Homework 10
APPM/MATH 4650 Fall '20 Numerical Analysis

Due date: Saturday, November 21, before midnight, via Gradescope. **Instructor**: Prof. Becker
Theme: Multistep methods

Instructions Collaboration with your fellow students is OK and in fact recommended, although direct copying is not allowed. The internet is allowed for basic tasks (e.g., looking up definitions on wikipedia) but it is not permissible to search for proofs or to *post* requests for help on forums such as `http://math.stackexchange.com/` or to look at solution manuals. Please write down the names of the students that you worked with.

An arbitrary subset of these questions will be graded.

Turn in a PDF (either scanned handwritten work, or typed, or a combination of both) to **Gradescope**, using the link to Gradescope from our Canvas page. Gradescope recommends a few apps for scanning from your phone; see the Gradescope HW submission guide.

We will primarily grade your written work, and computer source code is *not* necessary except for when we *explicitly* ask for it (and you can use any language you want). If not specifically requested as part of a problem, you may include it at the end of your homework if you wish (sometimes the graders might look at it, but not always; it will be a bit easier to give partial credit if you include your code).

Problem 1: In addition to the Adams-Bashforth, Adams-Moulton and Backward Differentiation formulas, there are other linear multistep methods, such as Milne's method (explicit), the midpoint method (explicit), and Simpson's method (implicit). We'll work with the ODE $y' = f(t, y)$.

a) Derive Simpson's *method* by applying Simpsons (quadrature) *rule* to the integral below (the integral coming from the FTC and the ODE)

$$y(t_{i+1}) - y(t_{i-1}) = \int_{t_{i-1}}^{t_{i+1}} f(t, y(t))dt.$$

Simpson's method is in our book Burden & Faires at the very end of chapter 5.6 (note: in some old editions of the text, there was a typo in the book here; it's fixed in the 9th and 10th editions).

b) What is the local truncation error $\tau(h)$? (The book gives the answer, but you should show how to get this; you *can* assume you know the error for Simpson's *rule*).

Problem 2: a) Assume $y \in C^3[t_i, t_{i+2}]$ where $t_i \in \mathbb{R}$ is arbitrary and $t_{i+1} = t_i + h$ and $t_{i+2} = t_i + 2h$. Show that

$$y'(t_i) = \frac{-3y(t_i) + 4y(t_{i+1}) - y(t_{i+2})}{2h} + O(h^2) \tag{1}$$

and that

$$y'(t_{i+2}) = \frac{3y(t_{i+2}) - 4y(t_{i+1}) + y(t_i)}{2h} + O(h^2). \tag{2}$$

Bonus: show

$$y'(t_i) = \frac{-3y(t_i) + 4y(t_{i+1}) - y(t_{i+2})}{2h} + \frac{h^2}{3} y^{(3)}(\xi) \quad \text{for some } \xi \in [t_i, t_{i+2}]. \tag{3}$$

b) Consider the initial value problem (IVP)

$$y' = f(t, y) \quad \forall t \in [a, b], \qquad y(a) = \alpha.$$

Let n be an integer and $h = \frac{b-a}{n}$ and $w_0 = \alpha$. Eq. (1) suggests the difference method

$$w_{i+2} = 4w_{i+1} - 3w_i - 2hf(t_i, w_i) \qquad i = 0, 1, \ldots, n-2 \tag{4}$$

and Eq. (2) suggests the difference method

$$w_{i+2} = \frac{1}{3}\left(4w_{i+1} - w_i + 2hf(t_{i+2}, w_{i+2})\right) \qquad i = 0, 1, \ldots, n-2. \tag{5}$$

The method in Eq. (4) is an explicit method and doesn't have a name (that I know of). The method in Eq. (5) is the backward differentiation method BD2 (and is implicit).

For both methods, what is the local truncation error $\tau(h)$, and are these *consistent* methods?

c) Analyze the *stability* and *convergence* for both methods. If the method is stable, specify if it is *strongly stable* or *weakly stable*.

d) Consider a specific IVP

$$y' = \cos(t) \quad \forall t \in [0, 0.1], \qquad y(0) = 0.$$

This is a very simple ODE and you should be able to work out the true solution by hand. Implement both the explicit method Eq. (4) and the implicit method Eq. (5) (BD2).

To check if your implementation is correct, when using $n = 10$ (i.e., $h = 10^{-2}$), the explicit method should give $w_n \approx 0.1097$ and the implicit method should give $w_n \approx 0.0998$. Report the values you get.

Details: As these are multi-step methods (and in particular, 2-step methods), we cannot use these methods to find w_1 since we'd need to have w_{-1}. So to find w_1, just evaluate the true solution at the relevant time (in a more realistic setting we would use something like RK4, but let's not involve RK4 for this problem).

For BD2, this is an implicit method, so we need to be able to solve a root-finding problem. However, because our ODE is so simple, this is not really a problem!

e) For both methods, make a plot of the error $e = |w_n - y(0.1)|$ as a function of the stepsize h. For both methods, can you find a stepsize h that gives $e \leq 10^{-10}$? (If so, about what value is that h?)

f) Comment on your findings