Modeling fluid flow

Stokes Equations
and
Method of Regularized Stokeslets

Mathematical Modeling, Computational Methods, and Biological Fluid Dynamics: Research and Training
NITMB, Chicago, Illinois

Developed in collaboration with Ricardo Cortez

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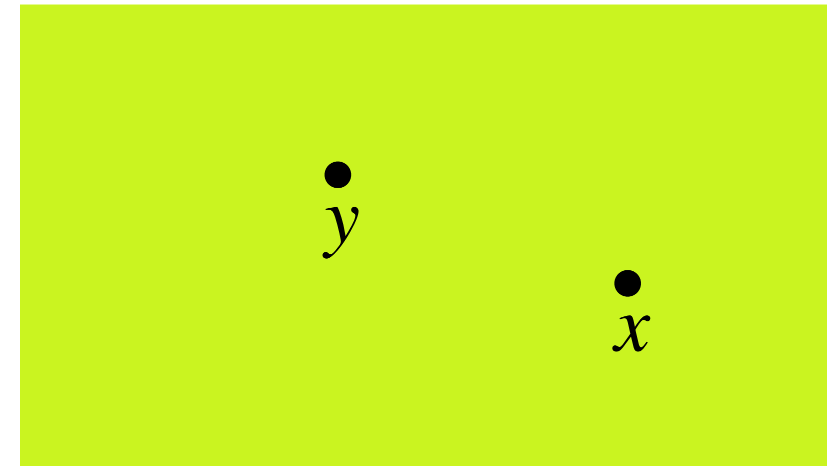
Stokes Equations (2D)

$$\nabla p = \mu \nabla^2 u + f\delta(x - y)$$
$$\nabla \cdot u = 0$$



position of solution: $x = (x_1, x_2)$

position of force: $y = (y_1, y_2)$

velocity: $u(x) = (u_1(x_1, x_2), u_2(x_1, x_2))$

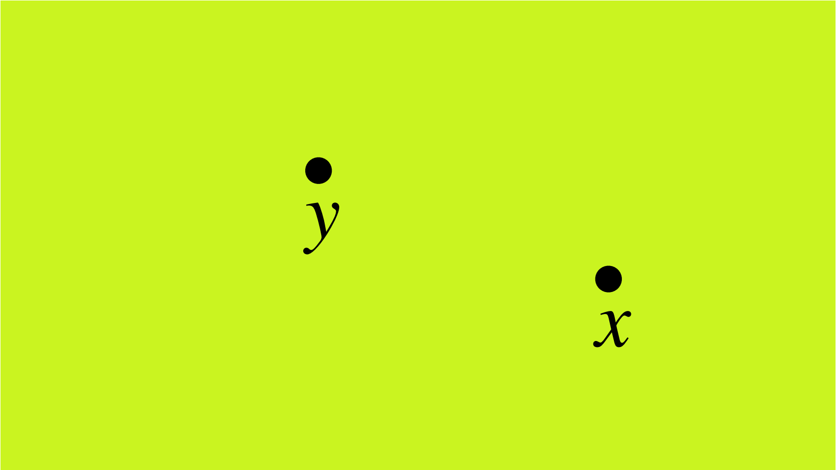
pressure: $p(x) = p(x_1, x_2)$

force applied on fluid: $f(y) = (f_1(y_1, y_2), f_2(y_1, y_2))$

viscosity: μ

Stokes Equations (3D)

$$\nabla p = \mu \nabla^2 u + f\delta(x - y)$$
$$\nabla \cdot u = 0$$



position of solution: $x = (x_1, x_2, x_3)$

position of force: $y = (y_1, y_2, y_3)$

velocity: $u(x) = (u_1(x_1, x_2, x_3), u_2(x_1, x_2, x_3), u_3(x_1, x_2, x_3))$

pressure: $p(x) = p(x_1, x_2, x_3)$

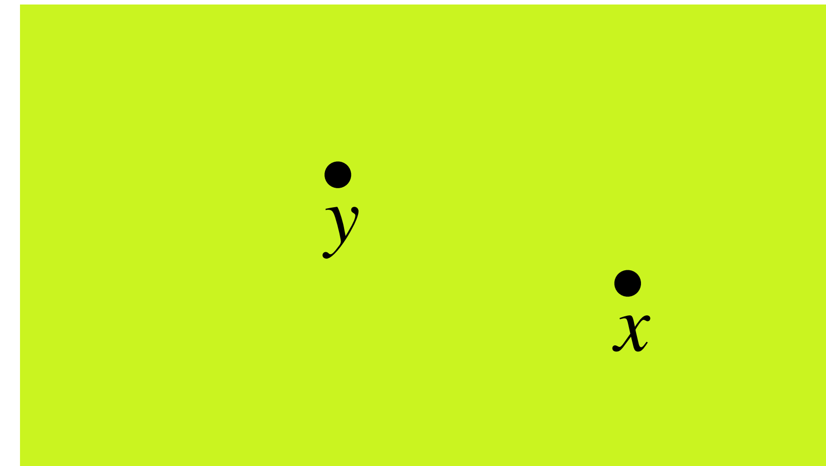
force applied on fluid:

$$f(y) = (f_1(y_1, y_2, y_3), f_2(y_1, y_2, y_3), f_3(y_1, y_2, y_3))$$

viscosity: μ

Stokes Equations (2D)

$$\nabla p = \mu \nabla^2 u + f\delta(x - y)$$
$$\nabla \cdot u = 0$$

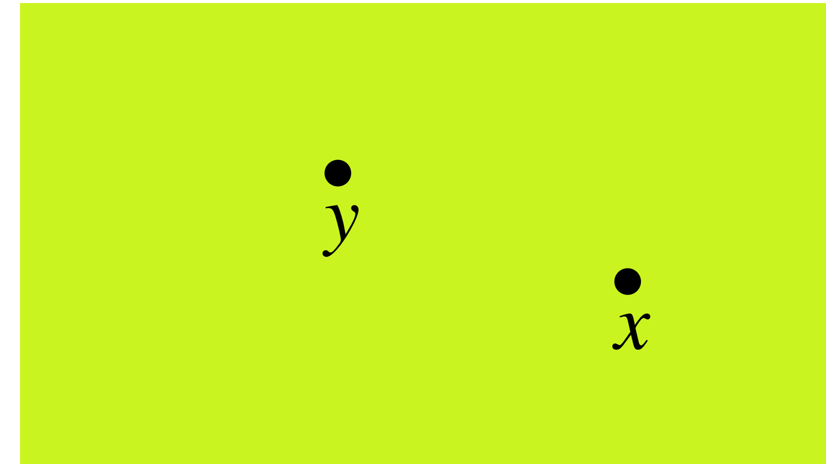


component-wise:

$$\frac{\partial p}{\partial x_1} = \mu \left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} \right) + f_1 \delta(x_1 - y_1, x_2 - y_2)$$
$$\frac{\partial p}{\partial x_2} = \mu \left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} \right) + f_2 \delta(x_1 - y_1, x_2 - y_2)$$
$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0$$

Stokes Equations (3D)

$$\nabla p = \mu \nabla^2 u + f\delta(x - y)$$
$$\nabla \cdot u = 0$$

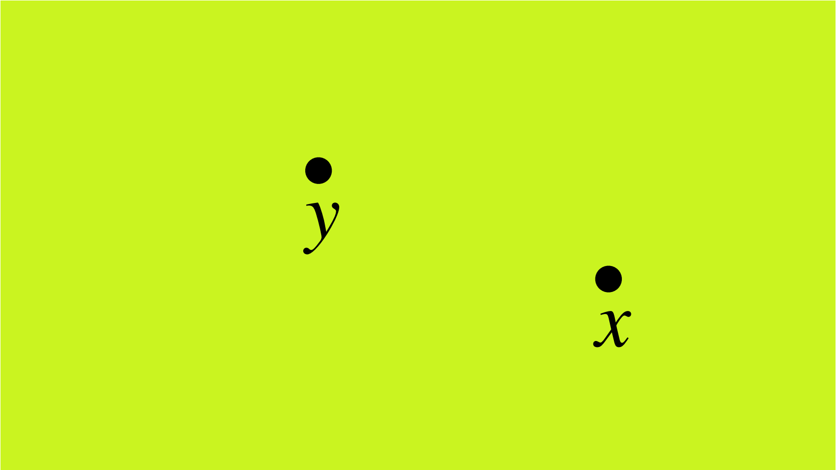


component-wise:

$$\frac{\partial p}{\partial x_1} = \mu \left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2} \right) + f_1 \delta(x_1 - y_1, x_2 - y_2, x_3 - y_3)$$
$$\frac{\partial p}{\partial x_2} = \mu \left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_3^2} \right) + f_2 \delta(x_1 - y_1, x_2 - y_2, x_3 - y_3)$$
$$\frac{\partial p}{\partial x_3} = \mu \left(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} + \frac{\partial^2 u_3}{\partial x_3^2} \right) + f_3 \delta(x_1 - y_1, x_2 - y_2, x_3 - y_3)$$
$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0$$

Stokes Equations: force

$$\nabla p = \mu \nabla^2 u + f\delta(x-y)$$
$$\nabla \cdot u = 0$$



Dirac delta function 2D: $\delta(x-y) = \delta(x_1 - y_1, x_2 - y_2)$

3D: $\delta(x-y) = \delta(x_1 - y_1, x_2 - y_2, x_3 - y_3)$

where

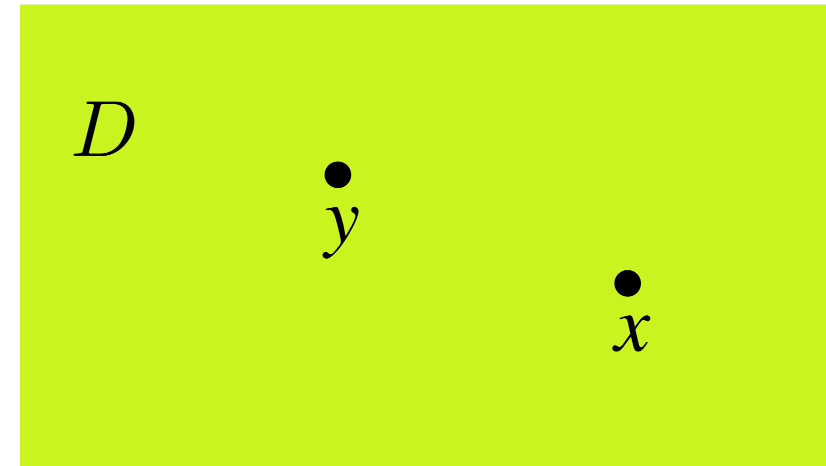
$$\int_{\mathbb{R}^n} \delta(x-y)\,dx = 1$$

$$\int_{\mathbb{R}^n} g(y)\delta(x-y)\,dy = g(x)$$

**time dependence can come in through the force term, f, depending on time or the location of the force depending on time

Stokes Equations: Solution with a point force

$$\nabla p = \mu \nabla^2 u + f\delta(x-y)$$
$$\nabla \cdot u = 0$$



Pressure and velocity at x due to a point force applied at y

$$p(x) = f(y) \cdot \nabla G(x-y)$$
$$u(x) = \frac{1}{\mu}\big((f(y) \cdot \nabla)\nabla B(x-y) - f(y)G(x-y) \big)$$

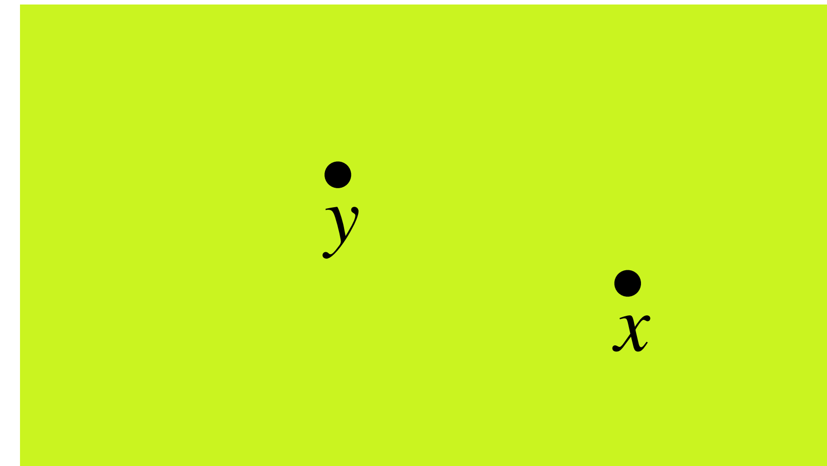
where

$$\nabla^2 G(x-y) = \delta(x-y)$$
$$\nabla^2 B(x-y) = G(x-y)$$

**can solve for this by taking divergence and solve for pressure first

Stokes Equations: Solution with a point force

$$\nabla p = \mu \nabla^2 u + f\delta(x - y)$$
$$\nabla \cdot u = 0$$



Pressure and velocity at x due to a point force applied at y

$$p(x) = f(y) \cdot \nabla G(x - y)$$
$$u(x) = \frac{1}{\mu}\Big(\left(f(y) \cdot \nabla\right) \nabla B(x - y) - f(y)G(x - y)\Big)$$

where

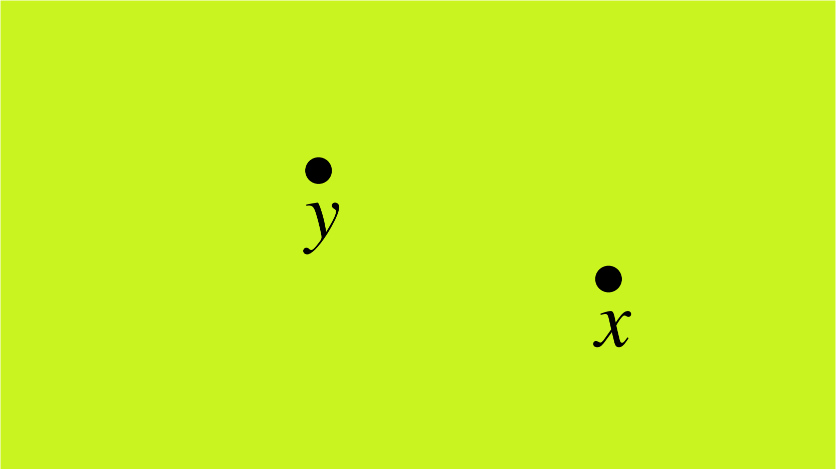
2D: $G(x - y) = \dfrac{1}{2\pi}\log(r)$ and $B(x - y) = \dfrac{r^2}{8\pi}(\log(r) - 1)$

3D: $G(x - y) = -\dfrac{1}{4\pi r}$ and $B(x - y) = -\dfrac{r}{8\pi}$

**$r = |x - y|$

Stokes Equations: Solution with a point force

$$\nabla p = \mu \nabla^2 u + f\delta(x - y)$$
$$\nabla \cdot u = 0$$



2D Green's function or Stokeslet:

$$S_{ij}(x, y) = \frac{(x_i - y_i)(x_j - y_j)}{|x - y|^2} - \delta_{ij} \log |x - y|$$

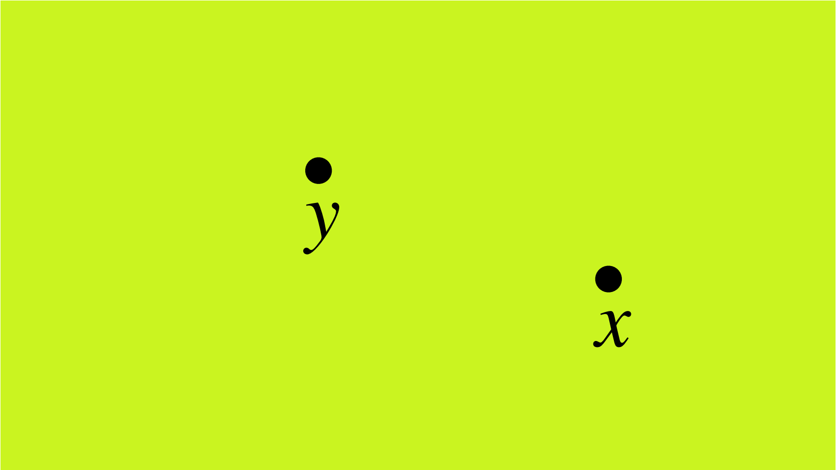
**δ_{ij} = Kronecker delta

Velocity at x due to a point force applied on the fluid at y

$$u(x) = \frac{1}{4\pi\mu} S(x, y) f = \frac{1}{4\pi\mu} \left(\frac{(f \cdot (x - y))(x - y)}{|x - y|^2} - f \log |x - y| \right)$$

Stokes Equations: Solution with a point force

$$\nabla p = \mu \nabla^2 u + f\delta(x-y)$$
$$\nabla \cdot u = 0$$



3D Green's function or Stokeslet:

$$S_{ij}(x,y) = \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^3} + \delta_{ij}\frac{1}{|x-y|}$$

**δ_{ij} = Kronecker delta

Velocity at x due to a point force applied on the fluid at y

$$u(x) = \frac{1}{8\pi\mu}S(x,y)f = \frac{1}{8\pi\mu}\left(\frac{(f\cdot(x-y))(x-y)}{|x-y|^3} + f\frac{1}{|x-y|}\right)$$

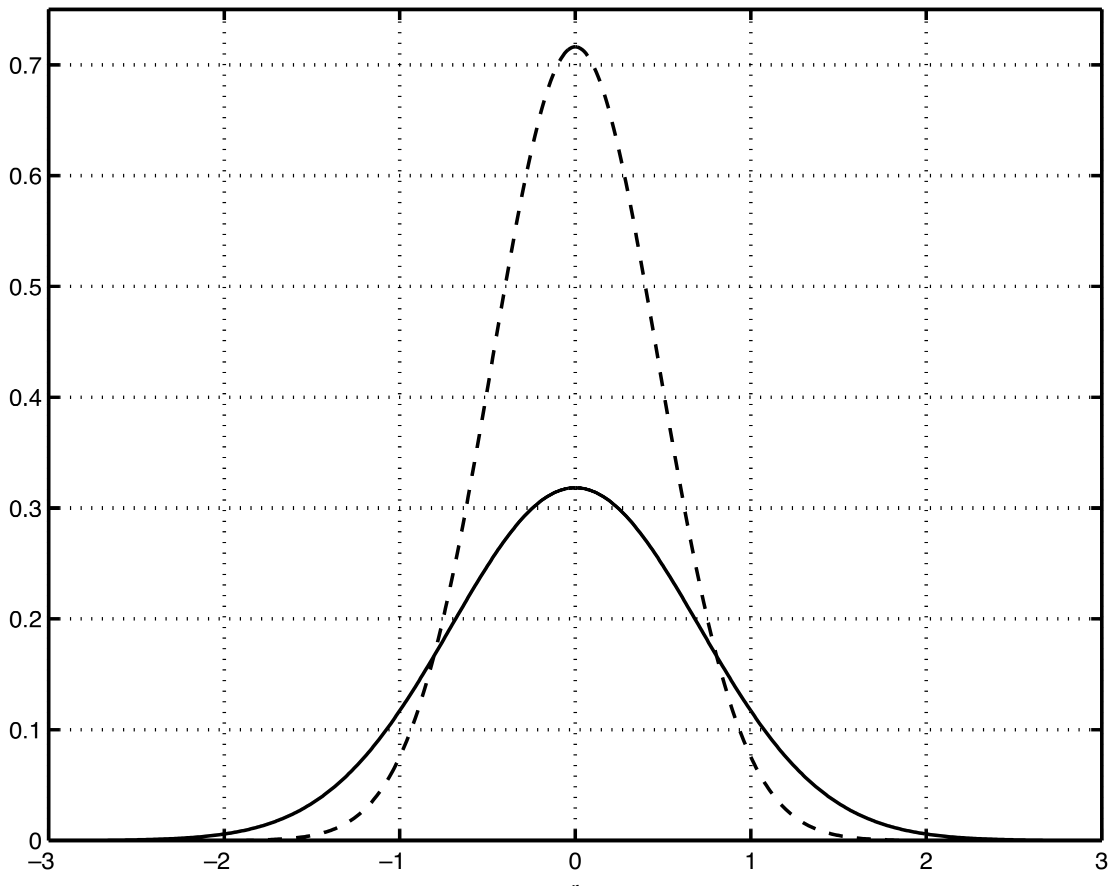
Regularization of point force

- Singularities are integrable (curves in 2D, surfaces in 3D)
- Challenges with these singularities when curves in 3D and points in 2D
- Numerical challenges dealing with integration of singularities

2D Method of Regularized Stokeslets

Idea: Regularize the Dirac delta function -> blob function

$$\phi_\epsilon(x, y) = \frac{3\epsilon^3}{2\pi(|x-y|^2 + \epsilon^2)^{\frac{5}{2}}}$$



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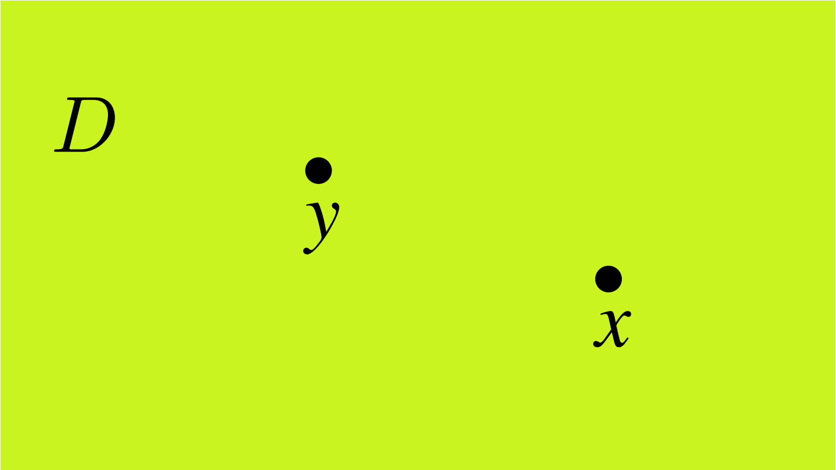
Stokes Equations: Solution with a blob force

$$\nabla p = \mu \nabla^2 u + f\phi_\epsilon(x-y)$$
$$\nabla \cdot u = 0$$



Pressure and velocity at x due to a blob force applied at y

$$p(x) = f(y) \cdot \nabla G_\epsilon(x-y)$$
$$u(x) = \frac{1}{\mu}\Big(\big(f(y)\cdot\nabla\big)\nabla B_\epsilon(x-y) - f(y)G_\epsilon(x-y)\Big)$$

where

$$\nabla^2 G_\epsilon(x-y) = \phi_\epsilon(x-y)$$
$$\nabla^2 B_\epsilon(x-y) = G_\epsilon(x-y)$$

**can solve for this by taking divergence and solve for pressure first

Stokes Equations: Solution with a blob force

Taking advantage of radial symmetry of the blobs
(which is the same as for the Dirac delta function)

Pressure and velocity at x due to a blob force applied at y

$$p(x) = f(y) \cdot \nabla G_\epsilon(x - y)$$

$$u(x) = \frac{1}{\mu}\Big(\big(f(y) \cdot \nabla\big) \nabla B_\epsilon(x - y) - f(y) G_\epsilon(x - y)\Big)$$

where

$$\nabla B_\epsilon(x - y) = B'_\epsilon(r) \frac{(x - y)}{r}$$

$$(f(y) \cdot \nabla) \nabla B_\epsilon(x - y) = f(y) \frac{B'_\epsilon(r)}{r} + (f(y) \cdot x) x \left(\frac{r B''_\epsilon(r) - B'_\epsilon(r)}{r^3} \right)$$

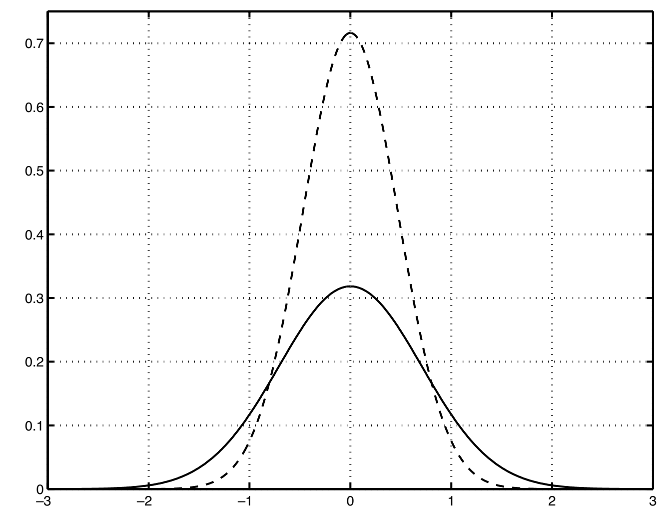
$$^{**} r = |x - y|$$

Regularization of point force

Method of Regularized Stokeslets 2D

Idea: Regularize the Direct delta function -> blob function

$$\phi_\epsilon(x, y) = \frac{3\epsilon^3}{2\pi(|x-y|^2+\epsilon^2)^{\frac{5}{2}}}$$



Solution in this case:

$$p(x) = \frac{1}{2\pi} f(y) \cdot (x-y) \left(\frac{R_\epsilon^2 + \epsilon^2 + \epsilon R_\epsilon}{(R_\epsilon + \epsilon) R_\epsilon^{3/2}} \right)$$

$$u(x) = \frac{-f}{4\pi\mu} \left(\log(R_\epsilon + \epsilon) - \frac{\epsilon(R + 2\epsilon)}{(R_\epsilon + \epsilon) R_\epsilon} \right) + \frac{1}{4\pi\mu} (f \cdot (x-y))(x-y) \left(\frac{R_\epsilon + 2\epsilon}{(R_\epsilon + \epsilon)^2 R_\epsilon} \right)$$

$$^{**}R_\epsilon = \sqrt{|x-y|^2 + \epsilon^2}$$

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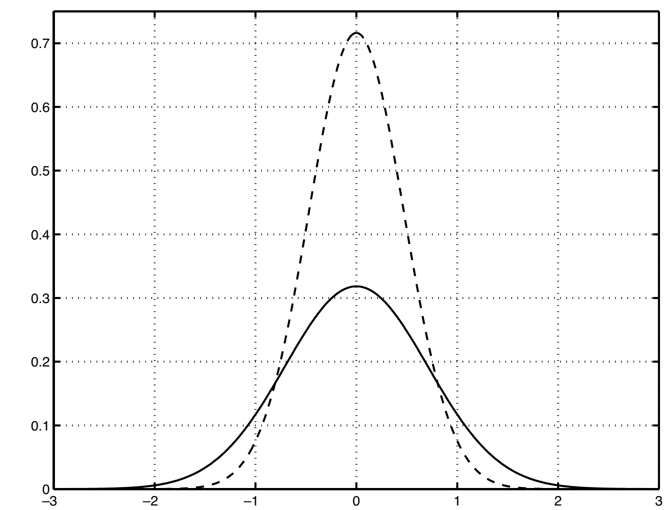
Another blob: more commonly used

Method of Regularized Stokeslets 2D

Idea: Regularize the Direct delta function -> blob function

$$\phi_\epsilon(x, y) = \frac{2\epsilon^4}{\pi(|x-y|^2 + \epsilon^2)^3}$$



Solution in this case:

$$p(x) = \frac{1}{2\pi} f(y) \cdot (x-y) \left(\frac{R_\epsilon^2 + \epsilon^2}{R_\epsilon^4} \right)$$

$$u(x) = \frac{f}{4\pi\mu} \left(-\log(R_\epsilon) + \frac{\epsilon^2}{R_\epsilon^2} \right) + \frac{1}{4\pi\mu} (f \cdot (x-y))(x-y) \left(\frac{1}{R_\epsilon^2} \right)$$

$$^{**}R_\epsilon = \sqrt{|x-y|^2 + \epsilon^2}$$

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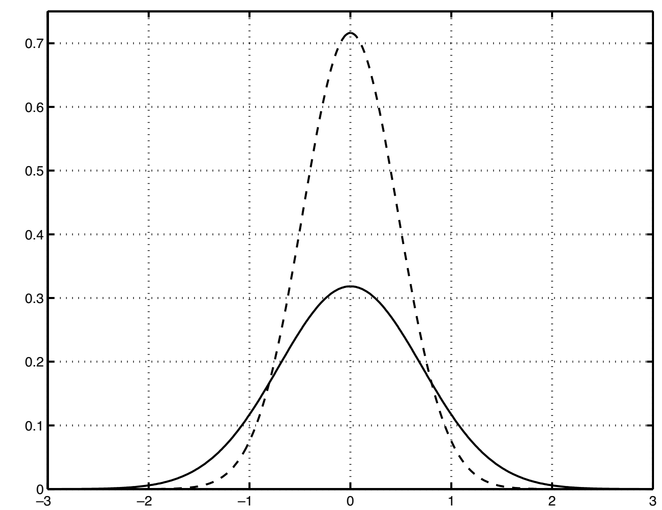
Regularization of point force

Method of Regularized Stokeslets 3D

Idea: Regularize the Direct delta function -> blob function

$$\phi_\epsilon(x,y) = \frac{15\epsilon^4}{8\pi(|x-y|^2+\epsilon^2)^{\frac{7}{2}}}$$



Solution in this case:

$$p(x) = \frac{1}{8\pi} f(y)\cdot(x-y)\left(\frac{2R_\epsilon^2+3\epsilon^2}{R_\epsilon^5}\right)$$

$$u(x) = \frac{f}{8\pi\mu}\left(\frac{R_\epsilon^2+\epsilon^2}{R_\epsilon^3}\right) + \frac{1}{8\pi\mu}(f\cdot(x-y))(x-y)\left(\frac{1}{R_\epsilon^3}\right)$$

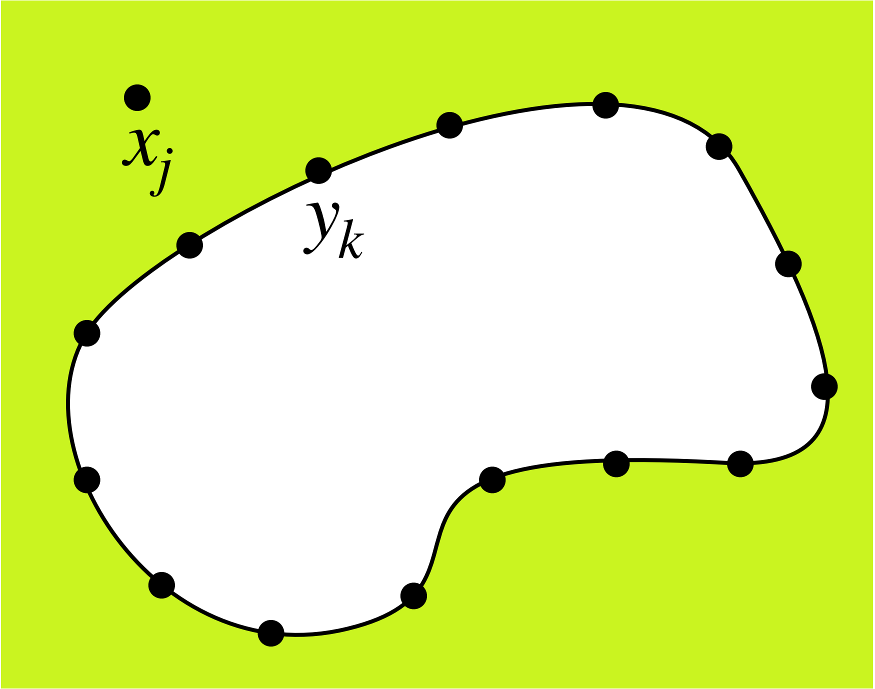
$$^{**}R_\epsilon = \sqrt{|x-y|^2+\epsilon^2}$$

Cortez et al. Phys. Fluids 2005

Discretization of ‘surface’ applying forces

$$\nabla p = \mu \nabla^2 u + \sum_{k=1}^{N} f_k \phi_\epsilon (x_j - y_k)$$

$$\nabla \cdot u = 0$$

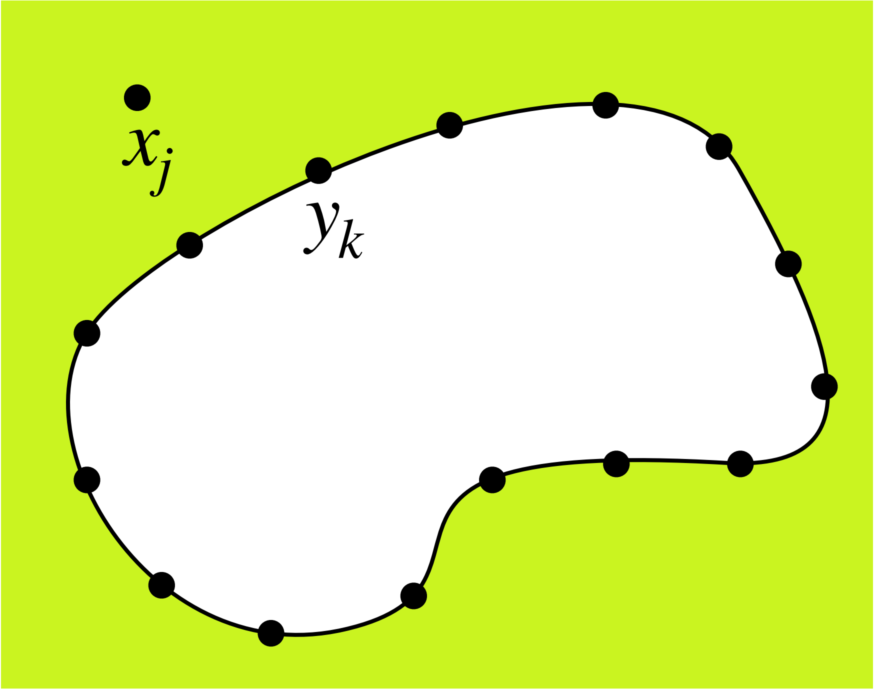


Solution (focusing on velocity):

$$u(x_j) = \frac{1}{\mu} \sum_{k=1}^{N} (f_k \cdot \nabla) \nabla B_\epsilon (x_j - y_k) - f_k G_\epsilon (x_j - y_k)$$ where $j = 1,2 \ldots M$

Discretization of 'surface' applying forces

Applying forces continuously:



$$\nabla p = \mu \nabla^2 u + \int g(y(s)) \phi_\epsilon(x_j - y(s))\, ds$$

$$\nabla p = \mu \nabla^2 u + \sum_{k=1}^{N} g_k \phi_\epsilon(x_j - y_k) ds$$

$$\nabla p = \mu \nabla^2 u + \sum_{k=1}^{N} f_k \phi_\epsilon(x_j - y_k)$$

where $g(y(s))$ and g_k are both force densities and $f_k = g_k ds$

Discretization of 'surface' applying forces

$$u(x_j) = \frac{1}{\mu}\sum_{k=1}^{N}(f_k \cdot \nabla)\nabla B_\epsilon(x_j - y_k) - f_k G_\epsilon(x_j - y_k) \quad \text{where } j = 1,2\ldots M$$

Rewrite in matrix form:

$$\vec{u} = M\vec{f} \quad \text{where } \vec{u} = \begin{bmatrix} u_{11} \\ u_{12} \\ \cdot \\ \cdot \\ \cdot \\ u_{1M} \\ u_{21} \\ u_{22} \\ \cdot \\ \cdot \\ \cdot \\ u_{2M} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \text{ and } \vec{f} = \begin{bmatrix} f_{11} \\ f_{12} \\ \cdot \\ \cdot \\ \cdot \\ f_{1N} \\ f_{21} \\ f_{22} \\ \cdot \\ \cdot \\ \cdot \\ f_{2N} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

and M is $2M \times 2N$ in 2D

M is $3M \times 3N$ in 3D

**Note you can go from forces to velocities or velocities to forces (well-posedness)

Background flows

can add constant background translational and angular velocity

$$u(x_j) = \frac{1}{\mu}\sum_{k=1}^{N} (f_k \cdot \nabla)\nabla B_\epsilon(x_j - y_k) - f_k G_\epsilon(x_j - y_k) + U_c + \Omega_c \times x$$

where $j = 1,2\ldots M$

When we do not include these (as in the previous slide) we choose

$U_c = \dfrac{1}{8\pi\mu}\displaystyle\sum_{k=1}^{N} f_k$ in 2D and $U_c = 0$ in 3D

$$\Omega = 0$$

This removes a constant translational flow from everywhere.

Biological Flows

In biological flows often want net force and net torque to be zero which may allow the organism to have a rigid body motion

$$u(x_j) + U_o + \Omega \times x_j = \frac{1}{\mu} \sum_{k=1}^{N} (f_k \cdot \nabla) \nabla B_\epsilon(x_j - y_k) - f_k G_\epsilon(x_j - y_k)$$

where $j = 1,2 \ldots M$

Add to system:

Net force $= \sum_{k=1}^{N} f_k = 0$ and

Net torque = $\sum_{k=1}^{N} y_k \times f_k = 0$

Solve for U_o and Ω additionally now. Usually, x_j's are the same as the y_k's.

Biological Flows: Augmented matrix

$$u(x_j) + U_o + \Omega \times x_j = \frac{1}{\mu} \sum_{k=1}^{N} (f_k \cdot \nabla) \nabla B_\epsilon(x_j - y_k) - f_k G_\epsilon(x_j - y_k)$$

where $j = 1,2 \ldots M$

Now,

$$\vec{u} = M\vec{f} \quad \text{where } \vec{u} = \begin{bmatrix} u_{11} \\ u_{12} \\ \cdot \\ \cdot \\ \cdot \\ u_{1M} \\ u_{21} \\ u_{22} \\ \cdot \\ \cdot \\ \cdot \\ u_{2M} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \text{ and } \vec{f} = \begin{bmatrix} f_{11} \\ f_{12} \\ \cdot \\ \cdot \\ \cdot \\ f_{1N} \\ f_{21} \\ f_{22} \\ \cdot \\ \cdot \\ \cdot \\ f_{2N} \\ \cdot \\ \cdot \\ \cdot \\ U_{o1} \\ U_{o2} \\ \cdot \\ \Omega \end{bmatrix}$$

and M is $2M \times (2N + 3)$ in 2D

M is $3M \times (3N + 6)$ in 3D

References and Acknowledgments

References:

- R. Cortez, The Method of Regularized Stokeslets, SIAM J. Sci Comp, 2001
- R. Cortez, L. Fauci and A. Medovikov The method of regularized Stokeslets in three dimensions: Analysis, validation, and application to helical swimming, Phys. Fluids, 2005

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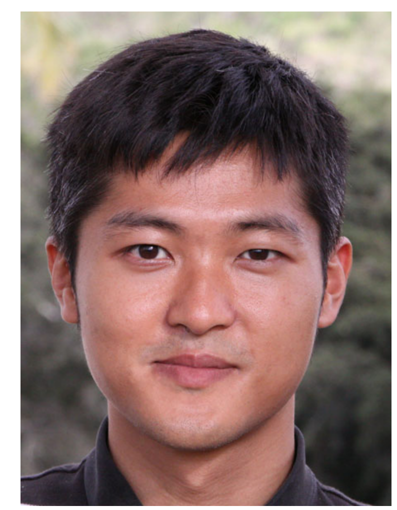

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