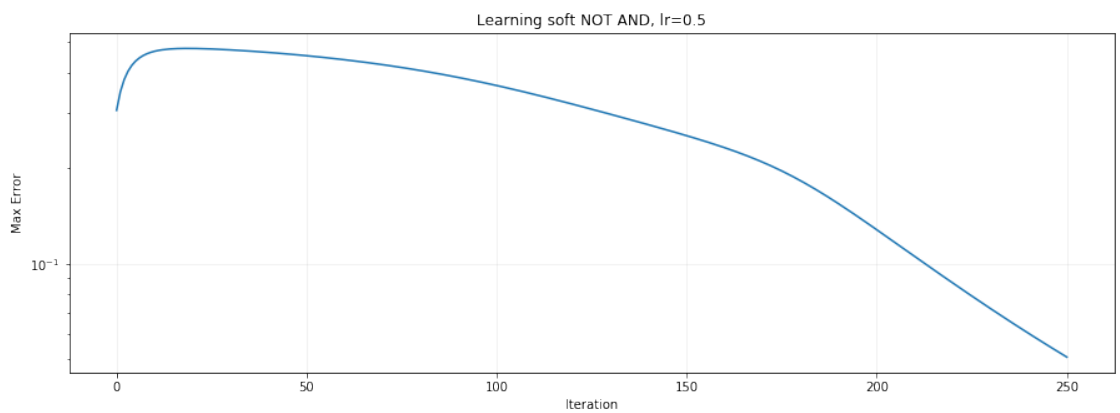
```
In [15]: rs = [20, 10, 5, 1, 0.5]
         errthresh = 0.05
         maxiter = 10000
         avg_num_iterations = []
         for r in rs:
             curr_num_iterations = []
             for i in range(10):
                 err = train_net(r, errthresh, maxiter)
                 curr_num_iterations.append(len(err))

             # just plot last one
             plot_err(err, r)

             # keep track of mean num. iterations to reach error thresh
             avg_num_iterations.append(np.mean(curr_num_iterations))
```












```
In [16]: avg_num_iterations
```

```
Out[16]: [8389.7, 20.6, 19.2, 87.0, 164.8]
```

On average, the learning rate of 5 is optimal. This is most likely due to the fact that a higher learning rate may not reach the error threshold, since it could overshoot the minumim value for the loss and never return. For example, the learning rate of 20 usually never convereged, since the update is so large it wouldn't reach the minimal error value. In order for the network to converge more rapidly, a different activation, such as a rectified linear function could be used, since its slope is greater than the sigmoid's.