Implicit Differentiation

Reading

- Sections 3.8

Practice problems

- Section 3.8: 3, 5, 7, 9, 11, 31, 33, 39
- To turn in (together with 3.9): 3.8 6, 10, 38

Notes

Implicit Differentiation

Implicit differentiation is used when we have a relation between two variables x, y, which would in theory allow us to write y as a function of x, but when actually doing so might be hard to do.

Example:

Let us look at a case that we can solve in two ways.

- Imagine we have the equation: $x^2 + y^2 = 1$.
- Then y can be thought of as a function of x.
- We could in fact solve it, and get $y = \sqrt{1 - x^2}$.
- Then we can compute the derivative by chain rule: $\frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}}$.
- Alternatively, we can differentiate the formula $x^2 + y^2 = 1$ **treating** $y = y(x)$ **as a function of** x.
- This gives us: $2x + 2y\frac{dy}{dx} = 0$
- Solving for $\frac{dy}{dx}$ gives us: $\frac{dy}{dx} = -\frac{x}{y}$
- Since $y = \sqrt{1 - x^2}$, these two formulas are the same.

> **Implicit Differentiation**
>
> Instead of a function $y = y(x)$, we are given a formula $F(x, y) = c$ that relates x and y.
>
> We differentiate that formula with respect to x, treating y as a function of x.
>
> Then we solve the resulting equation for $\frac{dy}{dx}$. The result will contain both x and y.

Key property: The chain rule tells us that if y is a function of x, then:

$$\frac{d}{dx}f(y) = f'(y)\frac{dy}{dx}$$

Example: Consider the relation $y^3 + xy^2 = x^2 + 1$. The point $(1, 1)$ belongs to this curve. We want to find the equation of the tangent line to this graph at that point.

The equation would be:

$$y - 1 = \left.\frac{dy}{dx}\right|_{(x,y)=(1,1)} (x - 1)$$

We use implicit differentiation to compute that derivative. We have:

$$3y^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} = 2x$$

Solving for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{2x - y^2}{3y^2 + 2xy}$$

Plugging in $(x, y) = (1, 1)$:

$$\frac{dy}{dx} = \frac{1}{5}$$

So tangent line equation becomes:

$$y - 1 = \frac{1}{5}(x - 1)$$