Project report on “Bifurcation diagram for logistic attractor”

Group 5

Avijit Ghosh (22PHPH04)

Phamei Rumthao (22PHPH14)

Bedanta Kumar Deka (22PHPH16)

Joseph Soren (22PHPH27)

I. The Logistic Map

The logistic map is given by the relation

$$x_{n+1} = rx_n(1 - x_n)$$

i.e. the $(n+1)^{th}$ value of x is determined by using the n^{th} value of x.

To understand this concept, consider the following function which has the same mathematical form as the logistic map

y = $rx(1 - x)$

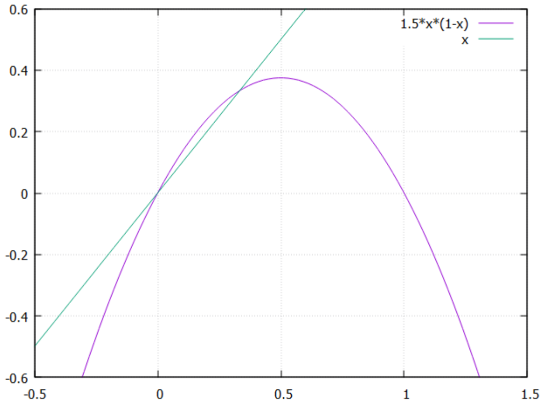


Fig 1. Plot of y = rx(1-x) with r =1.5. y = x is also plotted in the same graph.

Depending on the value of ‘a’, there will be one or two fixed points of this function.

- **Fixed Points**

For a function f(x), at the fixed points

$$f(x) = x$$

Fixed points can be determined by an iterative procedure.

(1) Take an initial guess value of x (say x_0).
(2) Put x_0 in f(x). We will get f(x_0).
(3) Put f(x_0) as the input of f(x). We will get f(f(x_0)).
(4) Continue this process which will result in the following sequence
    x_0, f(x_0), f(f(x_0)), ……

This sequence may or may not converge. If it does, it will converge at one of the fixed point.

From **graphical analysis**,

(a) For 0$\leq r \leq 1$, there will be one fixed point at x =0. This fixed point will be an attractor.
(b) For 1$\leq r \leq 3$, there will be two fixed points, one at x = 0 and the other at x_m (say). x_m will increase with a. The fixed point at x = 0 will repel and the fixed point at x_m will be an attractor.
(c) For r$>$ 3, there will again be two fixed points, one at x = 0 and the other at x_m. Like before, the value of x_m will increase with a. This time however, both the fixed points will repel.

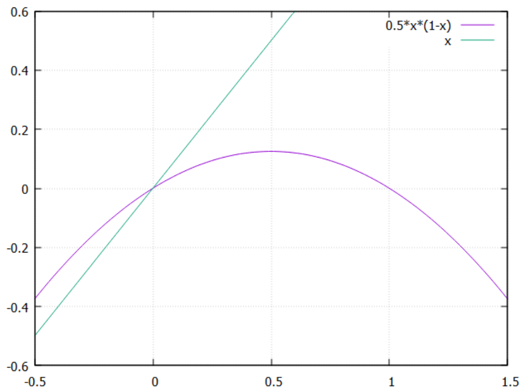


Fig 2(a). Plot for r = 0.5

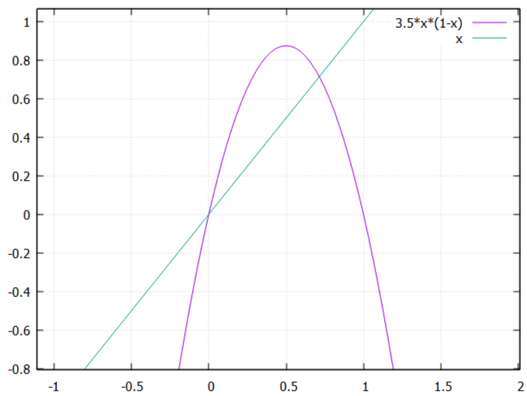


Fig 2(b). Plot for r = 3.5

Let us use the **iterative procedure** to find the fixed point for different values of a. A value between 0 and 1 is taken as the initial starting guess.

(a) For 0$\leq r \leq 1$, the iterative procedure converges to x = 0.
(b) For 1$\leq r \leq 3$, the iterative procedure converges to x_m. As 'a' increases, the value of x_m also increases.
(c) For r$>$ 3, the iterative procedure neither converges to x = 0 nor to x= x_m. As 'a' increases, the number of points to where this iterative procedure converges keeps doubling until chaos sets in at about r = 3.6.

The bifurcation diagram lets us visualize the points to where this iterative procedure converges to for different values of r.

II. The Bifurcation Diagram

The bifurcation diagram is essentially a plot between different value of 'r' and the values of 'x' to which the iterative procedure converges to for those values of 'r' .

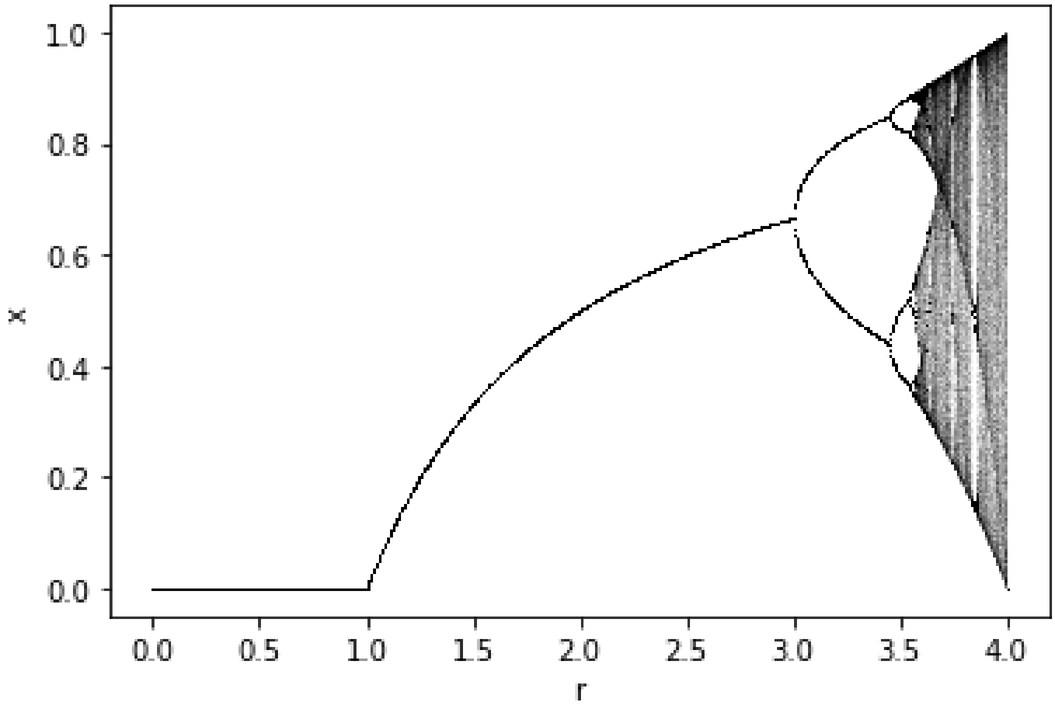


Fig 3. The bifurcation diagram of the logistic map.

For r <1, the iterative procedure converges to one point (x = 0).
For $1 \leq r \leq 3$, the iterative procedure again converges to one point (x_m, which increases with a).
For r> 3, the procedure neither converges to x = 0 nor to x = x_m, but initially to some other two points. As r increases, the number of points double until the onset of chaos.

If we zoom in the range of r (3.8, 3.9), we see mini bifercations. Zooming in that mini bifurcation, we see a repetition of the same pattern. This goes on as we keep zooming in further and further.

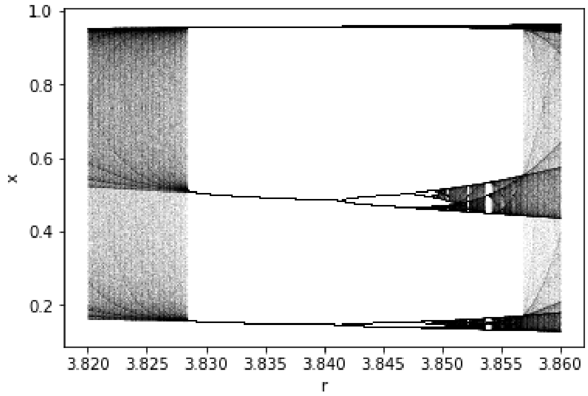


Fig 4. Mini bifurcations for 'r' in the range of around (3.8, 3.9)

III. Code to generate the bifurcation diagram

1) Create two empty lists to store the values to be plotted on the x-axis ('r' values) and the values to be plotted on the y-axis ('x' values)

   r_vals= [ ]

   x_vals = [ ]

2) 'r' values are created with the help of 'numpy.linspace' function in python which creates equally spaced values between two specified limits.

3) **For each value of 'r'**, using the initial value (seed value x_0) and the recurrence relation $x_{n+1} = rx_n(1 - x_n)$, x_{n+1} is determined (putting it inside a for-loop). We use this for-loop to arrive at the converged value(s) of x (if the iterative procedure does so for that value of 'r').

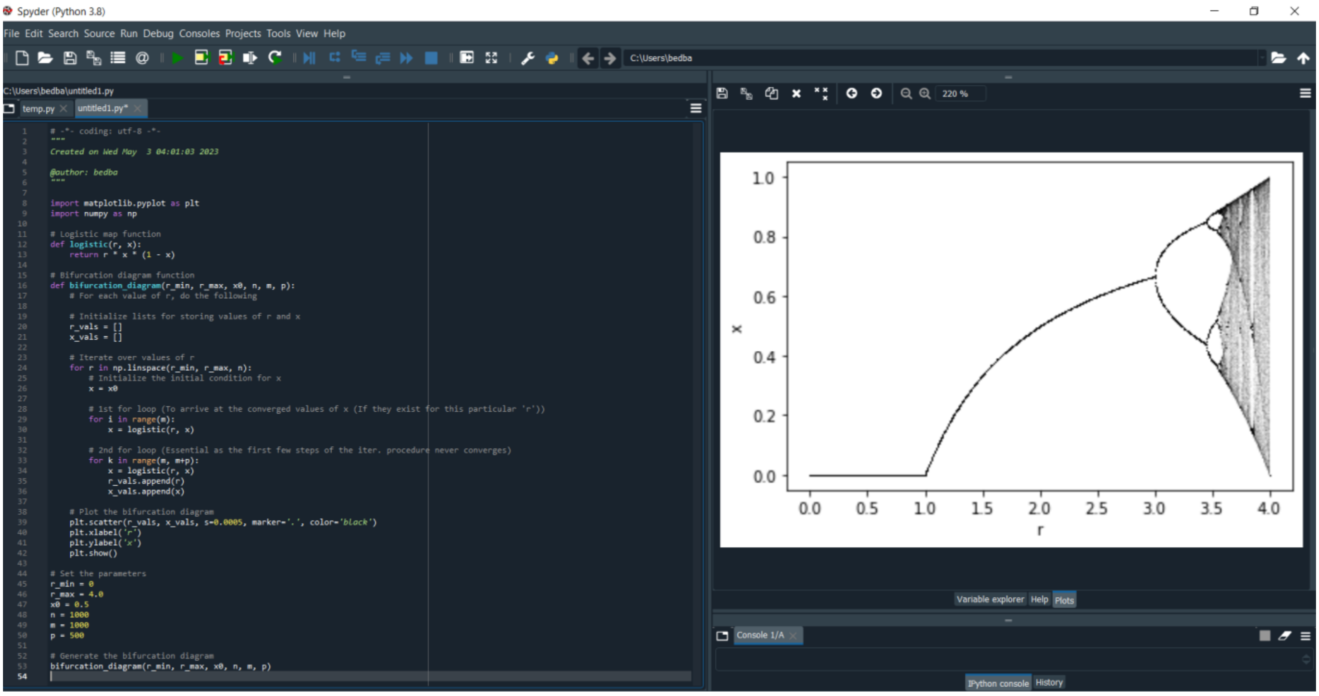
4) After we have arrived at the converged value(s) of x, we use one of those converged value as seed value to find x_{n+1} using the recurrence relation (putting it inside a 2nd for-loop). Using two for loops is essential for plotting the bifurcation diagram as the initial value of the iterative procedure do not mean anything and only after certain number of iterations does the iterative procedure starts to converge (if it does so for that value of 'r').

5) Append the values of x generated by the 2nd for-loop to the list 'x_vals'. Append the same value of 'r' in the list 'r_vals' that many times. So, for one value of 'r', there can be multiple values of 'x'

   Example : r_vals = [3.1, 3.1, 3.1, 3.1, 3.1]
   x_vals = [0.5580, 0.7645, 0.5580, 0.7645, 0.5580]

6) Plot 'r_vals' and 'x_vals' using 'matplotlib.pyplot' in python.

Python code generating the bifurcation diagram



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