Linear Approximation

Reading

- Section 4.1

Practice problems

- Section 4.1 13, 15, 17, 19, 23, 25, 33, 38, 45
- To turn in (together with 4.2): 4.1 14, 46, 52

Notes

Linear Approximation

Derivatives allow us to approximate changes to a function over a small interval.

Some notation:

- Δx represents a small change in x. So we consider our x changing from a to $a + \Delta x$.
- Denote by $\Delta f = f(a + \Delta x) - f(a)$. Namely how much f has changed as x changes from a to $a + \Delta x$.

> **Linear Approximation of** Δf: If f is differentiable at a, and Δx is small, then:
>
> $$\Delta f \approx f'(a)\Delta x$$
>
> In other words, we can approximate the *actual change* Δf with the quantity $f'(a)\Delta x$.
>
> In graphical terms, the linear approximation says that the tangent line is a good approximation to the function as long as we keep close to the point of tangency.

Example: Suppose $f(x) = \frac{1}{x}$, $a = 5$ and $\Delta x = 0.1$. Then:

$$\Delta f = f(5.1) - f(5) = \frac{1}{5.1} - \frac{1}{5} = -0.003921569$$

Instead we could estimate this quantity via:

$$f'(5) \cdot 0.1 = \frac{-1}{5^2} \cdot 0.1 = -0.004$$

The error we have made is only $-0.004 - (-0.003921569) = -0.000078431$, which is about a 2% error relative to the actual value.

Differential Notation

In this notation we write $dx = \Delta x$ and $dy = f'(a)dx$. Then:

$$\Delta y \approx dy$$

We can write the same approximation in terms of $f(x)$. For that we write $x = a + \Delta x$, so $\Delta f = f(x) - f(a)$, and $dy = f'(a)(x - a)$. We then have the following approximation:

Approximation of $f(x)$ by its linearization

If f(x) is differentiable at $x = a$, and x is a number is close to a, then:

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

Example: If $f(x) = \sqrt{x}$ and $a = 4$. Then $f'(x) = \frac{1}{2\sqrt{x}}$, and we can compute the linearization:

$$L(x) = f(4) + f'(4)(x - 4) = 2 + \frac{1}{4}(x - 4)$$

As an example, let us estimate the value of $\sqrt{4.1}$, which is in reality very close to 2.024846. Our linearization would instead give us:

$$L(x) = 2 + \frac{1}{4}(4.1 - 4) = 2.025$$

Linearizations have numerous applications in other sciences.