Chapter 3: The Four Fundamental Subspaces

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1 Vector Spaces

Definition 1.1. *A **vector space** V defined over $\mathbb{R}$ consists of a set on which addition and scalar multiplication are defined so that for each pair of elements $v, w \in V$, there is a unique element $v + 2 \in V$, and for each element $c \in \mathbb{R}$ and $v \in V$, there is a unique element $cv \in V$, such that the following conditions hold:*

1. *For all $v, w \in V$, $v + w = w + v$ (commutativity).*
2. *For all $u, v, w \in V$, $(u + v) + w = u + (v + w)$ (associativity under addition).*
3. *There exists an element $0 \in V$ such that $v + 0 = v$ for each $v \in V$ (identity element under vector addition).*
4. *For each element $v \in V$, there exists an element $-v \in V$ such that $v + (-v) = 0$ (additive inverse).*
5. *There exists an element $1 \in V$ such that $1v = v$ for each $v \in V$ (identity element under scalar multiplication).*
6. *For each pair of elements $c, d \in \mathbb{R}$, and each $v \in V$, $(cd)v = c(dv)$ (associativity under scalar multiplication).*
7. *For each element $c \in \mathbb{R}$, and each pair $v, w \in V$, $c(v + w) = cv + cw$ (distributivity of scalar multiplication over vector addition).*
8. *For each pair of elements $c, d \in \mathbb{R}$, and each $v \in V$, $(c + d)v = cv + dv$ (distributivity of scalar addition over scalar multiplication).*

2 Subspaces

Definition 2.1. *Let V be a vector space. A subset $S \subseteq V$ is a **subspace** of V if the following hold:*

1. *$\vec{0} \in S$.*
2. *If $\vec{x}, \vec{y} \in S$, then $\vec{x} + \vec{y} \in S$.*
3. *If $\vec{x} \in S$ and c is a scalar, then $c\vec{x} \in S$.*

3 Column Space

Definition 3.1. *The **column space** of a matrix consists of all linear combinations of its columns. So if $A = \begin{bmatrix} \vec{a_1} & \vec{a_2} & \cdots & \vec{a_n} \end{bmatrix}$ is an $m \times n$ matrix, where $\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n} \in \mathbb{R}^m$, then $ColA =$span$\{\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n}\}$. $ColA$ is a subspace of $\mathbb{R}^m$.*

Proposition 3.2. *The system $A\vec{x} = \vec{b}$ is solvable if and only if $\vec{b} \in C(A)$.*

4 Nullspace

Definition 4.1. *The **nullspace** of a matrix A consists of all solutions $\vec{x}$ to the system $A\vec{x} = \vec{0}$. So if $A = \begin{bmatrix} \vec{a_1} & \vec{a_2} & \cdots & \vec{a_n} \end{bmatrix}$ is an $m \times n$ matrix, where $\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n} \in \mathbb{R}^m$, then $NulA = \{\vec{x} : A\vec{x} = \vec{0}\}$. $NulA$ is a subspace of $\mathbb{R}^n$.*

5 Row Echelon Form (REF)

Definition 5.1. *An $m \times n$ matrix is in* ***row echelon form (REF)*** *if:*

1. *All rows consisting entirely of zeros lie beneath all nonzero rows.*
2. *The first nonzero element in any row, called a* ***pivot,*** *must lie to the right of any pivot above it.*

Algorithm 5.2. ***(REF)*** *To find the REF of a matrix A, find the pivots and use them to make all elements below them equal zero.*

Definition 5.3. *A* ***pivot column*** *in a matrix in REF is a column that contains exactly one pivot. A* ***free column*** *in a matrix in REF is a column that contains no pivots.*

6 Reduced Row Echelon Form (RREF)

Algorithm 6.1. ***(RREF)*** *To find the RREF of a matrix A ::*

1. *Find the REF of A using Algorithm 5.2.*
2. *Use the obtained pivots to make all elements above them equal zero.*
3. *Finally, make all pivots equal* 1.

Proposition 6.2. *The RREF of a matrix A has the same nullspace as the original matrix A.*

Algorithm 6.3. ***(Nullspace)*** *To find the nullspace of a matrix A,*

1. *Find the RREF of A.*
2. *Use the RREF and back substitution to solve the system $A\vec{x} = \vec{0}$.*
3. *NulA is the set of solutions $\vec{x}$.*

7 Rank

Definition 7.1. *The* ***rank*** *r of an $m \times n$ matrix A is the number of pivots in its REF.*

8 Complete Solutions

Theorem 8.1. *Let A be an $m \times n$ matrix such that $m < n$. In this case, we are guaranteed to have free columns, and the system $A\vec{x} = \vec{b}$ will have more unknowns than equations, so it will have free variables associated with the free columns. Thus, the system $A\vec{x} = \vec{b}$ will always have either an infinite number of solutions or no solutions.*

Definition 8.2. *The* ***particular solution*** *$\vec{x_p}$ of a system is obtained by setting the free variables to zero. $\vec{x_p}$ solves $A\vec{x_p} = \vec{b}$.*

Definition 8.3. *The* ***nullspace solution*** *$\vec{x_n}$ of a system is obtained by setting $\vec{b}$ to $\vec{0}$. There are $n - r$ nullspace solutions which solve $A\vec{x_n} = \vec{0}$, where r is the rank of A.*

Definition 8.4. *The* ***complete solution*** *to $A\vec{x} = \vec{b}$ can be expressed as $\vec{x} = \vec{x_p} + \vec{x_n}$.*

9 Ranks and Systems

Proposition 9.1. *Let A be an $m \times n$ matrix with rank r.*

1. *The r pivot columns of A are linearly independent.*
2. *A has $n - r$ free columns.*
3. *Since ColA is the span of the pivot columns of A, the column space spans an r-dimensional space.*
4. *The dimension of NulA is $n - r$.*

Definition 9.2. *Let A be an $m \times n$ matrix. Then A has* ***full column rank*** *$r = n$ if:*

1. *All columns of A are pivot columns.*
2. *All columns of A are linearly independent.*
3. *There are no free columns, which implies that there are no free solutions.*
4. *$NulA = \{\vec{0}\}$.*
5. *If $A\vec{x} = \vec{b}$ has a solution, then it has exactly one solution.*

Definition 9.3. *Let A be an $m \times n$ matrix. Then A has **full row rank** $r = m$ if:*

1. *All rows of A have pivot positions.*
2. *All rows of A are linearly independent.*
3. *There are $n - r = n - m$ nullspace solutions.*
4. *$ColA$ spans all of $\mathbb{R}^m$.*
5. *$A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$.*

Proposition 9.4. ***(Rank and Solvability)*** *Let A be an $m \times n$ matrix with rank r. The solutions to $A\vec{x} = \vec{b}$ can be classified as follows:*

1. *If $r = m = n$, then A is a square invertible matrix, so $A\vec{x} = \vec{b}$ has exactly one solution.*
2. *If $r = m, r < n$, then A is short and wide, so $A\vec{x} = \vec{b}$ has an infinite number of solutions.*
3. *If $r < m, r = n$, then A is tall and thin, so $A\vec{x} = \vec{b}$ has either no solution or exactly one solution.*
4. *If $r < m, r < n$, then A does not have full rank, so $A\vec{x} = \vec{b}$ has either no solutions or an infinite number of solutions.*

10 Row Space

Definition 10.1. *The **row space** $RowA$ of an $m \times n$ matrix A is the span of the nonzero rows of its REF.*

Theorem 10.2. *Let A be an $m \times n$ matrix. Then $RowA = ColA^T = span\{\text{linearly independent columns of } A^T\}$.*

11 Basis of a Vector Space

Definition 11.1. *A **basis** β for a vector space V is a set of vectors $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ such that*

1. *$vecv_1, \vec{v_2}, \ldots, \vec{v_n}$ are linearly independent.*
2. *$vecv_1, \vec{v_2}, \ldots, \vec{v_n}$ span V.*

Definition 11.2. *The **standard basis** for $\mathbb{R}^n$ is* $\beta = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$.

Proposition 11.3. *The pivot columns of a matrix A form a basis for $ColA$.*

Proposition 11.4. *The nullspace solutions of a matrix A form a basis for $NulA$.*

Theorem 11.5. *If V is a vector space and $\vec{v} \in V$, then there is a unique way to write $\vec{v}$ as a linear combination of the basis vectors of V.*

12 Dimension of a Vector Space

Definition 12.1. *The **dimension** of a vector space V is the number of vectors in a basis β for V.*

Proposition 12.2. *For an $m \times n$ matrix A with rank r,*

1. *$dim(ColA) = r$.*
2. *$dim(RowA) = r$.*
3. *$dim(NulA) = n - r$.*

Theorem 12.3. *If $\vec{v_1}, \ldots, \vec{v_m}$ and $\vec{w_1}, \ldots, \vec{w_n}$ are both bases for a vector space V, then $m = n$.*

13 Matrix Subspaces

Definition 13.1. *Let* $A = \begin{bmatrix} \vec{a_1} & \vec{a_2} & \cdots & \vec{a_n} \end{bmatrix}$ *be an* $m \times n$ *matrix, where* $\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n} \in \mathbb{R}^m$. *Then the* ***four fundamental subspaces*** *associated with* A *are:*

1. *The column space ColA =span{pivot columns}* $\subseteq \mathbb{R}^m$.
2. *The row space RowA =ColA*$^T \subseteq \mathbb{R}^n$.
3. *The nullspace NulA* $= \{\vec{x} : A\vec{x} = \vec{0}\} \subseteq \mathbb{R}^n$.
4. *The left nullspace NulA*$^T = \{\vec{y} : A^T\vec{y} = \vec{0}\} \subseteq \mathbb{R}^m$.

Proposition 13.2. *If* A *is an* $m \times n$ *matrix with rank* r, *then*

1. *dim(ColA)=r.*
2. *dim(RowA)=r.*
3. *dim(NulA)=n-r.*
4. *dim(NulA*^{T}*)=m-r.*