Contents

Acronyms

DES Discrete Event System.

Chapter 1

Introduction

Part I

Time-driven approach

Chapter 2

Model Description

In this chapter, first the basic financial quantities are introduced and the asset allocation problem is stated, then the same problem will be embedded in a dynamical control system framework which will allows us to develop the stochastic reachability approach to portfolio construction. We closely follow [16],[15] and [14].

2.1 Portfolio construction

In the financial industry, a group of securities that exhibits similar characteristics in the market place and is subject to the same regulation is called **asset class**. Typical asset classes include stocks, bonds, real estate, cash and commodities. The discipline consisting in allocating investor's wealth among different asset classes is called **asset allocation**. We will now introduce the financial quantities and a formal mathematical setting suitable for describing the asset allocation problem. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the underlying probability space and consider a discrete set of time indexed by $k \in \mathbb{N}$. Moreover, let us consider a universe of $m \in \mathbb{N}$ asset classes. Asset classes' performance at period k is described by a m-dimensional random vector $\boldsymbol{w}_k = [w_k(1), \ldots, w_k(m)]^T$ where

$$w_k(i) = \frac{z_k(i) - z_{k-1}(i)}{z_{k-1}(i)}, \quad i = 1, \ldots, m$$

is the rate of return of the i-th asset class and $\{z_k(i)\}_{k \in \mathbb{N}}$ the i-th asset class price process. In general, the correlation of $\boldsymbol{w}_k$ can be of two kinds:

- *synchronous* correlation, that is the correlation among different asset class at the same time period

- *time-lagged* correlation, that is the correlation among different asset class at different time period.

As the time-lagged correlation is usually negligible for short time period, $\boldsymbol{w}_k$ is a synchronous-correlated random vector. Standard notation is used for Expected Returns and Covariance Matrix

$$\mu_k(i) = \mathbb{E}[w_k(i)], \quad i = 1, \dots, m \quad k \in \mathbb{N}$$

$$\Sigma_k(i,j) = \mathbb{E}\Big[\big(w_k(i) - \mu_k(i)\big)\big(w_k(j) - \mu_k(j)\big)\Big], \quad i,j = 1, \dots, m \quad k \in \mathbb{N}$$

An asset allocation at period $k \in \mathbb{N}$ is a vector $\boldsymbol{u}_k \in \mathbb{R}^m$ whose i-th element indicates the percentage of wealth to be invested in asset class i. This vector is the leverage the asset manager has at his disposal for driving the portfolio value towards his goals. Finally, portfolio performance over the period $[k-1, k]$ is measured by the portfolio return

$$r_{k+1} = \frac{x_{k+1} - x_k}{x_k}$$

where $\{x_k\}_{k\in\mathbb{N}}$ is the portfolio value process. The portfolio return can also be expressed as a weighted average of each asset class return as

$$r_{k+1} = \boldsymbol{u}_k^T \boldsymbol{w}_{k+1}$$

By combining the two previous relations we get the following recursive equation

$$\boxed{x_{k+1} = x_k(1 + \boldsymbol{u}_k^T \boldsymbol{w}_{k+1})} \tag{2.1}$$

which describes the time evolution of portfolio value. In plain words, the **asset allocation problem** consists in choosing the vector $\boldsymbol{u}_k$ at each time period $k \in \mathbb{N}$ so as to achieve investor's goal (i.e. beating a benchmark, maximizing the return and so on). As a matter of fact, the asset allocation vector $\boldsymbol{u}_k$ is bound to stay within a feasible set U_k, for each $k \in \mathbb{N}$. In this work, the feasible set U_k is obtained by imposing the following constraints

- *budget* constraint: $\sum_{i=1}^m u_k(i) = 1$, all the wealth is invested in the portfolio
- *long-only* constraint: $u_k(i) \geq 0, \quad i = 1, \dots, m$, no short-selling are allowed
- *risk* constraint: the metric value-at-risk ($V@R$) is used to limit portfolio risk.

2.2 Stochastic Reachability Approach

In the previous section the financial setting has been laid, now it will be embedded in a more general framework by employing the theory of dynamical systems. We will see that this formalism will allow us to formulate the asset allocation problem as a **stochastic reachability** problem which will be solved by using **dynamic programming** (DP) techniques. Let us introduce the following stochastic discrete-time dynamic control system

$$x_{k+1} = f(x_k, \boldsymbol{u}_k, \boldsymbol{w}_{k+1}) = x_k(1 + \boldsymbol{u}_k^T \boldsymbol{w}_{k+1}) \tag{2.2}$$

where, for any $k \in \mathbb{N}$

- $x_k \in \mathcal{X} = \mathbb{R}$ is the system state (portfolio value), $\mathcal{X}$ the system space
- $\boldsymbol{u}_k \in U \subset \mathbb{R}^m$ is the control input (asset allocation vector), U the control input space
- $\boldsymbol{w}_k$ is a m-dimensional random vector (asset class returns) with density function $p_{\boldsymbol{w}_k}$

Let $\mathcal{U} = \big\{\mu : \mathcal{X} \times \mathbb{N} \to U\big\}$ be the class of controls. In other words, $\mu \in \mathcal{U}$ is a map such that for any $x \in \mathcal{X}$ and any $k \in \mathbb{N}$, associates an asset allocation vector $\boldsymbol{u}_k \in U$. Given $N \in \mathbb{N}$ we define the set of control sequences as $\mathcal{U}_N = \big\{\pi = \{\mu_k\}_{k=0,\dots,N} : \mu_k \in \mathcal{U}\big\}$ and call any $\pi \in \mathcal{U}_N$ a **control policy**. Denote by π^k a control policy starting at period k, that is $\pi^k = \{\mu_k, \dots, \mu_N\}$. We are now ready to formulate the asset allocation problem in stochastic reachability terms

***Problem* 2.2.1** (Optimal Dynamic Asset Allocation)**:** Given a finite time horizon $N \in \mathbb{N}$ and a sequence of target sets $\{X_1, \dots, X_N\}$ such that each target set is a subset of the state space $\mathcal{X}$, find the optimal control policy $\pi^\star \in \mathcal{U}_{N-1}$ that maximizes the following objective function

$$\mathbb{P}\big(\{\omega \in \Omega : x_0 \in X_0, \dots, x_N \in X_N\}\big) \tag{2.3}$$

The target sets $\{X_1, \dots, X_N\}$ represent the investor's goal and we can think of them as the "good" states where we want the portfolio value to belongs to. For instance, a target set could be $X_k = [\underline{x}_k, \infty)$. Problem (2.2.1) is solved by resorting to dynamic programming. To this end, let $p_{f(x,\boldsymbol{u},\boldsymbol{w}_{k+1})}$ be the probability density function of random variable $f(x_k, \boldsymbol{u}_k, \boldsymbol{w}_{k+1}) = x_k(1 + \boldsymbol{u}_k^T \boldsymbol{w}_{k+1})$.

***Definition* 2.2.1** (value function)**:** Given a sequence of target sets $\{X_k\}_{k\geq 0}$, the **value function** is the following real map

$$\begin{aligned} V\colon \mathbb{N}\times\mathcal{X}\times\mathcal{U} &\to [0,1] \\ (k,x,\pi^k) &\mapsto V(k,x,\pi^k) \end{aligned}$$

such that

$$V(k,x,\pi^k)=\begin{cases}\mathbb{1}_{X_N}(x) & \text{if} \quad k=N \\ \int_{X_{k+1}} V(k+1,z,\pi^{k+1})p_{f(x,\boldsymbol{u},\boldsymbol{w}_{k+1})}(z)\mathrm{d}z & \text{if} \quad k=N-1,\dots,0\end{cases}$$

It is now possible to link the objective function (2.3) to the value function (see [14]) in the following way

$$\mathbb{P}\big(\{\omega\in\Omega : x_0\in X_0,\dots,x_N\in X_N\}\big)=V(0,x_0,\pi).$$

This result is extremely important since it allows us to rewrite the ODAA problem in terms of the value function as follows

***Problem* 2.2.2** (Optimal Dynamic Asset Allocation 2)**:** Given a finite time horizon $N\in\mathbb{N}$ and a sequence of target sets $\{X_1,\dots,X_N\}$, find

$$\pi^\star=\underset{\pi\in\mathcal{U}_{N-1}}{\arg\max}\, V(0,x_0,\pi).$$

So far, we have reached an intermediate point where the ODAA problem has been restated in terms of a value function V. The tool of dynamic programming gives us the final step, which is synthesized in the following theorem

***Theorem* 2.2.1** (ODAA algorithm)**:** the optimal value of the ODAA Problem 2.2.2 is

$$p^\star=J_0(x_0),$$

where for any $x\in\mathcal{X}$, $J_0(x)$ is the final step of the following algorithm

$$\boxed{\begin{aligned} &J_N(x)=\mathbb{1}_{X_N}(x) \\ &J_k(x)=\sup_{\boldsymbol{u}_k\in U_k}\int_{X_{k+1}} J_{k+1}(z)p_{f(x,\boldsymbol{u}_k,\boldsymbol{w}_{k+1})}(z)\mathrm{d}z \\ &\qquad k=N-1,\dots,1,0 \end{aligned}} \tag{2.4}$$

The previous result provides us with a backward procedure (it starts at time N and ends at time 0) whose outputs are the optimal control policy $\pi^\star=\{\mu_0^\star,\dots,\mu_{N-1}^\star\}$ and the optimal probability $p^\star$ of reaching the target sets. It is worth pointing out some interesting features of algorithm in (2.4):

- $J_k(x)$ is a function of portfolio realization $x \in \mathcal{X}$ at time k. This dependence is hidden behind the probability density function $p_{f(x,\boldsymbol{u}_k,\boldsymbol{w}_{k+1})}$
- the constrained optimization must be numerically carried out in a space (U_k) of dimension $m \in \mathbb{N}$. At each iteration $k = N-1, \ldots, 1, 0$, the optimization has to be repeated for each x belonging to a suitable discretized grid
- the algorithm presented in theorem (2.2.1) does not depend on the particular distribution of random variable $f(x, \boldsymbol{u}_k, \boldsymbol{w}_{k+1})$. This fact gives us enough freedom to look outside the usual Guassian world
- given a period $k \in \mathbb{N}$ and a portfolio value realization $x \in \mathcal{X}$, $\mu_k^\star(x) \in U_k$ tells us which is the optimal allocation mix of our portfolio.

We now ask ourselves which probability distributions are suitable for vector $\boldsymbol{w}_{k+1}$; the answer to this question is the main objective of the next chapter.

Chapter 3

Asset Class Returns modeling

In this chapter, we address the asset class returns modeling issue. As it was noted above, the ODAA algorithm does not depend on a particular distribution for the asset class returns vector $\boldsymbol{w}_{k+1}$. However, by looking at equation (2.4) we see that we need the explicit analytical form for the density function $p_{f(x,\boldsymbol{u}_k,\boldsymbol{w}_{k+1})}$. For this reason, we will be dealing exclusively with probability distributions closed under linear combination. In this work, we propose three such distributions:

- Gaussian
- Gaussian Mixture
- Generelized Hyperbolic

For each of them, first we will give a theoretical introduction, secondly derive the portfolio value density function $p_{f(x,\boldsymbol{u}_k,\boldsymbol{w}_{k+1})}$ and an expression for the risk constraint (which depends on the distribution chosen).

3.1 Gaussian model

The first probability distribution we considered is the Gaussian

***Definition* 3.1.1** (Gaussian random vector)**:** A m-dimensional random vector $\boldsymbol{w} = [w_1, \ldots, w_m]^T$ is **Gaussian** if every linear combination $\sum_{i=0}^{m} u_i w_i = \boldsymbol{u}^T \boldsymbol{w}$ has a one-dimensional Gaussian distribution.

Let the asset class returns random vector $\boldsymbol{w}_{k+1}$ follow a Gaussian distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ $\big(\boldsymbol{w}_{k+1} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})\big)$. By

definition we have

$$x(1+\boldsymbol{u}_k^T\boldsymbol{w}_{k+1}) \sim \mathcal{N}\Big(\underbrace{x(1+\boldsymbol{u}_k^T\boldsymbol{\mu})}_{\tilde{\mu}}, \underbrace{x^2\boldsymbol{u}_k^T\Sigma\boldsymbol{u}_k}_{\tilde{\sigma}^2}\Big)$$

hence

$$\boxed{p_{f(x,\boldsymbol{u}_k,\boldsymbol{w}_{k+1})}(z) = \frac{1}{\sqrt{2\pi}\tilde{\sigma}} \exp\big\{-\frac{1}{2}\frac{(z-\tilde{\mu})^2}{\tilde{\sigma}^2}\big\}, \quad z \in \mathbb{R}.} \tag{3.1}$$

Let us now introduce the two important concepts of loss function and value-at-risk that we will use to derive the portfolio risk constraint

***Definition* 3.1.2** (loss function)**:** Denoting the value of our portfolio at time $k \in \mathbb{N}$ by x_{k+1}, the **loss function** of the portfolio over the period $[k, k+1]$ is given by

$$L_{k+1} := -\frac{(x_{k+1}-x_k)}{x_k} = -r_{k+1} = -\boldsymbol{u}_k^T\boldsymbol{w}_{k+1}.$$

***Definition* 3.1.3** (Value-at-risk)**:** Given some confidence level $1-\alpha \in (0,1)$ the **value-at-risk** ($V@R_{1-\alpha}$) of our portfolio is

$$V@R_{1-\alpha} = \inf\{l \in \mathbb{R} : \mathbb{P}\big(L_{k+1} \leq l\big) \geq 1-\alpha\}.$$

The V@R is a risk measure commonly use by financial institutions to asses the risk they run to carry a portfolio for a specified period of time (the portfolio must be kept constant during this time period). For instance, if our portfolio has a $V@R_{0.99} = 7\%$, this means that with a confidence level of 99% our portfolio does not suffer a loss greater or equal than 7% over per period $[k, k+1]$ (e.g. a month). In our case, we receive the V@R specification ($V@R_{0.99} = 7\%$) in input by the investor (it is an indicator of its risk-aversion) and we will construct an asset allocation $\boldsymbol{u}_k$ that satisfies this risk constraint at each $k \in \mathbb{N}$.

Using definition (3.1.1) we have

$$L_{k+1} \sim \mathcal{N}\big(\underbrace{-\boldsymbol{u}_k^T\boldsymbol{\mu}}_{\mu_p}, \underbrace{\boldsymbol{u}_k^T\boldsymbol{\Sigma}\boldsymbol{u}_k}_{\sigma_p^2}\big)$$

therefore

$$\begin{aligned}\mathbb{P}\big(L_{k+1} \leq V@R_{1-\alpha}\big) &= \mathbb{P}\Big(Z \leq \frac{V@R_{1-\alpha}-\mu_p}{\sigma_p}\Big)\\ &= 1-\alpha\end{aligned}$$

$$\implies \boxed{V@R_{1-\alpha} \geq -\boldsymbol{u}_k^T \boldsymbol{\mu} + z_{1-\alpha}\sqrt{\boldsymbol{u}_k^T \boldsymbol{\Sigma} \boldsymbol{u}_k}} \tag{3.2}$$

where Z is a standard normal random variable and $z_{1-\alpha}$ is the $1-\alpha$ quantile of the standard normal distribution. The *risk* constraint in equation (3.2), together with the *budget* and *long-only* constraint, define the control space U_k which is the feasible set of the constrained optimization problem given in theorem (2.2.1).

3.2 Gaussian Mixture model

In this section we present the second asset class returns model, the **Gaussian Mixture model** (GM). After introducing the GM distribution we will derive the density and the *risk* constraint, as we did for the Gaussian model. We closely follow [2].

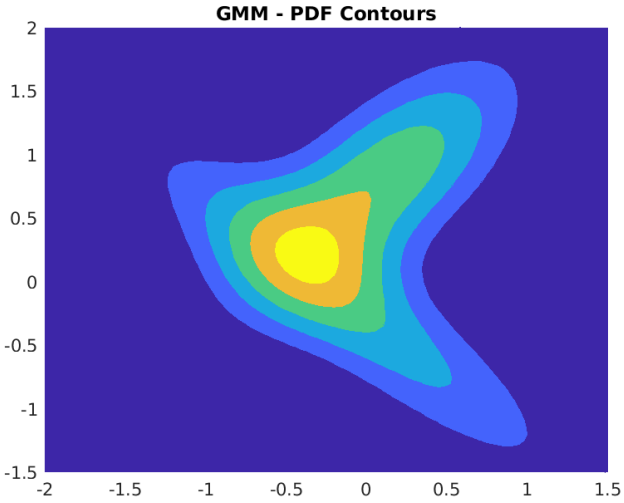
The standard assumption that asset returns have a multivariate Gaussian distribution is a reasonable first approximation to reality and it usually has the big advantage of generating analytically tractable theories (e.g. Markowitz portfolio theory). However, the Gaussian model does not capture two important key features of asset returns which are observed in the market real data:

1. the skewed (asymmetric around the mean) and leptokurtic (more fat-tailed than the Gaussian) nature of marginal probability density function

2. the asymmetric correlation between asset returns, that is the tendency of volatilities and correlations to depend on the prevailing market conditions.

To overcome this shortcomings, the Gaussian Mixture distribution is a validate alternative to the Gaussian one. Loosely speaking, the pdf of a GM random vector id a linear combination of Gaussian pdfs (called Gaussian regimes or mixing components). This closeness to the Normal distribution offers a good trade-off between analytical tractability and parsimony in the number of parameters. By adopting a GM model, it is possible to represent protuberances on the probability iso-density contour, as can be seen in the following figure

Figure 3.1: Example of a GM density contour plot with two mixing components.



To obtain this highly non-linear dependence structure, we would usually need cross-moments of all order; a big advantage of the GM distribution is that its dependence structure is fully and conveniently captured by the means, covariance matrices and weights of each Gaussian regime (as we will see in the following).

We start the more formal introduction on the GM distribution giving its definition

***Definition* 3.2.1** (GM distribution)**:** An m-dimensional random vector $\boldsymbol{Z}$ has a **multivariate GM distribution** if its probability density function is of the form

$$p_{\boldsymbol{Z}}(\boldsymbol{z}) = \sum_{i=1}^{n} \lambda_i \varphi_{(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)}(\boldsymbol{z}), \quad \boldsymbol{z} \in \mathbb{R}^m,$$

where $\varphi_{(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)}$ is the multivariate Gaussian density with mean vector $\boldsymbol{\mu}_i$ and covariance matrix $\boldsymbol{\Sigma}_i$ and λ_i are positive mixing weights summing to one.

The following proposition is crucial for our purposes since it tells us that linear combinations of GM random vector have a one-dimensional GM distribution

***Proposition* 3.2.1:** Linear combinations of GM random vectors follow a univariate GM distribution. In particular, if $\boldsymbol{Z} \sim GM$ then $Y = \boldsymbol{\theta}^T \boldsymbol{Z}$,

$\forall \boldsymbol{\theta} \in \mathbb{R}^m$, has a GM distribution with probability density function

$$p_Y(y) = \sum_{i=1}^{n} \lambda_i \varphi_{(\mu_i, \sigma_i^2)}(y), \quad y \in \mathbb{R}$$

where

$$\begin{cases} \mu_i & = \boldsymbol{\theta}^T \boldsymbol{\mu}_i \quad i = 1, \dots, m \\ \sigma_i^2 & = \boldsymbol{\theta}^T \boldsymbol{\Sigma}_i \boldsymbol{\theta} \quad i = 1, \dots, m \end{cases}$$

Proof. The characteristic function (CF) of a GM random vector is the linear combination of the CF of the Gaussian mixing components. Indeed

$$\begin{aligned} \phi_{\boldsymbol{Z}}(\boldsymbol{u}) = \mathbb{E}[\exp\{i\boldsymbol{u}^T \boldsymbol{Z}\}] = \int_{\mathbb{R}^m} \exp\{i\boldsymbol{u}^T \boldsymbol{z}\} p_{\boldsymbol{Z}}(\boldsymbol{z}) \mathrm{d}\boldsymbol{z} = \\ = \int_{\mathbb{R}^m} \exp\{i\boldsymbol{u}^T \boldsymbol{z}\} \sum_{i=1}^{n} \lambda_i \varphi_{(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)}(\boldsymbol{z}) \mathrm{d}\boldsymbol{z} = \\ = \sum_{i=1}^{n} \lambda_i \phi_{\boldsymbol{X}_i}(\boldsymbol{u}), \quad \boldsymbol{u} \in \mathbb{R}^m \end{aligned}$$

where $\boldsymbol{X}_i \sim \mathcal{N}\big(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i\big)$. Therefore, $\forall \boldsymbol{\theta} \in \mathbb{R}^m$ we have

$$\begin{aligned} \phi_{\boldsymbol{\theta}^T \boldsymbol{Z}}(u) = \mathbb{E}[\exp\{iu(\boldsymbol{\theta}^T \boldsymbol{Z})\}] = \mathbb{E}[\exp\{i(u\boldsymbol{\theta}^T)\boldsymbol{Z}\}] = \phi_{\boldsymbol{Z}}(u\boldsymbol{\theta}) = \\ = \sum_{i=1}^{n} \lambda_i \phi_{X_i}(u\boldsymbol{\theta}) = \sum_{i=1}^{n} \lambda_i \exp\{iu \underbrace{\boldsymbol{\theta}^T \boldsymbol{\mu}_i}_{\mu_i} - \frac{1}{2} u^2 \underbrace{\boldsymbol{\theta}^T \boldsymbol{\Sigma}_i \boldsymbol{\theta}}_{\sigma_i^2}\} = \\ = \sum_{i=1}^{n} \lambda_i \phi_{\widetilde{X}_i}(u), \quad u \in \mathbb{R} \end{aligned}$$

where $\widetilde{X}_i \sim \mathcal{N}\big(\mu_i, \sigma_i^2\big)$. Since the CF completely characterizes the distribution (see theorem 14.1 in [9]) we have the result. □

Suppose the asset class returns vector[1] $\boldsymbol{w}$ follow a Gaussian Mixture distribution ($\boldsymbol{w} \sim GM$). We want to compute the density of random variable $f(x, \boldsymbol{u}, \boldsymbol{w}) = x(1 + \boldsymbol{u}^T \boldsymbol{w})$. Thanks to proposition (3.2.1) we know that the random variable $\boldsymbol{u}^T \boldsymbol{w}$ follows itself a GM (univariate) distribution. Moreover, by integration we easily derive its cumulative distribution function (CDF) allowing us to write

$$\begin{aligned} F_{f(x,\boldsymbol{u},\boldsymbol{w})}(z) = \mathbb{P}\big(x(1 + \boldsymbol{u}^T \boldsymbol{w}) \leq z\big) = F_{x\boldsymbol{u}^T \boldsymbol{w}}(z - x) \\ = \sum_{i=1}^{n} \lambda_i \Phi\Big(\frac{z - x(1 + \boldsymbol{u}^T \boldsymbol{\mu}_i)}{\sqrt{x^2 \boldsymbol{u}^T \boldsymbol{\Sigma}_i \boldsymbol{u}}}\Big), \quad z \in \mathbb{R} \end{aligned}$$

[1] for the sake of clarity we drop the subscript $k + 1$ when it is not needed

where Φ is the standard normal CDF. Differentiating with respect to z, we have

$$\boxed{p_{f(x,\boldsymbol{u},\boldsymbol{w})}(z) = \sum_{i=1}^{n} \lambda_i \varphi_{(\mu_i,\sigma_i^2)}(z), \quad z \in \mathbb{R}} \tag{3.3}$$

where

$$\begin{cases} \mu_i &= x(1 + \boldsymbol{u}^T \boldsymbol{\mu}_i) \\ \sigma_i^2 &= x^2 \boldsymbol{u}^T \boldsymbol{\Sigma}_i \boldsymbol{u}. \end{cases}$$

We now turn to the problem of computing the *risk* constraint under the GM distribution assumption. We will follow two different approaches. Suppose we are given the $V@R_{1-\alpha}$ specification (e.g. 7%); by using definition (3.1.2) we have

$$\mathbb{P}\big(L \leq V@R_{1-\alpha}\big) = F_L(V@R_{1-\alpha}) \geq 1 - \alpha$$

as noted above, the CDF of $L = -\boldsymbol{u}^T \boldsymbol{w}$ is known, therefore

$$\sum_{i=1}^{n} \lambda_i \Phi\Big(\frac{V@R_{1-\alpha} - \mu_i}{\sigma_i}\Big) \geq 1 - \alpha \quad \Longrightarrow$$

$$\boxed{\sum_{i=1}^{n} \lambda_i \Phi\Big(-\Big\{\frac{V@R_{1-\alpha} - \mu_i}{\sigma_i}\Big\}\Big) \leq \alpha} \tag{3.4}$$

where

$$\begin{cases} \mu_i &= -\boldsymbol{u}^T \boldsymbol{\mu}_i \\ \sigma_i^2 &= \boldsymbol{u}^T \boldsymbol{\Sigma}_i \boldsymbol{u}. \end{cases}$$

We present also an alternative method to limit the risk exposure of our portfolio which turns out to be less computationally intensive. The idea is to set an upper bound to portfolio return volatility in the following way

$$(\mathrm{Var}\,[r_{k+1}])^{\frac{1}{2}} = (\boldsymbol{u}_k^T \boldsymbol{\Lambda} \boldsymbol{u}_k)^{\frac{1}{2}} \leq \sigma_{max} \tag{3.5}$$

where $\boldsymbol{\Lambda}$ is the covariance matrix of vector $\boldsymbol{w}_{k+1}$. Two questions are left open: how to compute $\boldsymbol{\Lambda}$ and how to link the upper bound σ_{max} to the $V@R_{1-\alpha}$ specification given in input by the investor. As far as the former is concerned, the following proposition gives us the answer

***Proposition* 3.2.2:** The covariance matrix of a random vector with the GM distribution can be expressed in terms of mean vectors, covariance matrices and weights of the mixing components in the following way

$$\boldsymbol{\Lambda} = \sum_{i=1}^{n} \lambda_i \boldsymbol{\Sigma}_i + \sum_{i=1, j<i}^{n,n} \lambda_i \lambda_j (\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T.$$

To answer the latter, we use a Guassian approximation and the fact that if the rebalancing frequency is relatively small (e.g. weekly) the portfolio return mean is negligible. In the end, the obtain

$$\sigma_{max} = \frac{V@R_{1-\alpha}}{z_{1-\alpha}}.$$

3.3 Generelized Hyperbolic model

The last distribution we propose for our asset class returns modeling purposes is the Generalized Hyperbolic (GH). Like the GM, in its general form also the GH presents a non-elliptical behaviour with asymmetric and fat-tailed marginals. We proceed to give the formal definition and then derive the density of $f(x, \boldsymbol{u}_k, \boldsymbol{w}_{k+1})$ and the *risk* constraint.

***Definition* 3.3.1** (GH distribution)**:** A m-dimensional random vector $\boldsymbol{X}$ is said to follow a **multivariate GH distribution** ($\boldsymbol{X} \sim GM_m(\lambda, \chi, \psi, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\gamma})$) if

$$\boldsymbol{X} = \boldsymbol{\mu} + W\boldsymbol{\gamma} + \sqrt{W}A\boldsymbol{Z}$$

where

- $\boldsymbol{Z} \sim \mathcal{N}\big(\mathbf{0}, I_d\big)$
- $A \in \mathbb{R}^{m\times d}$ is the Cholesky factor of dispersion matrix $\boldsymbol{\Sigma}$ ($A^T A = \boldsymbol{\Sigma}$)
- $\boldsymbol{\mu}, \boldsymbol{\gamma} \in \mathbb{R}^m$
- $W \sim \mathcal{N}^-(\lambda, \chi, \psi)$, $W \geq 0$ and $W \perp \boldsymbol{Z}$ (see Appendix A for the definition of the GIG distribution). W is sometimes called mixing random variable.

***Remark* 3.3.1:**

- λ, χ, ψ are shape parameters; the larger these parameters the closer the distribution is to the Gaussian
- $\boldsymbol{\gamma}$ is the skewness parameter. If $\boldsymbol{\gamma} = \mathbf{0}$ the distribution is symmetric around the mean
- $\boldsymbol{X}|W = w \sim \mathcal{N}\big(\boldsymbol{\mu} + w\boldsymbol{\gamma}, w\boldsymbol{\Sigma}\big)$

The GH distribution contains some special cases:

- If $\lambda = \frac{m+1}{2}$ we have a *Hyperbolic* distribution
- If $\lambda = -\frac{1}{2}$ the distribution is called *Normal Inverse Gaussian* (NIG)

- If $\chi = 0$ and $\lambda > 0$ we have the limiting case of the *Variance Gamma* (VG) distribution
- If $\psi = 0$ and $\lambda < 0$ the resulting distribution is called *Student-t*.

The following proposition gives us the closeness under linear transformation that we need for our modeling purposes

***Proposition* 3.3.1:** If $\boldsymbol{X} \sim GH_m(\lambda, \chi, \psi, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\gamma})$ and $\boldsymbol{Y} = B\boldsymbol{X} + \boldsymbol{b}$, where $B \in \mathbb{R}^{d\times m}$ and $\boldsymbol{b} \in \mathbb{R}^d$, then

$$\boldsymbol{Y} \sim GH_d(\lambda, \chi, \psi, B\boldsymbol{\mu} + b, B\boldsymbol{\Sigma}B^T, B\boldsymbol{\gamma}).$$

Suppose $\boldsymbol{w}_{k+1} \sim GH_m(\lambda, \chi, \psi, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\gamma})$. Applying the previous result to our case $\big(Y = f(x, \boldsymbol{u}_k, \boldsymbol{w}_{k+1}), \quad B = x\boldsymbol{u}_k^T, \quad b = x\big)$ we have

$$f(x, \boldsymbol{u}_k, \boldsymbol{w}_{k+1}) \sim GH_1(\lambda, \chi, \psi, \underbrace{x(1 + \boldsymbol{u}_k^T\boldsymbol{\mu})}_{\widetilde{\mu}}, \underbrace{x^2\boldsymbol{u}_k^T\boldsymbol{\Sigma}\boldsymbol{u}_k}_{\widetilde{\Sigma}}, \underbrace{x\boldsymbol{u}_k^T\boldsymbol{\gamma}}_{\widetilde{\gamma}})$$

and the density reads as (see Appendix A)

$$\boxed{p_{f(x,\boldsymbol{u}_k,\boldsymbol{w}_{k+1})}(z) = c\frac{K_{\lambda-\frac{1}{2}}\Big(\sqrt{\big(\chi + \widetilde{Q}(z)\big)\big(\psi + \widetilde{\gamma}^2/\widetilde{\Sigma}\big)}\exp\big\{(z-\widetilde{\mu})\widetilde{\gamma}/\widetilde{\Sigma}\big\}\Big)}{\Big(\sqrt{\big(\chi + \widetilde{Q}(z)\big)\big(\psi + \widetilde{\gamma}^2/\widetilde{\Sigma}\big)}\Big)^{\frac{1}{2}-\lambda}}} \tag{3.6}$$

where

$$c = \frac{\big(\sqrt{\chi\psi}\big)^{-\lambda}\psi^{\lambda}\big(\psi + \widetilde{\gamma}^2/\widetilde{\Sigma}\big)^{\frac{1}{2}-\lambda}}{(2\pi\widetilde{\Sigma})^{\frac{1}{2}}K_{\lambda}(\sqrt{\chi\psi})}$$

and $\widetilde{Q}(z) = (z - \widetilde{\mu})/\widetilde{\Sigma}$.

As far as the *risk* constraint is concerned, we adopt here the alternative approach expressed in Equation (3.5). The covariance matrix $\boldsymbol{\Lambda}$ is easily derived from the definition of a GH random vector (Definition (3.3.1)) and Equation (A.2) in Appendix A; in the end we obtain

$$\boldsymbol{\Lambda} = \mathrm{Var}\,[W]\,\boldsymbol{\gamma}\boldsymbol{\gamma}^T + \mathbb{E}[W]\boldsymbol{\Sigma} \tag{3.7}$$

where

$$\mathbb{E}[W] = \Big(\frac{\chi}{\psi}\Big)^{\frac{1}{2}}\frac{K_{\lambda+1}(\sqrt{\chi\psi})}{K_{\lambda}(\sqrt{\chi\psi})}$$
$$\mathrm{Var}\,[W] = \Big(\frac{\chi}{\psi}\Big)\frac{1}{K_{\lambda}(\sqrt{\chi\psi})}\Big\{K_{\lambda+2}(\sqrt{\chi\psi}) - \frac{K_{\lambda+1}^2(\sqrt{\chi\psi})}{K_{\lambda}(\sqrt{\chi\psi})}\Big\}.$$

Chapter 4

Model Calibration

In this chapter we show how to calibrate the models introduced in Chapter 3 to market data. The asset class menu we will consider consists in equity, bond and cash and a suitable index will be used to represent each of these markets (the dataset is discussed in Section **??**). We focus our attention only on GM and GH since calibrating the Gaussian model is trivial (it amounts to compute the sample mean and covariance matrix). As far as the GM model is concerned, we set the number of mixing Gaussian components to 2. In financial terms, the two mixing components could be interpreted as economic regimes, namely a *tranquil* regime and a distressed one (see [1]). Different calibration methods are available, namely the **Method of Moments** (MM), **Maximum Likelihood** (ML) estimation and the **Expectation-Maximization** (EM) algorithm. Each of them will be discussed in Section 4.1 and also a comparison between them will be provided. Finally, in Section 4.2 the GH model will be fitted to data using the multi-cycle expectation conditional estimation (MCECM) algorithm.

4.1 GM calibration

The problem of estimating the parameters of a Gaussian Mixture distribution dates back to [12] and still nowadays it raises in a wide spectrum of different disciplines (Finance and Classification just to name a few). Thanks to the computational power available today, the EM algorithm is considered to be the state-of-the-art method for fitting the GM distribution. Nevertheless, MM and ML are worth studying as they shed light on different aspect of the problem at hand and they could provide the starting point for the EM algorithm. The main reference for the MM method is [7], for the ML [4] and for EM [10].

4.1.1 Method of Moments

In this subsection we present the Method of Moments for calibrating a 3-dimensional Gaussian Mixture distribution with $n = 2$ mixing component. The idea behind MM is to match observed and theoretical moments; this translates into a system of polynomial equations that most of the times, for real-size problems, has to be solved numerically. Since we need to fit a 3-dimensional distribution, we will work component-wise: moment-matching equations will be written for each component together with unimodality on each marginal. In order to keep the number of parameters to a reasonable degree, we will suppose a common correlation matrix between the two Gaussian mixing components.

Let $\{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n\}$ be a random sample from a GM distribution whose density function is

$$f(\boldsymbol{z}) = \lambda\varphi_{(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)}(\boldsymbol{z}) + (1-\lambda)\varphi_{(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2)}(\boldsymbol{z}), \quad \boldsymbol{z} \in \mathbb{R}^3 \tag{4.1}$$

Our goal is to estimate $\{\lambda, \boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1, \boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2\}$ from the random sample. Due to the assumption of a shared correlation matrix, the number of actual parameters to estimate is 16: λ, 6 means, 6 standard deviations and 3 correlations. To set the notation we give the following definition

***Definition* 4.1.1** (theoretical and sample moments)**:** Let X be a random variable and $\{x_1, \ldots, x_n\}$ a realization of a random sample. The first four theoretical and sample moments are:

$$\mu_X = \mathbb{E}[X] \qquad \bar{x} = \frac{1}{n}\sum_{j=1}^{n} x_j$$

$$\sigma_X^2 = \mathbb{E}\big[(X-\mu_X)^2\big] \qquad s^2 = \frac{1}{n}\sum_{j=1}^{n}(x_j - \bar{x})^2$$

$$\gamma_X = \frac{1}{\sigma_X^3}\mathbb{E}\big[(X-\mu_X)^3\big] \qquad \widehat{\gamma} = \frac{\frac{1}{n}\sum_{j=1}^{n}(x_j-\bar{x})^3}{\left(\sqrt{\frac{1}{n}\sum_{j=1}^{n}(x_j-\bar{x})^2}\right)^3}$$

$$\kappa_X = \frac{1}{\sigma_X^4}\mathbb{E}\big[(X-\mu_X)^4\big] \qquad \widehat{\kappa} = \frac{\frac{1}{n}\sum_{j=1}^{n}(x_j-\bar{x})^4}{s^4}$$

Let $\boldsymbol{X}$ be a random vector with density (4.1), its i-th marginal is

$$f_{X_i}(z) = \varphi_{(\mu_{1i}, \sigma_{1i}^2)}(z) + (1-\lambda)\varphi_{(\mu_{2i}, \sigma_{2i}^2)}(z), \quad z \in \mathbb{R} \quad i \in \{1,2,3\}$$

where μ_{ji} and σ_{ji}^2 denote respectively the j-th element of the i-th mixing component mean vector and the j-th diagonal entry of the i-th mixing component covariance matrix, $i \in \{1,2,3\}$, $j \in \{1,2\}$ (namely the first subscripts

indicates the dimension, the second the mixing component). Computing explicitly the theoretical moments we obtain

$$\mu_{X_i} = \lambda\mu_{1i} + (1-\lambda)\mu_{2i}$$

$$\sigma^2_{X_i} = \lambda(\sigma^2_{1i} + \mu^2_{1i}) + (1-\lambda)(\sigma^2_{2i} + \mu^2_{2i})$$

$$\gamma_{X_i} = \frac{1}{\sigma^3_{X_i}}\Big\{\big[\lambda(\mu^3_{1i} + 3\mu_{1i}\sigma^2_{1i}) + (1-\lambda)(\mu^3_{2i} + 3\mu_{2i}\sigma^2_{2i})\big] - 3\mu_{X_i}\sigma^2_{X_i} - \mu^3_{X_i}\Big\}$$

$$\begin{aligned}\kappa_{X_i} = \frac{1}{\sigma^4_{X_i}}\Big\{&\big[\lambda(\mu^4_{1i} + 6\mu^2_{1i}\sigma^2_{1i} + 3\sigma^4_{1i}) + (1-\lambda)(\mu^4_{2i} + 6\mu^2_{2i}\sigma^2_{2i} + 3\sigma^4_{2i})\big] + \\ &- \mu^4_{X_i} - 6\mu^2_{X_i}\sigma^2_{X_i} - 4\gamma_{X_i}\sigma^3_{X_i}\mu_{X_i}\Big\}\end{aligned}$$

where $i \in \{1,2,3\}$. Equating them with their sample counterparts gives us the first twelve moment equations. The three correlation equations are derived equating the theoretical covariances (written as a function of correlation coefficients ρ_{ij})

$$\sigma_{X_iX_j} = \lambda\rho_{ij}\sigma_{1i}\sigma_{1j} + (1-\lambda)\rho_{ij}\sigma_{2i}\sigma_{2j} + \lambda(1-\lambda)(\mu_{1i} - \mu_{2i})(\mu_{1j} - \mu_{2j})$$

and the sample ones

$$\widehat{\sigma}_{X_iX_j} = \frac{1}{n}\sum_{s=1,t=1}^{n,n}(x_s - \bar{x})(x_t - \bar{x})$$

$i \in \{1,2,3\} \quad j < i$. So far, we have derived 15 equations in 16 unknown parameters. In order to have as many equations as unknown parameters, we solve the moment equation system by numerically minimizing the sum of square differences between theoretical and sample moments for different values of λ in a discretized grid of the interval $[0,1]$. The optimal λ will the one giving the smallest residual. Moreover, in the optimization process we also imposed the following uni-modality constraints on each marginal[1]

$$(\mu_{2i} - \mu_{1i})^2 \leq \frac{27}{4}(\sigma^2_{2i}\sigma^2_{1i})/(\sigma^2_{1i} + \sigma^2_{2i}) \quad i \in \{1,2,3\}$$

and positive-definiteness constraints on the standard deviation and correlation parameters. The uni-modality constraint is required since bi-modal return distributions are not observed in the market.

[1]see [6] for the proof of this sufficient condition for uni-modality for a 2-mixing-component GM density

4.1.2 Expectation-Maximization

In this section we introduce the EM algorithm for calibrating a GM model. Before diving into it, we need to define the maximum-likelihood estimator since the EM algorithm comes into play to solve difficulties in the ML method.

***Definition* 4.1.2** (Likelihood function)**:** Let $\boldsymbol{x} = \{x_1, \ldots, x_N\}$ be a realization of a random sample from a population with pdf $f(x|\boldsymbol{\theta})$ parametrized by $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_k]^T$. The **likelihood function** is defined by

$$L(\boldsymbol{\theta}|\boldsymbol{x}) = L(\theta_1, \ldots, \theta_N|x_1, \ldots, x_k) = \prod_{i=1}^{N} f(x_i|\boldsymbol{\theta}).$$

The following definition of a maximum likelihood estimator is taken from [4]

***Definition* 4.1.3** (maximum-likelihood estimator)**:** For each sample point $\boldsymbol{x}$, let $\widehat{\boldsymbol{\theta}}(\boldsymbol{x})$ be the parameters value at which $L(\boldsymbol{\theta}|\boldsymbol{x})$ attains its maximum as a function of $\boldsymbol{\theta}$, with $\boldsymbol{x}$ held fixed. A **maximum-likelihood estimator** (MLE) of the parameters vector $\boldsymbol{\theta}$ based on a random sample $\boldsymbol{X}$ is $\widehat{\boldsymbol{\theta}}(\boldsymbol{X})$

Intuitively, the MLE is a reasonable estimator since is the parameter point for which the observed sample is most likely. However, its main drawback is that finding the maximum of the likelihood function (or its logarithmic transformation) might be difficult both analytically and numerically. Consequently, the idea is to adopt an iterative procedure that converges to a local maximum. In order to focus on the idea behind the EM algorithm and not on technical details, we will present it in the simpler case of a uni-variate GM distribution with 2 mixing components (as presented in [8]). The interested reader can refer to [8] for the general case or [13] for a more throughout discussion.

Consider a mixture of two Gaussian random variables

$$X = (1 - \Delta)X_1 + \Delta X_2$$

where $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$, $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ and $\Delta \sim B(\lambda)$ is the mixing random variable. The density function of X, parametrized by $\boldsymbol{\theta} = [\lambda, \mu_1, \sigma_1^2, \mu_2, \sigma_2^2]^T$, is

$$f_X(x) = (1 - \lambda)\varphi_{(\mu_1, \sigma_1^2)}(x) + \lambda\varphi_{(\mu_2, \sigma_2^2)}(x), \quad x \in \mathbb{R}.$$

Our objective is to find an estimate $\widehat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$. Let $\boldsymbol{x} = \{x_1, \ldots, x_N\}$ be a realization of a random sample (our data at hand), the log-likelihood function

is

$$l(\boldsymbol{\theta}; \boldsymbol{x}) = \sum_{i=1}^{N} \log \left[(1-\lambda)\varphi_{(\mu_1,\sigma_1^2)}(x_i) + \lambda\varphi_{(\mu_2,\sigma_2^2)}(x_i) \right] \tag{4.2}$$

In higher dimensions, the direct maximization of (4.2) is difficult and prevent the ML method from being successful. Let us suppose to know the following latent random variables

$$\Delta_i = \begin{cases} 1 & \text{if } X_i \text{ comes from model 2} \\ 0 & \text{if } X_i \text{ comes from model 1} \end{cases}$$

for $i = 1, \ldots, N$. Model 1 or 2 is intended the population whose density is the first or second Gaussian component. In this hypothetical case, the log-likelihood function would be

$$\begin{aligned} l_0(\boldsymbol{\theta}; \boldsymbol{x}, \boldsymbol{\Delta}) = &\sum_{i=1}^{N} \left[(1-\Delta_i) \log \left(\varphi_{(\mu_1,\sigma_1^2)}(x_i) \right) + \Delta_i \log \left(\varphi_{(\mu_2,\sigma_2^2)}(x_i) \right) \right] + \\ &+ \sum_{i=1}^{N} \left[(1-\Delta_i) \log(1-\lambda) + \Delta_i \log(\lambda) \right]. \end{aligned}$$

If the Δ_i's were known, the maximum-likelihood estimate for μ_1 and σ_1^2 would be the sample mean and sample variance from the observations with $\Delta_i = 0$. The same holds true for μ_2, σ_2^2 and $\Delta_i = 1$. The estimate for λ would be the proportion of $\Delta_i = 1$. However, as the Δ_i's are not known, we use as their surrogates the conditional expectations

$$\gamma_i(\boldsymbol{\theta}) = \mathbb{E}[\Delta_i | \boldsymbol{\theta}, \boldsymbol{x}] = \mathbb{P}\big(\Delta_i = 1 | \boldsymbol{\theta}, \boldsymbol{x}\big) \quad i = 1, \ldots, N$$

called *responsability* of model 2 for observation i. The iterative procedure called EM algorithm consists in alternating an *expectation* step in which we assign to each observation the probability to come from each model, and a *maximization* step where these responsabilities are used to update ML estimates.

Algorithm 1 Expectation-Maximization (EM) for 2-component GM

1: take initial guesses for parameters $\widehat{\mu}_1, \widehat{\mu}_2, \widehat{\sigma}_1^2, \widehat{\sigma}_2^2, \widehat{\lambda}$

2: *Expectation* step: compute responsabilities

$$\widehat{\gamma}_i = \frac{\widehat{\lambda}\varphi_{(\widehat{\mu}_2,\widehat{\sigma}_2^2)}(x_i)}{(1-\widehat{\lambda})\varphi_{(\widehat{\mu}_1,\widehat{\sigma}_1^2)}(x_i) + \widehat{\lambda}\varphi_{(\widehat{\mu}_2,\widehat{\sigma}_2^2)}(x_i)}, \quad i = 1, \dots, N$$

3: *Maximization* step: compute weighted means and standard deviations

$$\widehat{\mu}_1 = \frac{\sum_{i=1}^N (1-\widehat{\gamma}_i)x_i}{\sum_{i=1}^N (1-\widehat{\gamma}_i)}, \qquad \widehat{\sigma}_1^2 = \frac{\sum_{i=1}^N (1-\widehat{\gamma}_i)(x_i-\widehat{\mu}_1)^2}{\sum_{i=1}^N (1-\widehat{\gamma}_i)}$$

$$\widehat{\mu}_2 = \frac{\sum_{i=1}^N \widehat{\gamma}_i x_i}{\sum_{i=1}^N \widehat{\gamma}_i}, \qquad \widehat{\sigma}_2^2 = \frac{\sum_{i=1}^N \widehat{\gamma}_i (x_i-\widehat{\mu}_2)^2}{\sum_{i=1}^N \widehat{\gamma}_i}$$

4: Iterate 2 and 3 until convergence.

A reasonable starting value for $\widehat{\mu}_1$ and $\widehat{\mu}_2$ is a random sample point x_i, both $\widehat{\sigma}_1, \widehat{\sigma}_2$ can be set equal to the sample variance and $\widehat{\lambda} = 0.5$. A full implementation of the EM algorithm is available in MATLAB.

4.1.3 MM vs ML vs EM

In this subsection we put the calibration methods into practice to see which one is better at recovering the parameters of a GM distribution. To this end, we simulated 10^4 observations from a GM distribution with the following parameters

$$\boldsymbol{\mu}_1 = \begin{bmatrix} 6.11\text{e}{-4} & 1.373\text{e}{-3} & 2.34\text{e}{-3} \end{bmatrix} \qquad \boldsymbol{\Sigma}_1 = \begin{bmatrix} 4.761\text{e}{-9} & 2.474\text{e}{-8} & 2.731\text{e}{-8} \\ & 3.21\text{e}{-5} & -2.55\text{e}{-6} \\ & & 3.656\text{e}{-4} \end{bmatrix}$$

$$\boldsymbol{\mu}_2 = \begin{bmatrix} 6.83\text{e}{-4} & -1.61\text{e}{-2} & -1.75\text{e}{-2} \end{bmatrix} \qquad \boldsymbol{\Sigma}_2 = \begin{bmatrix} 3.844\text{e}{-9} & 2.42\text{e}{-8} & 6.739\text{e}{-8} \\ & 3.804\text{e}{-5} & -7.644\text{e}{-6} \\ & & 2.757\text{e}{-3} \end{bmatrix}$$

and $\lambda = 0.98$. In order to have a fair comparison, the two Gaussian regimes have a common correlation matrix

$$R = \begin{bmatrix} 1 & 6.33\text{e}{-2} & 2.07\text{e}{-8} \\ & 1 & -2.36\text{e}{-2} \\ & & 1 \end{bmatrix}$$

Parameter	MM	e_{MM} (%)	ML	e_{ML} (%)	EM	e_{EM} (%)
$\widehat{\mu}_1$	6.167e−4	0.94			6.11e−4	0.0264
$\widehat{\mu}_2$	1.578e−3	14.98			1.368e−3	0.366
$\widehat{\mu}_3$	2.396e−3	2.40			2.174e−3	7.066
$\widehat{\Sigma}_{11}$	2.757e−8	479.2			4.704e−9	1.189
$\widehat{\Sigma}_{22}$	3.215e−5	0.14			3.155e−5	1.699
$\widehat{\Sigma}_{33}$	3.092e−4	15.4			3.661e−4	0.146
$\widehat{\Sigma}_{12}$	−3.554e−8	243.6			2.249e−8	9.123
$\widehat{\Sigma}_{13}$	−3.1e−8	213.5			3.487e−8	27.68
$\widehat{\Sigma}_{23}$	−4.805e−7	81.20			−2.756e−4	7.788

Table 4.1: estimates for the first mixing component and respective estimation errors

Parameter	MM	e_{MM} (%)	ML	e_{ML} (%)	EM	e_{EM} (%)
$\widehat{\mu}_1$	5.461e−4	20.03			6.843e−4	0.193
$\widehat{\mu}_2$	−7.26e−3	54.91			−1.554e−2	3.481
$\widehat{\mu}_3$	−8.11e−3	53.65			−1.953e−2	11.605
$\widehat{\Sigma}_{11}$	1.157e−8	201.07			3.223e−9	16.131
$\widehat{\Sigma}_{22}$	8.133e−5	113.79			4.156e−5	9.254
$\widehat{\Sigma}_{33}$	2.108e−3	23.52			2.941e−3	6.674
$\widehat{\Sigma}_{12}$	−3.662e−8	251.31			1.545e−8	36.164
$\widehat{\Sigma}_{13}$	−5.244e−8	177.81			2.693e−7	299.6
$\widehat{\Sigma}_{23}$	−1.996e−6	73.88			3.435e−5	549.4

Table 4.2: estimates for the second mixing component and respective estimation errors

Parameter	MM	e_{MM}	ML	$e_{ML}(\%)$	EM	e_{EM} (%)
$\widetilde{\lambda}$	0.94	4.08			0.9812	0.119
$\log L^\star$	1.3903e5				1.438e5	

Table 4.3: mixing proportion estimate and log-likelihood

The result is summarized in Table 4.1, 4.2 and 4.3.

From the tables above we see that the EM method is definitely the most accurate one. Therefore, we decide to adopt it for calibrating the GM model to market data. As far as the MM method is concerned, the formulation given in Section 4.1.1 relies on the assumption of a common correlation matrix between the two Gaussian regimes. Although this assumption reduces the number of parameter to be estimated, there is empirical evidence (see [3]) that this is not the case in global financial markets where correlation between asset classes is actually increased during bear markets. Nonetheless, even if MM is not as accurate as EM, it is still a valuable method since it does require full time series but only their sample statistics. This turns out to be particularly useful when distribution parameters are set via market hypothesis and economic views (e.g. bull market in the next investment period) instead of using historical data.

4.2 GH calibration

In this section we present a modified EM scheme (the MCECM algorithm) for fitting a GH model to data. In Definition (3.3.1) we introduced the GH distribution using the so-called $(\lambda, \chi, \psi, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\gamma})$-parametrization. Although this is the most convenient one from a modeling perspective, it comes with an identification issue: the distributions $GH(\lambda, \chi, \psi, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\gamma})$ and $GH(\lambda, \chi/k, k\psi, \boldsymbol{\mu}, k\boldsymbol{\Sigma}, k\boldsymbol{\gamma})$ are the same (it is easily seen by writing the density (3.6) in the two cases). To solve this problem, we require the mixing random variable W (see Definition (3.3.1)) to have expectation equal to 1. From Equation (A.2) we have

$$\mathbb{E}[W] = \sqrt{\frac{\chi}{\psi}} \frac{K_{\lambda+1}\big(\sqrt{\chi\psi}\big)}{K_\lambda\big(\sqrt{\chi\psi}\big)} = 1$$

and if we set $\bar{\alpha} = \sqrt{\chi\psi}$ it follows that

$$\psi = \bar{\alpha} \frac{K_{\lambda+1}\big(\bar{\alpha}\big)}{K_\lambda\big(\bar{\alpha}\big)}, \qquad \chi = \frac{\bar{\alpha}^2}{\psi} = \bar{\alpha} \frac{K_\lambda\big(\bar{\alpha}\big)}{K_{\lambda+1}\big(\bar{\alpha}\big)} \tag{4.3}$$

The relations above define the $(\lambda, \bar{\alpha}, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\gamma})$-parametrization, which will be used in the MCECM algorithm.

Let $\boldsymbol{X} \sim GH_m(\lambda, \chi, \psi, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\gamma})$ and $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ be a realization of an iid random sample. Our objective is to find an estimate of the parameters represented by $\boldsymbol{\theta} = [\lambda, \chi, \psi, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\gamma}]^T$. The log-likelihood function to be maximized is

$$\log L(\boldsymbol{\theta}; \boldsymbol{x}) = \log L(\boldsymbol{\theta}; \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = \sum_{i=1}^{n} \log f_{\boldsymbol{X}}(\boldsymbol{x}_i; \theta) \tag{4.4}$$

where $f_{\boldsymbol{X}}$ is the function in (3.6). It well-known that finding a maximizer of (4.4) might be difficult, therefore we resort to a different approach. The situation would look much better if we could observe the latent mixing variables $W_1, \ldots, W_n$. Let us suppose to be in this fortunate situation and define the augmented log-likelihood function

$$\begin{aligned} \log \widetilde{L}(\boldsymbol{\theta}; \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n, W_1, \ldots, W_n) = &\sum_{i=1}^{n} \log f_{\boldsymbol{X}|W}(\boldsymbol{x}_i | W_i; \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\gamma}) + \\ &+ \sum_{i=1}^{n} \log h_W(W_i; \lambda, \chi, \psi) \end{aligned} \tag{4.5}$$

where we used the fact that $f_{(\boldsymbol{X}_i, W_i)}(\boldsymbol{x}, w; \boldsymbol{\theta}) = f_{\boldsymbol{X}_i|W_i}(\boldsymbol{x}|w; \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\gamma}) h_{W_i}(w; \lambda, \chi, \psi)$ and h_{W_i} is the density in (A.1). The advantage of this augmented formulation is that the two terms in (4.5) can be maximized separately. Although counter-intuitive, the first term involving the difficult parameters (e.g. a matrix), is the easiest to maximize and it is done analytically; the second term has to be treated numerically instead. To overcome the latency of the mixing variables W_i's, the MCECM algorithm is used. The algorithm consists in alternating an *expectation* step (in which the W_i's are replaced by an estimate deducted from the data and the current parameters estimate) and a *maximization* step (where parameters estimates are updated). Suppose we are at iteration k and $\boldsymbol{\theta}^{(k)}$ is the current parameters estimate, the two steps are as follows

- **E-step**: compute the conditional expectation of the augmented log-likelihood function given the data and the current parameters estimate

$$Q(\boldsymbol{\theta}; \boldsymbol{\theta}^{(k)}) = \mathbb{E}[\log \widetilde{L}(\boldsymbol{\theta}; \boldsymbol{x}, \boldsymbol{W}) | \boldsymbol{x}, \boldsymbol{\theta}^{(k)}] \tag{4.6}$$

- **M-step**: maximize $Q(\boldsymbol{\theta}; \boldsymbol{\theta}^{(k)})$ to get $\boldsymbol{\theta}^{(k+1)}$.

In practice, the E-step amounts to numerically maximize the second term in (4.6), which is

$$\mathbb{E}\Big[\sum_{i=1}^{n}\log h_{W_i}(W_i;\lambda,\chi,\psi)\Big|\boldsymbol{x},\boldsymbol{\theta}\Big] = \sum_{i=1}^{n} -\lambda\log\chi+\lambda\log\sqrt{\chi\psi}+ \tag{4.7}$$
$$-\log 2K_\lambda(\sqrt{\chi\psi})+(\lambda-1)\underbrace{\mathbb{E}\big[\log W_i|\boldsymbol{x},\boldsymbol{\theta}^{(k)}\big]}_{\xi_i}-\tfrac{1}{2}\chi\underbrace{\mathbb{E}\big[W_i^{-1}|\boldsymbol{x},\boldsymbol{\theta}^{(k)}\big]}_{\delta_i}+$$
$$-\tfrac{1}{2}\psi\underbrace{\mathbb{E}\big[W_i|\boldsymbol{x},\boldsymbol{\theta}^{(k)}\big]}_{\eta_i} = n\big(-\lambda\log\chi+\lambda\log\sqrt{\chi\psi}-\log 2K_\lambda(\sqrt{\chi\psi})\big)+$$
$$+(\lambda-1)\sum_{i=1}^{n}\xi_i-\tfrac{1}{2}\chi\sum_{i=1}^{n}\delta_i-\tfrac{1}{2}\sum_{i=1}^{n}\eta_i.$$

In order to proceed further, we need to compute the conditional expectations ξ_i, δ_i and η_i. Thankfully, the following results holds (see Appendix E.1 in [1])

$$W_i|\boldsymbol{x}_i\sim\mathcal{N}^-\big(\underbrace{\lambda-\tfrac{1}{2}d}_{\widetilde{\lambda}},\underbrace{\chi+(\boldsymbol{x}_i-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}_i-\boldsymbol{\mu})}_{\widetilde{\chi}},\underbrace{\psi+\boldsymbol{\gamma}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}}_{\widetilde{\psi}}\big).$$

By using Equations (A.2) and (A.3) we end up with

$$\delta_i=\mathbb{E}[W_i^{-1}|\boldsymbol{x},\boldsymbol{\theta}^{(k)}]=\Big(\frac{\widetilde{\chi}}{\widetilde{\psi}}\Big)^{-\frac{1}{2}}\frac{K_{\lambda-1}(\sqrt{\widetilde{\chi}\widetilde{\psi}})}{K_\lambda(\sqrt{\widetilde{\chi}\widetilde{\psi}})} \tag{4.8}$$

$$\eta_i=\mathbb{E}[W_i|\boldsymbol{x},\boldsymbol{\theta}^{(k)}]=\Big(\frac{\widetilde{\chi}}{\widetilde{\psi}}\Big)^{\frac{1}{2}}\frac{K_{\lambda+1}(\sqrt{\widetilde{\chi}\widetilde{\psi}})}{K_\lambda(\sqrt{\widetilde{\chi}\widetilde{\psi}})} \tag{4.9}$$

$$\xi_i=\mathbb{E}[\log W_i|\boldsymbol{x},\boldsymbol{\theta}^{(k)}]=\frac{\mathrm{d}}{\mathrm{d}\alpha}\left\{\Big(\frac{\widetilde{\chi}}{\widetilde{\psi}}\Big)^{\frac{\alpha}{2}}\frac{K_{\lambda+\alpha}(\sqrt{\widetilde{\chi}\widetilde{\psi}})}{K_\lambda(\sqrt{\widetilde{\chi}\widetilde{\psi}})}\right\}_{\alpha=0} \tag{4.10}$$

We have now all the ingredients to present the MCECM algorithm as exposed in [1]

Algorithm 2 MCECM

1: Select reasonable starting points. For instance $\lambda^{(1)} = 1, \bar{\alpha}^{(1)} = 1$, $\boldsymbol{\mu}^{(1)} =$ sample mean, $\boldsymbol{\Sigma}^{(1)} =$ sample covariance and $\boldsymbol{\gamma}^{(1)} = \mathbf{0}$

2: Compute $\chi^{(k)}$ and $\psi^{(k)}$ using (4.3)

3: Compute the weights η_i and δ_i using (4.8) and (4.9). Average the weights to get

$$\bar{\eta}^{(k)} = \frac{1}{n}\sum_{i=1}^{n}\eta_i \qquad \bar{\delta}^{(k)} = \frac{1}{n}\sum_{i=1}^{n}\delta_i$$

4: If a symmetric model is to be fitted set $\boldsymbol{\gamma} = \mathbf{0}$, else

$$\boldsymbol{\gamma}^{(k+1)} = \frac{1}{n}\frac{\sum_{i=1}^{n}(\delta_i^{(k)}(\bar{\boldsymbol{x}} - \boldsymbol{x}_i)}{\bar{\eta}^{(k)}\bar{\delta}^{(k)} - 1}$$

5: Update $\boldsymbol{\mu}^{(k)}$ and $\boldsymbol{\Sigma}^{(k)}$

$$\boldsymbol{\mu}^{(k+1)} = \frac{1}{n}\frac{\sum_{i=1}^{n}\delta_i^{(k)}(\boldsymbol{x}_i - \boldsymbol{\gamma}^{(k+1)})}{\bar{\delta}^{(k)}}$$

$$\boldsymbol{\Sigma}^{(k+1)} = \frac{1}{n}\sum_{i=1}^{n}\delta_i^{(k)}(\boldsymbol{x}_i - \boldsymbol{\mu}^{(k+1)})(\boldsymbol{x}_i - \boldsymbol{\mu}^{(k+1)})^T - \bar{\eta}^{(k)}\boldsymbol{\gamma}^{(k+1)}\boldsymbol{\gamma}^{(k+1)T}$$

6: Set $\boldsymbol{\theta}^{(k,2)} = [\lambda^{(k)}, \bar{\alpha}^{(k)}\boldsymbol{\mu}^{(k+1)}, \boldsymbol{\Sigma}^{(k+1)}, \boldsymbol{\gamma}^{(k+1)}]$ and compute $\eta_i^{(k,2)}, \delta_i^{(k,2)}$ and $\xi_i^{(k,2)}$ using (4.9),(4.8) and (4.10)

7: Maximize (4.7) with respect to λ and $\bar{\alpha}$ (using relation (4.3)) to complete the calculation of $\boldsymbol{\theta}^{(k,2)}$. Go to step 2

Chapter 5

Numerical Results in the Time-Driven Approach

This chapter is dedicated to presenting the results obtained by applying the Stochastic Reachability approach (discussed in Chapter 2) to asset allocation. We recall that the output of the ODAA algorithm (see Theorem (2.2.1)) is a sequence of allocation maps $\pi^\star = \{\mu_0^\star, \dots, \mu_{N-1}^\star\}$. For any portfolio realization $x \in \mathbb{R}$ at time $k \in \mathbb{N}$, the maps $\mu_k^\star$ provides us with the optimal asset allocation $\mu_k^\star(x) = \boldsymbol{u}_k^\star$; for instance, if $\boldsymbol{u}_k^\star = \begin{bmatrix} 0.2 & 0.2 & 0.6 \end{bmatrix}^T$ this means that 20% of investor's wealth should be allocated to the first asset class, 20% to the second one and the remaining 60% to the third one. Objective of this chapter is to see what form these maps have at different time instant. The chapter unfolds as follows: in Section 5.1 the dataset is presented and summarized by some sample statistics, in Section 5.2 the parameters of the asset allocation problems are set and the allocation maps for the different models discussed in Chapter 3 are reported.

5.1 The Dataset

Our asset class menu consists of cash, bond and equity. To represent these markets we adopt the indexes presented in Table 5.1. The dataset is composed of weekly time series from 23 January 2010 to 15 April 2016. Data is downloaded from Yahoo Finance which is also where the reader is referred for more details of index composition. Asset class statistical indicators are summarized in Figure 5.1 and Table 5.2. By comparing the annualized Mean return, it is clear that asset class Equity leads to higher performance than Bond and Bond, in turn, ensures higher performance than Cash. The annualized volatility tells us that the same hierarchy holds true also in terms of risk-

Label	Asset Class	Index
C	Money Market	iShares Short Treasury Bond ETF
B	US Bond	Northern US Treasury Index
E	US Equity	S&P 500

Table 5.1: Asset class and relative index

iness, being Equity the riskiest investment and Cash the least. Higher sample moments (Skewness and Kurtosis) suggest that the multivariate return distribution diverges significantly from a Gaussian one. Indeed, a quantitaive proof of this fact is given us from the Henze-Zirkler multivariate normality test which exhibits a zero p-value.

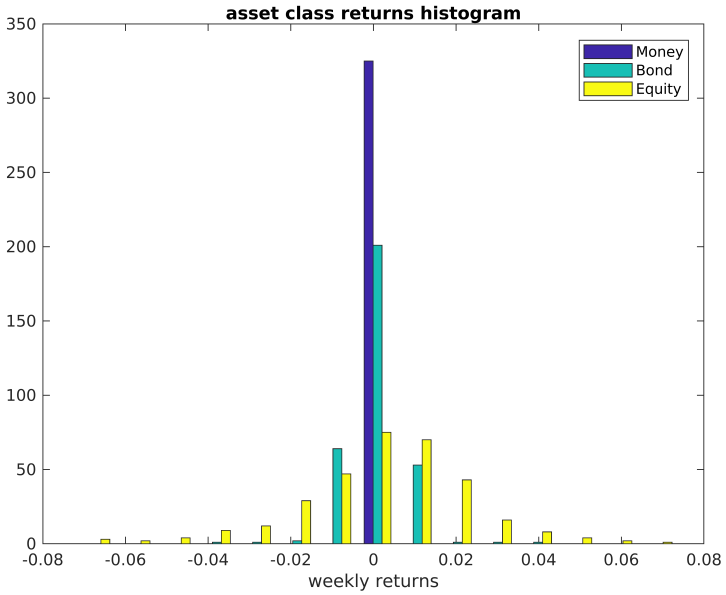


Figure 5.1: Weekly asset class returns histogram.

Finally, the sample correlation matrix is

$$\begin{bmatrix} 1 & 0.166 & -0.075 \\ & 1 & -0.454 \\ & & 1 \end{bmatrix}.$$

Statistic	C	B	E
Mean Return (ann)	0.064%	3.46%	12.11%
Volatility (ann)	0.113%	4.81%	14.81%
Median (ann)	0%	4.58%	17.74%
Skewnwss	0.262	-0.0621	-0.36
Kurtosis	3.90	10.62	4.42
Monthly $V@R_{0.95}$	0.0808%	3.73%	14.95%
Max Drawdown	0.106%	5.87%	23.98%
Mean Drawdown	0.020%	1.5%	4.62%
Sharpe ratio	0	0.692	0.767

Table 5.2: Asset class returns sample statistics

5.2 Optimal Allocation Maps

Let us consider an asset allocation problem characterized by the following parameters:

- 2-year investment horizon
- weekly rebalancing frequency, which means $N = 104$ portfolio rebalancings
- monthly value-at-risk equals to 7%
- target return $\theta = 7\%$ per year
- initial wealth $x_0 = 1$

The target sets we want our portfolio value to stay within are

$$\begin{aligned} X_0 &= \{1\} \\ X_k &= [0, \infty) \quad k = 1, \dots, 103 \\ X_{104} &= [(1+\theta)^2, \infty) = [1.07^2, \infty) \end{aligned}$$

In practice, these sets are discretized with a discretization step of 10^{-3} and truncated where the probability measure is negligible; the actual sets used in the implementation thus are $X_k = [0.5, 1.9]$ $k = 1, \dots, 103$, $X_{104} =$

$[(1.07)^2, 1.9]$. As stated in Problem 2.2.1, we are looking for a sequence of allocation maps which maximize the following joint probability

$$\mathbb{P}\big(\{\omega \in \Omega : x_0 \in X_0, \dots, x_{104} \in X_N\}\big).$$

The final choice to be made before running the algorithm is picking a model for the asset class returns. As a first example, we opt for the GM model which has been fitted to data applying the Expectation-Maximization method (see Subsection 4.1.2) which brings the following results:

$$\boldsymbol{\mu}_1 = \begin{bmatrix} 1.054\text{e}{-5} & 3.713\text{e}{-4} & 2.298\text{e}{-3} \end{bmatrix} \qquad \boldsymbol{\Sigma}_1 = \begin{bmatrix} 2.437\text{e}{-8} & 1.266\text{e}{-7} & -2.365\text{e}{-7} \\ & 3.596\text{e}{-5} & -5.944\text{e}{-5} \\ & & 4.232\text{e}{-4} \end{bmatrix}$$

$$\boldsymbol{\mu}_2 = \begin{bmatrix} 2.115\text{e}{-4} & 3.105\text{e}{-2} & -8.266\text{e}{-3} \end{bmatrix} \qquad \boldsymbol{\Sigma}_2 = \begin{bmatrix} 2.372\text{e}{-8} & -7.961\text{e}{-7} & 1.277\text{e}{-6} \\ & 2.9\text{e}{-5} & -4.411\text{e}{-5} \\ & & 6.949\text{e}{-5} \end{bmatrix}$$

and $\lambda = 0.9908$. By applying the backward algorithm enunciated in Theorem 2.2.1, we obtained the maps reported in Figure 5.2

Let us now take the time to analyze the kind of investment strategy these maps imply. At the beginning of the investment ($k = 0$), the optimal strategy prescribes that 25% of investor's wealth be invested in Bond and 75% in Equity. After 25 weeks, depending on the realization of portfolio value (x-axis in Figure 5.2), the optimal strategy tells us to allocate wealth as follows: if the portfolio is underperforming (its value is in the *danger* zone, approximately below \$1.029), the optimal allocation is a mix of Equity and Bond, that is the riskiest mix allowed (a 100% allocation in Equity is not allowed due to the risk constraint). As soon as performance gets better (the *neutral* zone is from \$1.029 to \$1.16) the Equity weight decreases in favor of more Bond and from a certain point on also Cash. When the portfolio is outperforming (the *safe* zone is above \$ 1.16), the whole wealth is invested in Cash, the least risky of the three asset classes. This kind of investing strategy is known in the literature with the name of *contrarian* strategy. The name stems from the fact that contrarian investors bet against the prevailing market trend, namely they try to sell "high" and buy "low". Contrarian strategies perform well is volatile markets and poorly in trending market due to their convex nature. The optimal strategy obtained by the ODAA algorithm exhibits the same pattern also at successive rebalancing times, the only difference is that it becomes more extreme while approaching the end of the investment; for instance, at time $k = 103$ there is no transition from the riskiest allocation to the least risky one (if the target has not been

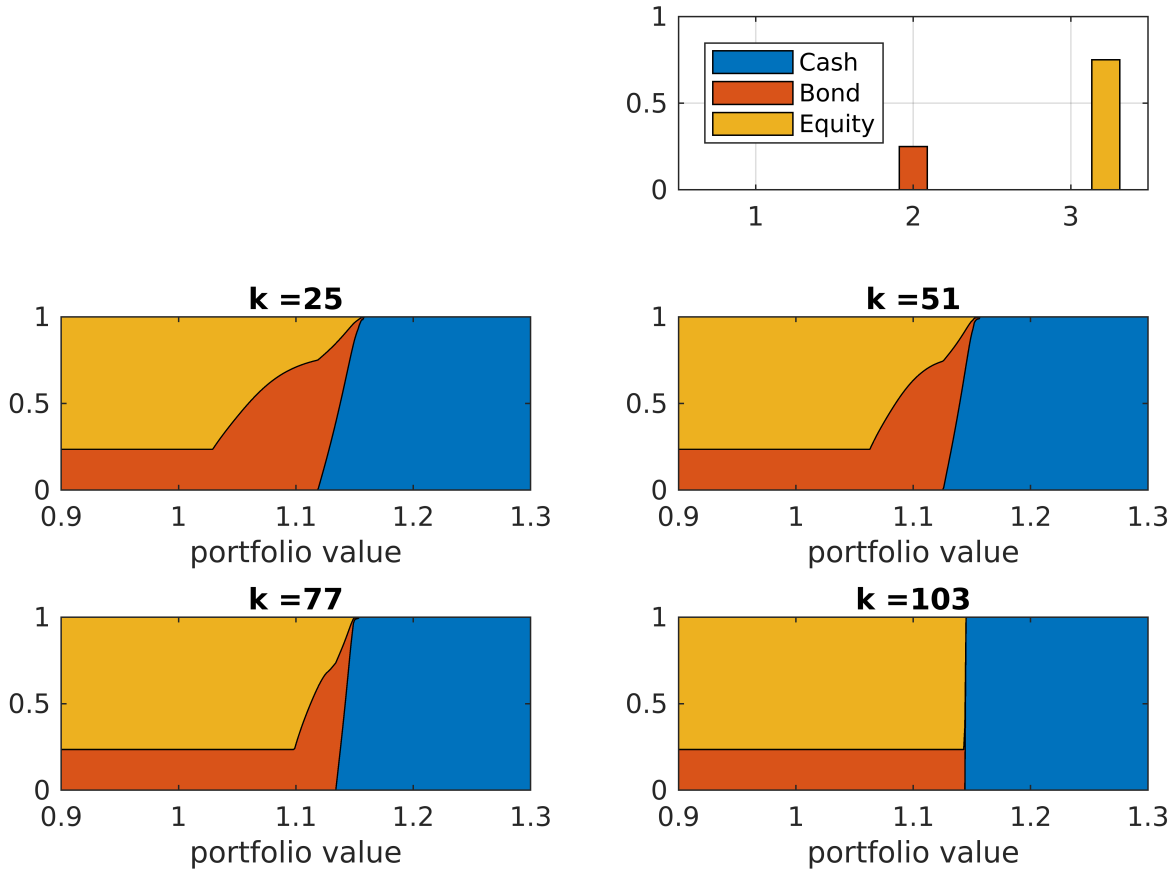


Figure 5.2: Optimal allocation maps, weekly rebalancing, GM model

	G			GM			NIG		
	wk	m	q	wk	m	q	wk	m	q
$p^\star$	79.67%	75.56%	73.26%	78.59%	73.20%	69.44%	78.53%	73.24%	69.47%
p_{MC}	79.77%	75.56%	73.28%	78.82%	73.21%	69.58%	78.76%	73.30%	69.32%
time[h]	0.712	0.157	0.050	0.857	0.316	0.283	6.131	1.467	0.371

Table 5.3: Probability of reaching the target set obtained via ODAA algorithm ($p^\star$) and Monte-Carlo simulation (p_{MC}) for the Gaussian (GM), Gaussian Mixture (GM) and Normal Inverse Gaussian (NIG) model. Time is the computational time of the ODAA algorithm in hours.

reached before the last week, the strategy will do whatever it takes to get there by assuming the riskiest exposure).

The joint probability of reaching investor's goal is $J(x_0) = p^\star = 78.72\%$. This result is verified by running a Monte-Carlo simulation with 10^5 draws at each rebalancing period, the joint probability obtained is $p_{MC} = 78.73\%$. Another interesting feature of the ODAA strategy is that $p^\star$ increases as the rebalancing frequency decreases. By looking at Table 5.3 it can be seen that as we move from a quarterly rebalancing frequancy[1] to a monthly one the optimal probability goes from 69.52% to 73.35%, and the same happens from monthly to weekly. This fact is rather intuitive since the more rebalancings the more chances to steer the portfolio within the target sets. It should be noted however, that in practice transaction costs have not a negligible impact when portfolio rebalancing is frequent.

	G	GM	NIG
$\log L^\star$	4396.2	4455.0	4481.4

[1]the problem of switching from a rebalancing frequency to another has been tackled as follows: the model is calibrated to weekly data, then linear returns are approximated by log-returns enabling us to write additive relations such as $w_{monthly} = w_{wk1} + \ldots + w_{wk4}$. Finally, using the hypothesis if iid returns we analytically derive the distribution of monthly and quarterly returns for the G, GM and NIG model. All this models are closed under convolution.

Part II

Event-Driven approach

Chapter 6

Discrete Event Systems and Asset Allocation

From this chapter on, we will adopt a different approach to the asset allocation problem: time will no longer be the driver of portfolio rebalancing but instead portfolio weights will be updated whenever a predefined **event** occurs. This new point of view stems from the fact that when developing the stochastic reachability approach in Chapter 2, we let the system dynamics be indexed by an independent variable $k \in \mathbb{N}$ which we interpreted as discrete time but, as a matter of fact, the theory did not rely on this particular interpretation. This gives us the freedom to think of k as an abstract index (for instance it could be an event counter). This observation is the basis for embedding the asset allocation problem in a Discrete Event System (DES) environment.

In this chapter we will present the basics of DES modeling (Section 6.1) which are essential for discussing the Event-driven (ED) approach to asset allocation (Section 6.2). As far as the DES is concerned, the main reference is the rich monograph [5], whereas the ED asset allocation model is taken from [17].

6.1 Introduction to DES

From a Control System point of view, the dynamical system introduced in (2.2) can be classified as *continuous-state* (the state space $\mathcal{X}$ is a proper subset of $\mathbb{R}$) and *discrete-time* ($k \in \mathbb{N}$). Informally, if the state space is a discrete set and state transitions are observed whenever an "event" occurs we will talk about Discrete Event Systems. Systems considered so far are time-driven, in that we could imagine the systems being synchronized to a

clock and at every clock tick an event e is drawn from an event space E causing the state to change. However, we could think of a different mechanism governing the state transition: at various time instants (not necessarily known in advanced), some event $e \in E$ occurs, making the state change. The following example will hopefully clarify the difference between Time-driven (TD) and ED systems. Imagine a particle bound to move on a plane; at every tick of a clock an event is drawn and the particle is allowed to move by a unit step in direction North, South, East or West. In this case we have a TD system whose events are $e_1 =$ "one step North", $e_2 =$ "one step South" and so on. On the other hand, suppose there are four players, each of them capable of making the particle move in his direction by issuing a signal. A player issues a signal at random times. The resulting system is ED since it is not synchronized to any clock and state transitions are caused by event like $e_k =$"player 1 issued a signal". The formal definition of a DES reads as follows

***Definition* 6.1.1** (Discrete Event System)**:** A **Discrete Event System** is a *discrete-state*, *event-driven* system, that is, its state evolution depends entirely on the occurrence of asynchronous discrete events over time.

A DES can be studied from three different levels of abstraction. We will present them in increasing order of complexity

1. *untimed*: the interest is on the sequences of events that the system could execute, without any time information. For instance, an untimed sequence could be
$$e_1, e_2, e_3, e_4, e_5, e_6$$
2. *timed*: in this representation each possible event sequence is coupled with time information, that is, not only the order of occurrence is given but also the exact time instant an event occurred. For example, in the *timed* setting, the following could be a system sample path
$$(e_1, t_1), (e_2, t_2), \ldots, (e_6, t_6)$$
3. *stochastic timed*: it is the most detailed description of a DES since it contains event information on all possible orderings, time information about the exact instant at which the event occurs and also statistical information about successive occurrences.

As our modeling purposes required the stochastic timed level of abstraction, we now give the definition of a *Stochastic Clock Structure* which is the tool used to include time and statistical information to a sequence of events.

***Definition* 6.1.2** (Stochastic Clock Structure)**:** The **Stochastic Clock Structure** associated with an event set E is a set of CDFs

$$\boldsymbol{G} = \{G_i \colon i \in E\}$$

characterizing the stochastic clock sequences

$$\boldsymbol{V}_i = \{V_{i,1}, V_{i,2}, \ldots\} \qquad i \in E$$

where $V_{i,k}$ is a random variable indicating the k-th occurrence time of event e_i.

***Remark* 6.1.1:** Sometimes, instead of modeling the exact time when an event occurs (as in the definition above), it is more convenient to model the elapsed time between two events, the so-colled **interevents time**. In this case we will write the Stochastic Clock Sequence in the following way

$$\boldsymbol{T}_i = \{T_{i,1}, T_{i,2}, \ldots\} \qquad i \in E.$$

If a deterministic clock sequence $\boldsymbol{v}_i = \{v_{i,1}, v_{i,2}, \ldots\}$ is given for each event in E, we will talk about a *Clock Structure* (this is the case in the *timed* case). The evolution of a DES needs to be described by a state equation of the form

$$x_{k+1} = f(x_k, e_{k+1}) \qquad k \in \mathbb{N} \tag{6.1}$$

where x_k is the current state and x_{k+1} the state once the event e_{k+1} has occurred. The above recursive equation is the event-driven equivalent of Equation (2.2). However, Equation (6.1) describes only the untimed dynamics, that is no time information is included. Conversely, in asset allocation applications, we are interested also in *when* an event occurs. For this reason, after introducing a Clock Structure $\boldsymbol{v} = \{v_i \colon i \in E\}$ associated with a finite event set $E = \{e_1, \ldots, e_n\}$, we seek a relationship of the form

$$e_{k+1} = h(x_k, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$$

so that we could replace (6.1) with

$$\begin{cases} x_{k+1} &= f(x_k, e_{k+1}) \\ e_{k+1} &= h(x_k, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n). \end{cases} \tag{6.2}$$

Equations (6.2) capture the *timed* dynamics of a DES.

As it was mentioned earlier, the Stochastic timed behavior is what interests us; therefore, we conclude this section by giving the definition of a Stochastic Timed Automaton which is the theoretical modeling structure of a DES (see [5] for a more detailed treatment)

***Definition* 6.1.3** (Stochastic Timed Automaton)**:** A **Stochastic Timed Automaton** is a six-tuple

$$(\mathcal{E}, \mathcal{X}, \Gamma, p, p_0, \boldsymbol{G})$$

where

$\mathcal{E}$ is a countable event set

$\mathcal{X}$ is a countable state space

$\Gamma(x)$ is the set of feasible events, defined $\forall x \in \mathcal{X}$

$p(x'; x, e')$ is the transition probability from state x to state x' given the occurrence of event e'

$p_0(x)$ is the pmf of the initial state X_0 (which is a random variable)

$\boldsymbol{G} = \{\boldsymbol{T}_i : i \in \mathcal{E}\}$ is a Stochastic Clock Time of interevent times.

A Stochastic Timed Automaton, together with the dynamics in Equation (6.2) (where the Clock Structure $\boldsymbol{v}$ is replaced by a Stochastic one $\boldsymbol{V}$) give the most complete description of a DES. We now move to the asset allocation application.

6.2 Event-Driven Asset Allocation

In this section we present the first event-driven model having in mind the objective to invest in the derivative market. In fact, we will consider a market consisting of a **risky asset** (a future index) and a **risk-free asset** (a bank account). The event-driven approach aims at modeling the industrial practice of rebalancing the portfolio weights whenever an "event" occurs. In the following, we suppose that an event has occurred every time the absolute value of the risky return hits a threshold (e.g. 7%). This policy could be beneficial in different aspects:

1. in low-volatile markets, when the risky asset price is quite steady, portfolio weights need not to be updated at predefined time instants but only when the market conditions have significantly changed. This cuts down on transaction costs.

2. in high-volatile markets, when the risky asset price repetitively increases or plummets in a short period of time (shorter than the rebalancing frequency), the event-driven policy can swiftly intervene by changing the portfolio exposure without having to wait the rebalancing time (when the loss could already be substantial).

The main goal of this section is first to derive a proper event-driven dynamics of the portfolio value and then find its density function which will be plugged in the ODAA algorithm. Let us now start off by investigating both the time-driven dynamics (which will allow us to estimate how long the investment is going to last) and the event-driven dynamics.

6.2.1 time-driven dynamics

A portfolio rebalancing is performed every time the absolute value of the risky asset cumulative return, starting from an initial baseline, hits a threshold. Let J be this threshold. We suppose the following time-driven discrete dynamics for the risky asset

$$S_{k+1} = S_k(1 + JN_{k+1}^{\Delta t}) \qquad k \in \mathbb{N}. \tag{6.3}$$

The random variable $N_{k+1}^{\Delta t}$ takes values in the discrete set $\{1, 0, -1\}$ and indicates whether the discrete price process has a positive, negative or null jump at the end of a time interval of length Δt. If it takes the value 1, the discrete price process $\{S_k\}$ experiences a positive jump at the end of time period $[t_k, t_{k+1}]$, if the value is -1 then the jump is negative and if the value is 0, the process has no jumps in this time interval. This is the same as saying that when the random variable takes the value 1 then the risky asset cumulative return is greater than J, when the value is -1 then the cumulative return is smaller than -J and when the value is 0, than it belongs to the interval $[-J, J]$. The superscript Δt indicates the length of the interval $[t_k, t_{k+1}]$. The next step is to find a proper distribution for $N_{k+1}^{\Delta t}$. Let the probability mass function (pmf) of this random variable have the following form

$$f_{N_{k+1}^{\Delta t}}(y) = \begin{cases} \exp\{-\lambda \Delta t\} & \text{if } y = 0 \\ \big(1 - \exp\{-\lambda \Delta t\}\big)p & \text{if } y = 1 \\ \big(1 - \exp\{-\lambda \Delta t\}\big)(1-p) & \text{if } y = -1 \end{cases} \tag{6.4}$$

where $\lambda \in \mathbb{R}^+$ and $p \in [0, 1]$. This functional form is particularly convenient since it implies an exponential distribution for the interevent times (also called **holding times** in a financial context). This fact is synthesized in the following proposition

***Proposition* 6.2.1:** Given the time-driven dynamics of the risky asset in (6.3) and the pmf (6.4) of random variable $N_{k+1}^{\Delta t}$, let τ_{k+1} be the random variable indicating the holding time between the k-th and the $(k+1)$-event. Then $\tau_{k+1} \sim \exp(\lambda)$

Proof. Let t_k be a realization of the random variable T_k, which is an element of a Stochastic Clock Sequence and therefore indicates when the k-th event occurs. From the definition of a Stochastic Clock Structure we have

$$G_{k+1}(t) = \mathbb{P}\big(\tau_{k+1} \leq t\big) = 1 - \mathbb{P}\big(\tau_{k+1} > t\big)$$

but

$$\begin{aligned}\mathbb{P}\big(\tau_{k+1} > t | T_k = t_k\big) &= \mathbb{P}\big(N_{k+1}^{(t_k+t)-t_k} = 0\big) \\ &= \mathbb{P}\big(N_{k+1}^{t} = 0\big) \\ &= \exp\{-\lambda t\}\end{aligned}$$

therefore $\mathbb{P}\big(\tau_{k+1} > t | T_k = t_k\big)$ is independent from t_k. Hence

$$\mathbb{P}\big(\tau_{k+1} > t | T_k = t_k\big) = \mathbb{P}\big(\tau_{k+1} > t\big) = \exp\{-\lambda t\}$$

which implies $G_{k+1}(t) = 1 - \exp\{-\lambda t\}$. Since this is the cdf of an exponential random variable, we have the result. □

***Remark* 6.2.1:** Given that $\mathbb{E}[\tau_{k+1}] = \frac{1}{\lambda}$, the parameter λ acquires the meaning of speed of the discrete dynamics. The larger it is, the more frequent portfolio rebalancings are.

6.2.2 event-driven dynamics

Dynamics (6.3) is still time-driven since the independent variable $k \in \mathbb{N}$ represents discrete time. Instead, in the event-driven framework, we let k indicate the number of events (portfolio rebalancings/trades). For example, S_{k+1} is the risky asset price after the k+1-th portfolio rebalancing is performed. The event-driven dynamics of the risky asset reads as follows

$$S_{k+1} = S_k(1 + J\widetilde{N}_{k+1}) \qquad k \in \mathbb{N} \tag{6.5}$$

where $\widetilde{N}_{k+1}$ is distributed according to

$$f_{\widetilde{N}_{k+1}}(y) = \begin{cases} p & \text{if } y = 1 \\ 1-p & \text{if } y = -1 \end{cases} \tag{6.6}$$

Let us understand how the pmf (6.6) follows from (6.4). First of all, the random variable $\widetilde{N}_{k+1}$ is Bernoullian. In fact, it models whether the jump is positive or negative. Therefore we are left to compute the probability of the jump to be positive (the parameter q of the Bernoulli distribution). By

applying the Bayes Theorem, the Law of Total Probability in the continuous case and the fact that τ_{k+1} is exponential with parameter λ, we obtain

$$\begin{aligned}
q &= \mathbb{P}\big(\widetilde{N}_{k+1} = 1\big) = \mathbb{P}\big(N_{k+1}^{\tau_{k+1}} = 1 | (N_{k+1}^{\tau_{k+1}} = 0)^C\big) \\
&= \frac{\mathbb{P}\big(N_{k+1}^{\tau_{k+1}} = 1, (N_{k+1}^{\tau_{k+1}} = 0)^C\big)}{\mathbb{P}\big((N_{k+1}^{\tau_{k+1}} = 0)^C\big)} \\
&= \frac{\mathbb{P}\big(N_{k+1}^{\tau_{k+1}} = 1\big)}{1 - \mathbb{P}\big(N_{k+1}^{\tau_{k+1}} = 0\big)} \\
&= \frac{\int_0^\infty \mathbb{P}\big(N_{k+1}^{\tau_{k+1}} = 1 | \tau_{k+1} = t\big) f_{\tau_{k+1}}(t) \mathrm{d}t}{1 - \int_0^\infty \mathbb{P}\big(N_{k+1}^{\tau_{k+1}} = 0 | \tau_{k+1} = t\big) f_{\tau_{k+1}}(t) \mathrm{d}t} \\
&= \frac{\int_0^\infty (1 - e^{-\lambda t}) p \lambda e^{-\lambda t} \mathrm{d}t}{1 - \int_0^\infty e^{-\lambda t} \lambda e^{-\lambda t} \mathrm{d}t} \\
&= p
\end{aligned}$$

Parameter p governs the trend of the discrete price process. The greater p, the more likely it is to have positive jumps.

6.2.3 Portfolio dynamics

We recall that the portfolio we are considering consists of a risky and a risk-free asset. The event-driven dynamics of the former has been given in (6.5). In this section the event-driven dynamics of portfolio value will be derived. For the sake of simplicity, we assume that the risk-free asset evolves in a deterministic way with interest rate r (continuously compounded). Throughout this section, let us fix two time instants, t_k and t_{k+1}, which are realizations of random variables T_k and T_{k+1}. These random variables indicate the time when the k-th and k+1-th trade takes place (or, in other words, when the k-th and k+1-th event occur).

In general, the event-driven portfolio dynamics is

$$x_{k+1} = x_k(1 + u_k^C w_{k+1}^C + u_k^S w_{k+1}^S) \qquad k \in \mathbb{N} \tag{6.7}$$

where u_k^C, u_k^S are the portfolio weights of the risk-free and risky asset respectively, w_{k+1}^C and w_{k+1}^S their return over the period $[t_k, t_{k+1}]$. It is important to remark that the length of the time interval $[t_k, t_{k+1}]$ is not deterministic, but it is a random variable exponentially distributed (see Proposition 6.2.1),

denoted by τ_{k+1}. Consequently, w^C_{k+1} and w^S_{k+1} are returns over a stochastic time period. From (6.5) we easily get

$$w^S_{k+1} = J\widetilde{N}_{k+1} \tag{6.8}$$

As far as the risk-free asset in concerned, denoting by C_{k+1} its price after the k+1-th trade, we have

$$\begin{aligned} C_{k+1} &= C_k(1 + w^C_{k+1})^{\tau_{k+1}} = C_k \exp\{r\tau_{k+1}\} \\ \implies \quad w^C_{k+1} &= \exp\{r\tau_{k+1}\} - 1 \end{aligned} \tag{6.9}$$

where r is the deterministic interest rate of the risk-free asset continuously compounded. By plugging (6.8) and (6.9) into (6.7) the portfolio dynamics becomes

$$\boxed{x_{k+1} = x_k(\exp\{r\tau_{k+1}\} + u_k J\widetilde{N}_{k+1})} \qquad k \in \mathbb{N} \tag{6.10}$$

where we dropped the superscript S from u^S_k and set $u^C_k = 1$. This reflexes what is usually done in the derivative trading practice, namely keeping a 100% cash position plus a long or short exposure to the derivative. Consequently, the weight u_k is allowed to take values in the compact set $[-1, 1]$.

6.2.4 the density of x_{k+1}

In order to apply the ODAA algorithm (see Theorem 2.2.1) the explicit form of the density of the random variable x_{k+1} is required. The result is given in the following lemma

***Lemma* 6.2.1:** The probability density function of random variable (6.10) is

$$f_{x_{k+1}}(z) = \begin{cases} 0 & \text{if } z < x - \xi \\ \frac{\lambda}{rx}(1-p)\big(\frac{z+\xi}{x}\big)^{-(\frac{\lambda+r}{r})} & \text{if } x - \xi \le z < x + \xi \\ \frac{\lambda}{rx}\big[(1-p)\big(\frac{z+\xi}{x}\big)^{-(\frac{\lambda+r}{r})} + p\big(\frac{z-\xi}{x}\big)^{-(\frac{\lambda+r}{r})}\big] & \text{if } z \ge x + \xi \end{cases}$$

where $\xi = xJu_{k+1}$.

Proof. Let $F_{\tau_{k+1}}(t) = (1 - e^{-\lambda t})\mathbb{1}_{[0,\infty)}(t)$ be the cdf of τ_{k+1}. The first step consists in finding the cdf F_Y of $Y = x\exp\{r\tau_{k+1}\}$. By simple calculations we obtain

$$F_Y(y) = \Big(1 - \Big(\frac{y}{x}\Big)^{-\frac{\lambda}{r}}\Big)\mathbb{1}_{[x,\infty)}.$$

Let us rewrite the portfolio value at the k+1-th trade in the following way

$$x_{k+1} = Y + xu_k J\widetilde{N}_{k+1} = Y + \xi\widetilde{N}_{k+1}$$

and by using the Law of Total Probability we have

$$\begin{aligned}
F_{x_{k+1}}(z) &= \mathbb{P}\big(Y + \xi\widetilde{N}_{k+1} \leq z\big) \\
&= \mathbb{P}\big(Y + \xi\widetilde{N}_{k+1} \leq z|\widetilde{N}_{k+1} = 1\big)\mathbb{P}\big(\widetilde{N}_{k+1} = 1\big) + \\
&\quad + \mathbb{P}\big(Y + \xi\widetilde{N}_{k+1} \leq z|\widetilde{N}_{k+1} = -1\big)\mathbb{P}\big(\widetilde{N}_{k+1} = -1\big) \\
&= F_Y(z-\xi)p + F_Y(z+\xi)(1-p) \\
&= \Big\{1 - \Big(\frac{z-\xi}{x}\Big)^{-\lambda/r}\Big\}\mathbb{1}_{[x+\xi,\infty)} + \Big\{1 - \Big(\frac{z+\xi}{x}\Big)^{-\lambda/r}\Big\}\mathbb{1}_{[x-\xi,\infty)}
\end{aligned}$$

We have the result differentiating the cdf:

$$\begin{aligned}
f_{x_{k+1}}(z) &= \frac{\mathrm{d}}{\mathrm{d}z}F_{x_{k+1}}(z) \\
&= \frac{\lambda}{rx}\Big\{p\Big(\frac{z-\xi}{x}\Big)^{-\left(\frac{\lambda+r}{r}\right)}\mathbb{1}_{[x+\xi,\infty)} + (1-p)\Big(\frac{z+\xi}{x}\Big)^{-\left(\frac{\lambda+r}{r}\right)}\mathbb{1}_{[x-\xi,\infty)}\Big\}
\end{aligned}$$

□

6.3 Model Calibration

6.4 Numerical Results

Appendix A

Probability Distributions

In this appendix we give further details about the probability distributions used in part I. Main references are [1], [11] and [10].

A.1 Generalized Inverse Gaussian

***Definition* A.1.1** (Bessel function)**:** The modified Bessel function of the third kind (simply called **Bessel function**) is defined as

$$K_\nu(x) = \frac{1}{2}\int_0^\infty t^{\nu-1}\exp\Big\{-\frac{1}{2}x(t+t^{-1})\Big\}\mathrm{d}t, \quad x>0.$$

***Definition* A.1.2** (Generalized Inverse Gaussian)**:** the density of a **Generalized Inverse Gaussian** (GIG) random variable W ($W\sim\mathcal{N}^-\big(\lambda,\chi,\psi\big)$) is

$$f_{GIG}(w) = \Big(\frac{\psi}{\chi}\Big)^{\frac{\lambda}{2}}\frac{w^{\lambda-1}}{2K_\lambda(\sqrt{\chi\psi})}\exp\Big\{-\frac{1}{2}\big(\frac{\chi}{w}+\psi w\big)\Big\} \tag{A.1}$$

with parameters satisfying

$$\begin{cases}\chi>0,\psi\geq 0, & \text{if} \quad \lambda<0\\ \chi>0,\psi>0, & \text{if} \quad \lambda=0\\ \chi\geq 0,\psi>0, & \text{if} \quad \lambda>0\end{cases}$$

Useful formulas The following formulas are used in the text:

$$\mathbb{E}[W^n] = \Big(\frac{\chi}{\psi}\Big)^{\frac{n}{2}}\frac{K_{\lambda+n}(\sqrt{\chi\psi})}{K_\lambda(\sqrt{\chi\psi})} \tag{A.2}$$

$$\mathbb{E}[\log W] = \Big\{\frac{\mathrm{d}\mathbb{E}[X^\alpha]}{\mathrm{d}\alpha}\Big\}_{\alpha=0} \tag{A.3}$$

A.2 Density Functions

We give here the probability density function for the general multivariate GH distribution and same special cases

A.2.1 GH

$$f(\boldsymbol{x}) = c\frac{K_{\lambda-\frac{m}{2}}\Big(\sqrt{\big(\chi+Q(\boldsymbol{x})\big)\big(\psi+\boldsymbol{\gamma}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}\big)}\Big)\exp\big\{(\boldsymbol{x}-\boldsymbol{\mu})^T\Sigma^{-1}\boldsymbol{\gamma}\big\}}{\Big(\sqrt{\big(\chi+Q(\boldsymbol{x})\big)\big(\psi+\boldsymbol{\gamma}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}\big)}\Big)^{\frac{m}{2}-\lambda}} \tag{A.4}$$

where $Q(\boldsymbol{x}) = (\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})$ and

$$c = \frac{\big(\sqrt{\chi\psi}\big)^{-\lambda}\psi^{\lambda}\big(\psi+\boldsymbol{\gamma}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}\big)^{\frac{m}{2}-\lambda}}{(2\pi)^{\frac{m}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}K_{\lambda}(\sqrt{\chi\psi})}$$

A.2.2 Student-t

Setting the degree of freedom $\nu = -2\lambda$ the density reads

$$f(\boldsymbol{x}) = c\frac{K_{\frac{\nu+m}{2}}\Big(\sqrt{\big(\nu-2+Q(\boldsymbol{x})\big)\big(\boldsymbol{\gamma}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}\big)}\Big)\exp\big\{(\boldsymbol{x}-\boldsymbol{\mu})^T\Sigma^{-1}\boldsymbol{\gamma}\big\}}{\Big(\sqrt{\big(\nu-2+Q(\boldsymbol{x})\big)\big(\boldsymbol{\gamma}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}\big)}\Big)^{\frac{\nu+m}{2}}} \tag{A.5}$$

where

$$c = \frac{\big(\nu-2\big)^{\frac{\nu}{2}}\big(\boldsymbol{\gamma}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}\big)^{\frac{\nu+m}{2}}}{(2\pi)^{\frac{m}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}\Gamma(\frac{\nu}{2})2^{\frac{\nu}{2}-1}}$$

A.2.3 VG

$$f(\boldsymbol{x}) = c\frac{K_{\lambda-\frac{m}{2}}\Big(\sqrt{Q(\boldsymbol{x})\big(2\lambda+\boldsymbol{\gamma}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}\big)}\Big)\exp\big\{(\boldsymbol{x}-\boldsymbol{\mu})^T\Sigma^{-1}\boldsymbol{\gamma}\big\}}{\Big(\sqrt{Q(\boldsymbol{x})\big(2\lambda+\boldsymbol{\gamma}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}\big)}\Big)^{\frac{m}{2}-\lambda}} \tag{A.6}$$

where

$$c = \frac{2\lambda^{\lambda}\big(2\lambda+\boldsymbol{\gamma}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}\big)^{\frac{m}{2}-\lambda}}{(2\pi)^{\frac{m}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}\Gamma(\lambda)}$$

Bibliography

[1] Breymann, W., and Luthi, D. ghyp: A package on generelized hyperbolic distributions.

[2] Buckley, I., Saunders, D., and Seco, L. Portfolio optimization when asset returns have the gaussian mixture distribution. *European Journal of Operational Research 185*, 3 (2008), 1434 – 1461.

[3] Campbell, R., Koedijk, K., and Kofman, P. Increased correlation in bear markets. *Financial Analysts Journal 58*, 1 (2002), 87–94.

[4] Casella, G., and Berger, R. *Statistical Inference*. Duxbury advanced series in statistics and decision sciences. Thomson Learning, 2002.

[5] Cassandras, C., and Lafortune, S. *Introduction to Discrete Event Systems*. SpringerLink Engineering. Springer US, 2009.

[6] Eisenberger, I. Genesis of bimodal distributions. *Technometrics 6*, 4 (1964), 357–363.

[7] Everitt, B., and Hand, D. *Finite Mixture Distributions*. Monographs on applied probability and statistics. Chapman and Hall, 1981.

[8] Hastie, T., Tibshirani, R., and Friedman, J. *The Elements of Statistical Learning: Data Mining, Inference, and Prediction, Second Edition*. Springer Series in Statistics. Springer New York, 2009.

[9] Jacod, J., and Protter, P. *Probability Essentials*. Hochschultext / Universitext. Springer, 2000.

[10] McNeil, A. J., Frey, R., and Embrechts, P. *Quantitative Risk Management: Concepts, Techniques and Tools*. Princeton University Press, Princeton, NJ, USA, 2005.

[11] Paolella, M. *Intermediate Probability: A Computational Approach.* Wiley, 2007.

[12] Pearson, K. Contributions to the mathematical theory of evolution. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences 185* (1894), 71–110.

[13] Plasse, J. H. The em algorithm in multivariate gaussian mixture models using anderson acceleration. `https://pdfs.semanticscholar.org/d8c1/650966042e2a58fedf6ce04ef002dd6fb3a8.pdf`, 2013.

[14] Pola, G., Lygeros, J., Domenica, M., and Benedetto, D. Invariance in stochastic dynamical control systems.

[15] Pola, G., and Pola, G. Optimal dynamic asset allocation: A stochastic invariance approach. In *Proceedings of the 45th IEEE Conference on Decision and Control* (Dec 2006), pp. 2589–2594.

[16] Pola, G., and Pola, G. A stochastic reachability approach to portfolio construction in finance industry. *IEEE Transactions on Control Systems Technology 20*, 1 (Jan 2012), 189–195.

[17] Specchio, R. An event driven/stochastic reachability approach to portfolio construction. final project at SCHOOL OF MANAGEMENT, Politecnico di Milano, 2011.