Quantifiers and Proofs

We discuss in this section how quantifiers are used, and how they affect the proving process.

Reading

Section 1.5

Practice Problems

1.5 2, 3, 4, 5, 8, 9, 10, 12

Notes

- There are two quantifiers:

  $\forall$ "for all". Called the **universal quantifier**.
  $\exists$ "exists". Called the **existential quantifier**.

  They are followed by a variable, and they are meant to suggest that the remaining statement is true for all values (respectively for at least one values).

- In every theorem or lemma, and variable needs to be quantified. If it is not, it is implicitly assumed that "forall" was intended.

- To prove a statement "for every integer n":

  - Say "let n be an integer". n is now a specific, fixed, but unknown number. It stands as a "placeholder" for any arbitrary number.
  - Continue with the proof.
  - If you end up proving the statement for this n, and since it could be absolutely any number, you have proven it for all numbers.
  - NOTE: the word "let" is also used to introduce new intermediate variables that we define to help us along.

- To prove a statement "there exists an integer n such that ..."

  - The simplest approach is to in fact demonstrate a specific number n that has the property. A single example suffices, even if there might be many numbers that have the property.
  - Some times you can prove by contradiction:
    - Assume the "there exists an integer such that ..." statement is false. Then it must be the case that for all integers n, whatever is in the "..." has to be false.

    * So we end up instead assuming a statement that says "for all integers n, ... is false".
    * If we can prove that this somehow produces a contradiction, that means there must be an integer.
  - One subtle point however. Here is what you should NOT do: Say we want to show that "there exists an integer n that is even".
    * Assume by contradiction that all integers are not even.
    * But for example 4 is in fact even. Contradiction.
    * Therefore our assumption that all integers are not even must be false. So there must be an integer that is even.
    * Can you spot why this reasoning is to be avoided?

- Order of quantifiers matters. Consider the following two statements:

  $\forall x \exists y \, x + y > 0$ This is true (why?)
  $\exists y \forall x \, x + y > 0$ This is false (why?)

- Negating quantified statements ends up switching the quantifiers:

  - The negation of "$\forall n \, P(n)$" is "$\exists n$ not$P(n)$"
  - The negation of "$\exists n \, P(n)$" is "$\forall n$ not$P(n)$"