

# Compound Interest

## Reading

Section 7.5

## Problems

Practice Exercises: 7.5 1, 2, 3, 4, 5

Exercises to turn in (along with those from 7.3 and 7.4): 7.5 6

## Compound Interest

In finance there is an interest concept called “compound interest”. The basic idea is simple:

You have a *principal*, your starting amount of money,  $P_0$ . Then over time you accrue interest based on a fixed annual *rate*  $r$ . So after a year the *balance* on the account would be:

$$P_0 + P_0 \times r = P_0(1 + r)$$

Now what would happen if the interest is instead accrued every semester? This typically means that you also have an interest amount  $\frac{r}{2}$  applied. After the first six months the balance would be:

$$P_0 \left(1 + \frac{r}{2}\right)$$

For the next 6 months, the interest would be accrued on this increased balance, so at the end of the year we would have made:

$$P_0 \left(1 + \frac{r}{2}\right) \left(1 + \frac{r}{2}\right) = P_0 \left(1 + \frac{r}{2}\right)^2$$

What if we accrue the interest more often, say  $M$  times during the year? Then the formula, with very similar logic, becomes:

$$P_0 \left(1 + \frac{r}{M}\right)^M$$

As  $M$  gets larger, this quantity grows. That is not immediately clear, as even though the exponent increases, the base gets closer and closer to 1. There must be a certain balance between the two. In fact there is. Turns out that the following is true:

$$e^r = \lim_{n \rightarrow +\infty} \left(1 + \frac{r}{n}\right)^n$$

This is what we do in the case of **compound interest**. We assume that the interest is accrued “continuously”, so as if the number  $M$  of times in a year is very very large. In that case, we can work out the formula for the balance as

$$P_0 e^r$$

and after time  $t$  years:

$$P(t) = P_0 e^{rt}$$

In other words, *continuously compounded interest grows exponentially*.

### **Proof of formula for exponential**

All this is based on the remarkable formula:

$$e^x = \lim_{n \rightarrow +\infty} \left(1 + \frac{x}{n}\right)^n$$

We will now prove this formula. The first observation is that we can take logarithms on both sides, and then we can pass the logarithm through the limit (because it is a continuous function). Using the properties of logarithms, the equation becomes instead:

$$x = \lim_{n \rightarrow +\infty} n \ln \left(1 + \frac{x}{n}\right)$$

We will now demonstrate this formula, at least for  $x > 0$ . We will do so by figuring out bounds for  $\ln \left(1 + \frac{x}{n}\right)$ . This follows from the fact that the logarithm is defined as an integral:

$$\ln \left(1 + \frac{x}{n}\right) = \int_1^{1+\frac{x}{n}} \frac{1}{t} dt$$

We can get bounds for that integral by figuring out what the largest and smallest values of the function  $\frac{1}{t}$  on that interval would be. This is easy since the function is decreasing. In our case, the smallest value is at  $1 + \frac{x}{n}$ , and it is equal to  $\frac{n}{n+x}$ , and the largest value is at 1, and it is  $\frac{1}{1} = 1$ . The general rule we will use is:

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

In our case this says:

$$\left(1 + \frac{x}{n} - 1\right) \times \frac{n}{n+x} \leq \ln \left(1 + \frac{x}{n}\right) \leq \left(1 + \frac{x}{n} - 1\right) \times 1$$

We multiply by  $n$ , as that is what interests us:

$$x \frac{n}{n+x} \leq n \ln \left(1 + \frac{x}{n}\right) \leq x$$

As  $n$  goes to infinity, the quantity on the left approaches  $x$ , as does the one on the right. By the squeeze theorem, the quantity in the middle must also. And this is what we were trying to establish.