Homework 5
APPM/MATH 4650 Fall '20 Numerical Analysis

Due date: Saturday, October 10, before midnight, via Gradescope. **Instructor**: Prof. Becker
Theme: Finite differences

Instructions Collaboration with your fellow students is OK and in fact recommended, although direct copying is not allowed. The internet is allowed for basic tasks (e.g., looking up definitions on wikipedia) but it is not permissible to search for proofs or to *post* requests for help on forums such as `http://math.stackexchange.com/` or to look at solution manuals. Please write down the names of the students that you worked with.

An arbitrary subset of these questions will be graded.

Turn in a PDF (either scanned handwritten work, or typed, or a combination of both) to **Gradescope**, using the link to Gradescope from our Canvas page. Gradescope recommends a few apps for scanning from your phone; see the Gradescope HW submission guide.

We will primarily grade your written work, and computer source code is *not* necessary except for when we *explicitly* ask for it (and you can use any language you want). If not specifically requested as part of a problem, you may include it at the end of your homework if you wish (sometimes the graders might look at it, but not always; it will be a bit easier to give partial credit if you include your code).

Problem 1: **3-point finite difference formulas for unequispaced grid** Consider a 3-point finite difference formula using the grid values $\{x, x+h, x+3h\}$. We'll assume that the function f we apply the formula to is in $C^3([x-1, x+1])$.

a) Using these nodes, our finite difference formula to approximate $f'(x)$ has the form

$$\frac{Af(x) + Bf(x+h) + Cf(x+3h)}{h}.$$

Using Taylor expansion, determine the values of A, B and C that are needed to make this an $O(h^2)$ accurate approximation of $f'(x)$. *Note:* If you need to solve a linear system, you may do this via a computer, but your final answers should be exact (like 1/3 not 0.3333).

b) Repeat the above exercise but this time find the coefficients A, B and C by constructing the Lagrange interpolating polynomial $p(x)$ and differentiating p at x. *Hint:* to simplify the derivation, you can assume without loss of generality that $x = 0$, since only the *spacing* between the nodes matters.

c) If we want an $O(h^2)$ approximation, are the choices for A, B and C unique? If we only want our formula to be an $O(h)$ approximation, now are the choices for A, B and C unique?

d) Now let's use these nodes to make a finite difference formula to approximate $f''(x)$, using the form

$$\frac{Af(x) + Bf(x+h) + Cf(x+3h)}{h^2}.$$

Using Taylor expansion, find the values of A, B and C to approximate $f''(x)$ such that the error in the approximation goes to 0 as $h \to 0$.

e) Repeat the above exercise but this time find the coefficients A, B and C by constructing the Lagrange interpolating polynomial $p(x)$ and differentiating p twice at x.

Problem 2: Let $f(x) = e^{3x}$ and $x = 0.3$. Approximate $f'(0.3)$ to within 10^{-12} absolute error using one of the finite difference rules in Table 1. Report which rule you used and what stepsize h. *Optional but recommended:* plot the errors for each method as a function of the stepsize.

Name	Order	Node location						
		$-3h$	$-2h$	$-h$	0	h	$2h$	$3h$
2-pt forward diff.	1	0	0	0	-1	1	0	0
3-pt forward diff.	2	0	0	0	$-3/2$	$4/2$	$-1/2$	0
4-pt forward diff.	3	0	0	0	$-11/6$	$18/6$	$-9/6$	$2/6$
3-pt centered diff.	2	0	0	$-1/2$	0	$1/2$	0	0
5-pt centered diff.	4	0	$1/12$	$-8/12$	0	$8/12$	$-1/12$	0
7-pt centered diff.	6	$-1/60$	$9/60$	$-45/60$	0	$45/60$	$-9/60$	1

Table 1: Forward and centered difference formula to approximate the first derivative

Problem 3: Consider the 3-pt centered difference formula $f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$ which is $O(h^2)$. Suppose we numerically calculate $f(x+h)$ with error bounded by ϵ, and similarly for $f(x-h)$, and suppose the truncation error is bounded by Mh^2. In particular, we are supposing we can numerically compute $\frac{f(x+h)-f(x-h)}{2h}$ up to an error $2\epsilon/h$ [we assume $h > 0$, otherwise we would write this as $2\epsilon/|h|$), and that $|f'(x) - \frac{f(x+h)-f(x-h)}{2h}| \le Mh^2$. That is, the output of our numerical implementation of the formula, which we'll call δ, is

$$|\delta - f'(x)| \le 2\epsilon/h + Mh^2.$$

Find the value of h that minimizes this error bound, as well as reporting the resulting error bound.

Problem 4: Misc. questions

a) Consider a finite difference formula that has truncation error $C \cdot h^n \cdot f^{(n+1)}(\xi)$ for some constant C and some $\xi \in [x, x+h]$ where x is the point of interest. What can you say about the truncation error of this finite difference formula if the function f is a polynomial f of degree n or less?

b) If we used the 3-point centered difference formula to approximate the derivative of a quadratic function, what kind of stepsize h should we use? (large? small? medium?)

c) Denote the 3-point centered difference formula as $N_1(h)$. Apply Richardson extrapolation to N_1 to get $N_2(h)$. What is the order of accuracy of N_2? Does $N_2(2h)$ remind you of another formula?