The Fundamental Theorem of Calculus

Reading

- Sections 5.3, 5.4

Practice problems

- Section 5.3: 5, 7, 9, 15, 35, 41, 45
- To turn in: 5.3 10, 20, 38, 46
- Section 5.4: TODO
- To turn in: 5.4 TODO

Notes

The Fundamental Theorem of Calculus, Part I

The Fundamental Theorem of Calculus (FTC) is one of the cornerstones of the course. It is a deep theorem that relates the processes of integration and differentiation, and shows that they are in a certain sense inverse processes.

The FTC has two parts. We will first discuss the first form of the theorem:

> **Fundamental Theorem of Calculus, Part I**
>
> If $F(x)$ is an antiderivative of $f(x)$ on $[a, b]$, i.e. $F'(x) = f(x)$ on $[a, b]$, then we have:
>
> $$\int_a^b f(x)dx = F(b) - F(a)$$
>
> In essence, we can directly compute an integral if we know of an antiderivative for the integrand.
>
> Shorthand notation: The difference $F(b) - F(a)$ is often written as:
>
> $$F(x)\Big|_a^b$$

Examples:

$$\int_1^4 x^2 dx = \frac{x^3}{3}\Big|_1^4 = \frac{4^3}{3} - \frac{1^3}{3} = \frac{63}{3} = 21$$

$$\int_0^\pi \cos x dx = \sin x\Big|_0^\pi = \sin\pi - \sin 0 = 0$$

Make sure to understand that second integral graphically, and why it should indeed equal 0.

Proof of the Fundamental Theorem of Calculus, Part I The idea of the theorem is to relate the difference $F(b) - F(a)$ to the Riemann sums.

We start with a partition P of the interval $[a, b]$:

$$\{a = x_0 < x_1 < x_2 < x_3 < \cdots < x_N = b\}$$

On each interval, we want to estimate the difference in the endpoints, $F(x_i) - F(x_{i-1})$. To do that, we use the mean value theorem, which says that this difference should equal $F'(c_i)(x_i - x_{i-1})$ for some point $c_i \in [x_{i-1}, x_i]$L

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1})$$

Since $F' = f$, we have:

$$F(x_i) - F(x_{i-1}) = f(c_i)\Delta x_i$$

If we write this for every i, then add up all the equations, the left-hand-side has many terms cancelling out:

$$[F(x_1) - F(x_0)] + [F(x_2) - F(x_1)] + [F(x_3) - F(x_2)] + \cdots + [F(x_N) - F(x_{N-1})]$$

Note that every term except for $F(x_0) = F(a)$ and $F(x_N) = F(b)$ appears twice, once with a plus sign and once with a minus sign. So they all cancel out. We end up with the formula:

$$F(b) - F(a) = \sum_{i=1}^{N} f(c_i)\Delta x_i$$

The right-hand-side is exactly a Riemann sum with those specific sample points. In other words:

> For every partition P there is a choice C of sample points such that:
>
> $$R(f, P, C) = \sum_{i=1}^{N} f(c_i)\Delta x_i = F(b) - F(a)$$

Since this happens for every partition, and since the right-hand-side is a constant independent of the partition, it follows that the limit should have the same relation, so:

$$\int_a^b f(x)dx = F(b) - F(a)$$

as desired.

The Fundamental Theorem of Calculus, Part II

TODO