Taylor Polynomials

Reading

Section 9.4

Problems

Practice Exercises: 9.4 1, 3, 9, 11, 21, 23, 33

Exercises to turn in: 9.4 6, 24, 36, 48, 52

Taylor Polynomials

Taylor Polynomials

Taylor Polynomials stem from a desire to approximate a function near a point via polynomials. We have already seen this idea when we studied derivatives and the idea of linearization:

$$L(x) = f(a) + f'(a)(x - a)$$

This is in effect a polynomial of degree 1 that is fairly similar to the function near $x = a$.

We will extend this idea further:

> The **Taylor Polynomial** of degree n **centered at** $x = a$ is defined as the unique polynomial of degree n that **agrees with** f **to order** n **at** $x = a$.
>
> Two functions f, g are said to agree to order n at $x = a$ if their derivatives at $x = a$ match up to the n-th derivative, so if $f^{(k)}(a) = g^{(k)}(a)$ for all $k = 0, \ldots, n$.
>
> The Taylor polynomial has formula:
>
> $$T_n(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}}{n!}(x - a)^n$$

We often abreviate this as:

$$T_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x - a)^j$$

> The **Maclaurin polynomial** is the Taylor polynomial centered at $x = 0$.

Practice:

1. Compute the Maclaurin polynomial of degree 5 for $f(x) = x^4$. Use it to estimate $e^{0.1}$ and $e = e^1$.

2. Compute the Maclaurin polynomial of degree 4 for $f(x) = \tan^{-1} x$, and use it to estimate $\frac{\pi}{4} = f(1)$.
3. Compute the Maclaurin polynomial of degree 3 for $f(x) = \sin x$, and use it to estimate $\sin 0.2$.
4. Compute the Taylor polynomial of degree 4 for $f(x) = \sqrt{x}$ at $x = 1$, and use it to estimate $\sqrt{1.2}$.
5. Compute the Maclaurin polynomial of degree 5 for $f(x) = \ln(1 + x)$ and use it to estimate $\ln 1.2$.

The Taylor remainder

In order to use Taylor polynomials effectively, we need a way to measure the error. The first step in that is the Taylor remainder:

$$R_n(x) = f(x) - T_n(x)$$

So the Taylor remainder measures how far the Taylor polynomial is from f.

Taylor's theorem gives us a more precise formulation of this remainder:

> **Taylor's Theorem**. If $f^{(n+1)}(x)$ exists and is continuous, then:
>
> $$R_n(x) = \frac{1}{n!} \int_a^x (x - u)^n f^{(n+1)}(u) du$$

Using Taylor's Theorem, we can obtain the following error bound:

> **Error Bound**. If $f^{(n+1)}(x)$ exists and is continuous, and K is such that $\left|f^{(n+1)}(u)\right| \leq K$ for all u between a and x, then:
>
> $$|f(x) - T_n(x)| = |R_n(x)| \leq K \frac{|x - a|^{n+1}}{(n + 1)!}$$

Practice: Compute the error bound where $f(x) = \sin(x)$, $a = 0$, $n = 5$, and at the point $x = 0.4$.