1) a.) $\dfrac{A f(x) + B f(x+h) + C f(x+3h)}{h}$

→ want $O(h^2)$ approx for $f'(x)$

T.S. expansion: $\frac{1}{h}\left[A f(x) + B\left(f(x) + h f'(x) + \frac{h^2}{2} f''(x) + O(h^3)\right)\right.$
$\left. + C\left(f(x) + 3h f'(x) + \frac{9h^2}{2} f''(x) + O(h^3)\right)\right]$

$= \frac{1}{h}\left[(A+B+C) f(x) + (B+3C) h f'(x) + \left(\frac{B}{2} + \frac{9C}{2}\right) h^2 f''(x) + O(h^3)\right]$

$= \frac{(A+B+C)}{h} f(x) + (B+3C) f'(x) + \left(\frac{B}{2} + \frac{9C}{2}\right) h f''(x) + O(h^2)$

↳ want this to be $f'(x) + O(h^2)$, so get system

$A+B+C = 0$

$B + 3C = 1$ $\qquad B - B/3 = 1 \Rightarrow B = 3/2$

$B + 9C = 0 \Rightarrow C = -B/9$ plug in

$C = -1/6$ plug in

$A = -3/2 + 1/6 = -8/6 = -4/3$

$A = -4/3$, $B = 3/2$, $C = -1/6$

b) $p(x) = \dfrac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} y_1 + \dfrac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} y_2 + \dfrac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} y_3$

Here $x_1 = 0$, $x_2 = h$, $x_3 = 3h$
$y_1 = f(0)$, $y_2 = f(h)$, $y_3 = f(3h)$

$p(x) = \dfrac{(x-h)(x-3h)}{(-h)(-3h)} f(0) + \dfrac{(x)(x-3h)}{(h)(-2h)} f(h) + \dfrac{(x)(x-h)}{(3h)(2h)} f(3h)$

$f'(x) \approx p'(x) = \dfrac{2x-4h}{3h^2} f(0) + \dfrac{2x-3h}{-2h^2} f(h) + \dfrac{2x-h}{6h^2} f(3h)$

w.l.o.g take $x=0$:

$$p'(0) = \frac{-4}{3h} f(0) + \frac{3}{2} f(0) + \frac{-1}{6} f(3h)$$

finite diff. at $x=0$: $\frac{A}{h} f(0) + \frac{B}{h} f(h) + \frac{C}{h} f(3h)$, match up terms:

$A = -4/3$, $B = 3/2$, $C = -1/6$

c.) yes, this is a unique choice. yes, still unique:

$$f'(x) = -f(x_0) + f(x_0+h), \quad A = 1, B = 1$$

d.) $$\frac{Af(x) + Bf(x+h) + Cf(x+3h)}{h^2} \approx f''(x)$$

$$= \frac{1}{h^2}\Big[A(f(x)) + B\Big(f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + O(h^4)\Big)$$
$$+ C\Big(f(x) + 3hf'(x) + \frac{9h^2}{2}f''(x) + \frac{27h^3}{6}f'''(x) + O(h^4)\Big)\Big]$$

$$= \frac{1}{h^2}\Big[(A+B+C)f(x) + (B+3C)hf'(x) + \Big(\frac{B}{2} + \frac{9C}{2}\Big)h^2 f''(x) + \Big(\frac{B}{6} + \frac{27C}{6}\Big)h^3 f'''(x)$$

$$= \underbrace{\frac{(A+B+C)}{h^2}}_{0} f(x) + \underbrace{\frac{B+3C}{h}}_{0} f'(x) + \underbrace{\Big(\frac{B}{2} + \frac{9C}{2}\Big)}_{1} f''(x) + \underbrace{\Big(\frac{B}{6} + \frac{27C}{6}\Big)}_{\neq 0} h f'''(x) + O($$

$B + 3C = 0 \rightarrow B = -3C$

$B + 9C = 2$

$6C = 2 \Rightarrow C = 1/3$, $B = -1$

$A = 2/3$, $B = -1$, $C = 1/3$

over

e.) As before, $p'(x) = \frac{2x-4h}{3h^2} f(0) + \frac{2x-3h}{-2h^2} f(h) + \frac{2x-h}{6h^2} f(3h)$

$p''(x) = \frac{1}{h^2}\left[\frac{2}{3} f(0) - f(h) + \frac{1}{3} f(3h)\right]$.

$f''(0) \approx \frac{1}{h^2}\left[A f(0) + B f(h) + C f(3h)\right]$

So, $A = 2/3$, $B = -1$, $C = 1/3$

2.) jupyter

3.) $f'(x) \approx \underbrace{\frac{f(x+h) - f(x-h)}{2h}}_{\delta}$

$f(x+h), f(x-h)$ have error bounded by ε

truncation error bounded by Mh^2

$\hookrightarrow \frac{f(x+h) - f(x-h)}{2h} \to 2\varepsilon/h$

$\left|f'(x) - \frac{f(x+h)-f(x-h)}{2h}\right| \leq Mh^2$

$|\delta - f'(x)| \leq 2\varepsilon/h + Mh^2$

minimize w.r.t h

$\frac{d}{dh}\left[\frac{2\varepsilon}{h} + Mh^2\right] = \frac{-2\varepsilon}{h^2} + 2Mh = 0 \Rightarrow 2Mh = \frac{2\varepsilon}{h^2}$

$2\varepsilon = 2Mh^3 \Rightarrow h^3 = \underbrace{\varepsilon/M}_{>0} \Rightarrow \boxed{h = \left(\frac{\varepsilon}{M}\right)^{1/3}}$ is min value.

bound: $\frac{2\varepsilon}{(\varepsilon/M)^{1/3}} + M\left(\frac{\varepsilon}{M}\right)^{2/3} = 2M^{1/3}\cdot\varepsilon^{2/3} + M^{1/3}\varepsilon^{2/3}$

$= \boxed{3M^{1/3}\varepsilon^{2/3}}$

4.) a.) $\underbrace{C\, h^n\, f^{(n+1)}(\xi)}_{\text{Error}}$ for some C, some $\xi \in (x, x+h]$.

what is error for function f w/ degree $\leq n$?

$|\text{error}| \leq C h^n \xi$

b.) 3 step centred finite diff. $\frac{f(x+h)-f(x-h)}{2h}$,

error is $\left|\frac{f'''([x+h,x-h])}{6}h^2\right|$, but for quadratic poly, $f'''(x)=0$, so can get exact value.

pick h to be medium to avoid rounding/truncation error.

c.) $N_1(h)=\frac{f(x+h)-f(x-h)}{2h}$

$$N_2(h)=N_1\left(\frac{h}{2}\right)+\left(N_1\left(\frac{h}{2}\right)-N_1(h)\right)$$

$$=\frac{f(x+h/2)-f(x-h/2)}{h}+\left(\frac{f(x+h/2)-f(x-h/2)}{h}-\frac{f(x+h)-f(x-h)}{2h}\right)$$

$$=\frac{f(x+h)+4f(x+h/2)-4f(x-h/2)-f(x-h)}{2h}$$

Order of accuracy: $f(x\pm\frac{h}{2})=f(x)\pm\frac{h}{2}f'(x)+\underbrace{\frac{h^2}{8}f''(x)\pm\frac{h^3}{48}f'''(x)}_{\text{Cancel out w/ } f(x+h),f(x-h)}+O(h^4)$

$\hookrightarrow O(h^4)$

$\hookrightarrow$ 5 point centred finite difference! $O(h^4)$
(w/ step size $h/2$) ...

hw5

October 10, 2020

1 Problem 2

```
In [1]: import numpy as np
```

```
In [7]: f = lambda x : np.exp(3*x)
        f_prime = lambda x : 3*np.exp(3*x)
```

```
In [20]: # 7 point formula: (-1, 9, -45, 0, 45, -9, 1) / 60
         finite_diff = lambda f, x, h : (-f(x-3*h) + 9*f(x-2*h) - 45*f(x-h) + 45*f(x+h) - 9*f(x
```

```
In [23]: x0 = 0.3
         h = 0.001
         np.abs(finite_diff(f, x0, h) - f_prime(x0))
```

```
Out[23]: 1.6608936448392342e-13
```

Using 7-point finitie difference with step size of 0.001, get a low enough absolute error (1.66e-13)