Study questions for APPM 4650

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1. You want to find the root of an equation $f(x) = 0$ using a fixed-point iteration $x_{n+1} = g(x_n)$. State and prove a condition when the fixed point method converges.

2. Prove that Newton's method for $f(x) = 0$ converges quadratically for a root with multiplicity one.

3. Write a Matlab program that solves $f(x) = 0$ using the secant method and with a relative error no larger than 10^{-8}.

4. Show that the operation count of Gaussian elimination for a system of size $n \times n$ is $\mathcal{O}(n^3)$.

5. Given the L and U in the $LU = A$ decomposition, write a program using only loops (no built in functions) that solves $Ax = b$.

6. Let:
$$A = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}.$$
Let the relative backward error, $\|b - Ax_{\text{computed}}\|_2 / \|b\|_2$, in solving the system $Ax = b$ be 1%. Give an upper bound on the relative forward error $\|x - x_{\text{computed}}\|_2 / \|x\|_2$.

7. (a) Find the $A = LU$ decomposition of the matrix
$$A = \begin{pmatrix} 10^{-10} & 1 \\ 1 & 0 \end{pmatrix}.$$
   (b) Now we will solve $A\bar{x} = \bar{b}$ with $\bar{b} = (1, \pi)^T$. To solve use forward substitution to find the solution to $L\bar{z} = \bar{b}$ and then backward substitution to find $\bar{x}$ from $U\bar{x} = \bar{z}$.
   (c) In double precision the relative error in the solution, $\bar{x}$, computed as above is $\sim 10^{-7}$ and the relative error in the right hand side is $\sim 10^{-16}$, that is the magnification is $\sim 10^9$. Compare this to the condition number $\kappa_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty$.[1]

8. Write a matlab program that interpolates $f(x)$ at n points $x_1 < x_2 < \ldots < x_n$ with a degree $n - 1$ polynomial and plots the interpolating polynomial a denser grid between x_1 and x_n and the function on the original grid. You may assume that the function values and the grid are given as column vectors as input from the user.

[1] In the computer it is better to first swap the first and second row (this is called LU decomposition with pivoting) of the matrix and the right hand side $\bar{b}$ and then compute the solution. Only when pivoting is used, the condition number is a good upper estimate for the magnification.

9. A spline is a piecewise degree three polynomial which is continuous and has continuous first and second derivative at break-points.

   (a) State the conditions / equations that determines the coefficients of the two spline polynomials

   $$s_1(x) = a_1 + b_1 x + c_1 x^2 + d_1 x^3, \quad x \in [-1, 0],$$

   $$s_2(x) = a_2 + b_2 x + c_2 x^2 + d_2 x^3, \quad x \in [0, 1],$$

   interpolating $\cos(\pi x)$ at $x = -1, 0, 1$, with first derivative coinciding with the derivative of $\cos(\pi x)$ at $x = -1, 1$ and satisfying the spline continuity conditions at $x = 0$.

   (b) Write a matlab program that solves the system of equations (using \) and plots the spline and $\cos(\pi x)$.

10. Given the data

x	y
1	1
2	2
4	3

    Find the interpolating Lagrange and Newton polynomial.

11. Assume that we use the three points $x = 1/2, 4/3, 1$ to interpolate $\exp(x)$ by a polynomial $p(x)$. Find an upper bound for the error $|\exp(x) - p(x)|$.

12. Describe the properties of cubic splines.

13. Describe the properties of Bezier curves.

14. Show that the approximation

    $$\frac{d^2 f(x)}{dx^2} \approx \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2},$$

    is first order accurate. Use the formula to find approximate values for $\frac{d^2 f(x)}{dx^2}\big|_{x=1}$ for $f(x) = x^2$ using $h = 1/4$ and $1/2$. Use Richardson extrapolation to find a better approximation from your computed values.

15. Write a Matlab program that computes an approximation to $\int_0^1 \frac{\exp(x)}{1+x} \mathrm{d}x$ using the composite trapezoidal rule with 100 subintervals.

16. Derive a formula for the error in the midpoint rule on one interval used to approximate $\int_a^b f(x)\mathrm{d}x$.

17. Find the nodes x_i and the weights w_i so that the Gaussian quadrature $\sum_{i=1}^{2} w_i f(x_i)$ approximating $\int_{-1}^{1} f(x)\mathrm{d}x$ is exact when $f(x)$ is a polynomial of as high degree as possible.

18. Use Euler's method to solve

    $$y'(t) = y(t), \quad t > 0 \;\; y(0) = 1,$$

    with $h = 1/4$ until time $t = 1$.

19. Write the differential equation $y'''' = -\frac{\sqrt{y'}}{y}$ as a system of first order ODEs.

20. Write a Matlab program that uses the classic 4th order Runge-Kutta method to solve the ODE

    $$\ddot{x}(t) + k\dot{x}(t) + \omega x(t) = \sin(t^2),$$

    with initial conditions

    $$\dot{x}(0) = x(0) = 1.$$

21. Find and sketch the region of absolute stability for the method

$$u_{n+1} = u_n + \frac{h}{2}(f(t_n, u_n) + f(t_n + h, u_n + hf(t_n, u_n))).$$

22. I recorded the temperature in Oslo for the first few days of November but unfortunately I spilled some coffee on my records. Help me find the missing temperature by interpolating a second degree polynomial to the known data.

Temp	63	46	♣	41
Day	Nov. 1	Nov. 3	Nov. 5	Nov. 7

(a) Formulate the interpolation problem using the naive (Vandermonde), Newton's and Lagrange's approach.

(b) Write a program that computes the missing temperature, using either of the three methods.

(c) Use linear interpolation and pen and paper to fill in the missing value.

(d) Discuss the suitability of the three approaches if we want to expand the table with today's temperature and use a cubic polynomial to find the missing temperature.

23. Use the formula

$$\tilde{T}_p(v) = 33 - (c_0 + c_1 v + c_2\sqrt{v} + c_3 e^{-v} + c_4 v^2)(33 - T),$$

with $T = 0$ to find values for c_0, c_1, c_2, c_3, c_4 so that $\tilde{T}_p$ interpolates the T_c values in the above table. Then set $T = -5$ and compute the perceived temperature for $T = -5$ and a wind speed of 7 m/s, i.e. $T_p(7)$.

v	2	5	8	11	14
T_c	0	-7.5	-12	-14.5	-16.5

24. Assume that we use the two points $x = 1, 2$ to interpolate x^2 by a polynomial $p(x)$. Find an upper bound for the error $|x^2 - p(x)|$ for $x \in [1, 2]$.

25. We are concerned with the approximation of the smooth function $f(x)$ on the domain $[x, x + h]$

(a) State the Taylor series for $f(x + h)$ around x, include powers of h up to h^2.

(b) Use the Taylor series to find approximate values for the derivative of $f(x) = \frac{1}{1+x}$ at $x = 0$ using $h = 2, 1, 1/2$.

(c) Use the formula

$$\frac{h}{2} \max_{z \in [0,h]} |f''(z)|,$$

to estimate the error in your approximations and compare it to your results.

(d) Outline a **derivative free** algorithm to compute the 100 first terms in the Maclaurin expansion of $f(x) = \frac{1}{1+x}$, that is find the first 100 coefficients a_n in $\sum_{n=0}^{\infty} a_n x^n = \frac{1}{1+x}$.

26. The following table displays the errors for a numerical method for solving the elastic wave equation.

h	error in u
$1/80$	0.16533
$1/160$	0.04245
$1/320$	0.01071
$1/640$	0.00269

(a) What is the order of accuracy of the method?

(b) If we want an error no larger than 0.001 (for the example in the table) how should h be chosen?

27. (a) State the Taylor expansion (with terms up to and including the second derivative + the reminder) of a function $f(x)$ around a point $x = x_0$.

(b) Use the result from (a) to derive the leading order error term for the one-sided approximation

$$\frac{df(t)}{dt} \approx \frac{1}{2\Delta t}\left(-3f(t) + 4f(t+\Delta t) - f(t+2\Delta t)\right),$$

that is find an expression for

$$\left|\frac{df(t)}{dt} - \frac{1}{2\Delta t}\left(-3f(t) + 4f(t+\Delta t) - f(t+2\Delta t)\right)\right|$$

Hint: substitute $x = t + \Delta t$ or $x = t + 2\Delta t$ and $x_0 = t$ in the answer from (a).

28. Assume that we use the four points $x = 0, 1/2, 4/3, 1$ to interpolate $\exp(x)$ by a polynomial $p(x)$. Find an upper bound for the error $|\exp(x) - p(x)|$.

29. Use Euler's method to solve

$$y'(t) = y(t), \quad t > 0 \quad y(0) = 1,$$

with $h = 1/4$ until time $t = 1$.

30. Write a Matlab program that computes an approximation to $\int_0^1 \frac{\exp(x)}{1+x}\mathrm{d}x$ using the composite trapezoidal rule with 100 subintervals.

31. Show that the approximation

$$\frac{d^2 f(x)}{dx^2} \approx \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2},$$

is first order accurate. Use the formula to find approximate values for $\left.\frac{d^2 f(x)}{dx^2}\right|_{x=1}$ for $f(x) = x^2$ using $h = 1/4$ and $1/2$. Use Richardson extrapolation to find a better approximation from your computed values.