

Cauchy-Riemann Equations

Reading

Section 3.1

Problems

Practice problems:

(Pages 41-42) 1, 2, 5, 6, 7, 8, 9, 10

Be ready to present propositions 3.6 and 3.7, problems 8, 9, 10

Topics to know

1. If f is complex-differentiable, then f_x and f_y both exist and relate to complex derivative f' :

- For real h : $\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = f_x$.
- For imaginary $h = it$, $\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \frac{f_y}{i}$.
- So $f_y = if_x$. Equivalently $u_x = v_y$ and $u_y = -v_x$.

2. Converse is true if the partial derivatives are continuous: If f_x and f_y exist in a neighborhood of z and $f_y = if_x$. Then f is complex-differentiable and $f'(z) = f_x - i f_y$.

- Rewrite of mean value theorem: $\frac{f(x+h) - f(x)}{h} = f'(x + \theta h)$ where $0 < \theta < 1$ is some specific but unknown number. ($x + \theta h$ is a way to specify an element between x and $x+h$)
- Suppose $h = h_1 + ih_2$, $z = x + iy$.
- $\frac{f(z+h) - f(z)}{h} = \frac{u(z+h) - u(z)}{h} + i \frac{v(z+h) - v(z)}{h}$.
- $u(z+h) - u(z) = u(x+h_1, y+h_2) - u(x+h_1, y) + u(x+h_1, y) - u(x, y)$ which equals $h_2 u_y(x+h_1, y+\theta_2 h_2) + h_1 u_x(x+\theta_1 h_1, y)$ where $\theta_1, \theta_2 \in (0, 1)$.
- That becomes $h_2 u_y(z_2) + h_1 u_x(z_1)$ where $z_2 = (x+h_1) + i(y+\theta_2 h_2)$, $z_1 = x + \theta_1 h_1 + iy$. In both cases $z_1, z_2 \rightarrow z$ as $h \rightarrow 0$.
- Similarly $v(z+h) - v(z) = h_2 v_y(z_4) + h_1 v_x(z_3)$ where $z_3, z_4 \rightarrow z$ as $h \rightarrow 0$.
- Therefore $\frac{f(z+h) - f(z)}{h} = \frac{h_2}{h} [u_y(z_2) + i v_y(z_4)] + \frac{h_1}{h} [u_x(z_1) + i v_x(z_3)]$
- Since $f_y = if_x$, we can write $f_x(z) = \frac{h_2}{h} f_y(z) + \frac{h_1}{h} f_x(z)$
- Take the difference: $\frac{f(z+h) - f(z)}{h} - f_x(z) = \frac{h_2}{h} [(u_y(z_2) - u_y(z)) + i(v_y(z_1) - v_y(z))] + \frac{h_1}{h} [(u_y(z_4) - u_y(z)) + i(v_y(z_3) - v_y(z))]$

- Because of continuity, each of the terms goes to 0. The factors in front are bound by 1. Therefore the whole thing goes to 0.
3. f is called **analytic** at z if it is complex-differentiable everywhere in a neighborhood of z . It is called analytic on a set S if it is complex-differentiable everywhere in an open set containing S .