Trigonometric Limits

Reading

- Sections 2.6

Practice problems

- Section 2.6: 5, 7, 14, 17, 19, 21, 33, 37, 43
- To turn in (together with 2.5): 2.6 10, 24, 40

Notes

The Squeeze Theorem

- The squeeze theorem allows us to deal with functions that have some parts that are hard to deal with, but end up not to not matter in the long run.
- Example: $\lim_{x\to 0} x \sin \dfrac{1}{x}$
  - $\sin \frac{1}{x}$ behaves erratically. But it stays within ± 1.
  - x goes to 0.
  - Their product would also go to 0, because it never exceeds $\pm x$.

  Squeeze Theorem:

  Assume that for all $x \neq c$ in some interval containing c we have:

  $$\ell(x) \leq f(x) \leq u(x)$$

  and also that we have that the limits of ℓ and u agree:

  $$\lim_{x\to c} \ell(x) = \lim_{x\to c} u(x) = L$$

  Then we have that the limit of $f(x)$ also exists and:

  $$\lim_{x\to c} f(x) = L$$

- Example of using the squeeze theorem:
  - $f(x) = x \sin \frac{1}{x}$, $\ell(x) = -x$, $u(x) = x$
  - Then $\ell(x) \leq f(x) \leq u(x)$
  - And $\lim_{x\to 0} \ell(x) = \lim_{x\to 0} -x = 0$, $\lim_{x\to 0} u(x) = \lim_{x\to 0} x = 0$
  - Therefore by the squeeze theorem we also get that $\lim_{x\to 0} f(x) = \lim_{x\to 0} x \sin \dfrac{1}{x} = 0$.

Trigonometric Limits

The squeeze theorem allows us to prove two important trigonometric limits:

Important Trigonometric Limits

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

The first of these limits follows from the following theorem:

Theorem For all $x \neq 0$ with $-\frac{\pi}{2} < x < \frac{\pi}{2}$, we have:

$$\cos x \leq \frac{\sin x}{x} \leq 1$$

- Proof of the theorem: Consider the unit circle, and compare the area of a sector and two triangles surrounding it.
- The first limit follows directly from this theorem.
- The second limit follows from the following:

$$\frac{1 - \cos x}{x} = \frac{1 - \cos^2 x}{x(1 + \cos x)} = \frac{\sin^2 x}{x(1 + \cos x)} = \frac{\sin x}{1 + \cos x} \cdot \frac{\sin x}{x}$$

Other trigonometric limits

The above limits can give rise to other limits as well:

- $\lim_{x \to 0} \frac{\sin(3x)}{x}$
- $\lim_{x \to 0} \frac{\sin(3x)}{\sin(5x)}$
- $\lim_{x \to 0} \frac{\sin(3x)}{\tan(x)}$