Graph Theory

- Read 1.1, pages 1-7
  - What are the elements that comprise a **graph**?
  - Can a graph have no vertices? Can it have no edges?
  - Can we have two edges between the same two vertices?
  - If we have a graph with 4 vertices, *at most* how many edges can it have?
  - What are the **order** and the **size** of a graph?
  - Consider example 1.3. Does every vertex have the same number of edges coming out from it? Explain.
- Read 1.2, pages 9-17
  - When do we say two vertices are **neighbors**?
    - What is the largest possible number of vertices that can be neighors to one specific vertex?
  - When do we say two edges are **adjacent**?
    - What is the largest possible number of edges that can be adjacent to one specific vertex?
  - What is a **subgraph** of a graph G?
    - When is it called a **proper** subgraph?
    - When is it called a **spanning** subgraph?
      - How many spanning subgraphs can a graph have?
    - Can we form a subgraph by choosing any subset of the vertices and any subset of the edges?
    - When is a subgraph called an **induced** subgraph?
      - How many (vertex-)induced subgraphs can a graph have?
  - If X is a set of edges in a graph G, what graph do we denote by $G - X$?
  - If U is a set of vertices in a graph G, what graph do we denote by $G - U$?
  - What is a **walk** in a graph G?
    - Does every graph have at least one walk?
    - Is there always a walk from w to v for any two vertices in a graph G?
    - If there is a walk from w to v in a graph G, is there also one from v to w?
    - Can a walk pass through the same vertex multiple times?
    - When is a walk called **closed**, and when is it called **open**?
    - What is the **length** of a walk?
      - For a given graph G, is there a bound to the maximum length of a walk? Is there one sometimes?
  - What is a **trail** in a graph G, and how does it differ from a walk?
    - Can a trail traverse the same vertex more than once?
  - What is a **path** in a graph G.

- Describe how under certain circumstances we may combine two walks into one longer walk.
  * When is this possible?
  * If we combine two trails this way, is the result necessarily a trail?
  * If we combine two paths this way, is the result necessarily a path?
- True or False: A walk from u to v that has the smallest length out of all walks from u to v must necessarily be a path.
- Prove theorem 1.6: If a graph contains a u-v walk, then it also contains a u-v path with length at most that of the walk.
- What is a **circuit**? What is a **cycle**?
- What graphs are called **connected**?
  * True of False: If a graph is connected, and we remove an edge, then it becomes disconnected.
  * What are the **components** of a disconnected graph.
  * True or False: A component of a disconnected graph must be an induced subgraph.
  * True or False: Two components of a disconnected graph cannot possibly have a vertex in common.
- Prove theorem 1.7: The relation "u relates to v iff it is connected to it" is an equivalence relation.
- Consider theorem 1.8: If a graph of order at least 3 has two distinct vertices u, v, such that $G - u$ and $G - v$ are connected graphs, then G is also connected.
  * Explain why the assumption that the vertices be distinct is needed, and why the order being at least 3 is needed. Show by examples that the theorem fails in those cases.
  * Prove the theorem.
  * State the converse of the theorem.
- What is the **distance** between two vertices in a graph?
  * Can the distance be 0? When?
  * Can the distance be 1? When?
  * What is the largest possible value for the distance? Can you think of an example where it is achieved?
- Which paths are called **geodesic**?
  * True or False: A subgraph of a geodesic path is also geodesic.
  * True or False: If there is a non-geodesic path in a graph, then the graph must contain a cycle.
- What is the **diameter** of a graph?
  * What is the largest possible value for the diameter?
  * Can the diameter be 0? When?
  * Can the diameter be 1? When?
- Prove Theorem 1.9: If a graph of order at least 3 is connected, then G contains two distinct vertices u, v such that $G - u$ and $G - v$ are both connected.

• Practice problems:
  - 1.11, 1.12, 1.15, 1.17