Assignment 10

True/False questions, not to turn in:

- If $p > 2$ is a prime, and if $2p+1$ and $9p+4$ are both prime, then $\left(\frac{2p+1}{9p+4}\right) = 1$. (Make sure you also have proof one way or the other)
- If $p > 2$ and $q = p+2$ are both prime, then $\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)$.
- The product of two "positive residues" is a "positive residue".
- Let a be relatively prime to the prime p. For each "negative residue" k compute $a \cdot k$, and denote by g' the number of those products that result in "positive residues". Then $g' = g$.
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Questions to turn in:

1. Using Euler's identity, we determined the quadratic character for -1, namely the Law of Quadratic Reciprocity for $\left(\frac{-1}{p}\right)$. Prove the same result by computing g instead, and using Gauss's Lemma.

2. During the proof that $T(a,p) = g \bmod 2$, we used that a is an odd number. What happens when a is even? Can you demonstrate an example with p odd, a even, where $T(a,p) \neq g \bmod 2$?

3. Demonstrate that $T(p,q) + T(q,p) = \frac{p-1}{2} \times \frac{q-1}{2}$ by computing both sides directly, for $p = 13$ and $q = 17$.

4. The Law of Quadratic Reciprocity can help us determine if the equation $x^2+bx+c = 0$ has a solution modulo p. The equation can be rewritten as:

$$\left(x + \frac{b}{2}\right)^2 = \frac{b^2 - 4c}{4}$$

   So it will only have a solution if the quantity $b^2 - 4c$ is a square modulo p. Use this on the equation $x^2 + 13x + 4 = 0$ to find: At least two primes p for which there is a solution and at least two primes p for which there is not. For the primes for which there is a solution, find it using the above formula. Note that for each prime the equation should have 2 solutions, so you will be finding in total 4 solutions, 2 each for each of the two primes.

5. Compute $\left(\frac{7}{17}\right)$ in a number of different ways:

   a. Computing all the squares $\mathrm{mod} 17$.
   b. Using Euler's Identity and computing $7^8 \bmod 17$.
   c. Using Gauss's Lemma and computing g.

d. Using Eisenstein's Lemma and computing $T(7, 17)$.
e. Using the Laws of Quadratic Reciprocity to reduce the symbol to smaller symbols.

You should of course get the same result in all cases.