Method of Partial Fractions

Reading

Section 8.5

Problems

Practice Exercises: 8.5 1, 2, 3, 5, 6, 9, 11, 15, 21, 27, 29

Exercises to turn in: 8.5 4, 8, 16, 30

Method of Partial Fractions

The main goal of this section is to learn the techniques involved in computing integrals of the form:

$$\int \frac{P(x)}{Q(x)} dx$$

where $P(x)$ and $Q(x)$ are polynomials.

We will outline here the main ideas. There are plenty of examples in the book.

The method of computing such integrals follows 4 steps:

1. If the degree of $P(x)$ is at least as much as the degree of $Q(x)$, first do polynomial division.
2. Break the denominator $Q(x)$ into factors. We need linear factors $x - a$ as well as quadratic factors $x^2 + bx + c$ where the quadratic has no real roots.
3. Perform partial fraction decomposition, where the quotient $\frac{P(x)}{Q(x)}$ breaks down into a sum of terms with denominators those factors from the previous step.
4. Compute the integrals of each of these smaller pieces.

We will now look at each of these steps.

Polynomial Division

> If $P(x)$, $Q(x)$ are two polynomials, and the degree of P is equal to or more than the degree of Q, then there are polynomials $R(x)$, $S(x)$, so that the degree of S is less than the degree of Q and so that:
>
> $$P(x) = R(x)Q(x) + S(x)$$
>
> or equivalently
>
> $$\frac{P(x)}{Q(x)} = R(x) + \frac{S(x)}{Q(x)}$$

Quotients where the numerator has degree less than the denominator are often called **proper**.

To find R and S we use **long division**. An example would be:

$$\frac{x^3+1}{x^2-4} = x + \frac{4x+1}{x^2-4}$$

You can find many examples of long division online.

Factoring out the denominator

The fact that we can break down the denominator into a product of simple factors is an important theorem in algebra:

> **Fundamental Theorem of Algebra** (for real polynomials)
>
> Every non-constant polynomial can be written as a product of terms of two types:
>
> - Linear terms $x - a$.
> - Quadratic terms $x^2 + bx + c$ with no real roots (irreducible).

We will not prove this theorem here, but we will discuss how to find at least linear terms.

> The linear term $x - a$ will be a factor of the polynomial $P(x)$ if and only if a is a root of of the polynomial, i.e. if $P(a) = 0$.
>
> Roots can be found for instance via Newton's method, or the bisection method, both usually seen in Calculus 1.
>
> If the polynomial coefficients are all integers, then good candidates for roots are rational numbers of the form $\pm\frac{p}{q}$, where p divides the constant coefficient and q divides the leading coefficient.

Once a linear term has been found, long division can simplify the original polynomial into $P(x) = (x - a)P_1(x)$, then we continue with $P_1(x)$.

In most problems, we will be providing the factorization for you (or in the case of a quadratic, you can easily find it yourself by solving the quadratic).

Partial Fraction Decomposition

The heart of the matter is the partial fraction decomposition. At this point we have a quotient that has the form:

$$\frac{P(x)}{(x-a_1)^{k_1}(x-a_2)^{k_2}\cdots(x^2+b_1x+c_1)^{n_1}(x^2+b_2x+c_2)^{n_2}\cdots}$$

where the terms appearing in the denominator are distinct and have total degree adding up to more than the numerator. Then we can write this expression as a sum of terms:

$$\frac{A}{(x-a)^k}$$

and terms

$$\frac{Bx+C}{(x^2+bx+c)^k}$$

where the terms are amongst those that appear in the original denominator, and the power k goes from 1 to the power appearing in the original denominator.

An example will make this clearer. Suppose we have the expression:

$$\frac{x+1}{(x-1)(x+2)}$$

Then we should be able to write it as:

$$\frac{x+1}{(x-1)(x+2)} = \frac{A_1}{x-1} + \frac{A_2}{x+2}$$

All that remains is to determine the coefficients A_1 and A_2. We do this by making common denominators, then equating the resulting numerators on the two sides:

$$x+1 = A_1(x+2) + A_2(x-1)$$

The key thing here is that this equality should hold *for all* x. So we treat it as an identity: The two polynomials on the two sides must be equal. There are two ways to find the coefficients from here:

1. Equate same terms: The right-hand side can be rewritten as $(A_1+A_2)x+(2A_1-A_2)$. So we must have that $A_1+A_2=1$ and $2A_1-A_2=1$. We then solve this system of equations.
2. Plug in appropriate values for x. This can make things a lot easier. For example if we plug in $x=1$, then the A_2 term goes away, and we are left with: $1+1=A_1(1+2)$, or $A_1=\frac{2}{3}$. Similarly, plugging in $x=-2$ makes the A_1 term go away, and gives us $A_2=\frac{1}{3}$.

So we obtain:

$$\frac{x+1}{(x-1)(x+2)} = \frac{2/3}{x-1} + \frac{1/3}{x+2}$$

Some times we deal with one of the terms showing up with a higher power, for example:

$$\frac{2x+1}{(x-1)^2(x+2)}$$

In that case we need to include in the terms on the right the various powers as options:

$$\frac{2x+1}{(x-1)^2(x+2)} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{A_3}{x+2}$$

To find out the three terms, we again make common denominators:

$$\frac{2x+1}{(x-1)^2(x+2)} = \frac{A_1(x-1)(x+2)+A_2+A_3(x-1)^2}{x-1}$$

We now equate the numerators:

$$2x+1 = A_1(x-1)(x+2)+A_2+A_3(x-1)^2$$

- First off we can find A_2 by setting $x=1$ on both sides. Then we get $A_2 = 3$.
- Now we plug in $x = -2$. This eliminates one of the terms on the right, and leads to: $-3 = A_2 + A_3(-3) = 3 - 3A_3$, or $A_3 = 2$.
- Next, we can compare the quadratic terms. This leads us to conclude that $A_1 + A_3 = 0$, or that $A_1 = -A_3 = -2$.

So finally we have:

$$\frac{2x+1}{(x-1)^2(x+2)} = \frac{-2}{x-1} + \frac{3}{(x-1)^2} + \frac{2}{x+2}$$

Computing the resulting integrals

All remaining integrals fall into one of two forms:

$$\int \frac{A}{(x-a)^k} dx$$

If $k=1$, this integral is simply a logarithm $A \ln|x-a|$. If $k>1$, we can integrate directly to $\frac{A}{k+1}\frac{1}{(x-a)^{k+1}}$

$$\int \frac{Ax+B}{(ax^2+bx+c)^k} dx$$

where the quadratic has no real roots. These integrals can be solved by an appropriate trigonometric substitution using $\tan t$, using the techniques we have examined in previous sections.