

Compound Interest

Reading

Section 7.5

Problems

Practice Exercises: 7.5 1, 2, 3, 4, 5

Exercises to turn in (along with those from 7.3 and 7.4): 7.5 6

Compound Interest

In finance there is an interest concept called “compound interest”. The basic idea is simple:

You have a *principal*, your starting amount of money, P_0 . Then over time you accrue interest based on a fixed annual *rate* r . So after a year the *balance* on the account would be:

$$P_0 + P_0 \times r = P_0(1 + r)$$

Now what would happen if the interest is instead accrued every semester? This typically means that you also have an interest amount $\frac{r}{2}$ applied. After the first six months the balance would be:

$$P_0 \left(1 + \frac{r}{2}\right)$$

For the next 6 months, the interest would be accrued on this increased balance, so at the end of the year we would have made:

$$P_0 \left(1 + \frac{r}{2}\right) \left(1 + \frac{r}{2}\right) = P_0 \left(1 + \frac{r}{2}\right)^2$$

What if we accrue the interest more often, say M times during the year? Then the formula, with very similar logic, becomes:

$$P_0 \left(1 + \frac{r}{M}\right)^M$$

As M gets larger, this quantity grows. That is not immediately clear, as even though the exponent increases, the base gets closer and closer to 1. There must be a certain balance between the two. In fact there is. Turns out that the following is true:

$$e^r = \lim_{n \rightarrow +\infty} \left(1 + \frac{r}{n}\right)^n$$

This is what we do in the case of **compound interest**. We assume that the interest is accrued “continuously”, so as if the number M of times in a year is very very large. In that case, we can work out the formula for the balance as

$$P_0 e^r$$

and after time t years:

$$P(t) = P_0 e^{rt}$$

In other words, *continuously compounded interest grows exponentially*.

Proof of formula for exponential

All this is based on the remarkable formula:

$$e^x = \lim_{n \rightarrow +\infty} \left(1 + \frac{x}{n}\right)^n$$

We will now prove this formula. The first observation is that we can take logarithms on both sides, and then we can pass the logarithm through the limit (because it is a continuous function). Using the properties of logarithms, the equation becomes instead:

$$x = \lim_{n \rightarrow +\infty} n \ln \left(1 + \frac{x}{n}\right)$$

We will now demonstrate this formula, at least for $x > 0$. We will do so by figuring out bounds for $\ln \left(1 + \frac{x}{n}\right)$. This follows from the fact that the logarithm is defined as an integral:

$$\ln \left(1 + \frac{x}{n}\right) = \int_1^{1+\frac{x}{n}} \frac{1}{t} dt$$

We can get bounds for that integral by figuring out what the largest and smallest values of the function $\frac{1}{t}$ on that interval would be. This is easy since the function is decreasing. In our case, the smallest value is at $1 + \frac{x}{n}$, and it is equal to $\frac{n}{n+x}$, and the largest value is at 1, and it is $\frac{1}{1} = 1$. The general rule we will use is:

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

In our case this says:

$$\left(1 + \frac{x}{n} - 1\right) \times \frac{n}{n+x} \leq \ln \left(1 + \frac{x}{n}\right) \leq \left(1 + \frac{x}{n} - 1\right) \times 1$$

We multiply by n , as that is what interests us:

$$x \frac{n}{n+x} \leq n \ln \left(1 + \frac{x}{n}\right) \leq x$$

As n goes to infinity, the quantity on the left approaches x , as does the one on the right. By the squeeze theorem, the quantity in the middle must also. And this is what we were trying to establish.