Topology if the Complex Plane

Reading

Section 1.4, part II

Problems

Practice problems:

1. True or False: Union of two open sets is open.
2. True or False: Intersection of two open sets is open.
3. True or False: Union of two closed sets is closed.
4. True or False: Intersection of two closed sets is closed.
5. What about questions 1-4 but for infinitely many sets rather than just 2?
6. An "open rectangle" $(a, b) \times (c, d)$ is defined as all complex numbers whose real part is between a and b and whose imaginary part is between c and d. Show that the open rectangle is in fact an open set.
7. Show that the "closed rectangle" is in fact a closed set.
8. True or False: A set can be both open and closed at the same time.
9. Suppose U is an open set, z_n a sequence, $z_n \to z$ and $z \in U$. Show that there is an N such that for all $n \geq N$ we have $z_n \in U$.
10. Show that an open set does not contain any of its boundary points.
11. Show that a closed set contains all of its boundary points.
12. Show that a single point is a closed set.
13. True or false: If a set contains all of its boundary points, then it is a closed set.
14. True or false: If a set contains none of its boundary points, then it is an open set.
15. Suppose $f\colon \mathbb{C} \to \mathbb{C}$ is continuous, and $U \subset \mathbb{C}$ is set. Recall the definition of the set $f^{-1}(U) = \{x \mid f(x) \in U\}$, which makes sense regardless of whether f is an invertible function. Show that if U is an open set then $f^{-1}(U)$ is an open set. (Use $\epsilon - \delta$ definition of continuity)
16. Show that if f is continuous and K is a closed set then $f^{-1}(K)$ is closed.
17. Find a continuous function $f\colon \mathbb{C} \to \mathbb{C}$ and a set U that is open but such that $f(U)$ is not open.
18. (Challenge) Find a continuous function $f\colon \mathbb{C} \to \mathbb{C}$ and a set K that is closed but such that $f(K)$ is not closed. (It will have to be an unbounded set)
19. Show that if K is a compact set and F is a closed set, then $K \cap F$ is compact.
20. Show that if f is continuous and K is a compact set then $f(K)$ is a compact set. (use definition via sequences)

Topics to know

0. Notions of boundary points and limit points.

1. Open disc of radius r around a point. Picture.
   - Also “closed disc”.
2. Notion of open set.
   - For each of its points, the set contains an open disc centered at each that point.
   - The open disc itself is an open set. This is not obvious.
   - What is the largest open set? The smallest?
   - What are the open sets in the real line?
3. Notion of a closed set.
   - Closed sets are the complements of open sets.
   - What is the smallest/largest closed set?
   - The closed disc is a closed set.
4. A set is closed iff the limit of every convergent sequence from the set is also in the set.
   - The closed disc is a closed set, proof by thinking of the characterization via sequences.
5. Closed and bounded sets are called compact.
   - Set is compact if and only if every sequence from the set has a subsequence converging within the set.
6. Polygonally-connected sets.
   - Open and polygonally-connected sets are called regions.
7. Continuous functions.
   - Definition via $\epsilon - \delta$.
   - Definition via sequences.