The Substitution Method

Reading

- Section 5.6

Practice problems

- Section 5.6: 7, 9, 13, 15, 19, 37, 41, 55, 69, 71
- To turn in: 5.6 14, 16, 38, 60, 72

Notes

The Substitution Method

The substitution method is a powerful tool in our efforts to compute integrals and antiderivatives. It is essentially the inverse to the chain rule.

Indefinite Integrals

> **Substitution Method: Indefinite Integrals**
>
> If $F'(x) = f(x)$, i.e. if F is an antiderivative for f, then:
>
> $$\int f(u(x))u'(x)dx = F(u(x)) + C$$
>
> Note that the RHS can also be thought of as $\int f(u)du$ with an understanding the at the end we would substitute $u = u(x)$. With that in mind, the substitution method is also written:
>
> $$\int f(u(x))u'(x)dx = \int f(u)du$$

This follows from direct observation: The derivative of $F(u(x))$ would equal $F'(u(x))u'(x) = f(u(x))u'(x)$.

Example: Suppose we had to compute $\int x\sin(x^2)dx$. We can then consider $u(x) = x^2$, and we would have:

$$\int x\sin(x^2)dx = \frac{1}{2}\int \sin(x^2)(2x)dx = \frac{1}{2}\int \sin u du = -\frac{1}{2}\cos u = -\frac{1}{2}\cos(x^2) + C$$

In general to carry out the substitution:

1. Identify a $u = u(x)$ transformation.

2. Replace $u'(x)dx$ with du.
3. Replace all other occurences of x via a suitable form with u. If that is not possible then a u-substitution is not possible.
4. Compute the resulting indefinite integral.
5. Substitute back for x.

As another example, consider the following:

$$\int \frac{x^3}{(1+x^2)^3}dx$$

Here the denominator is problematic, and so it is a good candidate for a substitution:

$$u = 1 + x^2$$

We then need to compute:

$$du = (1+x^2)'dx = 2xdx$$

This leaves an x^2 in the numerator, and we need to replace it with a suitable expression of u:

$$x^2 = u - 1$$

Finally, our integral becomes:

$$\int \frac{u-1}{u^3}\frac{du}{2}$$

We can then compute this integral by braking it up in two pieces:

$$\int \frac{u-1}{u^3}\frac{du}{2} = \frac{1}{2}\int \frac{u}{u^3}du - \frac{1}{2}\int \frac{1}{u^3}du = \frac{1}{2}\int \frac{1}{u^2}du - \frac{1}{2}\int \frac{1}{u^3}du = -\frac{1}{2u} + \frac{1}{4u^2} = \frac{-2u+1}{4u^2} + C$$

Finally, we put back in $u = 1 + x^2$:

$$\int \frac{x^3}{(1+x^2)^3}dx = \frac{-2(1+x^2)+1}{4(1+x^2)^2}$$

Definite Integrals There if a version of the substitution method for definite integrals. The main difference is that the *endpoints change*:

Substitution Method: Definite Integrals

$$\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(u)du$$

In other words:

1. Identify $u = u(x)$
2. Replace $u'(x)dx$ with du.

3. Replace all other occurences of x via a suitable form with u. If that is not possible then a u-substitution is not possible.
4. Change the endpoints from x values to corresponding u values.
5. Compute the resulting definite integral

As an example, let us compute the integral:

$$\int_0^{\frac{\pi}{2}} \cos x \sin x dx$$

We can do here a substitution $u = \sin x$. Then $du = \cos x dx$. The endpoints will change: When $x = 0$ we have $u = \sin 0 = 0$ and when $x = \frac{pi}{2}$ we have $u = \sin \frac{pi}{2} = 1$. So we get the integral:

$$\int_0^{\frac{\pi}{2}} \cos x \sin x dx = \int_0^1 u du = \frac{u^2}{2}\bigg|_0^1 = \frac{1}{2}$$