Cauchy-Riemann Equations

Reading

Section 3.1

Problems

Practice problems:

(Pages 41-42) 1. 2, 5, 6, 7, 8, 9, 10

Be ready to present propositions 3.6 and 3.7, problems 8, 9, 10

Topics to know

1. If f is complex-differentiable, then f_x and f_y both exist and relate to complex derivative f':
    - For real h: $\lim\limits_{h\to 0}\dfrac{f(z+h)-f(z)}{h}=f_x$.
    - For imaginary $h=it$, $\lim\limits_{h\to 0}\dfrac{f(z+h)-f(z)}{h}=\dfrac{f_y}{i}$.
    - So $f_y=if_x$. Equivalently $u_x=v_y$ and $u_y=-v_x$.
2. Converse is true if the partial derivatives are continuous: If f_x and f_y exist in a neighborhood of z and $f_y=if_x$. Then f is complex-differentiable and $f'(z) = f_x = -i f_y $.
    - Rewrite of mean value theorem: $\dfrac{f(x+h)-f(x)}{h}=f'(x+\theta h)$ where $0<\theta<1$ is some specific but unknown number. ($x+\theta h$ is a way to specify an element between x and $x+h$)
    - Suppose $h=h_1+ih_2$, z = x + i y.
    - $\dfrac{f(z+h)-f(z)}{h}=\dfrac{u(z+h)-u(z)}{h}+i\dfrac{v(z+h)-v(z)}{h}$.
    - $u(z+h)-u(z)=u(x+h_1,y+h_2)-u(x+h_1,y)+u(x+h_1,y)-u(x,y)$ which equals $h_2u_y(x+h_1,y+\theta_2h_2)+h_1u_x(x+\theta_1h_1,y)$ where $\theta_1,\theta_2\in(0,1)$.
    - That becomes $h_2u_y(z_2)+h_1u_x(z_1)$ where $z_2=(x+h_1)+i(y+\theta_2h_2)$, $z_1=x+\theta_1h_1+iy$. In both cases $z_1,z_2\to z$ as $h\to 0$.
    - Similarly $v(z+h)-v(z)=h_2v_y(z_4)+h_1v_x(z_3)$ where $z_3,z_4\to z$ as $h\to 0$.
    - Therefore $\dfrac{f(z+h)-f(z)}{h}=\dfrac{h_2}{h}\big[u_y(z_2)+iv_y(z_4)\big]+\dfrac{h_1}{h}\big[u_x(z_1)+iv_x(z_3)\big]$
    - Since $f_y=if_x$, we can write $f_x(z)=\dfrac{h_2}{h}f_y(z)+\dfrac{h_1}{h}f_x(z)$
    - Take the difference: $\frac{f(z+h)-f(z)}{h}-f_x(z)=\frac{h_2}{h}\big[(u_y(z_2)-u_y(z))+i(v_y(z_1)-v_y(z))\big]+\frac{h_1}{h}\big[(u_y(z_4)-u_y(z))+i(v_y(z_3)-v_y(z))\big]$

- Because of continuity, each of the terms goes to 0. The factors in front are bound by 1. Therefore the whole thing goes to 0.
- Geometrically, instead of trying to go straight from $z+h$ to z, we instead move vertically then horizontally.

3. f is called **analytic** at z if it is complex-differentiable everywhere in a neighborhood of z. It is called analytic on a set S if it is complex-differentiable everywhere in an open set containing S.