Louisville's Theorem

Reading

Section 5.2

Problems

Practice problems (page 74): 6b, 8, 9, 10, 12, 15

Challenge (optional): 13

Topics to know

1. Louisville's Theorem: An entire bounded function is constant.
   - Fix two points a, b. Consider $R \to \infty$, and focus on circle C around 0 with radius R.
   - $f(b) - f(a) = \frac{1}{2\pi i}\left(\int_C \frac{f(z)}{z-a}dz - \int_C \frac{f(z)}{z-b}dz\right) = \frac{1}{2\pi i}\int_C \frac{f(z)(b-a)}{(z-a)(z-b)}dz$
   - If $f \ll M$, then this integral is $\ll \frac{M(b-a)R}{(R-|a|)(R-|b|)}$.
   - As $R \to \infty$, this goes to 0.
   - So $f(b) - f(a)$ must equal 0.
2. Extended Louisville Theorem: If f is entire and $|f(z)| \leq A + B|z|^k$, then f is a polynomial of degree at most k.
   - Proof by induction on k, base case being Louisville's Theorem.
   - $|g(z)| = \left|\frac{f(z)-f(0)}{z}\right| \leq C + D|z|^{k-1}$
     - Near 0 it can be extended to be entire, so is bounded
     - For $|z| \geq 1$: $\frac{|f(z)-f(0)|}{|z|} \leq \frac{|f(z)|+|f(0)|}{|z|} \leq \frac{A+|f(0)|+B|z|^k}{|z|} \leq C + B|z|^{k-1}$
   - By induction g must be a polynomial of degree at most $k-1$.
   - $f(z) = f(0) + zg(z)$ is a polynomial of degree at most k.
3. Alternative proof of the above two facts (exercises 6, 7):
   - If f is bounded by M along the circle of radius R around 0 and a_k is the k-th coefficient in its power series expansion around 0, then $|a_k| \leq \frac{M}{R^k}$. This follows from $M-L$ formula on $a_k = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^{k+1}}dz$
   - So if f is bounded on the entire complex plane, R can be arbitrarily large, forcing $a_k = 0$ for all $k \geq 1$. So power series is just constant, so f is constant.
   - If $|f(z)| \leq A + B|z|^k$, and $j > k$, then $|a_j| \leq \frac{A+BR^k}{R^j}$ must be valid for all $R > 0$, so it must equal 0 if we consider $R \to \infty$. Therefore the power series terminates at the k-th term, hence a polynomial.
4. Fundamental Theorem of Algebra: Any non-constant polynomial P must have a zero. And by induction, it has exactly deg$P$ zeroes.

- Suppose not. Consider $f(z) = \frac{1}{P(z)}$.
- f is entire.
- $P(z) \to \infty$ as $z \to \infty$. So f is bounded.
- So f must be constant. Then P is constant, a contradiction.

5. Gauss-Lucas theorem (proof on page 68): The zeroes of the derivative of a polynomial lie within the convex hull of the zeros of the polynomial.