First Work in Progress

October 2, 2017

Abstract

work plan, problems and partial results

1 Introduction

The work has been divided in the following parts:

1. Data collection
2. Calibration
3. Dynamic Programming (DP) Algorithm
4. Validation
5. Comparisons (CPPI, constant-mix)
6. Extensions and ideas

2 Details

In this section we present what has been done so far

2.1 Data collection

The three asset classes (money, bond and equity market) are represented by the following indexes:

1. iShares Short Treasury Bond ETF (Money Market)
2. BlackRock US Government Bond (Bond Market)
3. S&P 500 (Equity Market)

Data were downloaded from Yahoo finance (temporary solution, Thomson Reuters will be used for the final work). Time-series have either a daily or weekly frequency

2.2 Calibration

Gaussian model Asset class returns follow a Gaussian distribution ($w_{k+1} \sim N(\mu, \Sigma)$), the calibration consists of estimating μ and Σ from the data

Mixture model Two possible approaches: EM (Expectation Maximization) and MM (Method of Moments). The references for the first are ([1]) while for the second are ([8], [9]). I went for EM since it's already implemented in Matlab. The drowback is that it is not possible to impose an unique correlation matrix or unimodality.

2.3 DP Algorithm

The general Stochastic Reachability theory is presented following [4]. The financial application of the algorithm is implemented following [2]. The first draft of the code can be found here: `https://github.com/skiamu/Thesis`.

As an example, we chose the following parameters:

1. 2-years investment, quarterly rebalancing policy ($N = 8$)
2. target set as in [3]

The biggest challenge was the V@R constraint: Let's define the portfolio loss as $L = L(k, k+1) := -(x_{k+1} - x_k)$, using the relation $x_{k+1} = x_k(1 + u_k^T w_{k+1})$ we get

$$L = -x_k(u_k^T \cdot w_{k+1})$$

or if we want to express the loss in return's terms we have

$$L = -(u_k^T \cdot w_{k+1})$$

. In the **Gaussian case** we have that $L \sim \mathcal{N}(\underbrace{-u^T\mu}_{\mu_p}, \underbrace{u^T\Sigma u}_{\sigma_p^2})$ then

$$V@R_{1-\alpha}^{\Delta} = \mu_p + z_{1-\alpha}\sigma_p$$

where Δ is the time horizon (in this case is the rebalancing period, 3 months).

In the **Mixture case** we need the property that linear combinations of (multivariate) Gaussian mixture are still (univariate) Gaussian mixture (see [5]). Therefore we have

$$\mathcal{P}(L \leq V@R_{1-\alpha}^{\Delta}) = 1 - \alpha \quad \Rightarrow \mathcal{P}(-u^T w_{k+1} \leq V@R_{1-\alpha}^{\Delta}) = 1 - \alpha$$

since [1]

$$u^T w_{k+1} \sim p_{u^T w_{k+1}}(z) = \sum_{i=1}^{n} \lambda_i \phi_{(\mu_i, \sigma_i)}(z)$$

[1] the function $\phi_{(\mu_i,\sigma_i)}(z)$ is the normal density with mean $\mu_i = u^T\delta_i$ and variance $\sigma_i^2 = u^T V_i u$ where δ_i and V_i are mean vector and covariance matrix of the mixture components

by integration we get the CDF

$$F_{u^T w_{k+1}}(z) = \sum_{i=1}^{n} \lambda_i \Phi_{(\mu_i,\sigma_i)}(z)$$

hence

$$\begin{aligned}\mathcal{P}(-u^T w_{k+1} \leq V@R^{\Delta}_{1-\alpha}) &= \mathcal{P}(u^T w_{k+1} \geq -V@R^{\Delta}_{1-\alpha}) \\ &= 1 - F_{u^T w_{k+1}}(-V@R^{\Delta}_{1-\alpha})) \\ &= 1 - \alpha\end{aligned}$$

and finally

$$\boxed{F_{u^T w_{k+1}}(-V@R^{\Delta}_{1-\alpha}) \leq \alpha}$$

Remark 1 *Let's suppose we want to impose a monthly V@R constraint,* $V@R^{1m}_{99} = 7\%$. *First we need to convert this V@R specification into our time horizon* $\Delta = 3m$. *Using the compound rule we have*

$$V@R^{3m}_{1-\alpha} = (1 + V@R^{1m}_{1-\alpha})^3 - 1$$

and then impose the above relation.

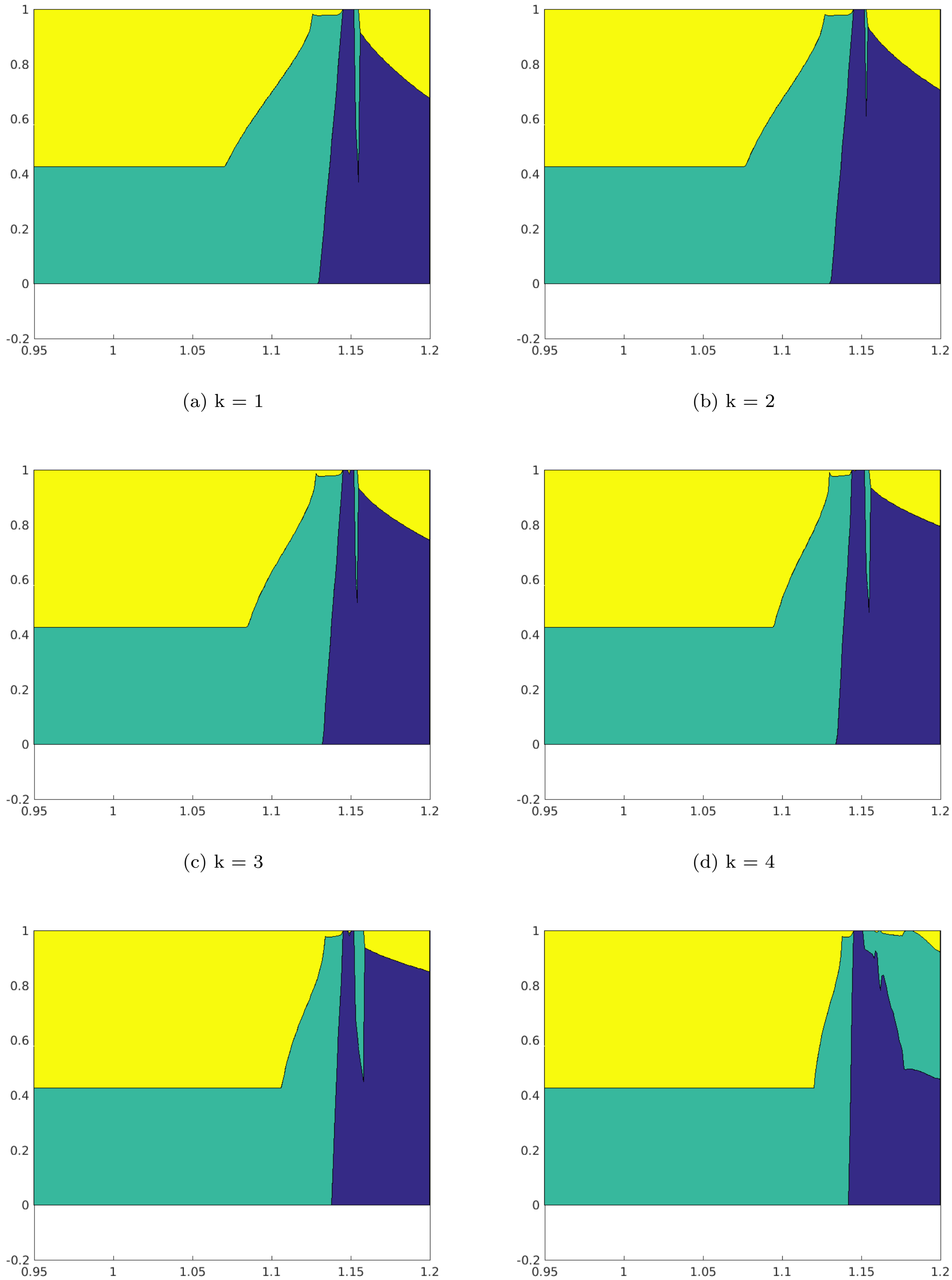
2.4 partial results

The target sets were approximated in the following way: $X_k \approx [0.5, 1.9] \quad k = 1, \ldots, 7$ and $X_8 \approx [(1+\theta)^2, 1.9]$. The discretization step $\eta = 10^{-3}$, $\theta = 7\%$, $V@R^{1m}_{99} = 7\%$.

Remark 2 *The results is very sensitive to the VaR constraint, the target return* θ *and the distributions of the asset classes. Another method for imposing the VaR constraint can be found in (e.g. [6]).*

Remark 3 *Another problem is the following: if the parameters are not carefully chosen it may happen that* $J(x_0) > 1$.

Here we report the allocations maps for different time instants:



(a) k = 1

(b) k = 2

(c) k = 3

(d) k = 4

(e) k = 5

(f) k = 6

Figure 1: 3-months-square-root scaling rule was used of converting the VaR

2.5 Validation

not started yet. Simulation only at the final period or at every time step? discuss paper on mixture simulation (I took a look at [7]).

2.6 Comparison

not started yet

2.7 Extensions

- Levy model for the return w_{k+1} (plus the associated calibration and simulation problem)
- transaction costs
- return estimation
- Python implementation for better performance
- ...

3 What next

- complete Comparison and Validation
- improve the code from any point of view
- study papers on Stochastic Hybrid System ([10]) for possible extensions
- ...

References

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