The Fundamental Theorem of Calculus

Reading

- Sections 5.3, 5.4

Practice problems

- Section 5.3: 5, 7, 9, 15, 35, 41, 45
- To turn in: 5.3 10, 20, 38, 46
- Section 5.4: 17, 21, 29, 37, 39
- In class: 5.4 17, 33, 35, 44, 45
- To turn in: 5.4 24, 28, 34, 36

Notes

The Fundamental Theorem of Calculus, Part I

The Fundamental Theorem of Calculus (FTC) is one of the cornerstones of the course. It is a deep theorem that relates the processes of integration and differentiation, and shows that they are in a certain sense inverse processes.

The FTC has two parts. We will first discuss the first form of the theorem:

> **Fundamental Theorem of Calculus, Part I**
>
> If $F(x)$ is an antiderivative of $f(x)$ on $[a,b]$, i.e. $F'(x) = f(x)$ on $[a,b]$, then we have:
>
> $$\int_a^b f(x)dx = F(b) - F(a)$$
>
> In essence, we can directly compute an integral if we know of an antiderivative for the integrand.
>
> Shorthand notation: The difference $F(b) - F(a)$ is often written as:
>
> $$F(x)\Big|_a^b$$

Examples:

$$\int_1^4 x^2 dx = \frac{x^3}{3}\Big|_1^4 = \frac{4^3}{3} - \frac{1^3}{3} = \frac{63}{3} = 21$$

$$\int_0^\pi \cos x dx = \sin x\Big|_0^\pi = \sin\pi - \sin 0 = 0$$

Make sure to understand that second integral graphically, and why it should indeed equal 0.

Proof of the Fundamental Theorem of Calculus, Part I The idea of the theorem is to relate the difference $F(b) - F(a)$ to the Riemann sums.

We start with a partition P of the interval $[a, b]$:

$$\{a = x_0 < x_1 < x_2 < x_3 < \cdots < x_N = b\}$$

On each interval, we want to estimate the difference in the endpoints, $F(x_i) - F(x_{i-1})$. To do that, we use the mean value theorem, which says that this difference should equal $F'(c_i)(x_i - x_{i-1})$ for some point $c_i \in [x_{i-1}, x_i]$L

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1})$$

Since $F' = f$, we have:

$$F(x_i) - F(x_{i-1}) = f(c_i)\Delta x_i$$

If we write this for every i, then add up all the equations, the left-hand-side has many terms cancelling out:

$$[F(x_1) - F(x_0)] + [F(x_2) - F(x_1)] + [F(x_3) - F(x_2)] + \cdots + [F(x_N) - F(x_{N-1})]$$

Note that every term except for $F(x_0) = F(a)$ and $F(x_N) = F(b)$ appears twice, once with a plus sign and once with a minus sign. So they all cancel out. We end up with the formula:

$$F(b) - F(a) = \sum_{i=1}^{N} f(c_i)\Delta x_i$$

The right-hand-side is exactly a Riemann sum with those specific sample points. In other words:

> For every partition P there is a choice C of sample points such that:
>
> $$R(f, P, C) = \sum_{i=1}^{N} f(c_i)\Delta x_i = F(b) - F(a)$$

Since this happens for every partition, and since the right-hand-side is a constant independent of the partition, it follows that the limit should have the same relation, so:

$$\int_a^b f(x)dx = F(b) - F(a)$$

as desired.

The Fundamental Theorem of Calculus, Part II

The Fundamental Theorem of Calculus has a second formulation, that is in a way the "other direction" than that described in the first part. In that part we started with a function $F(x)$, looked at its derivative $f(x) = F'(x)$, then took an integral of that, and landed back to F. This version goes the other way.

Fundamental Theorem of Calculus, Part II

Let $f(t)$ be a continuous function on $[a, b]$. Then for each $x \in [a, b]$ we can define:

$$G(x) = \int_a^x f(t)dt$$

This defines a function G on $[a, b]$. The theorem is that this function is *differentiable*, and its derivative is:

$$G'(x) = f(x)$$

So in a single formula we would write:

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$$

As an example, consider the function:

$$g(x) = \int_1^x \frac{1}{t}dt$$

Here's some things we can say about it:

- $g(x)$ is defined for all $x > 0$.
- $g(1) = \int_1^1 \frac{1}{t}dt = 0$.
- $g(x)$ is differentiable, with derivative $g'(x) = \frac{1}{x}$.
- As its derivative is always positive for $x > 0$, g is an increasing function.
- $g''(x) = \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$.
- Since the second derivative is always negative for $x > 0$, g is a concave down function.
- Since for $t \geq 1$ we have $\frac{1}{t} \leq 1$, we can also say that for $t \geq 1$ we would have:

$$g(x) = \int_1^x \frac{1}{t}dt \leq \int_1^x 1dt = x - 1$$

All this information can give us a good handle on the function, and we can for instance do a decent job graphing the function.

Let us look at some variations of the theorem, where the endpoints are more complicated. Suppose we have the following integral function:

$$h(x) = \int_1^{x^2} \frac{1}{t}dt$$

We have no theorem that deals with this function directly. Instead we will think of the function:

$$g(u) = \int_1^u \frac{1}{t}dt$$

For that function we know that $g'(u) = \frac{1}{u}$.

Now g and h are related:

$$h(x) = g(x^2)$$

Therefore we can use the chain rule to find the derivative of h. We would have:

$$h'(x) = g'(x^2)(2x) = \frac{1}{x^2}(2x) = \frac{2}{x}$$

So "compute the integrand function at the upper endpoint, then multiply by the derivative of the upper endpoint".

Similarly we can work with cases where the lower endpoint has x in it:

$$h_2(x) = \int_x^1 \frac{1}{t} dt$$

We can then write the integral as:

$$h_2(x) = -\int_1^x \frac{1}{t} dt$$

so the derivative would be:

$$h_2'(x) = -\frac{1}{x}$$

If we have both endpoints containing x, we can break the problem up in two:

$$h_3(x) = \int_{x^2}^{x^3} \frac{1}{t} dt = \int_1^{x^3} \frac{1}{t} dt - \int_1^{x^2} \frac{1}{t} dt$$

The derivative would then be, as before:

$$h_3'(x) = \frac{1}{x^3}(3x^2) - \frac{1}{x^2}(2x) = \frac{3}{x} - \frac{2}{x} = \frac{1}{x}$$

This is rather interesting, the derivative of this function turned out to be the same as the derivative of our original $\int_1^x \frac{1}{t} dt$. This won't happen in general of course, but it does happen for this particular integrand.

Practice: Compute the derivatives of $\int_0^{x^2} \sin(t^2) dt$ and $\int_x^{2x} \frac{\sin t}{t} dt$.

Proof of the Fundamental Theorem of Calculus, Part II We start with the function $f(t)$ and its integral function $G(x) = \int_a^x f(t) dt$. In order to determine if $G(x)$ is differentiable at a point x, we need to consider the expression:

$$\lim_{h \to 0} \frac{G(x+h) - G(x)}{h}$$

Throughout the rest of this section, we will treat this point x as a constant.

In order to compute this limit, we start by considering the difference:

$$G(x+h) - G(x) = \int_a^{x+h} f(t) dt - \int_a^x f(t) dt = \int_x^{x+h} f(t) dt$$

We therefore need to get a hold on this integral $\int_x^{x+h} f(t)dt$. In order to do that, we will make a simpifying assumption that f is an increasing function on the interval from x to $x+h$. With that in mind, we can say that for every $t \in [x, x+h]$ we have the inequalities:

$$f(x) \leq f(t) \leq f(x+h)$$

These inequalities are presented when we integrate:

$$\int_x^{x+h} f(x)dt \leq \int_x^{x+h} f(t)dt \leq \int_x^{x+h} f(x+h)dt$$

The first and third integral are the integrals of constants, as our variable is t. Therefore they become (since $(x+h) - x = h$:

$$hf(x) \leq \int_x^{x+h} f(t)dt \leq f(x+h)$$

And dividing by h we get:

$$f(x) \leq \frac{\int_x^{x+h} f(t)dt}{h} \leq f(x+h)$$

Going back to G, we have:

$$f(x) \leq \frac{G(x+h) - G(x)}{h} \leq f(x+h)$$

We next need to take the limit as $h \to 0$. Since f is a continuous function, then:

$$\lim_{h \to 0} f(x+h) = f(x)$$

So by the squeeze theorem we get that:

$$\lim_{h \to 0} \frac{G(x+h) - G(x)}{h} = f(x)$$

This proves that $G(x)$ is differentiable and its derivative if $G'(x) = f(x)$.