The Shell Method

Reading

Section 6.4

Problems

- Practice Exercises: 7, 9, 10, 15, 16, 21, 22, 23, 39, 41, 42
- Exercises to turn in (on Wednesday): 6.4 8, 40, 44, 46

The Shell Method

The Shell Method, like the Disc Method, deals with solids created by the revolution of some curve/function around some axis. But it works in sort of the opposite direction. Let's explain.

Consider the triangle formed by $x + y = 1$, $y = 0$ and $x = 0$. Say we rotate this region around the y-axis and we want to find out the area of the resulting solid.

With the disc method, we need to write our curve, $x + y = 1$, as a function of y, the variable that we are rotating around. So we have the function $x = 1 - y$. Then the cross-sections perpendicular to the y axis are circles with radius $1-y$, so we can write the integral, using the disc method, as:

$$\pi \int_0^1 (1-y)^2 dy$$

which we can compute to equal $\frac{\pi}{3}$.

Now the Shell method is different. It will try to integrating along the x-axis instead. To do that, imagine cutting the interval in x into many tiny segments. One such segment would be going from say x to $x + \Delta x$. The solid we are after that corresponds to this segment is the little area above this segment and below the $y = 1 - x$ line, which is roughly a rectangle with base Δx and height $1 - x$, but rotated around the y axis. This results in effect in two cylinders, the outer and the inner, and we look at their differences:

$$\pi(x + \Delta x)^2(1 - x) - \pi x^2(1 - x) = \pi(1 - x)\left((x + \Delta x)^2 - x^2\right)$$

We need to understand the quantity $(x + \Delta x)^2 - x^2$ more. Let's expand it out, as it is a difference of squares:

$$(x + \Delta x)^2 - x^2 = (x + \Delta x + x)(x + \Delta x - x) = (2x + \Delta x)\Delta x$$

Now here we are thinking of the Δx as a tiny, infinitesimal quantity. In which case $2x + \Delta x$ is really the same as $2x$. So at the end of the day the volume of the cylinder for that particular x is:

$$2\pi(1-x)x\Delta x$$

We then add up all these little cylinders and integrate:

$$V = 2\pi \int_0^1 x(1-x)dx = 2\pi \int_0^1 x - x^2 dx = \frac{\pi}{3}$$

Good, we get the same answer, as we should.

In general, the formula for the Shell method would read:

> Volume of the solid produced by rotating the area under $y = f(x)$ around the y axis.
>
> $$V = 2\pi \int_a^b x f(x) dx$$

Shell vs Disc

So when to use which method? That depends on which way of the function is easier. In general:

> You can compute a solid of revolution around the y axis via:
>
> - the shell method if you have the boundary as a function $y = f(x)$, or
> - the disc method if you have the boundary as a function $x = g(y)$.
>
> The Disc method uses the same variable of integration as the axis of rotation, the Shell method uses the other axis.