APPM 4650 | HW 6 | Soroush Khadem

1.) $\int_a^b f(x)dx$ using $x_0=a,\ x_1=a+\frac{1}{3}(b-a),\ x_2=b$

let $h=(b-a)/3$, $x_0=0$. Then:

$x_0=0,\ x_1=h,\ x_2=3h$

$$I=\int_a^b f(x)dx \approx \int_a^b P_n(x) = \int_a^b \sum_{i=0}^{n} f(x_i)L_{n,i}(x)$$
$$=\sum_{i=0}^{n} f(x_i)\int_a^b L_{n,i}(x)dx$$

For this problem, $= f(0)\int_0^{3h} L_0(x)dx + f(h)\int_0^{3h} L_1(x)dx$
$+f(3h)\int_0^{3h} L_2(x)dx$

$$L_0=\frac{1}{3h^2}(x^2-4hx+3h^2) \to \int_0^{3h}\frac{1}{3h^2}(x^2-4hx+3h^2)dx$$
$$=\frac{1}{3h^2}\left(\frac{x^3}{3}-2hx^2+3h^2x\right)\Big]_0^{3h}$$
$$=1/3h^2\left(\frac{27h^3}{3}-18h^3+9h^3\right)$$
$$=1/3h^2(0)=0$$

$$L_1=\frac{-1}{2h^2}(x^2-3hx) \to \int_0^{3h}\frac{-1}{2h^2}(x^2-3hx)dx=\frac{-1}{2h}\left(\frac{x^3}{3}-\frac{3}{2}hx^2\right)\Big]_0^{3h}$$
$$=\frac{-1}{2h^2}\left(9h^3-\frac{27}{2}h^3\right)=\frac{9}{4}h$$

$$L_2=\frac{1}{6h^2}(x^2-hx) \to \int_0^{3h}1/6h^2(x^2-hx)dx=\frac{1}{6h^2}\left(\frac{x^3}{3}-\frac{h}{2}x^2\right)\Big]_0^{3h}$$
$$=\frac{1}{6h^2}\left(9h^3-\frac{9h^3}{2}\right)=\frac{9}{(2)(6)}h=\frac{3}{4}h$$

$$I=0+\frac{9}{4}hf(h)+\frac{3}{4}hf(3h)$$

Numerically, this scheme can calculate quadratic integral exactly

hw6

October 17, 2020

```
In [1]: import numpy as np
        import matplotlib.pyplot as plt
        plt.style.use('seaborn-bright')
        plt.rcParams['figure.figsize'] = [15, 5]
        plt.rcParams['axes.grid'] = True
        plt.rcParams['grid.alpha'] = 0.25
```

```
In [2]: def midpoint(f, a, b, n):
            h = (b - a) / (n + 1)
            nodes = np.arange(a + h/2, b, step=h)
            return h*np.sum(f(nodes))

        def trapezoidal(f, a, b, n):
          nodes, h = np.linspace(a,b,n+1,retstep=True)
          y  = f(nodes)
          return h*(np.sum(y) - (y[0]+y[-1])/2)

        def simpson(f, a, b, n):
          n = int(n/2)*2
          nodes, h = np.linspace(a, b, n+1, retstep=True)
          y = f(nodes)
          return h/3*( y[0] + 2*np.sum(y[2:n:2]) + 4*np.sum(y[1:n:2]) + y[-1] )
```

0.1 Problem 2

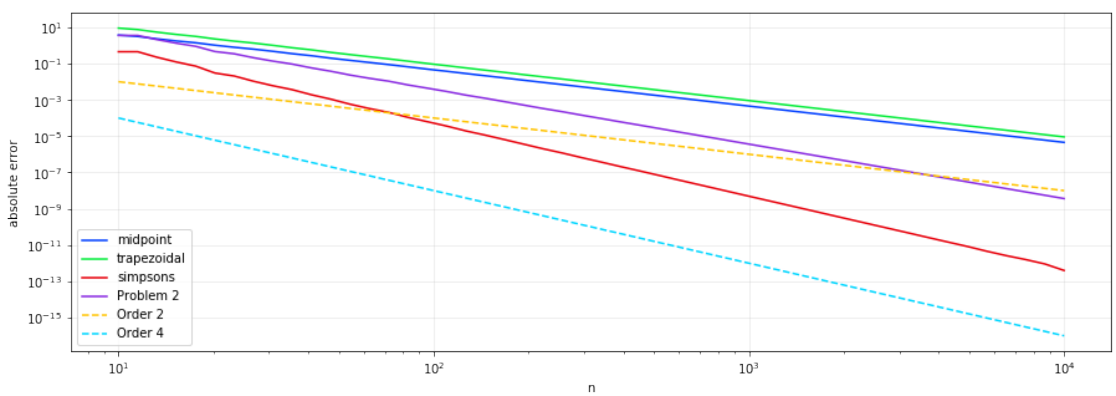
```
In [3]: def composite_prob1(f, a, b, n):
            n = int(n/2)*2
            nodes, h = np.linspace(a, b, num=int(3*n / 2) + 1, retstep=True)
            nodes = list(nodes)
            del nodes[2::3]
            nodes = np.array(nodes)
            y = f(nodes)
            return (h/4)*((9)*np.sum(y[1::2]) + (3)*np.sum(y[2::2]))
```

```
In [4]: f = lambda x : np.exp(3*x)
        F = lambda x : 1/3*np.exp(3*x)
        a = -1
        b = 2
        I = F(b) - F(a)
```

```
In [5]: ns = np.logspace(1, 4)
        midpoint_error = []
        trapezoidal_error = []
        simpson_error = []
        prob1_error = []

        for n in ns:
            n = int(n)
            midpoint_error.append(np.abs(I - midpoint(f, a, b, n)))
            trapezoidal_error.append(np.abs(I - trapezoidal(f, a, b, n)))
            simpson_error.append(np.abs(I - simpson(f, a, b, n)))
            prob1_error.append(np.abs(I - composite_prob1(f, a, b, n)))

        plt.plot(ns, midpoint_error)
        plt.plot(ns, trapezoidal_error)
        plt.plot(ns, simpson_error)
        plt.plot(ns, prob1_error)
        plt.plot(ns, 1/ns**2, '--')
        plt.plot(ns, 1/ns**4, '--')
        plt.legend(['midpoint', 'trapezoidal', 'simpsons', 'Problem 2', 'Order 2', 'Order 4'])
        plt.xscale('log')
        plt.yscale('log')
        plt.xlabel('n')
        plt.ylabel('absolute error')
        plt.show()
```



Based on the plot, the new method seems to be about second order. It performs better than midpoint and trapezoidal, but not as well as Simpson's.

0.2 Problem 3

```
In [6]: def create_table(f, a, b, nRows):
            n = 1
            table = []
```

```
    for i in range(nRows):
      I = midpoint(f, a, b, n)
      newRow = [I]
      if i > 0:
        oldRow = table[-1]
      for j in range(i):
        qj = 4**(j+1)
        newRow.append(  (qj*newRow[-1] - oldRow[j])/( qj-1) )
      table.append(newRow)
      n *= 2
    return table
```

```
In [7]: f = lambda x : np.cos(x)
        F = lambda x : np.sin(x)
        a = 0
        b = 50
        I = F(b) - F(a)

        table = create_table(f, a, b, 12)
        for row in table:
          for entry in row:
            print("{:.2e}  ".format(abs(entry-I)),end="") # suppress new line
          print("") # new line
```

```
4.97e+01
2.20e+00  1.95e+01
1.63e+00  2.91e+00  4.40e+00
1.79e+00  2.92e+00  3.31e+00  3.44e+00
1.25e-01  4.28e-01  6.52e-01  7.14e-01  7.31e-01
2.69e-02  5.96e-03  2.22e-02  3.29e-02  3.58e-02  3.66e-02
6.58e-03  1.86e-04  1.99e-04  1.50e-04  2.79e-04  3.14e-04  3.23e-04
1.65e-03  5.39e-06  1.81e-05  1.52e-05  1.59e-05  1.62e-05  1.63e-05  1.63e-05
4.14e-04  2.47e-06  2.28e-06  2.02e-06  1.97e-06  1.96e-06  1.95e-06  1.95e-06  1.95e-06
1.04e-04  4.24e-07  2.88e-07  2.56e-07  2.49e-07  2.48e-07  2.47e-07  2.47e-07  2.47e-07  2.47e
2.60e-05  6.04e-08  3.61e-08  3.21e-08  3.13e-08  3.11e-08  3.10e-08  3.10e-08  3.10e-08  3.10e
6.51e-06  8.01e-09  4.52e-09  4.02e-09  3.91e-09  3.88e-09  3.88e-09  3.88e-09  3.88e-09  3.88e
```

Based on the table values, there seems to be an adavantage to using composite trapezoidal rule.

0.3 Problem 4

```
In [8]: f = lambda x : np.exp(-x*1j)/(1 + np.cos(x)/2)
        a = -np.pi
        b = np.pi
        I = 2 - (4 / np.sqrt(3))
        I *= (np.pi * 2)
```

4.) $\hat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx$ ← Fourier coeffs.

$\cdot\ F_k = \frac{1}{n}\sum_{j=-n/2+1}^{n/2} f_j e^{-i2\pi jk/n}$, $k = -\frac{n}{2}+1, \dots 0, \dots n/2 - 1, n/2$

↳ DFT. $f_j = f(j \cdot h)$, $h = 2\pi/n$

a.) Assume $f(-\pi) = f(\pi)$, $|k| \leq n/2$.

Let $g(x) = f(x) e^{-ikx}$: $g(\pi) = f(\pi)(\cos(\pi k) - \underbrace{i\sin(\pi k)}_{0,\, k \in \mathbb{Z}})$

$g(-\pi) = f(\pi)(\cos(\pi k) + \underbrace{i\sin(\pi k)}_{0,\, k \in \mathbb{Z}})$

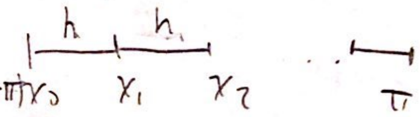
Since $g(\pi) = g(-\pi)$, trapezoidal quadrature simplifies

to:

$$\hat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx \approx \frac{h}{2\pi}\sum_{j=0}^{n-1} f(x_j) e^{-ikx_j}.$$

Reindex to: $-n/2+1$ to $(n-1)+(-n/2+1) = n/2$!

$\frac{h}{2\pi}\sum_{j=-n/2+1}^{n/2} f(x_j) e^{-ikx_j}$. Note that $x_j = x_0 + j\cdot h = -\pi + j \cdot h$.



$\frac{h}{2\pi}\sum_{j=-n/2+1}^{n/2} f(-\pi + j\cdot h) e^{-ik(-\pi + j\cdot h)}$. Since f is 2π periodic,

$f(\pi + x) = -f(x)$, same for e^{-ikx}. So, can add π to

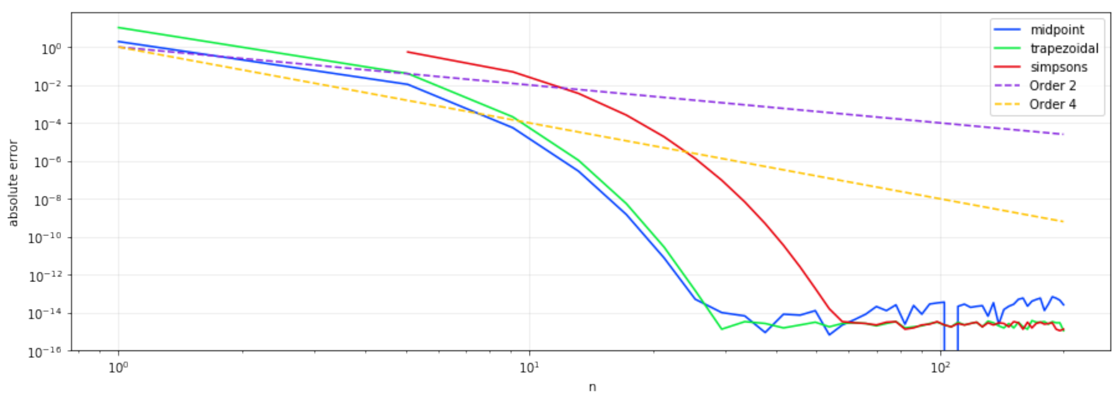
both : $\frac{h}{2\pi}\sum_{j=-n/2+1}^{n/2} -f(jh)\cdot -e^{-ikjh} = \frac{h}{2\pi}\sum_{-n/2+1}^{n/2} f(jh)\cdot e^{-ikjh}$

Since $h = 2\pi/n$: $\frac{1}{n}\sum_{-n/2+1}^{n/2} f_j e^{-ikj \cdot 2\pi/n}$

```
In [11]: ns = np.linspace(1, 200)
         midpoint_error = []
         trapezoidal_error = []
         simpson_error = []

         for n in ns:
             n = int(n)
             midpoint_error.append(np.abs(I - np.real(midpoint(f, a, b, n))))
             trapezoidal_error.append(np.abs(I - np.real(trapezoidal(f, a, b, n))))
             simpson_error.append(np.abs(I - np.real(simpson(f, a, b, n))))

         plt.plot(ns, midpoint_error)
         plt.plot(ns, trapezoidal_error)
         plt.plot(ns, simpson_error)
         plt.plot(ns, 1/ns**2, '--')
         plt.plot(ns, 1/ns**4, '--')
         plt.legend(['midpoint', 'trapezoidal', 'simpsons', 'Order 2', 'Order 4'])
         plt.xscale('log')
         plt.yscale('log')
         plt.xlabel('n')
         plt.ylabel('absolute error')
         plt.show()
```



There doesn't seem to be an advantage to using a higher-order method like Simpson's rule. Based on the plot, the midpoint and trapezoidal methods converge much faster. Theoretically, this is most likely because Simpson's method works better for polynomials, but when working with oscilatory functions like those used for DFT, the lower order methods work better.

```
In [ ]:
```