Power Series

Reading

Section 2.2, 2.3

Problems

Practice problems:

1. (Page 32) 8, 9b, 10, 13, 14, 21, 23

Challenge: 12, 17, 18

Topics to know

1. Definition of $\overline{\lim_{n\to\infty}} a_n = \lim_{n\to\infty}\left(\sup_{k\geq n} a_k\right)$. If $\overline{\lim_{n\to\infty}} a_n = L$, then:
   - For each N and $\epsilon > 0$ there is a $k > N$ such that $a_k \geq L - \epsilon$.
   - For each $\epsilon > 0$ there is an N such that for all $k \geq N$ we have $a_k \leq L + \epsilon$.
2. Radius of convergence for $\sum C_k z^k$ dependent on $L = \overline{\lim} |C_k|^{1/k}$.
   - If $L = 0$, then $L = \overline{\lim} |C_k|^{1/k} |z| = 0$, so $|C_k z^k| \leq \frac{1}{2^k}$. Series converges absolutely for all z.
   - If $L = \infty$, then $|C_k|^{1/k} \geq \frac{1}{|z|}$ for infinitely many k. Hence $|C_k z^k| \not\to 0$. Diverges for all $z \neq 0$.
   - If $L = 1/R \in (0, \infty)$:
     - If $|z| = R(1-2\delta) < R$, then $\overline{\lim} |C_k|^{1/k} |z| = 1 - 2\delta$, so $|C_k|^{1/k} |z| < 1 - \delta$ for all k sufficiently large. Series converges.
     - If $|z| > R$ then $\overline{\lim} |C_k|^{1/k} |z| > 1$ so $C_k z^k \not\to 0$.
3. Examples 1-7 (page 27)
4. Differentiability of power series:
   - If $\sum C_n z^n$ converges on some disc $D(0, R)$, then so does $\sum n C_n z^{n-1}$, since $\overline{\lim} |nC_n|^{1/(n-1)} = \overline{\lim} |C_n|^{1/n}$.
   - Key idea for case where $R = \infty$: $\dfrac{f(z+h) - f(z)}{h} - \sum_{n=0}^{\infty} n C_n z^{n-1} = \sum_{n=2}^{\infty} C_n b_n$ where $b_n = \frac{(z+h)^n - z^n}{h} - n z^{n-1} \leq |h| \left(|z| + 1\right)^n$.
   - So $\left| \dfrac{f(z+h) - f(z)}{h} - \sum_{n=0}^{\infty} n C_n z^{n-1} \right| \leq A|h|$ (because $\sum |C_n|(|z|+1)^n$ converges).
   - Let $h \to 0$.

- If $R < \infty$ the proof is technically more difficult (see page 29).

5. Power series are infinitely differentiable.
6. Power series coefficients depend on the higher order derivatives at 0.
7. Uniqueness theorem: If series is zero when evaluated at all points of a sequence $z_n \to 0$, then series is identically zero.
   - Inductively compute $C_n = \lim_{z\to 0} \frac{f(z)}{z^n} = \lim_{k\to\infty} \frac{f(z_k)}{z_k^n} = 0$
8. If two series agree on a sequence that goes to 0, then they are identical series.