Stability continued (multi-step methods)

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Recall our 3 notions: (all for $h \to 0$)

1) Consistency Does the method locally approximate the ODE?

2) Convergence Does the approximation w_i converge to $y(t_i)$? MAIN GOAL

3) Stability Links consistency to convergence

Recap for 1-step methods (RK, etc.):

Define a general notion of stability to mean w_i depends continuously on the initial data w_0.

Thm 5.20 (abbreviated) Most reasonable 1-step methods are

(i) stable

(ii) convergent iff consistent

(iii) global error is proportional (in terms of h) to the local truncation error $\tau(h)$

This lecture: multistep methods (AB, AM, BD, midpoint, trapezoidal, Simpson's, etc.)

More complicated picture than 1-step.

Not all reasonable multistep methods are stable!

Setup IVP: $y' = f(t,y)$, $a \le t \le b$, $y(a) = \alpha$

$w_0 = \alpha$, $w_1 = \alpha_1$, $w_2 = \alpha_2$, $\ldots$, $w_{m-1} = \alpha_{m-1}$] initialization

$$w_{i+1} = a_{m-1} w_i + a_{m-2} w_{i-1} + \cdots + a_0 w_{i+1-m}$$

$$+ h \cdot \underbrace{\left(b_m f_{i+1} + b_{m-1} f_i + \cdots + b_0 f_{i+1-m} \right)}_{F(t_i, h, w_{i+1}, w_i, \ldots, w_{i+1-m})}, \quad f_i := f(t_i, w_i)$$

So we can express the local truncation error as $y_i := y(t_i)$

$$\tau_{i+1}(h) = \frac{1}{h}\left(y_{i+1} - a_{m-1} y_i - \cdots - a_0 y_{i+1-m} \right) - F(t_i, h, y_{i+1}, y_i, \ldots, y_{i+1-m})$$

Our analysis will mostly ignore the F part (why? ch 5.10 is about $h \to 0$ so $h \cdot F(\dots)$ has less effect)

as long as F is reasonable, namely

① $f=0 \Rightarrow F=0$

② F satisfies a Lipschitz-like condition (see book)

Can we prove things like we did in Thm 5.20?

- For (iii) "global error is proportional (in terms of h) to the local truncation error $\tau(h)$"

yes, we can prove a result. The book gives a very specific result:

Thm 5.21 Global rate of convergence for Adams predictor-corrector methods.

Use a m-step Adams-Bashforth predictor, w/ local truncation error $\tau_{i+1}(h)$ and a $m-1$-step Adams-Moulton corrector, w/ local truncation error $\tilde{\tau}_{i+1}(h)$, and suppose $f(t,y)$ and $\frac{d}{dy} f(t,y)$ are continuous $(a \le b \le t)$ and $\frac{d}{dy} f(t,y)$ bounded ($\Rightarrow f$ Lipschitz in y), then

(paraphrasing) (i) the local truncation error of the combined predictor-corrector scheme is $\sigma_{i+1}(h) = O(\tau_{i+1}(h) + \tilde{\tau}_{i+1}(h))$

(ii) the global error $|w_i - y_i|$ $(\forall i = 1, \dots, n)$ is $O(\sigma(h))$

- What about more general statements?

yes, this will be our focus.

The characteristic polynomial $P(\lambda)$

$$w_{i+1} = a_{m-1} w_i + a_{m-2} w_{i-1} + \dots + a_0 w_{i+1-m} + h F(t_i, h, w_{i+1}, w_i, \dots, w_{i+1-m})$$

An observation:

For Adams methods, $a_{m-1} = 1$, all other $a_j = 0$

For backward differentiation,

Name	Order	Steps m	a_{m-1}	a_{m-2}	a_{m-3}	a_{m-4}	b_m
BD1 = Backward Euler	1	1	1				1
BD2	2	2	$4/3$	$-1/3$ $\quad \Sigma=1$			$2/3$
BD3	3	3	$18/11$	$-9/11$	$2/11$ $\quad \Sigma=1$		$6/11$
BD4	4	4	$48/25$	$-36/25$	$16/25$	$-3/25$	$12/25$

... so for AB, AM, BD, we have

$$(*) \quad \sum_{j=1}^{m} a_{m-j} = 1$$

and in fact you can show that $(*)$ is a necessary condition for Consistency (eg., assuming ① $f=0 \Rightarrow F=0$, then for the ODE $y'=0$, $y(a)=\alpha$ the local truncation error is $\tau_{i+1}(h) = \frac{1}{h}\left(1 - \sum_{j=1}^{m} a_{m-j}\right)\alpha$)

Let's look at the IVP $y'=0$, $y(a)=\alpha$ ($\Rightarrow y(t)=\alpha$ is the unique sol'n)

$f=0 \Rightarrow F=0$ so our numerical method is

$$w_{i+1} = a_{m-1} w_i + a_{m-2} w_{i-1} + \dots + a_0 w_{i+1-m}$$

(i.e. we can ignore the F term)

By the property $(*)$, if $w_i = w_{i-1} = w_{i-2} = \dots = \alpha$ then $w_{i+1} = \alpha$ too.

But is this stable to perturbations? (in our simple ODE, all perturbations are due to roundoff, but generally also due to $\tau_i(h) \neq 0$)

Rewrite our method as

$$(**) \quad w_{i+1} - \left(a_{m-1} w_i + a_{m-2} w_{i-1} + \dots + a_0 w_{i+1-m}\right) = 0$$

A "difference equation", which shares many features of a "differential eq'n" (aka ODE)

$m=3$ example: $w_{i+1} - a_2 w_i - a_1 w_{i-1} - a_0 w_{i-2} = 0$

Reminiscent of 3rd order linear homogeneous ODEs?

$$y''' - a_2 y'' - a_1 y' - a_0 y = 0$$

Ansatz: $y(t) = e^{\lambda t}$ so solve $\lambda^3 - a_2\lambda^2 - a_1\lambda - a_0 = 0$
(and special considerations if get a root w/ multiplicity > 1)

We'll do something similar:

Def The characteristic polynomial $P(\lambda)$ associated w/ the numerical method $(**)$ (it does not depend on what F looks like) is

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - a_{m-3}\lambda^{m-3} - \dots - a_0$$

$m=3$ example $P(\lambda) = \lambda^3 - a_2\lambda^2 - a_1\lambda - a_0$

Observe if λ is a root of P then $w_i = \lambda^i$ is a solution to the difference equation (not yet worrying about initial conditions)

To see this, let's work with our $m=3$ example:

⚠ ok if $\lambda \in \mathbb{C}$

$$w_{i+1} - a_2 w_i - a_1 w_{i-1} - a_0 w_{i-2} = 0, \quad \text{plug in } w_i = \lambda^i$$

$$\lambda^{i+1} - a_2\lambda^i - a_1\lambda^{i-1} - a_0\lambda^{i-2} = 0$$

$$\lambda^{i-2}\left(\lambda^3 - a_2\lambda^2 - a_1\lambda - a_0\right) = 0$$

$$\underbrace{\qquad\qquad\qquad}_{= P(\lambda) = 0} \quad \text{since } \lambda \text{ is a root}$$

The difference equation is homogeneous $\Rightarrow$ if w_i and $\tilde{w}_i$ are both solutions then so is $w_i + \tilde{w}_i$ (superposition)

So, if P has m distinct* roots $\{\lambda_j\}_{j=1}^{m}$ then all sol'n to the difference equation can be written as

$$w_i = \sum_{j=1}^{m} c_j \lambda_j^i \quad \text{for some coefficients } c_j$$

(* If not all roots are distinct, see Eq. 5.63
Ex: λ is a double root, then $w_i = \lambda^i$ and $w_i = i\lambda^{i-1}$ are solutions)

Now, throw in initial condition.
Also, via (*), we observe $\underbrace{\lambda = 1}_{\text{call this } \lambda_1}$ is always a root (if the method is consistent)

$m=3$ ex: $P(\lambda) = \lambda^3 - a_2\lambda^2 - a_1\lambda - a_0$

$$a_2 + a_1 + a_0 = 1 \quad \text{via } (*)$$

$\Rightarrow$ if $\lambda = 1$,

$$P(1) = 1 - a_2 - a_1 - a_0 = 0$$

So $w_i = 1^i = 1$ is a solution to the difference equation,
and $w_i = 1 \cdot \alpha$ is a solution to the difference eq'n and initial condition.

So all sol'n look like $w_i = \alpha + \underbrace{\sum_{j=2}^{m} c_j \lambda_j^i}_{=0}$ i.e., $c_2 = c_3 = \ldots = c_m = 0$

But, due to roundoff error, we might have $c_j \neq 0$ $(j \geq 2)$

Is this "catastrophic"? If $|\lambda_j| < 1$ then $\lim_{i\to\infty} c_j \lambda_j^i = 0$ so no big deal.

but if $|\lambda_j| > 1$ then $\lim_{i\to\infty} c_j \lambda_j^i$ diverges! Big deal (and bad news)

and if $|\lambda_j| = 1$

CASE: not a simple root, then have $c_j \cdot \underbrace{i}_{\text{deal breaker}} \underbrace{\lambda_j^{i-1}}_{\text{no big deal}} \to \infty$

CASE: $\lambda_j = 1$ (and simple), then that's good since this is the true solution, not an error

CASE: $|\lambda_j| = 1$ but $\lambda_j \neq 1$ (and simple)
then perturbation doesn't grow but also doesn't go away

This motivation was based on the ODE $y'=0$

What about other ODEs? $y'=f(t,y)$

For linear ODEs, we always can write $y = \underbrace{y_{\text{homogeneous}}}_{\text{solves } y'=0} + y_{\text{particular}}$

and it turns out, even for nonlinear ODEs, our analysis is still useful (we won't prove it though)

Def Root condition and strongly stable and weakly stable

Let $\{\lambda_j\}_{j=1}^{m}$ be the roots of the characteristic polynomial P associated with a multi-step scheme. Then we say the multi-step method satisfies the root condition if Ⓐ $|\lambda_j| \le 1 \quad \forall j$

and Ⓑ $|\lambda_j| = 1 \Rightarrow \lambda_j$ is a simple root

and if a method satisfies the root condition, then we call it either

strongly stable if $|\lambda_j| \Rightarrow \lambda_j = 1$

weakly stable otherwise.

If it doesn't satisfy the root condition, we call it unstable

Note: Sometimes you might hear of zero-stability which is that $\exists B$ such that $\exists h_0 > 0$ and $\forall\ 0 < h < h_0$, $n := \frac{b-a}{h}$, $|w_i| \le B$ $\forall i = 1, \dots, n$

i.e., the numerical solution is bounded $\forall\ a \le t_i \le b$ regardless of how small h is (some texts use slightly different definitions)

This zero-stability is basically equivalent to the root condition

Thm 11.4 (Quarteroni et al.)

For a consistent multistep method, the root condition is equivalent to zero-stability

Main result

Stability links consistency to convergence

Thm 5.24 "Equivalence Thm"

① A multistep method is stable iff it satisfies the root condition

i.e. output depends continuously on input data

② If the multistep method is consistent, then it is convergent iff it's stable

("iff" = if and only if)

This is a version of the Dahlquist equivalence theorem (mid 1950s, for all ODEs, linear and nonlinear)

very similar to the Lax (aka Lax–Richtmeyer) equivalence theorem
(mid 1950s, also for PDEs but only linear ones)

and related

not on exam

Thm First Dahlquist Stability Barrier — cf. Driscoll + Braun Thm 6.8.3

If a <u>linear</u> multistep method (eg AB, AM, BD) with m steps has global error $O(h^p)$

then if it is a stable method,

$$p \le \begin{cases} m+2 & m \text{ even} \\ m+1 & m \text{ odd} \\ m & \text{if method is explicit} \end{cases}$$