

Assignment 6

1. Describe a PDA and a CFG for the language $A = \{b^i a^j b^k a^l b^l \mid j > 0, i, k, l \geq 0\}$
2. Consider the language A of all words in $\Sigma = \{a, b\}$ such that every prefix of w has at least as many a 's as b 's. Construct a PDA for this language. As extra credit, also construct a CFG for it.
3. This problem concerns the language $C = A/B = \{w \mid wx \in A \text{ for some } x \in B\}$. In other words a string is in $C = A/B$ if it can be extended to a string in A , where the extension belongs to B . This part has an easy question for you, the next part will be more challenging. There is an obvious relation between the languages CB (concatenation of C and B) and A (one is a subset of the other). Which relation is that?
4. In this problem we will sketch the proof of the following fact: If A is a context-free language, and B is regular, then A/B is context-free. You may want to think about it on your own first, get a bit of a feel for the question before reading on.

We will outline the proof. It is your assignment to provide the details. Assume P is a PDA for A , and D is a DFA for B . We will construct a PDA O for A/B .

- The states of O are pairs of states from P and D , i.e. $Q_O = Q_P \times Q_D$. So at any given point a state in O keeps track of some state in P and some state in D .
- You will need to figure out the start state of O , though nothing fancy is needed here.
- The stack alphabet for O is the same as the one for P . Any moves we take from P , we will update our stack the same way they would.
- You will need to formally define the transitions in O . The idea is as follows:
 - As long as we have not moved in the DFA yet, i.e. as long as the state we are in in O is a pair that contains the start state of D , then our PDA has two types of choices:
 1. It can take a valid step from P , and not move in D . This corresponds to us still trying to recognize the “ w ” part of the wx in the definition of A/B .
 2. It can take a valid step in P that consumes some input, and take a move in D corresponding to that same input. This commits the PDA to stop working on the w part and start working on the x part.
 - Once we have moved D , so we are no longer on its start state, our transitions become more limited. Essentially you only move if you can do it as an ϵ -move in the PDA, or if it is a simultaneous move in both. At this point we are tracing the x part in the wx from the definition.
 1. We can make a valid step from P that does NOT consume any input, and stay in the same state in D .
 2. We can make a valid step from P that consumes an input, then move in D using that same input.

- If we make our accept states those pairs that correspond to both an accept state in P and an accept state in D , this recognizes a different language. Describe that language.
 - Describe which states in O we should call accept states, if we want to obtain A/B .
 - Sketch a proof of why/how this PDA achieves the desired objective of recognizing the language A/B .
5. If A and B are languages, we define a new language $C = A \diamond B = \{xy \mid x \in A, y \in B, |x| = |y|\}$, i.e. the new language is formed out of the concatenation of a string from A with a string from B , whenever those strings have the same length. Show that if A and B are regular languages, then $A \diamond B$ has to be context-free. (Hint: Construct the PDA for $A \diamond B$. Start with DFAs for A and for B , use the union of their states, with some epsilon transitions thrown in, and maybe extra start/end states for stack cleanup. Use the stack to keep track of how many elements you've seen in x , then count them backwards in y .)

Extra credit problem: If $C = A/B$ as above, part 3 already asked you to show that one of CB and A is a subset of the other. As an extra credit problem, determine the status of the other direction: Are A and CB always equal? Either prove that they are always equal or provide a counterexample of languages A and B where if we define $C = A/B$ then A and CB are not equal.

Extra credit problem 2: What about $A \diamond B$ when A is a CFL and B is regular?