Hamiltonian Graphs

- Read section 6.2, pages 140-150
- What is a **Hamiltonian cycle** in a graph? What is a **Hamiltonian graph**?
  - Do path-/cycle-/complete- graphs have Hamiltonian cycles?
  - Do complete bipartite graphs have Hamiltonian cycles?
  - Can a Hamiltonian graph contain a cut vertex? Can it contain a bridge?
- What is an **Hamiltonian path**?
  - Give examples of graphs that have an Hamiltonian path but no Hamiltonian cycle. Can you find a smallest possible example?
- Show that $K_{3,3}$ has a Hamiltonian cycle.
  - Do all complete bipartite graphs have a Hamiltonian cycle? If not, which ones do?
- Theorem 6.4: The Petersen graph is non-Hamiltonian.
  - What about the graph that is similar, but where the structure of the inner 5 vertices matches that of the outer 5?
- Theorem 6.5: If G is a Hamiltonian graph, then for every non-empty proper subset S of vertices of G we have that the number of components of $G - S$ is equal to $|S|$.
  - For each component G_i, the vertex in the Hamiltonian cycle following the last vertex in G_i must belong to S.
  - What is the contrapositive statement? Show an example of its use (e.g. bipartite graphs).
  - How does the contra-positive relate to a cut vertex?
  - Can the contrapositive be applied to prove that the Petersen graph is non-Hamiltonian?
- Theorem 6.6: If G has order at least 3, and the degrees of any two nonadjacent vertices add up to at least the order n, then G is Hamiltonian.
  - Start with such a graph that is not Hamiltonian, add as many edges as possible without making the graph Hamiltonian.
  - Look at a pair of non-adjacent vertices x, y in this graph. Adding the edge xy gives a Hamiltonian cycle, which must contain the edge xy. So there is a Hamiltonian path from x to y in the graph.
  - For each vertex on this path that is adjacent to x, there is one (its predecessor) that is not adjacent to y (Figure 6.17). Then the degrees don't add to more than $n - 1$
  - Show this theorem's bound is sharp.
- Corollary 6.7: If G has order $n \geq 3$ and for each vertex v we have $\deg(v) \geq (n/2)$, then G is Hamiltonian.

- Theorem 6.8: If u, v are non-adjacent vertices of a graph G, and their degrees add to at least n, then G is Hamiltonian if and only of $G + uv$ is Hamiltonian.
- What is the **closure** $C(G)$ of a graph G?
- A graph is Hamiltonian if and only if its closure is Hamiltonian.
- Practice Problems: 6.10, 6.11, 6.12, 6.13, 6.15, 6.16