The Coupled Logistic Map

Anas Gauba
Dept. of Computer Science
Albuquerque, New Mexico
mgauba@unm.edu

Siri Khalsa
Dept. of Computer Science
Albuquerque, New Mexico
skhalsa10@unm.edu

***Abstract*—The coupled logistic map and the major evolutionary transitions are critical in understanding the collective and complex behavior among bottom-up and top-down populations. To consider an emerging behavior among different top-down and their causal impact on the bottom-up, this paper first studied the logistic map itself and how changing the reproductive fitness r to be chaotic vs. non-chaotic can result in the population being either divergence and convergence respectively. Then, this paper studied the figures from Walker paper and analyzed whether the assertions made by Walker are true or not. The results show us that indeed the transition from bottom-up to top-down happens and the causal information flow between top-down and bottom-up emerges as the coupling strength increases. The results also suggest that Walker's claim about the emergence of organizations at epsilon 0.075 was wrong. This paper determined that the selection of the growth rate r is critical and deserves special attention for the model used by Walker.**

I. Introduction

What is the cause of most evolutionary transitions? When is there a change from bottom-up causation to top-down causation? Are they related? What is Chaos? How does it relate to the creation of new entities? How can we measure this phenomenon? These are the primary ideas behind Sara Walker's article *Evolutionary Transitions and Top-Down Causation* [8]. The Walker paper attempts to explore how individual entities behave and share information and how this eventually leads to the creation of a hierarchical structure. A new entity is born at the top of this structure and can transfer information to the entities below it. This paper will focus on analyzing the Walker paper and evaluating the results that it generated. Before this can be done, there is a need to explore some fundamental ideas that will help to analyze the results.

Part 1 of the "Methods & Results" section will explore the **logistic map**, which will be defined in that section, It will use an example to showcase **sensitive dependence on initial conditions**, which means that a system can showcase extremely different results from only slightly perturbed initial input. It will also explain how to measure the information of systems with Shannon entropy and how we can measure the relationships between multiple systems. Overall, Part 1 is critical in setting up the necessary foundation to understand the rest of the paper.

Part 2 will attempt to replicate the figure from the Walker paper [8], showing the collective behavior of the mean populations for all the subpopulations over generations for different values of coupling strength. Part 3 will take the results of the Walker paper [8] and analyze the causal behavior among the metapopulatons to subpopulations with more details to either confirm or challenge the results. Finally, part 4 will challenge the use of the growth rate used in all calculations in the Walker paper [8].

II. Methods & Results

A. Part 1

This paper begins by exploring some fundamental concepts that will be used to understand the rest of the paper. The starting point of these concepts is the logistic map. What exactly is the **logistic map**? It can be defined with the following mathematical equation [1] [3] [2]:

$$x_{t+1} = r(x_t - \frac{x_t^2}{K}) \tag{1}$$

It can be simplified even further with some algebra:

$$x_{t+1} = rx_t(1 - x_t) \tag{2}$$

What do these equations mean in plain English? Well, the first point to bring up is this is an iterative function. the equation models the dynamics of a population [2]. In other words, how does this population change over time. The t represents time. The x represents the current population. The x_{t+1} represents the population at the next time step. K represents the carrying capacity. That is the maximum population that can be obtained. Finally, we have r. this represents the growth rate. It is the birth rate minus the death rate. The population at the next time step relies on the previous population, growth rate, and the carrying capacity. This is called a map because it maps the population of each generation to the next generation's population [1].

Equation (2) simplifies equation (1) by eliminating the need for K. It instead represents the population as a percentage of the carrying capacity. x is mapped to a real number between 0 and 1. The most interesting part of the logistic map is the r variable. depending on the r, the map will show some interesting behavior. It will either converge to a fixed point, alternate among several values or show chaotic behavior. Let's see what this looks like. Let us begin with a value of r that converges.

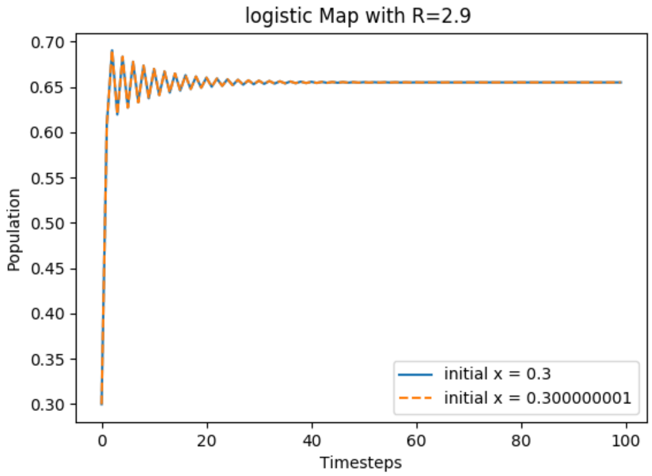

Fig. 1. Example of the logistic map with an R that leads to a fixed point

This converges to a fixed point. The graph includes two lines, one is a solid blue and the other is dotted orange. The starting values for each line have a slightly perturbed initial value. It contains two sets of data because it is critical in understanding the chaotic behavior which is explained next. As r increases, the graph will showcase a behavior that oscillates between values. The number of values that it oscillates between will double as r increases until $r >= 3.5$ [1]. At this point, it can be thought that there is an oscillation between an infinite amount of values [2]. In other words, there is chaotic behavior. Let's take a look at an example when $r = 3.9$.

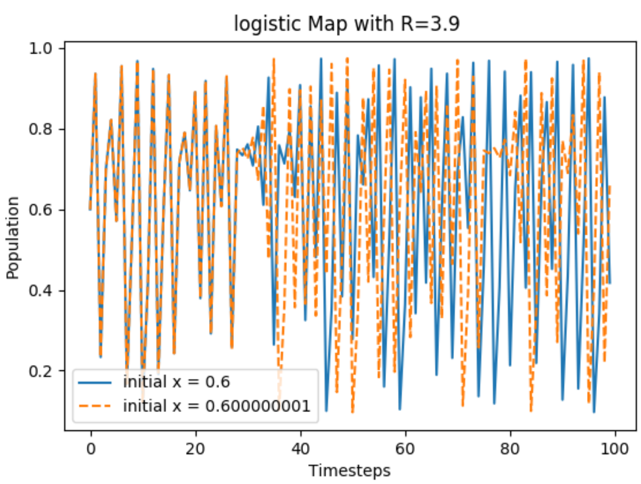

Fig. 2. Example of the logistic map where R leads to chaotic behavior. Note that the initial conditions are only slightly perturbed but lead to dramatically different results.

This is incredibly different. All that was changed was the r. The first thing to notice is that the behavior is immediately chaotic for both of the data sets. There is no pattern to the population over time. Something is interesting though. The two data sets with only slightly perturbed initial values seem to follow the same path much like the other graph but eventually diverge from each other. This is an example of **sensitive dependence on initial conditions**. This is why it is hard to predict the future of a chaotic system. The tools that we have to measure phenomena can only have so many degrees of accuracy.

Is it possible to effectively measure the information stored in the data shown in the graphs above?Yes. This paper will utilize the JIDT tool [6] to measure **Shannon Entropy and mutual information** of the data sets above, which is a way to measure the information contained in these figures. It can be seen that in the figure representing the chaotic behavior that the two separate time series diverge at about 31 time steps. The following Venn diagrams will showcase the Shannon entropy and mutual information. The overlapping circle represents mutual information between the two time series. Each circle by itself will represent the total Shannon entropy in the specific time series. There will be a total of four Venn diagrams. There will be two for each figure above. One will showcase the Shannon entropy and mutual information up to the time step 31. The other Venn diagram will showcase the same information after time step 31. To accurately depict this information, the data will be calculated with 10000 time steps instead of 100.

To calculate this information, all data must be discretized. This paper will use numpy.digitize() and map the current values from 0 to 1 into 100 bins. The goal is to represent the data as percent of the population. This is already what the data represents and this meaning of the data can still be captured this way by using 100 bins. Keep in mind that the maximum amount of entropy bits needed to represent 100 bins is 7. this is the max number of entropy possible from the calculations that follow.

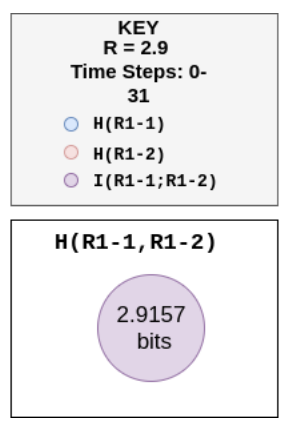

Fig. 3. entropy of each time series for time steps 0 to 31 from the logistic map in fig 1 above. **Note: figure above stops at time series 100.** All information from both time series is mutual.

Figure 3 above is stating that up to time step 31 of figure 1, there is 2.9157 bits of entropy in each time series. These are the time series represented by the solid orange line and the striped blue line. Out of the entropy from both time series, all of it is mutual information. this is why there is only one circle in the Venn diagram. This is stating that the series is the same. Since there is some entropy, it means that the population does vary over time but there is a pattern to the data. The entropy is relatively low compared to the maximum amount that is 7.

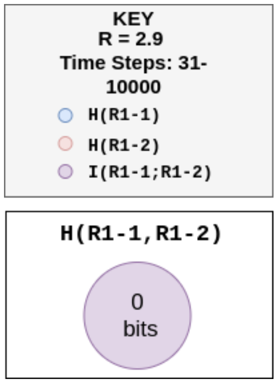

Fig. 4. entropy of each time series for time steps 32 to 10000. from the logistic map in fig 1 above.**Note: figure above stops at time series 100.**

Figure 4 gets even less interesting as it analyzes time steps from 32 to 10000. It is essentially measuring the entropy once it reaches the fixed point. This can be understood by the 0 bits of information. This means that the data is constant.

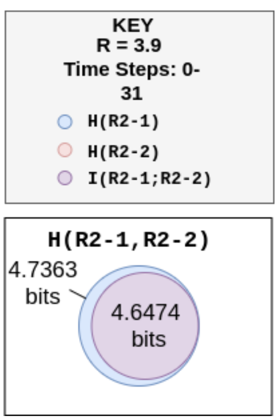

Fig. 5. entropy of each time series for time steps 0 to 31 from the logistic map in fig 2 above. **Note: figure above stops at time series 100.**

Figure 5 analyzes the time series from figure 2 above that leads to chaos, but it analyzes the section where the two time series are essentially identical. This can be seen in the Venn diagram because there is essentially only one circle that represents the mutual information. Another thing to note is that although this Venn diagram looks similar to figure 3, there is more entropy in general, so, the time series are more erratic and unpredictable.

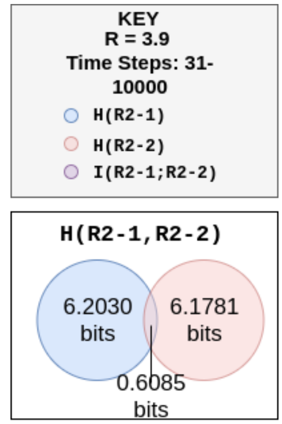

Fig. 6. entropy of each time series for time steps 0 to 31 from the logistic map in fig 2 above. **Note: figure above stops at time series 100.**

Figure 6 gets really interesting. It explores the section of the time series in figure 2 after they diverge from each other. Intuitively it seems that, if you know anything about one time series, that you will not gain any information about the other time series. This intuition is expressed in the Venn diagram. There is very little mutual information between each time series. Another thing to note is the entropy of each time series is extremely high. it is over 6 bits and very close to the maximum of 7. This number expresses the chaotic behavior of figure 2 above.

The last thing that will be calculated in this section is the transfer entropy between the two time series in figure 1 and figure 2 above. Again using the JIDT [6] tool, the following calculations have been determined. Transfer entropy between each time series from figure 1 above is 0. Transfer entropy between each time series from figure 2 above is 0.5799. The low transfer entropy expresses that knowing the history of either of each time series we do not understand any information about either time series.

With the information expressed in this part, part 2 will begin by attempting to replicate the results from Walker's paper [8]

B. Part 2

Understanding the fundamental concepts expressed in part 1 of this paper will set up the foundation for the rest of the paper. In part 2, this paper tries to replicate Walker et al. paper [8] and shows the causal behavior of how subpopulations get emerged and create a causal impact on the metapopulations to behave differently over time. This paper also tries to show how the information flow gets impacted based on global coupling strength. To better understand the coupled logistic map and the behavior that this section is trying to replicate, it is important for us to first understand what equations Walker used in her paper [8]. To define the subpopulations at the next time step, the paper [8] used the equation:

$$x_{i,n+1} = (1-\epsilon)f_i(x_{i,n}) + \epsilon m_n \tag{3}$$

where i represents the specific subpopulation, ϵ represents the coupling strength, $f_i(x_{i,n})$ represents the logistic map (1) at time step n, m_n represents the mean field of instantaneous logistic map dynamics, and $x_{i,n+1}$ represents the subpopulation i at the next time step $n+1$.

To represent instantaneous dynamics mean field, the paper [8] used the equation:

$$m_n = \frac{1}{N}\sum_{j=1}^{N} f_j(x_{j,n}) \tag{4}$$

In order to get the metapopulation from all subpopulations, the paper [8] introduced another equation:

$$M_n = \frac{1}{N}\sum_{j=1}^{N} x_{j,n} \tag{5}$$

where M_n represents the mean field or metapopulation of all subpopulations at a given time step n.

With the help of these equations, this paper generated the coupled logistic map showing different metapopulations M_n vs. M_{n+1}, comparing the behavior of the mean of subpopulations at time step/generation n vs. $n+1$ for all given epsilons ϵ. For the number of time steps and total number of subpopulations, this paper used $n = 10000$ and $N = 1000$ respectively. For the r values, this paper used uniformly random values between the chaotic range of $3.9 - 4.0$. Let's take a look at the generated figure with mentioned details above:

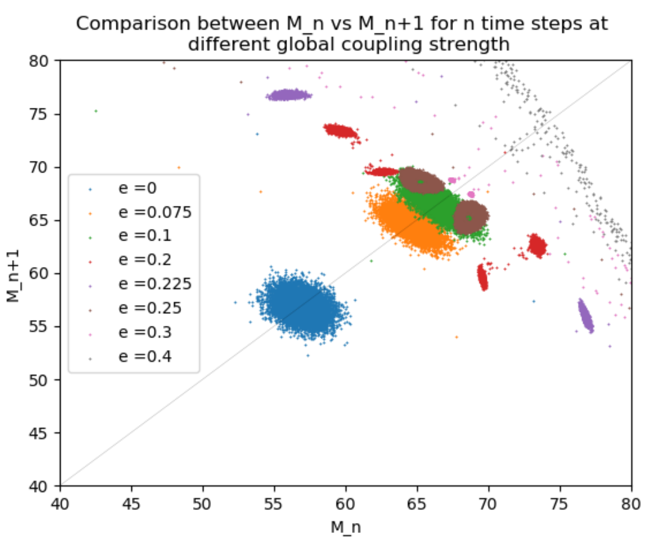

Fig. 7. shows the behavior of metapopulations M at different ϵ values over different generations of time n.

When looking at this generated figure 7, dots can be seen representing the mean field/metapopulations of $N = 1000$ logistic maps for $n = 10000$ time steps. For each epsilon, it can be seen that the behavior of metapopulations is different. For $\epsilon = 0(blue)$, there is no collective behavior among the mean of subpopulations as they are sticking to a fixed point, i.e. around a range of $M = 54$ to $M = 62$. The same behavior exists for $\epsilon = 0.075(orange)$ and $\epsilon = 0.1(green)$ where no such collective behavior can be seen among metapopulations for all subpopulations. As the ϵ values increase, it can be seen that there is starting to be some sort of collectivity among the metapopulations. For $\epsilon = 0.2(red)$, it can be seen that four clusters have been made which shows that now the metapopulations are somewhat behaving collectively, meaning that the metapopulations have started to influence the subpopulations to behave in a certain way. As the ϵ increases more, the collective behavior of metapopulations increases and becomes chaotic, telling us that the metapopulation has a significant causal impact on the subpopulations.

After analyzing the collective behavior of the metapopulations for all the subpopulations over time for different epsilons, it is now important for us to understand how this collective behavior among metapopulations created a causal impact on the subpopulations and vice versa. To understand this, part 3 talks about Transfer Entropy and how causality changes as epsilon increases.

C. Part 3

In part 3, this paper goes one step further by analyzing how the metapopulations caused the subpopulations to behave collectively and vice versa. It also analyzes whether the claims made by Walker [8] are true, whether the metapopulations' causation dominates the subpopulations for increasing epsilon values, primarily $0.2 < \epsilon < 0.7$? Is there any emerging behavior that happens when $\epsilon = 0.075$? To answer these questions, we first need to understand the causal information flow and the concept of transfer entropy is critical. The general concept of transfer entropy here is, let's say that we have metapopulations for all n generations, and we want to see the amount of uncertainty reduced in the future values of N subpopulation. Then, for the causal impact to happen, we need to know the history of metapopulations given that we know the history of a specific generation of subpopulation N within a metapopulation [3]. The more transfer entropy there is, the more uncertainty is reduced, meaning that the more causal impact it can have. The same idea can be calculated the other way around where we check the causality of a subpopulation on the metapopulation. The equations discussed in detail for transfer entropy can be seen in the Walker paper [8]. This paper uses the JIDT tool [6] for calculating transfer entropy.

To provide statistical significance, for each increasing epsilon value, this paper calculated 10 different metapopulations and sampled 3 random subpopulations in each metapopulation, primarily calculating 30 different values for both top-down (metapopulation to three subpopulations) and bottom-up (three subpopulations to metapopulation) transfer entropies. After calculating 30 values, both top-down and bottom-up values were then averaged together to represent the transfer entropy for each epsilon on the figure 8. The difference between bottom-up and top-down transfer entropies was also calculated to see the significant behavior of top-down to bottom-up. The shaded regions for both entropies were calculated using top-down $+/-$ standard deviation and bottom-up $+/-$ standard deviation.

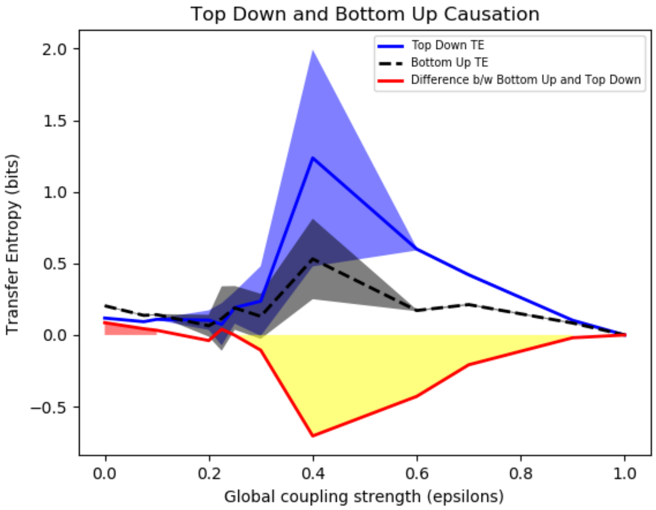

Fig. 8. shows top-down and bottom-up transfer entropy values for increasing epsilons.

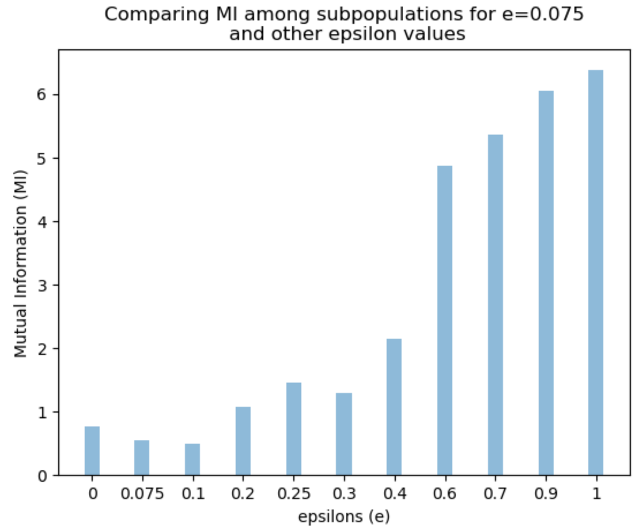

Fig. 9. shows the bar graph representing mutual information for epsilon 0.075 vs other values of epsilons.

Figure 8 analyzes the causation between top-down and bottom-up and vice versa. For epsilon $\epsilon = 0$, the bottom-up has a slightly more causal impact on the top-down, creating a reversal of flow of information. As the values of epsilon increase, the top-down start to have a significant causal impact on the bottom-up and the flow of information tend to increase. One way to look at this behavior is by taking a difference between bottom-up and top-down and seeing whether the difference is > 0, if it is > 0, then it becomes trivial to analyze the significance of bottom-up and same happens vice versa when the difference is < 0. So, as the epsilon gets in the range $0.2 < \epsilon < 0.8$, the difference(red line) gets significant, meaning the top-down is significantly causing the bottom-up to behave collectively. To be exact, at $\epsilon = 0.4$, that difference is the most significant. For $\epsilon = 1$, the transfer entropy becomes zero as both systems are synchronized and certain. So, there is no need for the uncertainty to be reduced.

Walker mentioned in her paper that the organization of subpopulations emerges at $\epsilon = 0.075$ [8]. To test her claim, this paper compares the Mutual Information (MI) among subpopulations at epsilon $\epsilon = 0.075$ with other values of epsilon.

Figure 9 shows the MI values for epsilon $\epsilon = 0.075$ vs other values of epsilon. As can be seen from the figure above, the subpopulations don't emerge at $\epsilon = 0.075$ as there is low MI and less amount of uncertainty is reduced if we look at the transfer entropy. The MI among subpopulations for other epsilon values is high as the subpopulations emerge as collective behavior. When the subpopulations become completely synchronized, for example, at $\epsilon = 1$, the mutual information is high and the subpopulations are completely certain so, the transfer entropy there is zero.

What can be found from this experiment is that Walker's claim about the emerging behavior at $\epsilon = 0.075$ was wrong. It is valid that the subpopulations emerge as the coupling strength increases but the subpopulation doesn't necessarily emerge at $\epsilon = 0.075$ and the evidence that is shown in Figure 9 contradicts her claim.

D. Part 4

Part 4 will focus on the use of the growth variable r in the sections above. The Walker paper [8] states that they are using a randomly chosen r at any point that r is needed for a calculation. The Walker paper almost dismisses the importance of r, but it may be the most critical variable related to the results. Small changes in r reveal a vastly different set of results. Part 4 will make two changes to the original calculations.

The first change will be to eliminate the random generation of r. How the growth rate changes over generations is a candidate for more research that is not performed here. This paper makes the assumption that the growth rate changing so drastically from generation to generation may be an unnatural model. This paper will apply a Perlin noise function to the number generation for r. This will give a smoother, gradient behavior to how r is generated compared to the previous generation. The inspiration for applying this idea can be found from Daniel Shiffman's book *The Nature of Code* [7]. Perlin noise was originally created by Ken Perlin to create a natural looking texture for the original *Tron* movie. [4] The next figure will showcase the type of behavior to expect from Perlin noise vs a traditional Random Number Generator.

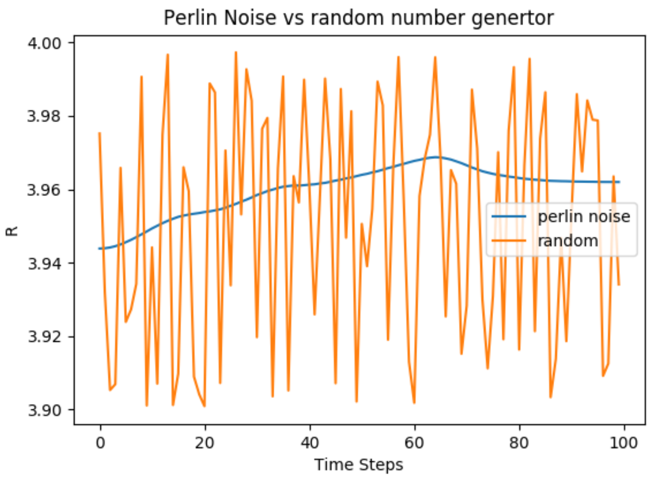

Fig. 10. This figure showcases the difference between a Perlin noise number generator and a traditional random number generator

The second change that will be made is to broaden the range of which r will be in. The Walker paper [8] forces r to be contained in the range $[3.9, 4.0]$. This does not encompass the full range at which chaos can be found in the logistic map. As part 1 explains this can be found when $r > 3.5$ [1]. The paper will expand the range of r from Walker's paper to be contained in $[3.5, 4.0]$.

To begin, this part will showcase how limiting the range changes the results. The next figure showcases the original range $[3.9, 4.0]$ using Perlin noise.

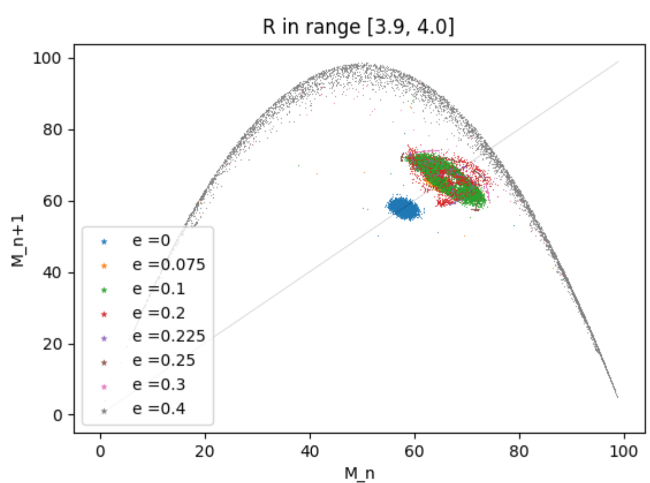

Fig. 11. Perlin noise generated r in range 3.9 to 4.0

Now what happens if we change this range slightly? The next figure showcases $[3.8, 3.9]$ using Perlin noise.

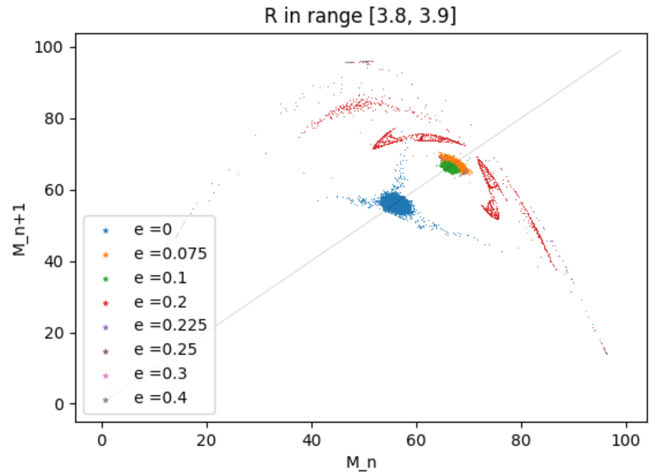

Fig. 12. Perlin noise generated r in range 3.8 to 3.9

Let's shift the range one more time. The next figure showcases $[3.6, 3.7]$ using Perlin noise.

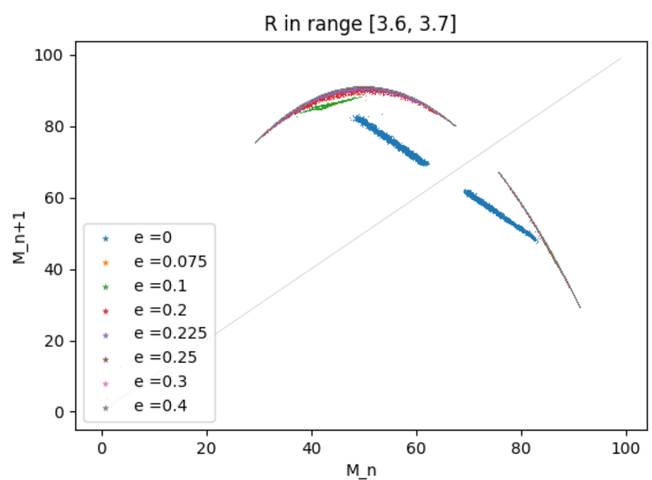

Fig. 13. Perlin noise generated r in range 3.6 to 3.7

It can be seen that the results vary greatly per range. The graphs also change slightly every time they are run as well. The full range of $[3.5, 4.0]$ should have been accounted for in the Walker paper [8]. The next figure will show the results for the full range.

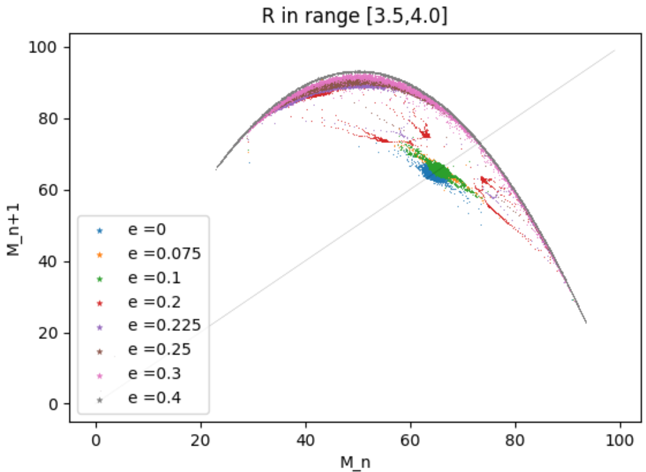

Fig. 14. Perlin noise generated r in range 3.5 to 4.0

Interestingly, the results in this last figure seem to stay similar even after performing the calculations 10 times. If only analyzing this last figure, it would seem that there is something interesting beginning at coupling strength of .2 and the the top-down causation takes over almost immediately at this point.

III. Discussion & Conclusions

When it comes to information flow and the emergence of collective behavior, it can be seen in Figure 8 that initially, the bottom-up was essentially driving the top-down but as the epsilon values increased, the top-down drove the emergence of collective behavior among bottom-up (subpopulations). Throughout the history of evolution, this behavior can be seen everywhere. For example, the transition of the stock in the stock market. The emergence of information flow is critical in all areas. If we relate these ideas to computer systems, the Walker paper [8] displayed and discovered an extremely creative way to analyze the birth of a new entity by analyzing and measuring bottom-up and top-down causation through the coupled logistic map and information flow. However, there are some obvious flaws in the Walker paper [8]. For example, her claim about the emergence of organization at epsilon = 0.075 was misleading. As mentioned above in part 3 figure 9, the amount of uncertainty reduced and the mutual information among subpopulations at epsilon = 0.075 doesn't give us the signs of an emergence among subpopulations. Although the emergence and transition occur for increasing values of epsilon, there wasn't any visible sudden emergence that occurred at epsilon = 0.075 from our analysis.

Part 4 determined that the value of r is extremely important to the output of these results. By limiting the range of r to $[3.9, 4.0]$ the Walker paper [8] may have found a range that produced interesting data to analyze, but it does not encompass the full range of possible r's. to accurately analyze the results in the Walker paper [8], the full range of r must be used. Another point is that since r is so important, there should be a lot of attention placed on how to model the change in r over each generation. The Walker paper used a random r and part4 utilized a Perlin noise with vastly different results. Perlin Noise may not be the right choice, but it highlights the importance of finding a better model. The last point is that the calculations being analyzed change per run, and to get the most accurate results, it should be run many times before it can be analyzed.

References

[1] D. Doretsays, B. Waymansays, Q. G. A.- zaidisays, M. F. Ferdoussays, Lorenz, B. Nelsonsays, and A. R. Chowdhurysays, "Chaos Theory and the Logistic Map," Geoff Boeing, 21-Oct-2016. [Online]. Available: https://geoffboeing.com/2015/03/chaos-theory-logistic-map/. [Accessed: 20-Feb-2020].

[2] W. G. Flake, The computational beauty of nature: Computer explorations of fractals, chaos, complex systems and adaptation. Massachusetts: M.I.T. Press, 1998.

[3] M. Mitchell, Complexity: a guided tour. Oxford: Oxford University Press, 2011.

[4] "Perlin noise," Wikipedia, 12-Jan-2020. [Online]. Available: https://en.wikipedia.org/wiki/Perlin_noise. [Accessed: 20-Feb-2020].

[5] "Perlin noise (article) — Noise," Khan Academy. [Online]. Available: https://www.khanacademy.org/computing/computer-programming/programming-natural-simulations/programming-noise/a/perlin-noise. [Accessed: 20-Feb-2020].

[6] Joseph T. Lizier, "JIDT: An information-theoretic toolkit for studying the dynamics of complex systems", Frontiers in Robotics and AI 1:11, 2014; doi:10.3389/frobt.2014.00011 (pre-print: arXiv:1408.3270)

[7] D. Shiffman and S. Fry, The nature of code. United States: D. Shiffman, 2012.

[8] S. I. Walker, L. Cisneros, and P. C. W. Davies, "Evolutionary Transitions and Top-Down Causation," Artificial Life 13, pp. 1–8, Feb. 2012.

Contributions

Siri Khalsa wrote the code for Part 1 and Part 4. He also wrote the methods and results for Part 1, Part 4, and the introduction of this paper. Anas Gauba wrote the code for Part 3 and wrote the associated methods and results paragraphs. He also wrote the methods and results for Part 2 and the abstract of this paper. Both authors collaborated to write the code for Part 2 and Discussions and Conclusions section of this paper.