# Project 1: Arms Race Modeling

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## Abstract

In this project, we analyzed a simple Richardson arms race model to determine whether it provides realistic results and useful insights. We did an eigen-analysis on the model to predict how the stability and overall behavior of the system is related to the model's parameters. We discovered that the system's stability is heavily dependent upon the relationship between the "defense" parameters and the "fatigue" parameters, whereas the position of the fixed point is influenced by the interaction of all parameters. The model proved to be a good starting point for determining whether an arms race might occur. However, a more sophisticated model is required to make realistic predictions of the expenditure levels during an in process arms race.

## 1. Introduction

Throughout history, countries have been swept up into arms races with their rivals. Often times, the result has been war and destruction. The arms expenditures of one country can scare another country into increasing their own expenditures, and vice versa, resulting in what can seem like positive feedback loop. Most countries would prefer to avoid such arms races if possible. An informative model could help leaders to modify their behavior in order to avoid future arms races and their associated dangers. For this reason, we will explore a model that predicts the stability of arms expenditures by individual countries. A truly accurate arms expenditure model would require a huge number of variables and quickly become unwieldy. That is why we have selected a fairly simple model that relies upon two first order differential equations.

## 2. Model Description

Our basic arms race model is based on the Lewis Richardson model proposed in 1960. For the sake of simplicity, we have limited our model to two countries. The two-nation case of the theory is as follows. Let x be the expenditures on armaments of nation A (e.g. the United States), and let y be that of nation B (e.g. Russia). Ordinary differential equations governing the change in x and y are derived from the following considerations: (1) If y is large, then nation A will tend to increase; conversely for nation B. Parameters b and c are the "defense parameters" for two nations, and both are typically positive. (2) The expense of maintaining armaments tends to retard their growth. Parameters a and d are the "fatigue parameters" for two nations, and both are typically negative. (3) Other factors is being taken into account by grievance parameters, m and n, which may be positive or negative to represent the influence.

The systems of equations is as follows:

$\frac{dx}{dt} = ax + by + m$
$\frac{dy}{dt} = cx + dy + n$

Alternatively, these equations could model the stockpile of arms held by a country. Parameters a and c represent a nation's tendency to spend proportionally to another country's stockpile size. Parameters b and d represent the rate at which the stockpile that are used up or expired, or might capture how much a nation will spend in relation to its current stockpile. For this project, we will use the model to describe expenditure rather than stockpile.

Defense parameters are typically positive because most countries are competitive with and wary of other countries' expenditures. It makes little sense for defense parameters to be negative unless one country spends inversely to another country. This could be the case if one country is submissive to another country, or if one country protects another country.

# 3. Analysis

## 3.1 Background

If we limit the model by setting the all the fatigue and grievance parameters to zero, the system reduces to
$\frac{dx}{dt} = by \qquad \frac{dy}{dt} = cx$
which implies a runaway (exponential) growth of x and y, which could equate with war.

If we limit the model by setting the grievance parameters to zero, the model becomes identical to the "love affair" model, where:
$\frac{dx}{dt} = ax + by$
$\frac{dy}{dt} = cx + dy$

These can be rewritten in the matrix form as:
$\frac{d}{dt}\overline{x} = A\overline{x}$
where $A$ is the matrix of parameters $[a\ b; c\ d]$.

Before we begin our analysis of our full model with grievance parameters, we must recognize that the eigen-analysis only viable near the fixed points. There, the behavior of the system is captured by the eigenvalues of the Jacobian matrix evaluated at the fixed points.

We find the fixed points by setting
$f(x,y) \;=\; ax + by + m = 0$
$g(x,y) \;=\; cx + dy + n = 0$
Solving, we get

$x^* = \frac{bn-dm}{ad-bc} \quad y^* = \frac{an-cm}{bc-ad}$

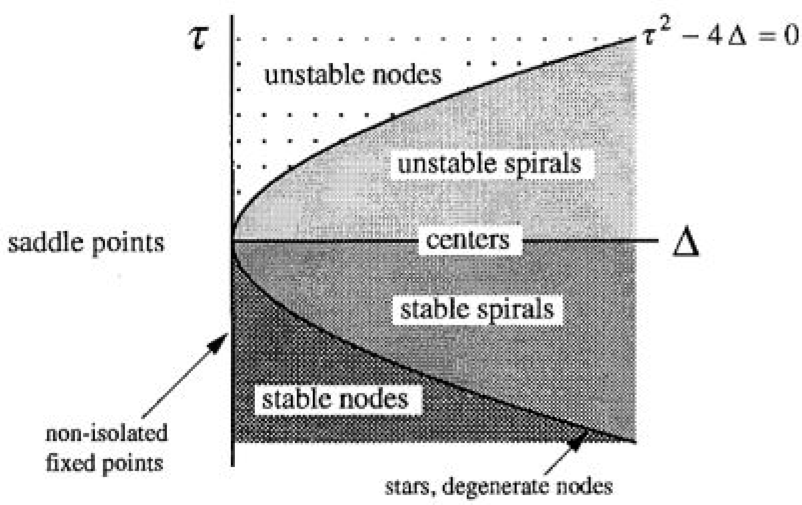
Using the Taylor expansion, we find that as we vary our position near to the fixed points,
$\frac{d}{dt}\overline{\delta x} = [J]|_{x^*,y^*}\overline{\delta x} = A\overline{\delta x}$

We can therefore use the eigenvalues of the Jacobian matrix to describe the system's behavior near the fixed points. Because the grievance parameters are constants, the Jacobian reduced to [a b;c d]. This means that we can solve for the eigenvalues as follows:
$\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$
where
$\tau = a + d$
$\Delta = det(A) = ad - bc$

The stability and oscillation of the system can be predicted by considering the real and imaginary components of the eigenvalues.

## 3.2 Parameter Sensitivity

The $\Delta - \tau$ portrait shown below summarizes the relationship between the quadratic equation parameters $\tau$ and $\Delta$ and their eigenvalue behavior profile. We will discuss this relationship in further detail throughout the remainder of the analysis.




*Source: https://canvas.harvard.edu/courses/11980/files/2434966?module_item_id=163793*

In reality, the fatigue parameters will always be negative, and so
$\tau < 0$

This means that we are only interested in the lower half of the $\Delta - \tau$ portrait. This makes it easy to learn a great deal by simply inspecting the state of $\Delta$. Therefore, we analyze two cases:

1. $\Delta$ is negative because $ad < bc$
2. $\Delta$ is positive because $ad > bc$
3. $\Delta$ is 0 because $ad = bc$

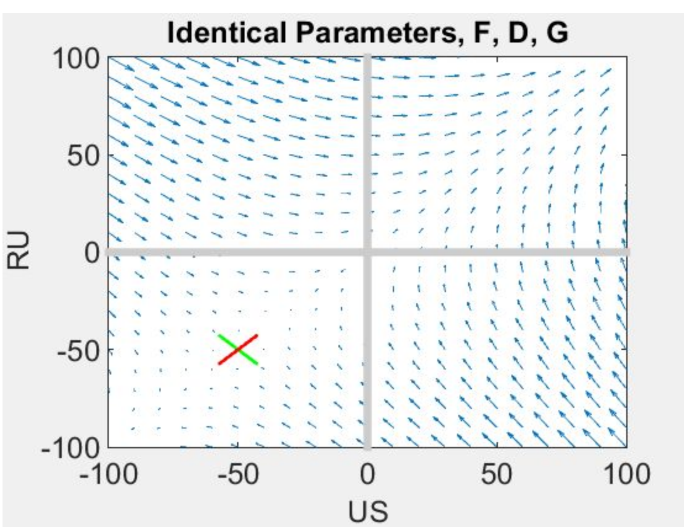
## 3.3 Analysis of ad<bc

In cases of $ad < bc$, product of defense parameters outweighs the product of fatigue parameters. This includes situations when countries are war mongering and resilient. Since $\Delta < 0$, we know that $\sqrt{\tau^2 - 4\Delta} > |\tau|$, which means that one eigenvalue will be positive and real and a second eigenvalue will be negative and real. These eigen-conditions mean that our fixed point is a saddle point.

Let us first examine a basic scenario where both countries have similar parameters. To best mimic what we might see in the real world, we use low fatigue parameters, high defense parameters, and positive grievance parameters, which is what you might expect in the real world. This scenario could be used to model a situation where two similar countries are hostile with each other. As expected, this scenario results in an unstable arms race. We used the following equations:

$$\frac{dx}{dt} = (-0.1)x + (0.2)y + (5)$$
$$\frac{dy}{dt} = (0.2)x + (-0.1)y + (5)$$

We can calculate that there is only one fixed point at (-50,-50). Furthermore, we calculate the eigenvalues to be -0.3 and 0.1 along the eigenvectors (0.707 and -0.707) and (0.707 and -0.707). Therefore, we might see instability in the direction of the upper right quadrant of the phase portrait. Our expectations are verified below.

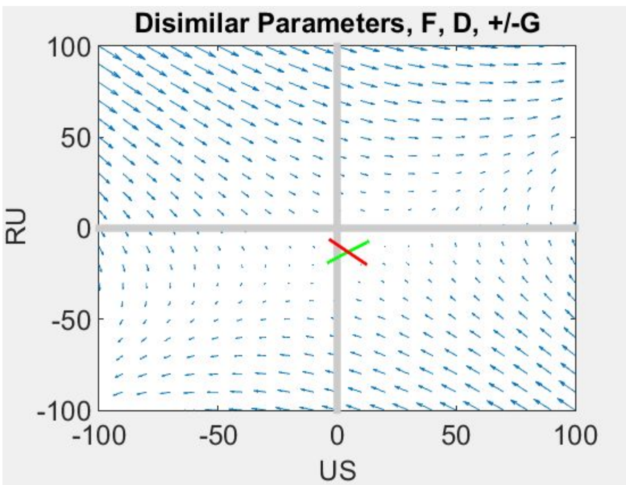


We are only interested in the upper right quadrant because the other quadrants have no meaning in the real world. In the upper right quadrant, the system is strictly unstable.

Even if we change the parameters so that they are dissimilar, we still get the same saddle point behavior, so long as $ad < bc$ and our fatigue parameters are negative. For example:

$\frac{dx}{dt} = (-0.1)x + (0.5)y + (7)$

$\frac{dy}{dt} = (0.3)x + (-0.2)y + (-4)$

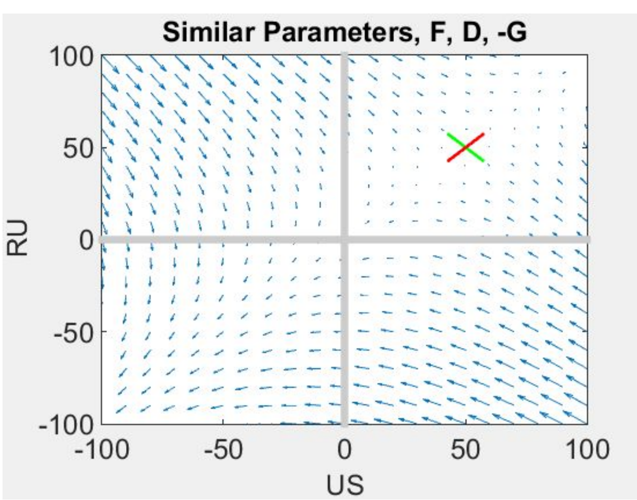



The behavior of the system in the upper right quadrant is still unstable at all points because the saddle point does not cross above the $y = -x$ line.

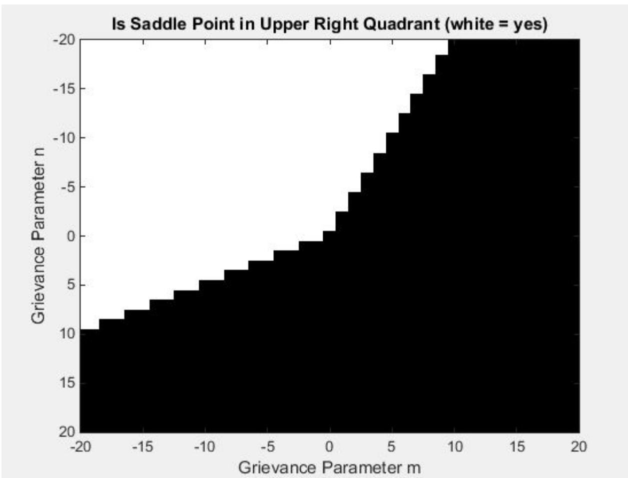
The location of the fixed point is heavily dependent on the grievance parameters, so let us try changing their sign to negative.

$\frac{dx}{dt} = (-0.1)x + (0.2)y + (-5)$

$\frac{dy}{dt} = (0.2)x + (-0.1)y + (-5)$



Changing the grievance parameters to be negative has pushed the saddle point across the $y=-x$ line and into the upper right quadrant. We have fundamentally changed the behavior of the system in the upper right quadrant so that de-escalation becomes possible! If we start with low enough expenditure rates, both expenditure rates will go down until one reaches zero. In other words, disarmament is very possible if both countries have a natural reduction tendency.
It would be interesting to know what values of the grievance parameters would bring the saddle point into the upper right quadrant, where the system has a realistic meaning. We have done this calculation in the figure below, using $a=-0.1,\ b=0.2,\ c=0.2,\ and\ d=-0.1$




This tells us that there are no positive values of $m$ and $n$ that push the saddle point into the positive quadrant. Note that the y-axis is reversed. Therefore, the system is always increasing in an unstable fashion unless both grievance parameters are negative, meaning that both countries naturally tend towards reduction in spend.

## 3.4 Analysis of ad>bc

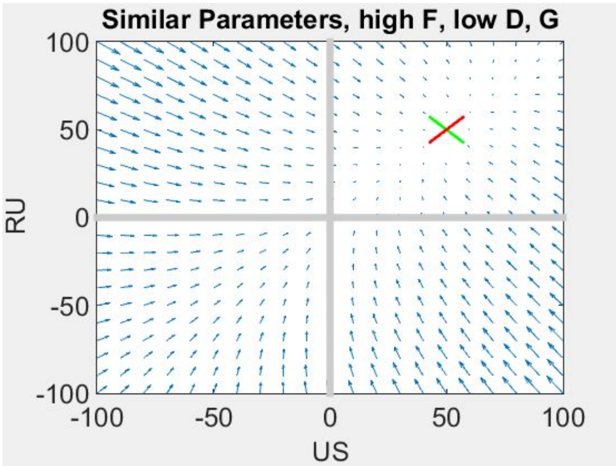
In cases of $ad > bc$, by definition the fatigue parameters are high and the defense parameters are low. Since $\Delta > 0$, we know that $\sqrt{\tau^2 - 4\Delta} < |\tau|$, which means that the eigenvalues will not have a positive real component. The only possible eigen-conditions are negative real and imaginary eigenvalues, meaning that there will either be stable nodes or stable spirals. However, because the defense parameters $b$ and $c$ are always positive in real life, it is impossible for the discriminant to be negative.

$$\tau^2 - 4\Delta = (a-d)^2 + 4bc \ > \ 0$$

This means that we cannot have imaginary eigenvalue components, and therefore only stable nodes are possible, not stable spirals.
We try a simple example:

$$\frac{dx}{dt} = (-0.2)x + (0.1)y + (5)$$
$$\frac{dy}{dt} = (0.1)x + (-0.2)y + (5)$$

Since the discriminant is equal to 0.04, we are not surprised to see a stable node in the phase portrait below.

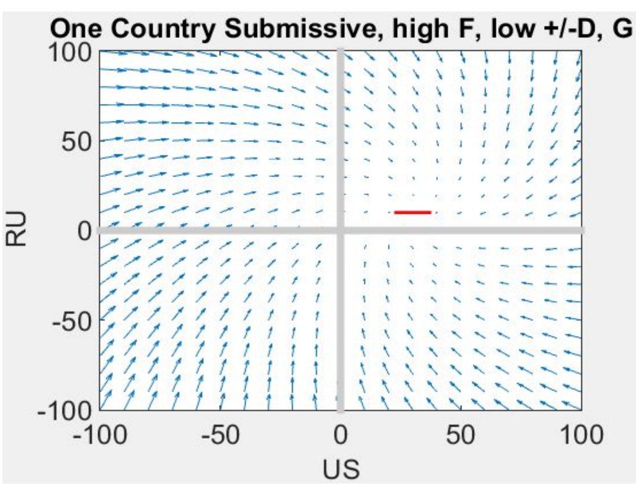



This tells us that when the fatigue parameters are high relative to the defense parameters, then the system will head towards a stable equilibrium. There will be no arms race in this situation.

In the bizarre situation where one country has a negative defense parameter (maybe because it is submissive when the other country has a high expenditure), it becomes possible for a stable spiral to occur, since the discriminant can now become negative.

$\frac{dx}{dt} = (-0.2)x + (0.1)y + (5)$

$\frac{dy}{dt} = (-0.1)x + (-0.2)y + (5)$




In the figure, Russia is submissive and has a negative defense parameter, resulting in an equilibrium where Russia spends very little and the US spends slightly more.

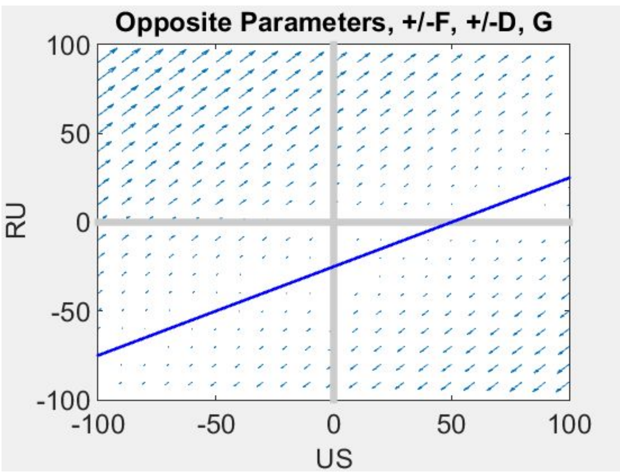
## 3.5 Analysis of ad=bc

When the defense and fatigue parameters are balanced, so that $ad = bc$, the discriminant reduces to $\tau^2$, which means that the eigenvalues are either zero or $-\tau$. Since $\tau$ is negative, the second eigenvalue is positive. This means that the system has non-isolated fixed points, but will become unstable if there is any perturbation away from the non-isolated fixed points.

We demonstrate this as follows:

$\frac{dx}{dt} = (-0.1)x + (0.2)y + (5)$

$\frac{dy}{dt} = (-0.1)x + (0.2)y + (5)$




The non-isolated fixed points are represented by the blue line.

## 4. Discussion

The analysis section of this paper has already provided an analytical explanations for why the behavior of the model makes sense. Therefore, this section will attempt to explain the results in a more intuitive and physical manner. Since we restricted the fatigue parameters to be negative, we were able to divide the parameter space into three regions.

The first region is $ad < bc$, which resulted in unstable growth in the positive direction for both countries. The instability revealed a weakness in the model, because it is unrealistic for arms expenditures to grow infinitely. However, it was no surprise that the system grew since we had modeled countries whose defensiveness outweighed their internal fatigue. The only way to ever achieve de-escalation was to provide a constant deceleration constant to outweigh the defensiveness. Even then, only low enough starting points would actually result in de-escalation, since a higher starting point would inspire too much defensive spending.

In the second region, $ad > bc$, we saw stable nodes. This makes sense given that the fatigue outweighed the defensiveness. The constant acceleration terms were the only things keeping the expenditures from sinking back to zero.

The last region is $ad = bc$, we saw non-isolated fixed points, as well as regions of unstable growth and regions of unstable shrinking. This makes sense since each country has a force pushing it forward and another force pushing it back, it makes sense that there is a direction in which these forces cancel out.

Interestingly, when we changed the sign of the grievance parameters, we saw that the fixed point location changed dramatically, which led to an equilibrium between two nations despite defense constantly overshadowed fatigue factors. The behavior of the system around the fixed point, however, remained unchanged. This makes sense since the Jacobian matrix is not dependent on the grievance parameters in this model, which means that the eigen analysis is identical no matter what the grievance parameters are.

Clearly, the many situations in which the system becomes unstable are somewhat unrealistic, especially since there are upper and lower limits to how much a country can spend on its arms. It may have been more realistic to include an upper limit $N$ as follows:

$$\frac{dx}{dt} = (ax + by + m)(1 - \frac{x}{N})$$
$$\frac{dy}{dt} = (cx + dy + n)(1 - \frac{y}{N})$$

Since expenditures often depend on total stockpiles as well as expenditures, it may be more accurate to use a second order differential equation where $X(t) = stockpile,\ x(t) = expenditures$:

$$x = \frac{dX}{dt} + rX$$
$$y = \frac{dY}{dt} + pY$$
$$\frac{dx}{dt} = ax + by + gY + m$$
$$\frac{dy}{dt} = cx + dy + hX + n$$

## 5. Summary

The model that we used may have been too simple to realistically predict expenditures in an arms race, especially when the phase portraits were often unstable. However, the model may be suitable for simply predicting whether an arms race will occur or not. We learned a great deal about the relationship between model parameters and the way that the grievance parameter shifts the phase portrait. A more realistic model might involve a second order dynamical model with some upper and lower expenditure limits, taking into account the dynamic nature of the parameters themselves.

## 6. Attribution of Effort

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## 7. References

Moses, L. E., A review, Lewis F. Richardson, *Arms and Insecurity* and Statistics of Deadly Quarrels, 1961, Journal of Conflict Resolution