Line Integrals

Reading

Section 4.1

Problems

Practice problems (page 56): 1, 2, 3, 4

Topics to know

1. Notions of piecewise-differentiable curves and of smooth curves (4.2)
2. Integral of a complex-valued function of a real variable (4.1).
3. Integrate $1/z$ and $1/z^2$ over a circle around 0. (page 48)
4. Integral of $f(z)$ along a smooth curve.
5. Integrals of $f(z)$ over equivalent curves are equal.
6. Integral over the opposite curve is the negative of the integral over the curve.
7. Linearity of the integral (proposition 4.8).
8. $\displaystyle\int_a^b G(t)dt \ll \int_a^b |G(t)|dt.$
   - $\int_a^b G(t)dt = Re^{i\theta}$
   - $\int_a^b e^{i\theta}G(t)dt = R$ is real
   - $e^{i\theta}G(t) = A(t) + iB(t)$
   - Then $R = \int_a^b A(t)dt = \int_a^b \text{Re}\left(e^{-\theta}G(t)\right) dt \leq \int_a^b |G(t)|dt$.
9. ML-formula: If $f \ll M$ along a curve C of length L, then:

$$\int_C f(z)dz = \int_a^b f(z(t))\dot{z}(t) \ll \int_a^b |f(z(t))\dot{z}(t)|dt \ll M \int_a^b |\dot{z}(t)|dt = ML$$

10. If $f_n \to f$ uniformly on C, then $\int_C f_n(z)dz \to \int_C f(z)dz$.
    - Notion of uniform convergence: For every $\epsilon > 0$ there is an N such that for all $n \geq N$ and for all $x \in C$ we have $|f_n(x) - f(x)| < \epsilon$.
    - Key idea: n depends only on $epsilon, but works for all x.
11. If $F'(z) = f(z)$ then $\int_C f(z)dz = F(z(b)) - F(z(a))$.
    - If $\lambda(t)$ is the curve, $\gamma(t) = F(\lambda(t))$
    - Then $\dot{\gamma}(t) = F'(\lambda(t))\dot{\lambda}(t) = f(\lambda(t))\dot{\lambda}(t)$
    - $\int_C f(z)dz = \int_a^b f(\lambda(t))\dot{\lambda}(t)dt = \int_a^b \dot{\gamma}(t) = \gamma(b) - \gamma(a) = F(z(b)) - F(z(a))$