APPM 4650 — Exam 2 Review Sheet

1. (a) Write down the Lagrange intepolating polynomial through x_0, x_1 and x_2 that approximates $f(x)$.
   (b) What is the error in the approximation?
   (c) Apply the interpolation to $f(x) = 10\sin(x)$ and $x_0 = -\pi$, $x_1 = 0$ and $x_2 = \pi$. What is the error bound? What does this tell you about interpolation?

2. What are the key properties of Hermite interpolation? What is the highest degree polynomial that can be intergrated exactly with Hermite interpolation?

3. (a) Define a cubic spline.
   (b) What does it mean for a spline to be clamped? Natural?

4. Determine the order of the following finite difference approximation.

$$f'(x) \sim \frac{f(x) - 4f(x-h) + f(x-2h)}{2h}$$

5. Assume the error in a numerical method has the asymptotic expansion

$$M - N_1(h) = C_1h^2 + C_2h^4 + C_3h^6 + C_4h^8 + \cdots$$

Derive the general form for Richardson extrapolation to create high order estimates of M. Assume that you are able to evaluate $N_1(h)$, $N_1(h/2)$, etc as needed.

6. Consider the integral $\int_a^b \int_c^d f(x,y)dydx$ being evaluated via Composite Simpson's rule in both the x and y direction. Assume f is $C^5([a,b]\times[c,d])$.
   (a) Write down the error in the approximation using the Composite Simpson's rule to approximate this integral.
   (b) Use this error term to figure out what how many nodes are needed to approximate the integral

$$\int_0^{0.5}\int_0^{0.5} e^{x-y}dydx$$

with an error of approximately 10^{-6}. You can assume uniform spacing in the x and y direction. (i.e. $h = h_x = h_y$)

7. Use composite Simpson's rule to approximate the following integral

$$\int_0^1 \frac{e^{-x}}{\sqrt{1-x}}dx$$

with $n = 6$ points.

8. Use composite Simpson's rule to approximate the following integral

$$\int_1^{\infty} \frac{1}{1+x^4} dx$$

with $n = 4$ points.

9. (a) Write down the definition of the Lipschitz condition for $f(t, y)$ on $D = \{(t, y) : 0 \le t \le \pi, -\infty \le y \le \infty\}$.
   (b) Show that the function $f(t, y) = \sin(ty)$ is Lipschitz in D via the definition.

10. Show that the following initial value problem has a unique solution in $D = \{(t, y) : 2 \le t \le 3, -\infty \le y \le \infty\}$.

$$y' = -y + ty^{-1/2} \quad 2 \le t \le 3$$
$$y(2) = 2$$

11. (a) Is the function $f(t, y) = \frac{1+y}{1+t}$ Lipschitz with respect to the y variable on $D = \{(t, y) : 0 \le t \le 1, -\infty \le y \le \infty\}$?
    (b) Use Theorem 5.6 to show whether or not the following initial value problem is well-posed on D.

$$y' = \frac{1+y}{1+t} \quad 0 \le t \le 1$$
$$y(0) = 1$$