The Fibonacci Numbers

Reading

Section 2.3

Practice Problems

2.3 1, 2, 4
Challenge 2.3 (optional) 6, 7, 8, 9, 10, 11

Notes

- The Fibonacci numbers are a well known sequence of numbers, defined via a recursion:
  - $F_1 = 1$
  - $F_2 = 1$
  - $F_n = F_{n-1} + F_{n-2}$ for every $n \geq 2$
- In other words, every subsequent number is the sum of the two previous numbers.
- Student TODO: Compute the next 8 terms.
- Student TODO: Compute successive sums:
  - $F_1 + F_2$
  - $F_1 + F_2 + F_3$
  - $F_1 + F_2 + F_3 + F_4$
  - Do you notice a pattern in how those number relate, to the Fibonacci sequence perhaps?
  - Together TODO: Prove our guess using induction
- The Fibonacci numbers have a close relation to the "golden ratio", which is equal to $\frac{1+\sqrt{5}}{2}$ and is denoted by φ.
- The golden ratio is the positive solution to the equation $x^2 = x + 1$ which can also be written as $x = 1 + \frac{1}{x}$.
- In fact it turns out that the golden ratio equals the limit of the ratios of successive Fibonacci numbers:

$$\varphi = \lim_{n \to \infty} \frac{F_{n+1}}{F_n}$$

- Try the first couple of quotients by hand.

- We now will discuss the following wonderful theorem:

  > Every natural number n can be written as a sum of distinct Fibonacci numbers.

  - First off, try some numbers and convince yourselves that you can write them.
  - Here is a sketch of a proof:
    * Take the largest Fibonacci number that is not bigger than n, say it is F_k.
    * Look at $n - F_k$. By induction, we can assume that this number can be written as a sum of Fibonacci numbers, say $n - F_k = F_a + F_b + F_c + \ldots$.
    * Moving F_k to the other side would do it, provided there isn't an F_k there already.
    * But we can show that $n - F_k < F_{k-1}$ (THINK ABOUT IT!), so in fact the Fibonacci numbers used to write $n - F_k$ cannot include F_{k-1} nor anything bigger (e.g. not F_k either).

- Food for thought: We can always assume that the Fibonacci numbers we use are non-consecutive. Why?

- In fact with that extra assumption, there is a uniqueness component:

  > Every number can be written as a sum of non-consecutive Fibonacci numbers in a unique way.