

Graph Theory

Reading

- Read 1.1, pages 1-7
 - What are the elements that comprise a **graph**?
 - Can a graph have no vertices? Can it have no edges?
 - Can we have two edges between the same two vertices?
 - If we have a graph with 4 vertices, *at most* how many edges can it have?
 - What are the **order** and the **size** of a graph?
 - Consider example 1.3. Does every vertex have the same number of edges coming out from it? Explain.
- Read 1.2, pages 9-17
 - When do we say two vertices are **neighbors**?
 - * What is the largest possible number of vertices that can be neighbors to one specific vertex?
 - When do we say two edges are **adjacent**?
 - * What is the largest possible number of edges that can be adjacent to one specific vertex?
 - What is a **subgraph** of a graph G ?
 - * When is it called a **proper** subgraph?
 - * When is it called a **spanning** subgraph?
 - How many spanning subgraphs can a graph have?
 - * Can we form a subgraph by choosing any subset of the vertices and any subset of the edges?
 - * When is a subgraph called an **induced** subgraph?
 - How many (vertex-)induced subgraphs can a graph have?
 - If X is a set of edges in a graph G , what graph do we denote by $G - X$?
 - If U is a set of vertices in a graph G , what graph do we denote by $G - U$?
 - What is a **walk** in a graph G ?
 - * Does every graph have at least one walk?
 - * Is there always a walk from w to v for any two vertices in a graph G ?
 - * If there is a walk from w to v in a graph G , is there also one from v to w ?
 - * Can a walk pass through the same vertex multiple times?
 - * When is a walk called **closed**, and when is it called **open**?
 - * What is the **length** of a walk?

- For a given graph G , is there a bound to the maximum length of a walk? Is there one sometimes?
- What is a **trail** in a graph G , and how does it differ from a walk?
 - * Can a trail traverse the same vertex more than once?
- What is a **path** in a graph G .
- Describe how under certain circumstances we may combine two walks into one longer walk.
 - * When is this possible?
 - * If we combine two trails this way, is the result necessarily a trail?
 - * If we combine two paths this way, is the result necessarily a path?
- True or False: A walk from u to v that has the smallest length out of all walks from u to v must necessarily be a path.
- Prove theorem 1.6: If a graph contains a u - v walk, then it also contains a u - v path with length at most that of the walk.
- What is a **circuit**? What is a **cycle**?
- What graphs are called **connected**?
 - * True or False: If a graph is connected, and we remove an edge, then it becomes disconnected.
 - * What are the **components** of a disconnected graph.
 - * True or False: A component of a disconnected graph must be an induced subgraph.
 - * True or False: Two components of a disconnected graph cannot possibly have a vertex in common.
- Prove theorem 1.7: The relation " u relates to v iff it is connected to it" is an equivalence relation.
- Consider theorem 1.8: If a graph of order at least 3 has two distinct vertices u, v , such that $G-u$ and $G-v$ are connected graphs, then G is also connected.
 - * Explain why the assumption that the vertices be distinct is needed, and why the order being at least 3 is needed. Show by examples that the theorem fails in those cases.
 - * Prove the theorem.
 - * State the converse of the theorem.
- What is the **distance** between two vertices in a graph?
 - * Can the distance be 0? When?
 - * Can the distance be 1? When?
 - * What is the largest possible value for the distance? Can you think of an example where it is achieved?
- Which paths are called **geodesic**?
 - * True or False: A subgraph of a geodesic path is also geodesic.

- * True or False: If there is a non-geodesic path in a graph, then the graph must contain a cycle.
- What is the **diameter** of a graph?
 - * What is the largest possible value for the diameter?
 - * Can the diameter be 0? When?
 - * Can the diameter be 1? When?
- Prove Theorem 1.9: If a graph of order at least 3 is connected, then G contains two distinct vertices u, v such that $G - u$ and $G - v$ are both connected.
- Practice problems:
 - 1.11, 1.12, 1.15, 1.17