# Cauchy-Riemann Equations

## Reading

Section 3.1

## Problems

Practice problems:

(Pages 41-42) 1. 2, 5, 6, 7, 8, 9, 10

Be ready to present propositions 3.6 and 3.7, problems 8, 9, 10

## Topics to know

1. If $f$ is complex-differentiable, then $f_x$ and $f_y$ both exist and relate to complex derivative $f'$:
   - For real $h$: $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = f_x$.
   - For imaginary $h = it$, $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \frac{f_y}{i}$.
   - So $f_y = if_x$. Equivalently $u_x = v_y$ and $u_y = -v_x$.
2. Converse is true if the partial derivatives are continuous: If $f_x$ and $f_y$ exist in a neighborhood of $z$ and $f_y = if_x$. Then $f$ is complex-differentiable and $f'(z) = f_x = -if_y$.
   - Rewrite of mean value theorem: $\frac{f(x+h)-f(x)}{h} = f'(x+\theta h)$ where $0 < \theta < 1$ is some specific but unknown number. ($x+\theta h$ is a way to specify an element between $x$ and $x+h$)
   - Suppose $h = h_1 + ih_2$, z = x + i y.
   - $\frac{f(z+h)-f(z)}{h} = \frac{u(z+h)-u(z)}{h} + i\frac{v(z+h)-v(z)}{h}$.
   - $u(z+h)-u(z) = u(x+h_1, y+h_2) - u(x+h_1, y) + u(x+h_1, y) - u(x,y)$ which equals $h_2 u_y(x+h_1, y+\theta_2 h_2) + h_1 u_x(x+\theta_1 h_1, y)$ where $\theta_1, \theta_2 \in (0,1)$.
   - That becomes $h_2 u_y(z_2) + h_1 u_x(z_1)$ where $z_2 = (x+h_1) + i(y+\theta_2 h_2)$, $z_1 = x+\theta_1 h_1 + iy$. In both cases $z_1, z_2 \to z$ as $h \to 0$.
   - Similarly $v(z+h)-v(z) = h_2 v_y(z_4) + h_1 v_x(z_3)$ where $z_3, z_4 \to z$ as $h \to 0$.
   - Therefore $\frac{f(z+h)-f(z)}{h} = \frac{h_2}{h}\left[u_y(z_2) + iv_y(z_4)\right] + \frac{h_1}{h}\left[u_x(z_1) + iv_x(z_3)\right]$
   - Since $f_y = if_x$, we can write $f_x(z) = \frac{h_2}{h} f_y(z) + \frac{h_1}{h} f_x(z)$
   - Take the difference: $\frac{f(z+h)-f(z)}{h} - f_x(z) = \frac{h_2}{h}\left[(u_y(z_2) - u_y(z)) + i(v_y(z_1) - v_y(z))\right] + \frac{h_1}{h}\left[(u_y(z_4) - u_y(z)) + i(v_y(z_3) - v_y(z))\right]$

- Because of continuity, each of the terms goes to $0$. The factors in front are bound by $1$. Therefore the whole thing goes to $0$.
- Geometrically, instead of trying to go straight from $z+h$ to $z$, we instead move vertically then horizontally.

3. $f$ is called **analytic** at $z$ if it is complex-differentiable everywhere in a neighborhood of $z$. It is called analytic on a set $S$ if it is complex-differentiable everywhere in an open set containing $S$.