Review of Calculus 1 Concepts

This will be a review of topics and concepts from Calculus 1, along with a list of suggested problems.

Practice Problems

- 2.4 4, 2.8 7
- 3.1 26, 3.2 17, 3.3 18, 3.5 26, 3.6 28, 3.7 44
- 4.1 19, 4.2 8, 4.2 32, 4.4 38, 4.6 9, 4.8 17
- 5.2 9, 5.2 44, 5.3 11, 5.3 38, 5.4 13, 5.4 16, 5.6 30, 5.6 71

Limits and Continuity

> The **limit** of a function $f(x)$ at a point a, denoted:
>
> $$\lim_{x \to a} f(x)$$
>
> is the number that the values $f(x)$ approach as the points x get closer and closer to (but are distinct from) the point a.

This is the important thing to remember, that the limit talks about the behavior of the function *near a point*, and *not at the point*.

When the behavior near a point agrees with the value at the point, we have continuity:

> The function $f(x)$ is **continuous** at the point a exactly when the limit at a agrees with the value there:
>
> $$\lim_{x \to a} f(x) = f(a)$$

Intuitively, the value at the point a is what is *expected* by looking at the behavior of the function nearby. In practice, when dealing with continuous functions we can compute a limit by **plugging in**.

Example: Find the limit of $\lim_{x \to 2} \frac{x^2-4}{x-2}$.

> Various types of discontinuities:
>
> - Removable discontinuity: The value $f(a)$ is simply not defined, but the limit exists. We can fix the discontinuity by defining the value $f(a)$ via the limit.
> - Jump discontinuity: The *left limit* and the *right limit* both exist but differ.
> - Infinite discontinuity: The left limit and/or right limit are infinite.

An important property of continuous functions is the **intermediate value theorem**:

> If $f(x)$ is a continuous function on an interval $[a, b]$, and L is a value anywhere between $f(a)$ and $f(b)$, then L can be obtained from f, i.e. there is a value $c \in [a, b]$ such that $f(c) = L$.

This has tremendous consequences, amongst them the fact that the square roots of numbers exist, by applying the intermediate value theorem on power functions.

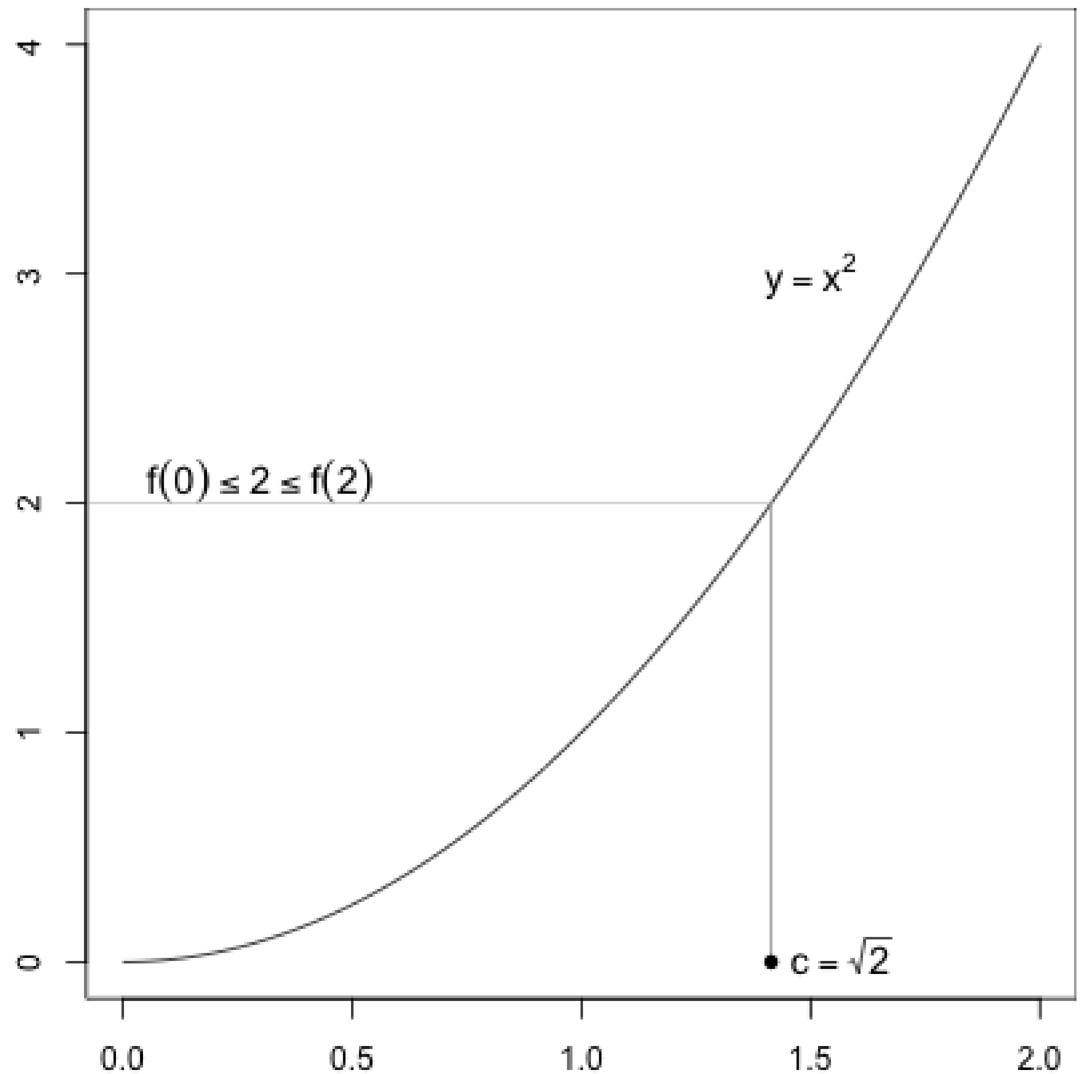

Figure 1: Proof that square root of 2 exists (IVT)

Derivatives

The **derivative** of a function $f(x)$ at a point a is defined as a *limit of slopes of secant lines*:

$$\left.\frac{df}{dx}\right|_{x=a} = f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h}$$

If we do this at each point a, we build a new *function*, $f'(x)$.

When computing derivatives, we rarely have to rely on the above definition, and we typically rely instead on a number of rules:

Derivative Rules:

$$(f \pm g)'(a) = f'(a) \pm g'(a)$$
$$(cf)'(a) = cf'(a)$$
$$(fg)'(a) = f'(a)g(a) + f(a)g'(a)$$
$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}$$
$$(x^n)' = nx^{n-1}$$
$$\sin'(x) = \cos(x), \quad \cos'(x) = -\sin(x)$$
$$\tan'(x) = \sec^2(x), \quad \sec'(x) = \sec(x)\tan(x)$$
$$(f(g(x)))' = f'(g(x))g'(x)$$

Derivatives tell us about the behavior of a function near a point, approximated as a line, namely the **tangent line**:

Tangent Line for $f(x)$ at a point $x = a$:

$$y - f(a) = f'(a)(x - a)$$

Example: Find the tangent line to the function $f(x) = \sqrt{x}$ at the point $a = 25$.

This can in particular be used to obtain an approximation for $f(x)$ near a point a:

Approximation for $x = a$:

$$f(x) \approx f(a) + f'(a)(x - a)$$

Example: Find an approximate value to $\sqrt{27}$ by using $a = 25$ and $f(x) = \sqrt{x}$.

One of the main applications of derivatives is in locating **maxima and minima for functions**:

If $f(x)$ is continuous on $[a, b]$, then:

- There is a maximum and a minimum for $f(x)$ on $[a, b]$
- Those extremal values occur at one of the following:
  - the endpoints a, b
  - points where $f'(c) = 0$
  - points, if any, where $f'(c)$ does not exist.

These last two categories are called **critical points**

Example: Find the minimum and maximum of $f(x) = x^3 - x$ from 0 to 1.

The mean value theorem for derivatives is one of the main results, with important consequences:

Mean Value Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a point $c \in (a, b)$ where:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Consequences of MVT:

- If $f'(x) > 0$ on an open interval, then f is increasing there
- If $f'(x) < 0$ on an open interval, then f is decreasing there
- If $f'(x) = 0$ on an open interval, then f is constant there

Integrals

The **integral** of a continuous function $f(x)$ on an interval $[a, b]$ is intuitively defined as the area between the x-axis and the function. A formal equation usually looks something like this:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

Integrals are often computed via the **antiderivatives**, based on the fundamental theorem of Calculus:

Fundamental Theorem of Calculus

1. If $f(x)$ is continuous on $[a, b]$, and $F(x)$ is a function such that $F'(x) = f(x)$, then:
$$\int_a^b f(x)dx = F(b) - F(a)$$
2. If $f(x)$ is continuous on $[a, b]$ and we defined the integral function:
$$F(x) = \int_a^x f(t)dt$$
is differentiable and $F'(x) = f(x)$.

One of the key techniques for computing integrals, and antiderivatives, is the substitution method:

Substitution Method:

- If $F'(x) = f(x)$, then $F(g(u))$ is an antiderivative for $f(g(u))g'(u)$.
- $\displaystyle\int_a^b f(g(u))g'(u)du = \int_{g(a)}^{g(b)} f(u)du$.