The function families used for the Lyness-Kaganove[1] test are

$$\int_0^1 |x-\lambda|^\alpha \,\mathrm{d}x, \qquad \lambda \in [0,1],\ \alpha \in [-0.5,0] \tag{1}$$

$$\int_0^1 (x>\lambda)e^{\alpha x} \,\mathrm{d}x, \qquad \lambda \in [0,1],\ \alpha \in [0,1] \tag{2}$$

$$\int_0^1 \exp(-\alpha|x-\lambda|) \,\mathrm{d}x, \qquad \lambda \in [0,1],\ \alpha \in [0,4] \tag{3}$$

$$\int_1^2 10^\alpha/((x-\lambda)^2+10^\alpha) \,\mathrm{d}x, \qquad \lambda \in [1,2], \alpha \in [-6,-3] \tag{4}$$

$$\int_1^2 \sum_{i=1}^4 10^\alpha/((x-\lambda_i)^2+10^\alpha) \,\mathrm{d}x, \qquad \lambda_i \in [1,2], \alpha \in [-5,-3] \tag{5}$$

$$\int_0^1 2\beta(x-\lambda)\cos(\beta(x-\lambda)^2) \,\mathrm{d}x, \qquad \lambda \in [0,1],\ \alpha \in [1.8,2], \quad \beta = 10^\alpha/\max\{\lambda^2,(1-\lambda)^2\} \tag{6}$$

where the boolean expressions are evaluated to 0 or 1. The integrals are computed to relative precisions of $\tau = 10^{-3}$, 10^{-6}, 10^{-9} and 10^{-12} for 1 000 realizations of the random parameters λ and α. The results of these tests are shown in Table 1. For each function, the number of correct and incorrect integrations is given with, in brackets, the number of cases each where a warning (either explicit or whenever an error estimate larger than the requested tolerance is returned) was issued.

The functions used for the "battery" test are

$$\begin{array}{rclrcl}
f_1 &=& \int_0^1 e^x \,\mathrm{d}x & f_{14} &=& \int_0^{10} \sqrt{50}e^{-50\pi x^2} \,\mathrm{d}x \\
f_2 &=& \int_0^1 (x>0.3) \,\mathrm{d}x & f_{15} &=& \int_0^{10} 25e^{-25x} \,\mathrm{d}x \\
f_3 &=& \int_0^1 x^{1/2} \,\mathrm{d}x & f_{16} &=& \int_0^{10} 50(\pi(2500x^2+1))^{-1} \,\mathrm{d}x \\
f_4 &=& \int_{-1}^1 (\frac{23}{25}\cosh(x)-\cos(x)) \,\mathrm{d}x & f_{17} &=& \int_0^1 50(\sin(50\pi x)/(50\pi x))^2 \,\mathrm{d}x \\
f_5 &=& \int_{-1}^1 (x^4+x^2+0.9)^{-1} \,\mathrm{d}x & f_{18} &=& \int_0^\pi \cos(\cos(x)+3\sin(x)+2\cos(2x)+3\cos(3x)) \,\mathrm{d}x \\
f_6 &=& \int_0^1 x^{3/2} \,\mathrm{d}x & f_{19} &=& \int_0^1 \log(x) \,\mathrm{d}x \\
f_7 &=& \int_0^1 x^{-1/2} \,\mathrm{d}x & f_{20} &=& \int_{-1}^1 (1.005+x^2)^{-1} \,\mathrm{d}x \\
f_8 &=& \int_0^1 (1+x^4)^{-1} \,\mathrm{d}x & f_{21} &=& \int_0^1 \sum_{i=1}^3 \left[\cosh(20^i(x-2i/10))\right]^{-1} \,\mathrm{d}x \\
f_9 &=& \int_0^1 2(2+\sin(10\pi x))^{-1} \,\mathrm{d}x & f_{22} &=& \int_0^1 4\pi^2 x\sin(20\pi x)\cos(2\pi x) \,\mathrm{d}x \\
f_{10} &=& \int_0^1 (1+x)^{-1} \,\mathrm{d}x & f_{23} &=& \int_0^1 (1+(230x-30)^2)^{-1} \,\mathrm{d}x \\
f_{11} &=& \int_0^1 (1+e^x)^{-1} \,\mathrm{d}x & f_{24} &=& \int_0^3 \lfloor e^x \rfloor \,\mathrm{d}x \\
f_{12} &=& \int_0^1 x(e^x-1)^{-1} \,\mathrm{d}x & f_{25} &=& \int_0^5 (x+1)(x<1)+(3-x)(1\le x\le 3) \\
f_{13} &=& \int_0^1 \sin(100\pi x)/(\pi x) \,\mathrm{d}x & & & \quad +2(x>3) \,\mathrm{d}x
\end{array}$$

where the boolean expressions in f_2 and f_{25} evaluate to 0 or 1. The functions are taken from Gander & Gautschi (2008)[2] with the following modifications:

- No special treatment is given to the case $x=0$ in f_{12}, allowing the integrand to return NaN.
- f_{13} and f_{17} are integrated from 0 to 1 as opposed to 0.1 to 1 and 0.01 to 1 respectively, allowing the integrand to return NaN for $x=0$.
- No special treatment of $x < 10^{-15}$ in f_{19} allowing the integrand to return −Inf.
- f_{24} was suggested by J. Waldvogel as a simple yet tricky test function with multiple discontinuities.
- f_{25} was introduced in Gander & Gautschi, yet not used in the battery test therein.

[1] J. N. Lyness and J. J. Kaganove, *A technique for comparing automatic quadrature routines*. Comp. J. 20, 2, 170–177, 1977.

[2] W. Gander and W. Gautschi, *Adaptive quadrature — revisited*, Tech. Rep. 306, Department of Computer Science, ETH Zurich, Switzerland, 1998.

$\tau = 10^{-3}$	quad			quadl			quadgk			integral			quadcc		
$f(x)$	✓	×	n_{eval}	✓	×	n_{eval}	✓	×	n_{eval}	✓	×	n_{eval}	✓	×	n_{eval}
Eqn (1)	88 (0)	912 (0)	19.22	406 (0)	594 (0)	95.13	454 (0)	546 (0)	238.23	454 (0)	546 (0)	238.23	1000 (0)	0 (0)	280.15
Eqn (2)	101 (0)	899 (0)	26.46	894 (0)	106 (0)	117.15	993 (0)	7 (0)	267.27	993 (0)	7 (0)	267.27	1000 (1)	0 (0)	175.34
Eqn (3)	468 (0)	532 (0)	14.42	868 (0)	132 (0)	45.39	1000 (0)	0 (0)	150.00	1000 (0)	0 (0)	150.00	1000 (0)	0 (0)	112.72
Eqn (4)	16 (0)	984 (0)	18.74	359 (0)	641 (0)	89.67	719 (0)	281 (0)	404.46	719 (0)	281 (0)	404.46	1000 (0)	0 (0)	342.12
Eqn (5)	0 (0)	1000 (0)	13.17	303 (0)	697 (0)	308.76	917 (0)	83 (1)	1300.89	917 (0)	83 (0)	1300.89	997 (0)	3 (0)	976.35
Eqn (6)	643 (0)	357 (0)	368.36	994 (0)	6 (0)	746.19	1000 (0)	0 (0)	491.85	1000 (0)	0 (0)	491.85	1000 (0)	0 (0)	873.80

$\tau = 10^{-6}$	quad			quadl			quadgk			integral			quadcc		
$f(x)$	✓	×	n_{eval}	✓	×	n_{eval}	✓	×	n_{eval}	✓	×	n_{eval}	✓	×	n_{eval}
Eqn (1)	31 (0)	969 (0)	118.66	398 (0)	602 (0)	368.70	305 (0)	695 (0)	894.90	305 (0)	695 (0)	894.90	1000 (0)	0 (0)	866.68
Eqn (2)	92 (0)	908 (0)	66.22	886 (0)	114 (0)	236.25	957 (0)	43 (0)	552.24	957 (0)	43 (0)	552.24	1000 (0)	0 (0)	315.75
Eqn (3)	308 (0)	692 (0)	36.72	770 (0)	230 (0)	103.41	899 (0)	101 (0)	240.09	899 (0)	101 (0)	240.09	1000 (0)	0 (0)	313.69
Eqn (4)	112 (0)	888 (0)	184.12	992 (0)	8 (0)	481.80	1000 (0)	0 (0)	788.43	1000 (0)	0 (0)	788.43	1000 (0)	0 (0)	614.82
Eqn (5)	0 (0)	1000 (0)	16.04	982 (0)	18 (0)	1297.56	1000 (2)	0 (0)	2073.30	1000 (0)	0 (0)	2073.30	1000 (0)	0 (0)	1811.84
Eqn (6)	843 (0)	157 (2)	1486.79	998 (0)	2 (2)	2029.44	1000 (0)	0 (0)	909.45	1000 (0)	0 (0)	909.45	1000 (0)	0 (0)	1196.39

$\tau = 10^{-9}$	quad			quadl			quadgk			integral			quadcc		
$f(x)$	✓	×	n_{eval}	✓	×	n_{eval}	✓	×	n_{eval}	✓	×	n_{eval}	✓	×	n_{eval}
Eqn (1)	38 (0)	962 (0)	530.14	367 (0)	633 (31)	1051.22	245 (22)	755 (406)	9987.36	279 (13)	721 (271)	144652.44	910 (26)	90 (88)	1849.60
Eqn (2)	116 (0)	884 (0)	111.64	886 (0)	114 (0)	356.58	915 (0)	85 (0)	825.66	915 (0)	85 (0)	825.66	1000 (0)	0 (0)	460.60
Eqn (3)	289 (0)	711 (0)	100.98	780 (0)	220 (0)	184.98	877 (0)	123 (0)	383.31	877 (0)	123 (0)	383.31	1000 (0)	0 (0)	520.52
Eqn (4)	657 (0)	343 (0)	756.48	999 (0)	1 (0)	1211.85	1000 (1)	0 (0)	1332.06	1000 (1)	0 (0)	1661.88	1000 (0)	0 (0)	1077.16
Eqn (5)	0 (0)	1000 (0)	36.98	998 (0)	2 (0)	3336.06	1000 (4)	0 (0)	3352.65	1000 (2)	0 (0)	4317.84	1000 (0)	0 (0)	3254.49
Eqn (6)	857 (0)	143 (71)	5677.96	959 (0)	41 (41)	5186.43	1000 (1)	0 (0)	1599.66	1000 (1)	0 (0)	1924.68	1000 (2)	0 (0)	1392.89

$\tau = 10^{-12}$	quad			quadl			quadgk			integral			quadcc		
$f(x)$	✓	×	n_{eval}	✓	×	n_{eval}	✓	×	n_{eval}	✓	×	n_{eval}	✓	×	n_{eval}
Eqn (1)	37 (0)	963 (238)	2152.67	261 (0)	739 (375)	2843.43	62 (14)	938 (893)	17611.11	113 (16)	887 (788)	366606.78	568 (103)	432 (432)	8593.84
Eqn (2)	114 (0)	886 (0)	186.15	879 (0)	121 (0)	488.10	813 (50)	187 (55)	1078.92	813 (50)	187 (55)	1078.92	1000 (0)	0 (0)	606.37
Eqn (3)	325 (0)	675 (0)	322.63	795 (0)	205 (0)	315.78	845 (0)	155 (0)	525.15	845 (0)	155 (0)	525.15	1000 (0)	0 (0)	736.10
Eqn (4)	978 (0)	22 (0)	2999.06	928 (0)	72 (0)	3217.08	888 (206)	112 (112)	7074.06	997 (311)	3 (3)	120443.04	1000 (426)	0 (0)	18000.30
Eqn (5)	0 (0)	1000 (0)	139.02	741 (0)	259 (259)	8697.84	1000 (35)	0 (0)	8274.06	1000 (20)	0 (0)	15607.95	1000 (326)	0 (0)	12835.15
Eqn (6)	8 (0)	992 (991)	10008.64	249 (0)	751 (751)	9601.47	982 (73)	18 (18)	4297.38	990 (60)	10 (10)	32893.77	993 (471)	7 (7)	19961.92

Table 1: Results of the Lyness-Kaganove tests for $\tau = 10^{-3}, 10^{-6}, 10^{-9}$ and 10^{-12}. The columns marked with ✓ and × indicate the number of correct and incorrect results respectively, out of 1 000 runs. The numbers in brackets indicate the number of runs in which a warning was issued. The column n_{eval} contains the average number of function evaluations required for each run.

	$\tau = 10^{-3}$					$\tau = 10^{-6}$					$\tau = 10^{-9}$					$\tau = 10^{-12}$				
$f(x)$	`quad`	`quadl`	`quadgk`	`integral`	`quadcc`	`quad`	`quadl`	`quadgk`	`integral`	`quadcc`	`quad`	`quadl`	`quadgk`	`integral`	`quadcc`	`quad`	`quadl`	`quadgk`	`integral`	`quadcc`
f1	**13**	18	150	150	33	**13**	18	150	150	33	41	**18**	150	150	33	161	**18**	150	150	33
f2	~~21~~	**108**	240	240	161	~~61~~	**198**	540	540	301	~~105~~	**318**	840	840	441	~~149~~	**408**	1170	1170	581
f3	~~13~~	**48**	150	150	101	~~41~~	**108**	150	150	429	~~149~~	258	**150**	**150**	799	~~609~~	~~618~~	**150**	**150**	1191
f4	**13**	18	150	150	33	**17**	18	150	150	33	73	**18**	150	150	33	297	48	150	150	**33**
f5	**13**	18	150	150	33	**33**	48	150	150	95	129	**48**	150	150	95	497	168	**150**	**150**	219
f6	**13**	18	150	150	33	**25**	48	150	150	159	**97**	108	150	150	359	385	288	**150**	**150**	607
f7	~~118~~	289	**150**	**150**	269	~~186~~	439	**150**	**150**	709	~~550~~	~~889~~	**150**	**150**	1409	~~2142~~	~~2429~~	**150**	**150**	2179
f8	**13**	18	150	150	33	**17**	18	150	150	33	73	**48**	150	150	95	289	138	150	150	**95**
f9	~~25~~	198	**150**	**150**	261	**189**	468	300	300	587	~~689~~	1038	**570**	**570**	991	2757	2808	**1200**	**1200**	1425
f10	**13**	18	150	150	33	**17**	18	150	150	33	65	48	150	150	**33**	257	48	150	150	**33**
f11	**13**	18	150	150	33	**13**	18	150	150	33	33	**18**	150	150	33	125	48	150	150	**33**
f12	**14**	19	150	150	47	**14**	19	150	150	55	26	**19**	150	150	63	90	**19**	150	150	63
f13	~~110~~	~~589~~	**780**	**780**	1403	~~1026~~	1519	**1500**	**1500**	2347	4330	4879	**1950**	**1950**	2459	~~10018~~	~~10039~~	3780	3780	**2521**
f14	**33**	78	150	150	151	**65**	138	180	180	183	~~193~~	228	300	300	**225**	717	588	450	450	**369**
f15	**33**	78	150	150	135	**65**	168	180	180	159	229	288	210	210	**191**	833	708	300	300	**277**
f16	**13**	18	150	150	33	**13**	18	150	150	33	**13**	18	150	150	33	**13**	18	150	150	33
f17	~~34~~	**79**	330	330	903	~~362~~	~~949~~	**840**	**840**	1491	**1658**	2839	1740	1740	2419	6558	6469	3120	3120	**2451**
f18	**41**	108	150	150	145	~~129~~	228	**150**	**150**	209	549	738	**240**	**240**	395	2173	1758	**390**	**390**	581
f19	~~38~~	**109**	150	150	255	~~94~~	229	**210**	**210**	717	~~318~~	~~499~~	**360**	**360**	1323	~~1214~~	1369	**510**	**510**	1943
f20	**13**	18	150	150	33	**33**	48	150	150	**33**	121	**48**	150	150	95	489	168	**150**	**150**	219
f21	~~25~~	~~138~~	~~240~~	~~240~~	~~203~~	~~141~~	~~348~~	~~270~~	~~270~~	~~391~~	~~629~~	**1158**	~~510~~	~~510~~	~~653~~	2437	2748	~~870~~	~~870~~	**1839**
f22	~~113~~	228	**150**	**150**	371	477	888	**330**	**330**	627	1881	2508	660	660	**627**	7565	5568	930	930	**627**
f23	~~33~~	**108**	210	210	191	**141**	258	360	360	365	541	588	**510**	**510**	569	2181	1608	**840**	**840**	957
f24	~~13~~	~~138~~	~~1380~~	~~1380~~	**4515**	~~253~~	~~1878~~	~~7290~~	~~7290~~	**11433**	~~529~~	~~3738~~	~~12690~~	~~12690~~	**18503**	~~797~~	~~5538~~	~~17010~~	~~17010~~	**25199**
f25	~~21~~	**108**	300	300	277	~~85~~	**348**	840	840	593	~~137~~	**528**	1350	1350	933	~~205~~	**678**	1950	1950	1253

Table 2: Results of battery test for $\tau = 10^{-3}, 10^{-6}, 10^{-9}$ and 10^{-12}. The columns contain the number of function evaulations required by each integrator for each tolerance. For each test and tolerance, the best result (least function evaluations) is in bold and unsuccessful runs are stricken through.

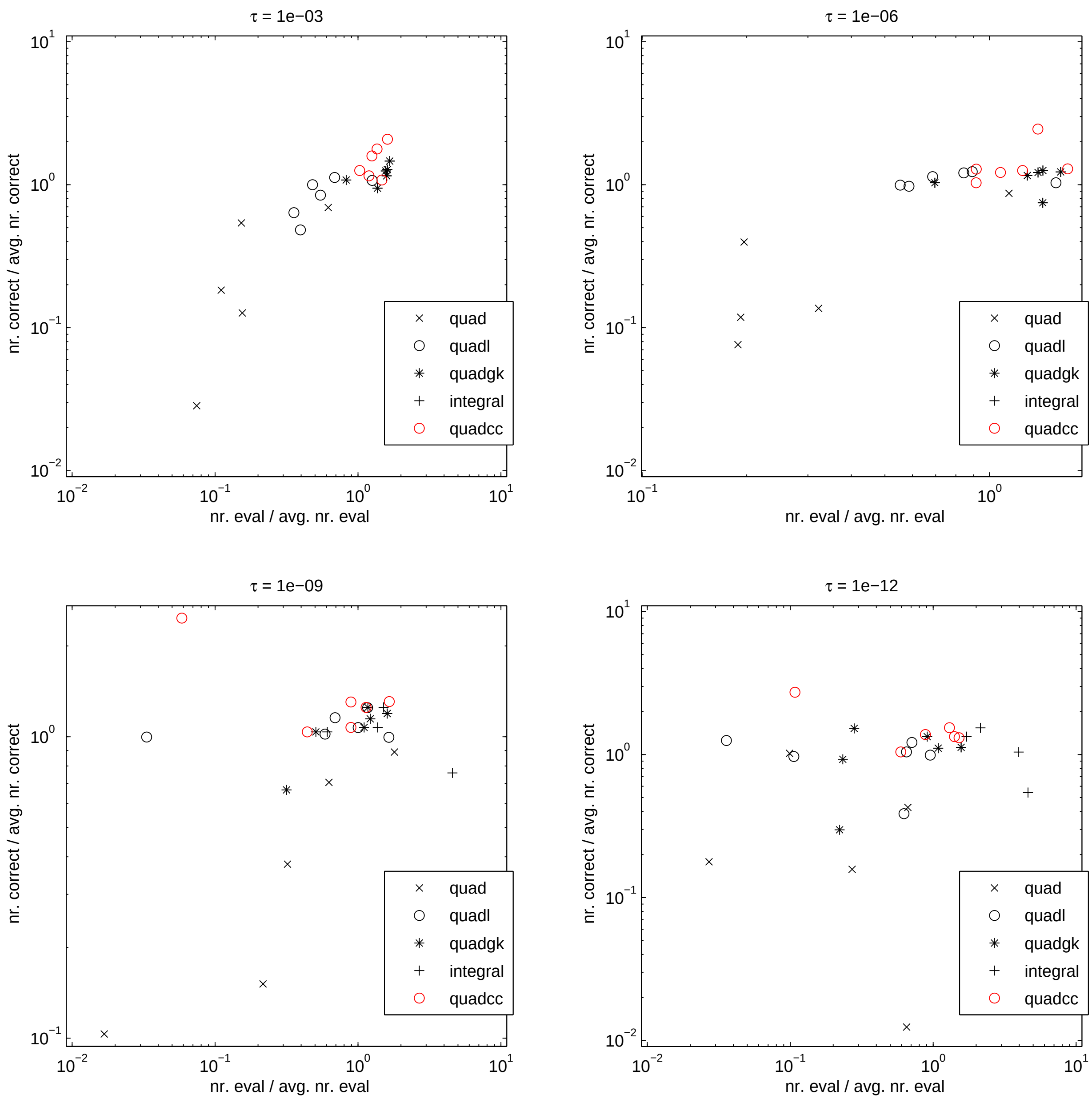

Figure 1: Scatter-plots of the results of the Lyness-Kaganove testsuite for each tolerance. Each point represents one of the test functions (Equations 1 to 6). Its location is determined by the relative number of function evaluations (on the x-axis) and the relative number of correct evaluations (on the y-axis). Algorithms in the top left corner are (relatively) efficient and reliable whereas algorithms in the lower right corner are (relatively) inefficient and unreliable.