Stability (examples)

Tuesday, November 17, 2020 8:52 PM

Recall our 3 notions: (all for $h \to 0$)

1) Consistency — Does the method locally approximate the ODE?

2) Convergence — Does the approximation w_i converge to $y(t_i)$? MAIN GOAL

3) Stability — Links consistency to convergence

This lecture: examples of stability/convergence for multistep methods

Setup IVP: $y' = f(t,y)$, $a \le t \le b$, $y(a) = \alpha$

$w_0 = \alpha$, $w_1 = \alpha_1$, $w_2 = \alpha_2$, $\ldots$, $w_{m-1} = \alpha_{m-1}$] initialization

$$w_{i+1} = a_{m-1} w_i + a_{m-2} w_{i-1} + \cdots + a_0 w_{i+1-m}$$ general linear multistep method

$$+ h \cdot \underbrace{\left(b_m f_{i+1} + b_{m-1} f_i + \cdots + b_0 f_{i+1-m} \right)}_{F(t_i, h, w_{i+1}, w_i, \ldots, w_{i+1-m})}, \quad f_i := f(t_i, w_i)$$

Def The characteristic polynomial $P(\lambda)$ associated w/ the numerical method (**) (it does not depend on what F looks like) is

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - a_{m-3}\lambda^{m-3} - \cdots - a_0$$

$m=3$ example $P(\lambda) = \lambda^3 - a_2\lambda^2 - a_1\lambda - a_0$

Def Root condition and strongly stable and weakly stable

Let $\{\lambda_j\}_{j=1}^m$ be the roots of the characteristic polynomial P associated with a multi-step scheme. Then we say the multi-step method satisfies the root condition if (A) $|\lambda_j| \le 1$ $\forall j$

and (B) $|\lambda_j| = 1 \Rightarrow \lambda_j$ is a simple root

} root condition = zero-stability

and if a method satisfies the root condition, then we call it either

strongly stable if $|\lambda_j| = 1 \Rightarrow \lambda_j = 1$

weakly stable otherwise.

If it doesn't satisfy the root condition, we call it unstable

Thm 5.24 "Equivalence Thm"

i.e. output depends continuously on input data

① A multistep method is stable iff it satisfies the root condition

② If the multistep method is consistent, then it is convergent iff it's stable

New examples

① Forward (explicit) Euler $\quad w_{i+1} = w_i + h f(t_i, w_i)$

Is this a convergent method?

1^{st}, check if it's consistent (if it isn't, then it's not convergent)

i.e., is $\lim_{h\to 0} \tau(h) = 0$? yes, we've already seen $\tau(h) = O(h)$

In extra detail: i.e., is $\displaystyle\lim_{h\to 0} \frac{y(t+h) - \big(y(t) + h f(t, y(t))\big)}{h} = 0$?

$$= \lim_{h\to 0} \frac{y(t+h) - y(t)}{h} - f(t, y(t))$$

$$= y'(t) - f(t, y(t)) = 0 \quad \text{since } y' = f(t, y)$$

(this is our ODE!)

2^{nd}, check is it stable

i.e., check the root condition, so form the characteristic polynomial

As $h \to 0$, so ignoring $h f(t_i, w_i)$, we have $\quad w_{i+1} = w_i$

i.e., $1 w_{i+1} - 1 w_i = 0$

So $\quad P(\lambda) = 1 \cdot \lambda - 1$, which has only one root, $\lambda = 1$

(recall $\lambda = 1$ is always a root if method is consistent)

Quick sanity check: degree $(P) = m$ (# steps)

$m = 1$ since it's a 1-step method

So, it satisfies the root condition

and is in fact strongly stable

3^{rd}, deduce it's convergent via the Dahlquist equivalence theorem

Note: for forward Euler, we already knew this from specialty theorems

(Thm. 5.9 for Euler, or Thm. 5.20 for all 1-step methods)

② Backward (implicit) Euler $\quad w_{i+1} = w_i + h f(t_{i+1}, w_{i+1})$

You can check this is consistent, so then convergence follows either from Thm 5.20 or since we know it has the same characteristic polynomial as forward Euler

③ AB4 (Adams-Bashforth 4th order / 4 step) $\quad f_i := f(t_i, w_i)$

$$w_{i+1} = w_i + \underbrace{h \tfrac{1}{24}\left(55 f_i - 59 f_{i-1} + 37 f_{i-2} - 9 f_{i-3}\right)}_{F(t_i, h, w_{i+1}, w_i, w_{i-1}, w_{i-2}, w_{i-3})}$$

This is consistent since we know it's $\tau(h) = O(h^4)$

Does it satisfy the root condition? TRY ON YOUR OWN!

The characteristic polynomial is $P(\lambda) = \lambda^4 - \lambda^3 = \lambda^3(\lambda - 1)$

(*) $w_{i+1} - w_i = 0$ $\quad \Rightarrow \lambda = 1$

$\lambda = 0$ (w/ multiplicity 3)

root condition ✓

strongly stable ✓

⚠ It's tempting to write

$P(\lambda) = \lambda - 1$ since (*)

is just like it was for Euler's.

In this case it wouldn't have led to a false conclusion, but it's not the right thing. P should always have degree m

(m = # steps = how far back we go)

④ All Adams methods (Adams-Bashforth, Adams-Moulton)

For m-steps, $P(\lambda) = \lambda^m - \lambda^{m-1} = \lambda^{m-1}(\lambda - 1)$

$\Rightarrow \lambda = 1$

$\lambda = 0$ w/ multiplicity $m-1$

$\Rightarrow$ All Adams method satisfy the root condition and are strongly stable hence convergent since they're all consistent

⑤ 4th order Milne's method

$$w_{i+1} = w_{i-3} + \underbrace{h \tfrac{4}{3}\left(2 f_i - f_{i-1} + 2 f_{i-2}\right)}_{F}$$

This is consistent since we know it's 4th order, i.e., $\tau(h) = O(h^4)$ so $\lim_{h \to 0} \tau(h) = 0$

Does it satisfy the root condition?

Think of difference equation

$$w_{i+1} + 0 \cdot w_i + 0 \cdot w_{i-1} + 0 \cdot w_{i-2} - w_{i-3} = 0$$

$P(\lambda) = \lambda^4 - 1 \Rightarrow$ roots are $\lambda = \{ \pm 1, \pm\sqrt{-1} \}$

I won't use "i" here since we use i as an index

All $|\lambda| \leq 1$, no multiple roots

$\Rightarrow$ root condition ✓

but... it's only weakly stable not strongly stable

So, people usually prefer something like AB4 over Milne's

(6) Backward Differentiation

Name	Order	Steps m	a_{m-1}	a_{m-2}	a_{m-3}	a_{m-4}	b_m
BD1 = Backward Euler	1	1	1				1
BD2	2	2	4/3	-1/3			2/3
BD3	3	3	18/11	-9/11	2/11		6/11
BD4	4	4	48/25	-36/25	16/25	-3/25	12/25

All are consistent

BD1, $P(\lambda) = \lambda - 1 \Rightarrow \lambda = 1 \Rightarrow$ strongly stable as we already knew

BD3, $P(\lambda) = \lambda^3 - \frac{18}{11}\lambda^2 + \frac{9}{11}\lambda^1 - 2/11 \Rightarrow \lambda = ?$ How do you solve a cubic?

1. cubic formula (Cardano, 1545, but due to del Ferro, $\underbrace{\text{Tartaglia}}_{\text{tried to keep secret}}$)
2. Mathematica, sympy, etc
3. Tricks

$\rightarrow$ we know $\lambda = 1$ is a root since BD is consistent

wlog work with $11\lambda^3 - 18\lambda^2 + 9\lambda - 2 = 0$

factor out $(\lambda - 1)$

$$\begin{array}{r|l} & 11\lambda^2 - 7\lambda + 2 \\ \hline (\lambda-1) & 11\lambda^3 - 18\lambda^2 + 9\lambda - 2 \\ & 11\lambda^3 - 11\lambda^2 \\ & \quad -7\lambda^2 + 9\lambda \\ & \quad -7\lambda^2 + 7\lambda \\ & \qquad 2\lambda - 2 \\ & \qquad 2\lambda - 2 \\ & \qquad\quad 0 \end{array}$$

so $11\lambda^3 - 18\lambda^2 + 9\lambda - 2 = (\lambda - 1)\underbrace{(11\lambda^2 - 7 + 2)}_{\text{find roots via quadratic formula}}$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{7 \pm \sqrt{49-88}}{22} = \frac{7 \pm \sqrt{-39}}{22}$$

Also, what is $\left| \frac{7 \pm \sqrt{-39}}{22} \right| = \left| 7/22 \pm \sqrt{39}/22 \, i \right|$

$i = \sqrt{-1}$ for now (not an index)

$$= \sqrt{7^2/22^2 + 39/22^2}$$

$$= \sqrt{\frac{49+39}{22^2}} = \sqrt{\frac{88}{22 \cdot 22}}$$

$$= \sqrt{\frac{8}{2 \cdot 22}}$$

$$= \sqrt{\frac{2}{11}}$$

$$= \frac{\sqrt{22}}{11} \approx 0.426 < 1$$

So, roots are $\lambda \in \left\{ 1, \frac{7 \pm \sqrt{-39}}{22} \right\}$ and is strongly stable

It turns out BD1, BD2, ..., BD6 are strongly stable

BD7 and higher are unstable which is why you never see them!

(7) LIAF (this is the name given to it by Driscoll + Braun... it doesn't really have a name because no one uses it!)

$$w_{i+1} = -4w_i + 5w_{i-1} + h(4f_i + 2f_{i-1})$$

is 3rd order accurate, so consistent. It has just $m=2$ steps yet 3rd order... nice! Better order of accuracy than Adams or BD (to get this better accuracy, it uses all available a_i, b_i terms)

But... 1st Dahlquist stability barrier:

$$\text{order of accuracy} \le \begin{cases} m+2 & \text{if } m \text{ even} \\ m+1 & \text{if } m \text{ odd} \\ m & \text{if explicit} \end{cases} \quad \text{if the method is stable}$$

$\Rightarrow$ LIAF can't be stable.

Verify: $P(\lambda) = \lambda^2 + 4\lambda - 5 = (\lambda - 1)(\lambda + 5)$

$$\Rightarrow \lambda \in \{1, -5\}$$

$\Rightarrow$ UNSTABLE