Trigonometric Substitutions

Reading

Section 8.3

Problems

Practice Exercises: 8.3 1, 2, 4, 5, 7, 9, 10, 16, 19, 37, 38, 39

Exercises to turn in (along with 8.2): 8.3 6, 8

Trigonometric Substitutions

Trigonometric substitutions are used to handle certain integrals containing square roots. There are roughly 3 forms to consider:

$$\sqrt{a^2 - x^2}$$
$$\sqrt{x^2 + a^2}$$
$$\sqrt{x^2 - a^2}$$

Each case is handled by a corresponding substitution, which allows us to eliminate the square root and gives us a trigonometric integral like those discussed in the previous section.

The sine transform

The sine transform is used to handle integrals involving $\sqrt{a^2 - x^2}$. It is based on the following:

$$x = a\sin(\theta)$$
$$dx = a\cos(\theta)d\theta$$
$$\sqrt{a^2 - x^2} = \sqrt{a^2\cos^2(x)} = a\cos(x)$$

where we assume $a > 0$, and we can assume that $\cos(x) > 0$ because the variable θ is basically restricted to the domain of $\sin^{-1}(x)$.

Let us see this method in action for the integral:

$$\int \frac{x^4}{\sqrt{9 - x^2}^5} dx$$

In this case $a = 3$, so we set $x = 3\sin(\theta)$, or if you prefer $\theta = \sin^{-1}\left(\frac{x}{3}\right)$. The integral them becomes:

$$\int \frac{3^4 \sin^4(\theta)}{(3\cos(\theta))^5} 3\cos(\theta) d\theta = \int \frac{1}{3}\tan^4(\theta) d\theta$$

We would then compute this integral using the methods shown in the section on trigonometric integrals.

The tangent transform

The tangent transform is used for integrals of the form $\sqrt{a^2 + x^2}$. It is based on:

$$x = a\tan(\theta)$$
$$dx = a\sec^2(\theta) d\theta$$
$$\sqrt{x^2 + a^2} = \sqrt{a^2 \sec^2(\theta)} = a\sec(\theta)$$

For example, if we had to deal with $\int \sqrt{4x^2 + 9}^3 dx$, we would do $x = \frac{3}{2}\tan(\theta)$. Then $\sqrt{4x^2 + 9} = 3\sec(\theta)$, so the integral becomes:

$$\int 3^5 \sec^5(\theta) \frac{3}{2} \sec^2(\theta) d\theta = \frac{3^6}{2} \int \sec^7(\theta) d\theta$$

The secant transform

The secant transform is used for integrals of the form $\sqrt{a^2 - x^2}$. It is based on:

$$x = a\sec(\theta)$$
$$dx = a\sec(\theta)\tan(\theta) d\theta$$
$$\sqrt{x^2 - a^2} = \sqrt{a^2 \tan^2(\theta)} = a\tan(\theta)$$

For example, consider the integral $\int \frac{x^2}{\sqrt{x^2-4}} dx$. We would let $x = 2\sec(\theta)$, and then we would have:

$$\int \frac{4\sec^2(\theta)}{2\tan(\theta)} 2\sec(\theta)\tan(\theta) d\theta = 4 \int \sec^3(\theta) d\theta$$

Completing the square

Cases where the quantity under the square root is a general quadratic can always be brought into one of the forms above via a method of computing the square. Here is an example:

$$\int x^2\sqrt{x^2-4x+5}$$

We start with by rewriting:

$$x^2-4x+5 = x^2-2\cdot 2x+2^2+1 = (x-2)^2+1$$

So the role of x is played by $x-2$. And $a=1$, so we would do a substitution:

$$x-2=\tan(\theta)$$

$$dx = \sec(\theta)\tan(\theta)d\theta$$

$$(x-2)^2+1=\sec^2(\theta)$$

The integral therefore becomes:

$$\int (2+\tan(\theta))^2\sqrt{\sec^2(\theta)}\sec(\theta)\tan(\theta)d\theta = \int (2+\tan(\theta))^2\sec^2(\theta)\tan(\theta)d\theta$$