

# Topology of the Complex Plane

## Reading

Section 1.4, part II

## Problems

Practice problems:

1. True or False: Union of two open sets is open.
2. True or False: Intersection of two open sets is open.
3. True or False: Union of two closed sets is closed.
4. True or False: Intersection of two closed sets is closed.
5. What about questions 1-4 but for infinitely many sets rather than just 2?
6. An “open rectangle”  $(a, b) \times (c, d)$  is defined as all complex numbers whose real part is between  $a$  and  $b$  and whose imaginary part is between  $c$  and  $d$ . Show that the open rectangle is in fact an open set.
7. Show that the “closed rectangle” is in fact a closed set.
8. True or False: A set can be both open and closed at the same time.
9. Suppose  $U$  is an open set,  $z_n$  a sequence,  $z_n \rightarrow z$  and  $z \in U$ . Show that there is an  $N$  such that for all  $n \geq N$  we have  $z_n \in U$ .
10. Show that an open set does not contain any of its boundary points.
11. Show that a closed set contains all of its boundary points.
12. Show that a single point is a closed set.
13. Suppose  $f: \mathbb{C} \rightarrow \mathbb{C}$  is continuous, and  $U \subset \mathbb{C}$  is set. Recall the definition of the set  $f^{-1}(U) = \{x \mid f(x) \in U\}$ , which makes sense regardless of whether  $f$  is an invertible function. Show that if  $U$  is an open set then  $f^{-1}(U)$  is an open set. (Use  $\epsilon - \delta$  definition of continuity)
14. Show that if  $f$  is continuous and  $K$  is a closed set then  $f^{-1}(K)$  is closed.
15. Find a continuous function  $f: \mathbb{C} \rightarrow \mathbb{C}$  and a set  $U$  that is open but such that  $f(U)$  is not open.
16. Find a continuous function  $f: \mathbb{C} \rightarrow \mathbb{C}$  and a set  $K$  that is closed but such that  $f(K)$  is not closed. (It will have to be an unbounded set)
17. Show that if  $K$  is a compact set and  $F$  is a closed set, then  $K \cap F$  is compact.
18. Show that if  $f$  is continuous and  $K$  is a compact set then  $f(K)$  is a compact set. (use definition via sequences)

## Topics to know

0. Notions of boundary points and limit points.
1. Open disc of radius  $r$  around a point. Picture.
  - Also “closed disc”.

## 2. Notion of open set.

- For each of its points, the set contains an open disc centered at each that point.
- The open disc itself is an open set. This is not obvious.
- What is the largest open set? The smallest?
- What are the open sets in the real line?

## 3. Notion of a closed set.

- Closed sets are the complements of open sets.
- What is the smallest/largest closed set?
- The closed disc is a closed set.

## 4. A set is closed iff the limit of every convergent sequence from the set is also in the set.

- The closed disc is a closed set, proof by thinking of the characterization via sequences.

## 5. Closed and bounded sets are called compact.

- Set is compact if and only if every sequence from the set has a subsequence converging within the set.

## 6. Polygonally-connected sets.

- Open and polygonally-connected sets are called regions.

## 7. Continuous functions.

- Definition via  $\epsilon - \delta$ .
- Definition via sequences.