Uniqueness, Mean Value, Maximum Modulus Theorems

Reading

Section 6.3

Problems

Practice problems (page 90): 4, 5, 6, 7, 8, 10

Challenge: 12, 13

Topics to know

1. Uniqueness Theorem: On a domain U, if there is a sequence $z_n \subset U$ and $z \in U$ such that $z_n \to z$, and $f(z_n) = 0$, then f is identically zero on U.
   - Note that power series uniqueness says: If $y_n \to y \in U$ and $f(y_n) = 0$ and $D(y, r) \subset U$, then $f(z) = 0$ for all $z \in D(y, r)$.
   - Define $A = \{a \in U \mid a$ is the limit point of a sequence of zeros $\}$ (in particular $f(z) = 0$ for all $z \in A$).
   - Define B be the rest.
   - We will show they are both open sets. Because U is connected one of them has to be empty. Since A is nonempty, B must be empty.
   - A is open because of the power series uniqueness: If $z \in A$ then we can find a sequence of zeros going to z. Since U is open there is an $D(z, r) \subset U$, and f is identically 0 there. All these points are limit points of zeros of f, so $D(z, r) \subset A$.
   - B is open: If $z \in B$, then there must be an $r > 0$ such that $D(z, r) \subset B$. If not, we will build a sequence of zeros going to z:
     - For each $\epsilon > 0$ there is a $w \in D(z, \epsilon/2)$ with $w \notin B$.
     - This w must be the limit point of a sequence of zeros, so there is a zero y such that $y \in D(w, \epsilon/2)$.
     - Then y is in $D(z, \epsilon)$.
     - So for each $\epsilon > 0$ there is a zero of f within ϵ of z.
     - So z would be a limit point of zeros. Contradiction.
   - NOTE: A function *can* have infinitely many zeros. But their limit point must be outside the domain of analyticity U.
2. Corollary: If two functions agree on a converging sequence in a domain U, then they must be equal throughout the domain U.
   - Use case: An identity between analytic functions, that holds for real numbers, must also hold for complex numbers.

3. Corollary: If f is an entire function and $f(z) \to \infty$ as $z \to \infty$, then f must be a polynomial.

   - There is an M so that $|f(z) \geq 1|$ for all $|z| \geq M$.
   - All zeros are restricted to $D(0, M)$. So there must be finitely many of them.
   - $g(z) = \frac{f(z)}{(z-a_1)(z-a_2)\cdots(z-a_n)}$ is entire, and has no zeros.
   - $h(z) = 1/g(z)$ satisfies $|h(z)| \leq A + |z|^n$. So is polynomial.
   - But $h(z) \neq 0$, so it must be constant. So $f(z) = C(z - a_1)(z - a_2) \cdots (z - a_n)$.

4. Mean Value Theorem: $f(a)$ equals the mean value taken around the boundary of a disc: $f(a) = \frac{1}{2\pi} \int_0^{2\pi} f\left(a + re^{i\theta}\right) d\theta$

   - Follows directly by Cauchy Integral Formula.

5. Maximum Modulus Theorem: A non-constant analytic function has no interior maximum points: For each $z \in D$ and $\delta > 0$ there is a $w \in D_z(\delta)$ such that $|f(w)| > |f(z)|$.

   - From mean value theorem: $|f(z)| \leq \max_\theta |f\left(z + re^{i\theta}\right)$
   - Since f cannot be constant on any such circle, there must be a point such that $|f(z)| < |f(w)|$.

6. Minimum Modulus Theorem: An interior point for an analytic function can only be a relative minimum for the modulus if it is actually 0.

   - Apply Maximum Modulus to $1/f$.