Proof Techniques

In this section we will practice some basic proof techniques.

Reading

Section 1.3

Practice Problems

1.3 Direct 1, 5, 6, 11, 12
1.3 Indirect 2, 4, 9, 10, 13
1.3 Advanced 18, 19
1.3 Challenge 21, 23, 24

Notes

- Proofs are the bread and butter of a mathematician's work. Every assertion we make needs to be proven.
- All assertions have two parts:
  - **hypotheses** are the things we assume to be true
  - **conclusions** are the things are trying to deduce as being true *under the assumption* that the hypotheses are true.
- This is paramount: A proof is simply evidence that if the hypotheses were to be true, then the conclusions would also *have* to be true. It does not concern itself at all with the validity of the hypotheses.
- Proofs fall into various categories. We will start with direct proofs.

Direct proofs

- Direct proofs simply start from their hypotheses, and move in a logical progression towards their conclusions.
- Example 1: Let m and n be two integers. If they are both odd, then their product is also odd.
  - To be odd means we can write the number as $2k + 1$ where k is an integers (why is it important to say that last part?)
  - We start with the hypothesis: m and n are odd, so we can write them as $m = 2k_1 + 1$ and $n = 2k_2 + 1$ where k_1 and k_2 are integers.
  - Now we compute $mn = ... = 2(2k_1k_2 + k_1 + k_2) + 1$

  - Since the parenthesized part is an integer (why?), we get that mn has the form required to be an odd integer.

- Example 2: Show that for every integer n, the expression $n^2 + n$ is necessarily an even number.

  - Exercise for the students. You have two cases to deal with: If n is odd, and if n is even. Do each separately.
  - Together in class: Every integer can be written as $2n + \epsilon$ where ϵ is either 0 or 1. This can bring the two cases together in this case.

- Food for thought: How do we know that each integer is either even or odd?

Indirect proofs

- There are many kinds of indirect proofs, but some techniques stand out.
- The most standard amongst them is **contradiction**:

  - We want to show that "if P then Q".
  - We instead assume that P is true but Q is false, and derive a contradiction: Something that is impossible.
  - Since we saw that if P is true an Q is not true, we would get something impossible, the only alternative is that if P is true then Q must also be true.

- It is very useful for proving negative statements.
- Contradiction Example 1: Show that there is no smallest positive rational number.

  - By contradiction: We assume there is one and derive an absurd statement from that assumption.
  - Say q is this "smallest positive rational number".
  - Can we construct a number that is positive and smaller? If we can, that is a contradiction, so q could not have existed in the first place.

- Food for thought:

  - Where does this proof break down if we try to apply it to the integers?
  - Is that fact enough to conclude that for the integers there *is* a smallest positive rational number?

- Contradiction Example 2: Show that for integers m, n, if mn is odd, then both m and n must be odd.

  - By contradiction, assume that mn is odd and one of m or n is not odd, hence even. Without loss of generality, assume it is m.
  - We know (show separately) that if m is even then mn is also even. This contradicts the assumption that mn was odd.

- This is actually best seen as an example of the **contrapositive**:

  - Showing "if P then Q" is the same as showing "if not Q then not P"

  - In the above example, this would read: If m or n is even, then the product mn is also even.
  - Food for thought: Understand why these two are equivalent (page 25 from the book but think about it first).

- The **converse** of "if P then Q" is "if Q then P". These are in general not equivalent statements, one could be true while the other is false. (Students: come up with examples)