Compound Interest

Reading

Section 7.5

Problems

Practice Exercises: 7.5 1, 2, 3, 4, 5

Exercises to turn in (along with those from 7.4): 7.5 6, 8

Compound Interest

In finance there is an concept called "compound interest". The basic idea is simple:

You have a *principal*, your starting amount of money, P_0. Then over time you accrue interest based on a fixed annual *rate* r. So after a year the *balance* on the account would be:

$$P_0 + P_0 \times r = P_0(1+r)$$

Now what would happen if the interest is instead accrued every semester? This typically means that you also have an interest amount $\frac{r}{2}$ applied. After the first six months the balance would be:

$$P_0\left(1+\frac{r}{2}\right)$$

For the next 6 months, the interest would be accrued on this increased balance, so at the end of the year we would have made:

$$P_0\left(1+\frac{r}{2}\right)\left(1+\frac{r}{2}\right) = P_0\left(1+\frac{r}{2}\right)^2$$

What if we accrue the interest more often, say M times during the year? Then the formula, with very similar logic, becomes:

$$P_0\left(1+\frac{r}{M}\right)^M$$

As M gets larger, this quantity grows. That is not immediately clear, as even though the exponent increases, the base gets closer and closer to 1. There must be a certain balance between the two. In fact there is. Turns out that the following is true:

$$e^r = \lim_{n\to+\infty}\left(1+\frac{r}{n}\right)^n$$

This is what we do in the case of **compound interest**. We assume that the interest is accrued "continuously", so as if the number M of times in a year is very very large. In that case, we can work out the formula for the balance as

$$P_0 e^r$$

and after time t years:

$$P(t) = P_0 e^{rt}$$

In other words, *continously compounded interest grows exponentially*.

Proof of formula for exponential

All this is based on the remarkable formula:

$$e^x = \lim_{n \to +\infty} \left(1 + \frac{x}{n}\right)^n$$

We will now prove this formula. The first observation is that we can take logarithms on both sides, and then we can pass the logarithm through the limit (because it is a continuous function). Using the properties of logarithms, the equation becomes instead:

$$x = \lim_{n \to +\infty} n \ln\left(1 + \frac{x}{n}\right)$$

We will now demonstrate this formula, at least for $x > 0$. We will do so by figuring out bounds for $\ln\left(1 + \frac{x}{n}\right)$. This follows from the fact that the logarithm is defined as an integral:

$$\ln\left(1 + \frac{x}{n}\right) = \int_1^{1+\frac{x}{n}} \frac{1}{t} dt$$

We can get bounds for that integral by figuring out what the largest and smallest values of the function $\frac{1}{x}$ on that interval would be. This is easy since the function is decreasing. In our case, the smallest value is at $1 + \frac{x}{n}$, and it is equal to $\frac{n}{n+x}$, and the largest value is at 1, and it is $\frac{1}{1} = 1$. The general rule we will use is:

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

In our case this says:

$$\left(1 + \frac{x}{n} - 1\right) \times \frac{n}{n+x} \leq \ln\left(1 + \frac{x}{n}\right) \leq \left(1 + \frac{x}{n} - 1\right) \times 1$$

We multiply by n, as that is what interests us:

$$x\frac{n}{n+x} \leq n \ln\left(1 + \frac{x}{n}\right) \leq x$$

As n goes to infinity, the quantity on the left approaches x, as does the one on the right. By the squeeze theorem, the quantity in the middle must also. And this is what we were trying to establish.