Taylor Polynomials

Reading

Section 9.4

Problems

Practice Exercises: 9.4 1, 3, 9, 11, 21, 23, 33

Exercises to turn in: 9.4 6, 24, 36, 48, 52

Taylor Polynomials

Taylor Polynomials

Taylor Polynomials stem from a desire to approximate a function near a point via polynomials. We have already seen this idea when we studied derivatives and the idea of linearization:

$$L(x) = f(a) + f'(a)(x - a)$$

This is in effect a polynomial of degree 1 that is fairly similar to the function near $x = a$.

We will extend this idea further:

> The **Taylor Polynomial** of degree n **centered at** $x = a$ is defined as the unique polynomial of degree n that **agrees with** f **to order** n **at** $x = a$.
>
> Two functions f, g are said to agree to order n at $x = a$ if their derivatives at $x = a$ match up to the n-th derivative, so if $f^{(k)}(a) = g^{(k)}(a)$ for all $k = 0, \ldots, n$.
>
> The Taylor polynomial has formula:
>
> $$T_n(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}}{n!}(x - a)^n$$

We often abreviate this as:

$$T_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x - a)^j$$

> The **Maclaurin polynomial** is the Taylor polynomial centered at $x = 0$.

Practice:

1. Compute the Maclaurin polynomial of degree 5 for $f(x) = x^4$. Use it to estimate $e^{0.1}$ and $e = e^1$.

2. Compute the Maclaurin polynomial of degree 4 for $f(x) = \tan^{-1} x$, and use it to estimate $\frac{\pi}{4} = f(1)$.
3. Compute the Maclaurin polynomial of degree 3 for $f(x) = \sin x$, and use it to estimate $\sin 0.2$.
4. Compute the Taylor polynomial of degree 4 for $f(x) = \sqrt{x}$ at $x = 1$, and use it to estimate $\sqrt{1.2}$.
5. Compute the Maclaurin polynomial of degree 5 for $f(x) = \ln(1 + x)$ and use it to estimate $\ln 1.2$.

The Taylor remainder

In order to use Taylor polynomials effectively, we need a way to measure the error. The first step in that is the Taylor remainder:

$$R_n(x) = f(x) - T_n(x)$$

So the Taylor remainder measures how far the Taylor polynomial is from f.

Taylor's theorem gives us a more precise formulation of this remainder:

Taylor's Theorem. If $f^{(n+1)}(x)$ exists and is continuous, then:

$$R_n(x) = \frac{1}{n!} \int_a^x (x-u)^n f^{(n+1)}(u) du$$

Using Taylor's Theorem, we can obtain the following error bound:

Error Bound. If $f^{(n+1)}(x)$ exists and is continuous, and K is such that $\left|f^{(n+1)}(u)\right| \leq K$ for all u between a and x, then:

$$|f(x) - T_n(x)| = |R_n(x)| \leq K \frac{|x-a|^{n+1}}{(n+1)!}$$

Practice: Compute the error bound where $f(x) = \sin(x)$, $a = 0$, $n = 5$, and at the point $x = 0.4$.

Proof of Taylor's Theorem

Taylor's theorem essentially follows from integration by parts. We start with the following simple observation:

$$f(x) = f(a) + \int_a^x f'(u) du$$

This is valid as long as $f'(u)$ exists and is continuous, and follows straight for the fundamental theorem of Calculus. We can in fact rewrite this as saying

$$f(x) = T_0(x) + R_0(x)$$

where

$$R_0(x) = \int_a^x f'(u)du = \frac{1}{0!}\int_a^x (x-u)^0 f^{(0+1)}(u)du$$

is exactly what is required by Taylor's theorem for $n = 0$.

From that point on, we perform integration by parts on the previous term. We use as a second term the constant 1, which we choose to integrate as **a function of** u to $-(x-u)$.

$$\int_a^x f'(u)du = \int_a^x f'(u)\cdot 1du = -f'(u)(x-u)|_a^x + \int_a^x f''(u)(x-u)du = f'(a)(x-a) + \int_a^x f''(u)(x-u)du$$

Putting this back in the original equation we find:

$$f(x) = f(a) + f'(a)(x-a) + \int_a^x f''(u)(x-u)du$$

This is exactly the formula in Taylor's theorem for $n = 1$.

To obtain the next iteration of the theorem, we each time integrate the $(x-u)^k$ term:

$$\int_a^x f''(u)(x-u)du = -\frac{1}{2}f''(u)(x-u)^2\Big|_a^x + \frac{1}{2}\int_a^x f'''(u)(x-u)^2du = \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{2}\int_a^x f'''(u)(x-u)^2du$$

Practice: Try the next step of this iteration.