Growth of Functions

This section starts off with the main ideas near the end of section 7.7 but goes on a bit further.

The interest in this section is how functions behave as $x \to \infty$. There are lots of functions that go to infinity: x, x^2, e^x, x^x, $\ln(x)$, $\ln(x)^2$ and so on. The question we will focus on in this section is: *How fast do these functions go to infinity? Do some of them go to infinity "faster" than others?*

Let us start with a definition:

> We say that a function $g(x)$ **grows faster** than a function $f(x)$, if:
>
> $$\lim_{x\to\infty} \frac{g(x)}{f(x)} = \infty$$
>
> We denote this by
>
> $$f(x) \ll g(x)$$
>
> We could also instead check that:
>
> $$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$$

Notice that a lot of those quotients will end up being indeterminate forms of the type $\frac{\infty}{\infty}$.

For example, $2x$ is not considered to grow faster than x, because $\lim_{x\to\infty} \frac{2x}{x} = 2 \neq \infty$. On the other hand, x^2 does grow faster than x.

Here are some facts about the relative growth of functions:

$$x^n \ll x^m \text{ when } m > n$$

$$x^n \ll e^x \ll e^{2x} \ll e^{x^n} \text{ for all } n$$

$$\ln(x)^n \ll x^\alpha \text{ for all } n, \alpha > 0$$

Let us demonstrate these various assertions. We start with the first, which is easy to see by looking at $\lim_{x\to\infty} \frac{x^m}{x^n} = \lim_{x\to\infty} x^{m-n} = \infty$.

The limits that compare the various exponentials follow from the formula:

$$\frac{e^{f(x)}}{e^{g(x)}} = e^{f(x)-g(x)}$$

So as long as $\lim_{x\to\infty} f(x) - g(x) = \infty$, then $e^{f(x)} \gg e^{g(x)}$. This helps us show for example that $e^x \ll e^{2x} \ll e^{x^2} \ll e^{x^3}$ and so on.

The hardest comparison is between x^n and e^x. Let us work on that now:

$$\lim_{x\to\infty} \frac{e^x}{x^n}$$

As long as $n > 0$, we have a $\frac{\infty}{\infty}$ form, and we can apply L'Hopital's rule to get:

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \lim_{x\to\infty} \frac{e^x}{nx^{n-1}}$$

This is again an indeterminate form, but where the x^n power is now of lower degree, but the e^x has remained unchanged. Continuing this way, we will eventually get to a point where we have e^x on the numerator, and a number ($n(n-1)(n-2)\cdots 2\cdot 1$) in the denominator. This limit would then be infinity.

Let us try out some of the remaining comparisons, for instance comparing different exponentials:

$$\lim_{x\to\infty} \frac{e^{f(x)}}{e^{g(x)}} = e^{\lim_{x\to\infty} f(x) - g(x)}$$

> So $e^{f(x)} \ll e^{g(x)}$ if $\lim_{x\to\infty} f(x) - g(x) = -\infty$ and $e^{f(x)} \gg e^{g(x)}$ if $\lim_{x\to\infty} f(x) - g(x) = +\infty$.

Use this to compare e^x, e^{2x} and e^{x^2}.

Finally, for the logarithm, we use L'Hopital's rule:

$$\lim_{x\to\infty} \frac{\ln(x)}{x^\alpha} = \lim_{x\to\infty} \frac{1/x}{\alpha x^{\alpha-1}} = \lim_{x\to\infty} \frac{1}{\alpha x^\alpha} = 0$$

as long as $\alpha > 0$.