Sequences and Series in the Complex Plane

Reading

Section 1.4, part I

Problems

Practice problems:

1. Show that $z_n \to z$ if and only if $\bar{z}_n \to \bar{z}$.
2. Suppose that $|z| < 1$. Show that $a_n = z^n \to 0$.
3. True or False: The series $\sum_{n=0}^{\infty} a_n$ converges if and only if the series $\sum_{n=0}^{\infty} \mathrm{Re}(a_n)$ and $\sum_{n=0}^{\infty} \mathrm{Im}(a_n)$ both converge.
4. True or False: If $\sum_{n=0}^{\infty} a_n$ converges then $\sum_{n=0}^{\infty} \bar{a}_n$ also converges.
5. If $\sum_{n=0}^{\infty} a_n$ converges, show that $\sum_{n=0}^{\infty} a_n^2$ also converges.

Topics to know

1. Sequence z_n converges to z iff $|z_n - z| \to 0$.
   - Alternative definition
   - Laws for convergent sequences
2. Sequence converges if and only if its real parts converge and its imaginary parts converge.
   - Key inequality: $|\text{Re(z)}|, |\text{Im(z)}| \leq |z| \leq |\text{Re(z)}| + |\text{Im(z)}|$
3. Cauchy sequences, for both real numbers and complex numbers.
   - Laws for Cauchy sequences
4. Convergent sequences are Cauchy.
5. Cauchy sequences of real numbers converge. Key steps:
   - Cauchy sequences are bounded.
   - Any sequence contains a monotone subsequence.
     - Key statement: There is an N such that for all $M \geq N$ there is a $K \geq M$ such that $a_K \geq a_M$.
     - If that is true, then can build an increasing subsequence.
     - If it is false, then we can build a decreasing subsequence.
   - If a Cauchy sequence has a convergent subsequence then it converges.
6. Cauchy sequences of complex numbers converge.
   - Go through their real/imaginary parts.

7. Series of complex numbers. Review of results from Calculus 3. How do they carry over to complex numbers?

   - Definition of Convergent Series.
   - Divergence test.
   - Geometric series.
   - Definition of absolute/conditional convergence.
   - Alternating series test.
   - Absolute convergence implies conditional convergence.
   - Root and ratio tests.