Interpolation (advanced): how to think of it

Tuesday, September 22, 2020 11:27 AM

We've discussed Lagrange interpolation and Newton Divided Difference interpolation.

There are other forms too.

What's the difference? The basis

we can change this

Interpolating polynomial p can be written as $p = \sum_{i=0}^{n} c_i p_i$

Assume degree is n or less.

i.e., $(\forall x)\ p(x) = \sum_{i=0}^{n} c_i p_i(x)$

Fact: $\dim(\underbrace{\text{all polynomials of degree } n \text{ or less}}_{\text{vector space } V_n}) = n+1$

proof: $\underbrace{\{1, x, \dots x^n\}}_{\text{"monomial basis"}}$ is clearly a basis

V_n = vector space of all degree $\leq n$ polynomials

Fact: the $\underbrace{\text{interpolating}}_{\text{on } n+1 \text{ points}}$ polynomial p is degree n or less

So we know we can represent it as a combination of basis functions for V_n

So,

LAGRANGE INTERPOLATION

uses the basis for V_n consisting of Lagrange polynomials

$\mathcal{B}_{Lag.} = \{ L_{n,k} : k = 0, 1, \dots, n \}$ is a basis for V_n

$$L_{n,k}(x) := \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)}$$

Scaling doesn't affect whether it's a basis (but can make it more convenient to work with)

How to prove $\mathcal{B}_{Lag}$ is a basis for V_n?

1. Check $\mathcal{B}_{Lag.} \subseteq V_n$ ✓ (ie., make sure we don't have degree too large)
2. Check $|\mathcal{B}_{Lag.}| = \dim(V_n)$ ✓
3. check $\mathcal{B}_{Lag.}$ is linearly independent (yes, we'll discuss later)

MONOMIAL BASIS

$\mathcal{B}_{mon.} = \{1, x, x^2, \dots x^n\}$ a basis for V_n

but doesn't lead to <u>stable</u> algorithms

Note: $\tilde{B}_{mon} = \{1, (x-x_0), (x-x_0)^2, \dots, (x-x_0)^n\}$ is also a basis.

NEWTON BASIS

$$B_{New.} = \{ 1, (x-x_0), (x-x_0)(x-x_1), \dots, (x-x_0)\cdot(x-x_1)\cdot\ldots\cdot(x-x_{n-1}) \}$$

i.e., elements either 1 or $\prod_{i=1}^{k} (x-x_{i-1})$ for $k=1,2,\dots,n$

Is this a basis?

① $B_{Newton} \subseteq V_n$? ✓

② $|B_{Newton}| = \dim(V_n) = n+1$? ✓

③ Lin. Independent? Yes

LINEAR INDEPENDENCE

if $B = \{v_0, v_1, \dots, v_n\}$ is a subset of a vector space V, we say

B is "linearly independent" if $\sum_{i=0}^{n} c_i \cdot v_i = 0 \Rightarrow c_i = 0,\ i=0,1,\dots,n$

What does this mean if our "vector" is a function or polynomial?

Same thing: if $\sum_{i=0}^{n} c_i\, v_i(x) = 0 \quad \forall x \Rightarrow c_i = 0,\ i=0,1,\dots,n$

How do we verify in practice?

<u>Method 1</u>

true $\forall x$, so in particular it's true for $\{x_0, x_1, \dots, x_n\}$

So we have $(n+1)$ equations: $\sum_{i=0}^{n} c_i\, v_i(x_j) = 0$ for $j=0,1,\dots,n$

i.e., solve the linear system

$\vec{A} \cdot \vec{c} = \vec{0}$ for $A_{j,i} = v_i(x_j)$

(Sorry, I should have switched i, j notation)

In the special case that B is monomials, then A is the Vandermonde matrix.

If $\underbrace{A \text{ is invertible}}_{\text{iff } \det(A) \neq 0}$, then there's a <u>unique solution</u> $\vec{c} = \underbrace{A^{-1} \cdot \vec{0}}_{=0}$

$\Rightarrow c_i = 0$

$\Rightarrow$ the set is linearly independent.

<u>Ex</u> Newton Basis

(this completes the claim we made in those lecture notes, since it proves it is a basis)

$$B_{New} = \{1, x-x_0, (x-x_0)(x-x_1), \dots, (x-x_0)(x-x_1)\cdot\ldots\cdot(x-x_{n-1})\}$$

plug in the points $\underbrace{\{x_0, x_1, \dots, x_n\}}_{\text{other points work but then it's messy and complicated.}}$ and our matrix A is

$$\begin{array}{l} \text{row } j \\ \text{is plugging} \\ \text{in } x = x_j \end{array} \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & & 0 \\ 1 & x_1 - x_0 & 0 & 0 & \cdots & 0 \\ 1 & x_2 - x_0 & (x_2-x_0)(x_2-x_1) & 0 & & 0 \\ \vdots & \vdots & \vdots & \vdots & & \\ 1 & \underbrace{x_n - x_0 \quad (x_n-x_0)(x_n-x_1)}_{\text{column } i = \text{ function } p_i \in \mathcal{B}_{\text{Newton}}} & & & & (x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1}) \end{bmatrix} \right.$$

then A is lower triangular, so $\det(A)$ = product of diagonal entries

and all diagonal terms are nonzero since $\{x_0, x_1, \ldots, x_n\}$ are distinct.

$\Rightarrow \det(A) \neq 0 \Rightarrow A$ is invertible $\Rightarrow \mathcal{B}_{\text{Newton}}$ is lin. independent. ✓

Method 2

If $\underbrace{\sum c_i v_i}_{f} = 0$, then $f' = 0$, i.e., $\sum c_i v_i' = 0$

and similarly $\sum c_i v_i'' = 0$, etc.

So,

pick any point x, and make

$$A = \begin{bmatrix} v_0(x) & v_1(x) & \cdots & v_n(x) \\ v_0'(x) & v_1'(x) & \cdots & v_n'(x) \\ \vdots & & & \\ v_0^{(n)}(x) & v_1^{(n)}(x) & \cdots & v_n^{(n)}(x) \end{bmatrix}$$ } the Wronskian at point x.

So if $A(x)$ is invertible at any point x then

you can deduce $\vec{c} = 0$ and hence $\{v_0, v_1, \ldots, v_n\}$ is lin. independent.

Back to ~~basis~~ bases,

1) Lagrange polynomials ← (all have degree n)

2) Monomials or Shifted Monomials

3) Newton polynomials

} each polynomial in the basis has a different degree

NEW (

4) Legendre polynomials (Gram-Schmidt orthogonalize monomials, $\int_{-1}^{1} v_i(x) v_j(x)\,dx = \delta_{ij}$)

5) Chebyshev polynomials (orthogonal with a new meaning of "inner product",

$$\int_{-1}^{1} v_i(x) v_j(x)\, w(x)\,dx = \delta_{ij}$$

for a weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ Chebyshev of the 1st kind

or $w(x) = \sqrt{1-x^2}$ Chebyshev of the 2nd kind)

we'll discuss more later in this class