

# Proof Techniques

In this section we will practice some basic proof techniques.

## Reading

Section 1.3

## Practice Problems

**1.3 Direct** 1, 5, 6, 11, 12

**1.3 Indirect** 2, 4, 9, 10, 13

**1.3 Advanced** 18, 19

**1.3 Challenge** 21, 23, 24

## Notes

- Proofs are the bread and butter of a mathematician's work. Every assertion we make needs to be proven.
- All assertions have two parts:
  - **hypotheses** are the things we assume to be true
  - **conclusions** are the things we are trying to deduce as being true *under the assumption* that the hypotheses are true.
- This is paramount: A proof is simply evidence that if the hypotheses were to be true, then the conclusions would also *have* to be true. It does not concern itself at all with the validity of the hypotheses.
- Proofs fall into various categories. We will start with direct proofs.

## Direct proofs

- Direct proofs simply start from their hypotheses, and move in a logical progression towards their conclusions.
- Example 1: Let  $m$  and  $n$  be two integers. If they are both odd, then their product is also odd.
  - To be odd means we can write the number as  $2k + 1$  where  $k$  is an integer (why is it important to say that last part?)
  - We start with the hypothesis:  $m$  and  $n$  are odd, so we can write them as  $m = 2k_1 + 1$  and  $n = 2k_2 + 1$  where  $k_1$  and  $k_2$  are integers.
  - Now we compute  $mn = \dots = 2(2k_1k_2 + k_1 + k_2) + 1$
  - Since the parenthesized part is an integer (why?), we get that  $mn$  has the form required to be an odd integer.

- Example 2: Show that for every integer  $n$ , the expression  $n^2 + n$  is necessarily an even number.
  - Exercise for the students. You have two cases to deal with: If  $n$  is odd, and if  $n$  is even. Do each separately.
  - Together in class: Every integer can be written as  $2n + \epsilon$  where  $\epsilon$  is either 0 or 1. This can bring the two cases together in this case.
- Food for thought: How do we know that each integer is either even or odd?

## Indirect proofs

- There are many kinds of indirect proofs, but some techniques stand out.
- The most standard amongst them is **contradiction**:
  - We want to show that “if  $P$  then  $Q$ ”.
  - We instead assume that  $P$  is true but  $Q$  is false, and derive a contradiction: Something that is impossible.
  - Since we saw that if  $P$  is true and  $Q$  is not true then we would get something impossible, the only alternative is that if  $P$  is true then  $Q$  must also be true.

- It is very useful for proving negative statements.

- Contradiction Example 1:

Show that there is no smallest positive rational number.

- By contradiction: We assume there is one and derive an absurd statement from that assumption.
- Say  $q$  is this “smallest positive rational number”.
- Can we construct a number that is positive and smaller? If we can, that is a contradiction, so  $q$  could not have existed in the first place.
- Food for thought:
  - Where does this proof break down if we try to apply it to the integers?
  - Is that fact enough to conclude that for the integers there is a smallest positive integer number?

- Contradiction Example 2:

Show that for integers  $m, n$ , if  $mn$  is odd, then both  $m$  and  $n$  must be odd.

- By way of contradiction, assume that  $mn$  is odd and one of  $m$  or  $n$  is not odd, hence even. Without loss of generality, assume it is  $m$ .
- We know (show separately) that if  $m$  is even then  $mn$  is also even. This contradicts the assumption that  $mn$  was odd.

- This is actually best seen as an example of the **contrapositive**:
  - Showing “if  $P$  then  $Q$ ” is the same as showing “if not  $Q$  then not  $P$ ”
  - In the above example, this would read: If  $m$  or  $n$  is even, then the product  $mn$  is also even.
  - Food for thought: Understand why these two are equivalent (page 25 from the book but think about it first).
- The **converse** of “if  $P$  then  $Q$ ” is “if  $Q$  then  $P$ ”. These are in general not equivalent statements, one could be true while the other is false. (Students: come up with examples)