# Euler's Theorem

## Reading

- Section 9.3

## Practice Problems

**9.3** 1-12, 15-18

## Notes

Euler's Theorem is an extension of Fermat's Little Theorem to the case of non-prime numbers.

#### Euler's Theorem

> **Euler's Theorem**
>
> If $a$ is relatively prime to $n$, then:
>
> $$a^{\phi(n)} \equiv 1 \bmod n$$
>
> Since when $n$ is prime we have $\phi(n) = n - 1$, this reverts back to Fermat's theorem.

The proof is very similar to the one for Fermat's:

- Consider the set of reduced residues (those relatively prime to $n$). Let us denote them by $b_1$, $b_2$, $b_3$, ..., $b_{\phi(n)}$.
- Because $a$ is also a reduced residue, multiplying those reduced residues by $a$ gives back reduced residues.
- Multiplying by the (invertible) $a$ is also 1-1, so it will simply permute them.
- We can then multiply them together: $(ab_1)(ab_2)\cdots(ab_{\phi(n)}) = b_1b_2\cdots b_{\phi(n)}$ in $\mathbb{Z}_n$.
- Pulling the $a$'s out, we would have: $a^{\phi(n)}b_1b_2\cdots b_{\phi(n)} = b_1b_2\cdots b_{\phi(n)}$
- The $b$'s are invertible, so we can cancel them out from both sides.
- So we end up with $a^{\phi(n)} = 1$.

#### Repeated Squaring for fast exponentiation

It is time to discuss how to quickly compute powers of a number. The trick here is repeated squaring. But first:

If $a$ is relatively prime to $n$, and $k \equiv r \bmod \phi(n)$, then:

$$a^k \equiv a^r \bmod n$$

So Euler's and Fermat's theorems can considerably lessen the work involved in computing $a^k \bmod n$ by reducing the power $k$ to a number less than $\phi(n)$. This might still be a very high number however. This is where repeated squaring comes in.

Let us suppose for example that we want to compute $6^{91} \bmod 715$. The trick to answer that is the **binary representation** of $91$: We can find that representation by looking at the powers of $2$ and taking one power out at a time:

$$91 = 64 + 27 = 64 + 16 + 11 = 64 + 16 + 8 + 3 = 64 + 16 + 8 + 2 + 1$$

Based on this, we can tell that:

$$6^{91} = 6^{64+16+8+2+1} = 6^{64}6^{16}6^{8}6^{2}6^{1}$$

So if we have computed $6$ raised to those powers of 2, we can put those answers together to get the answer for all other powers. Let us compute them:

$$\begin{gathered}
6^1 = 6 \\
6^2 = 36 \\
6^4 = 36^2 = 1296 \equiv 581 \\
6^8 = 581^2 = 337561 \equiv 81 \\
6^{16} = 81^2 = 6561 \equiv 126 \\
6^{32} = 126^2 = 15876 \equiv 146 \\
6^{64} = 146^2 = 21316 \equiv 581
\end{gathered}$$

As we compute those, we would at the same time multiply those that we need:

$$\begin{gathered}
6^1 = 6 \\
6^2 \cdot 6^1 = 36 \cdot 6 = 216 \\
6^8 \cdot 6^2 \cdot 6^1 = 81 \cdot 216 = 17496 \equiv 336 \\
6^{16} \cdot 6^8 \cdot 6^2 \cdot 6^1 = 126 \cdot 336 = 42336 \equiv 151 \\
6^{64} \cdot 6^{16} \cdot 6^8 \cdot 6^2 \cdot 6^1 = 581 \cdot 151 = 87731 \equiv 501
\end{gathered}$$

So that's our final answer, $6^{91} \equiv 501 \bmod 715$.

This may have been a lot of work, but let us count the operations: We had to do 7 multiplications to compute $6$ raised to the powers of $2$, and at most another 7 multiplications to combine those powers to get our answer. So a total of $2 \times 7 = 14$ multiplications (instead of $91$).

Exponentiation via repeated squaring requires $\log_2(k)$ multiplications.

This scales very well as the power $k$ grows.

For completeness, let us describe an algorithm for carrying the fast exponentiation out. Those of you with programming inclinations should try implement the algorithm.

**Algorithm for fast exponentiation**

Inputs: $n$, $k$, $a$

Output: $a^k \bmod n$

Local variables:

- "prod" accumulates the final product
- "$b$" keeps track of the power $a^{2^x}$ as we compute each new one

Steps:

- Initialize: prod $= 1$, $b = a$
- Repeat:
  - If $a = 0$ we are done. Return "prod".
  - Divide $a$ by $2$: $a = 2q + r$. (You wouldn't need to do this step as is, can rely on using the operators for mod and div. But conceptually it happens.)
  - If $r = 1$ we need to use this $b$. So $prod \leftarrow prod \cdot b \bmod n$.
  - $b \leftarrow b \cdot b \bmod n$.
  - $a \leftarrow q$.