Case study answers

July 13, 2022

Abstract

In this document we provide the solutions to the case study problems. Supporting code for numerical results and plots is available at this github repo.

1 Question 1

Let $X \sim \mathrm{logN}\big(\mu, \sigma^2\big)$. Here some standard results about the lognormal distribution which will be used in the following:

$$m = \mathbb{E}[X] = \exp\left(\mu + \sigma^2/2\right) \tag{1}$$

$$v = \mathrm{Var}\left[X\right] = \left[\exp(\sigma^2) - 1\right] \exp\left(2\mu + \sigma^2\right) \tag{2}$$

$$m^\star = \mathrm{Med}[X] = \exp(\mu) \tag{3}$$

1.1 Question 1.1

In order compute the median $m^\star$ we need to derive the parameters μ and σ^2 first. That amounts to solve the following system

$$\begin{cases} m = \exp\left(\mu + \sigma^2/2\right) \\ v = \left[\exp(\sigma^2) - 1\right] \exp\left(2\mu + \sigma^2\right). \end{cases} \tag{4}$$

By noting that the second factor in v is equal to m^2 we get that

$$\begin{cases} \sigma^2 = \log\left(\frac{v}{m^2} + 1\right) \\ \mu = \log(m) - \sigma^2/2. \end{cases} \tag{5}$$

By plugging the numbers we get $\sigma^2 = 0.02257$ and $\mu = 0.0469$. It follows from (3) that $m^\star = 1.0481$.

1.2 Question 1.2

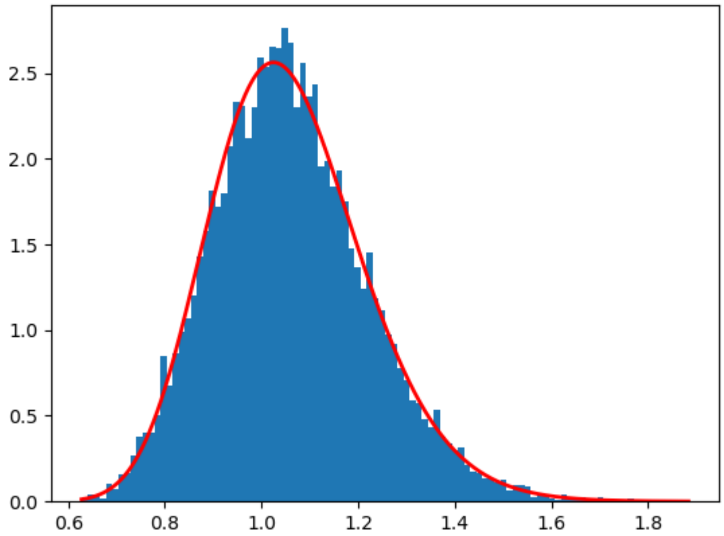

Figure 1: Probability density function of X and the histogram of 10000 samples drawn from the distribution of X.

2 Question 2

In this section we answer question 2. Before doing that, we briefly recall the underlying factor model for asset returns in order to set notation and recall its main properties.

***Definition* 2.1:** Let R be an n-dimensional random vector of assets returns over a generic period $[t, T]$. We suppose R follows a linear factor model

$$R = \beta R_f + \epsilon \tag{6}$$

where

- R_f is a f-dimensional random vector describing factors returns
- $\beta \in \mathbb{R}^{n \times f}$ is the factor loadings matrix
- ϵ is the n-dimensional random vector of residuals.

The following standard hypothesis apply

- $\epsilon_i \overset{\text{iid}}{\sim} (0, \Omega_{ii}), \forall i = 1, \dots, n$
- R_f and ϵ are uncorrelated (i.e. $\text{Cov}\left[R_f, \epsilon\right] = 0_{f \times n}$)

***Proposition* 2.1:** If R follows the factor model (6), then

$$\text{Cov}\left[R\right] := \Sigma_n = \beta \Sigma_f \beta^T + \Omega, \tag{7}$$

where Σ_f is the covariance matrix of R_f and Ω is the diagonal covariance matrix of ϵ

Proof. The expected value of R is $\mathbb{E}[R] = \beta \bar{R}_f$. Therefore

$$\begin{aligned}
\text{Cov}\left[R\right] &= \mathbb{E}\left[\left(R - \beta \bar{R}_f\right)\left(R - \beta \bar{R}_f\right)^T\right] &(8)\\
&= \mathbb{E}\left[\left(\beta R_f + \epsilon - \beta \bar{R}_f\right)\left(R_f^T \beta^T + \epsilon^T - \bar{R}_f^T \beta^T\right)\right] &(9)\\
&= \beta \mathbb{E}[R_f R_f^T] \beta^T + \mathbb{E}[\epsilon \epsilon^T] - \beta \bar{R}_f \bar{R}_f^T \beta^T &(10)\\
&= \beta \Sigma_f \beta^T + \Omega, &(11)
\end{aligned}$$

where in (10) we used the fact that R_f and ϵ are uncorrelated and ϵ has zero mean. □

2.1 Question 2.1

Let $w \in \mathbb{R}^n$ be the vector of portfolio weights. Then

$$\sigma(w) = \sqrt{\text{Var}\left[w^T R\right]} = \sqrt{w^T \Sigma_n w} = \sqrt{w^T \beta \Sigma_f \beta^T w + w^T \Omega w}. \tag{12}$$

Plugging the numbers we get $\sigma(w) = 40.71\%$

2.2 Question 2.2

Let us recall the main results about risk decomposition that will be used in this section.

***Definition* 2.2:** Let $f(x)$ be a continous continuous and differentiable function of $x \in \mathbb{R}^n$. f is said homogenous of degree one if

$$f(cx) = cf(x), \tag{13}$$

$\forall c \in \mathbf{R}$, $c > 0$.

***Theorem* 2.1:** (Euler) Let $f(x)$ be a continuous, differentiable and homogenous function of order one. Then

$$f(x) = \sum_{i=1}^{n} x_i \frac{\partial f(x)}{\partial x_i} = x^T \nabla f(x) \tag{14}$$

It can be shown that portfolio volatility $\sigma = (w^T \Sigma_n w)^{1/2}$ is an homogenous function of degree one in w. Applying Euler's Theorem we have

$$\sigma(w) = \sum_{i=1}^{n} w_i \frac{\partial \sigma(w)}{\partial w} = \sum_{i=1}^{n} w_i \mathrm{MRC}_i = \sum_{i=1}^{n} \mathrm{RC}_i, \tag{15}$$

where MRC stands for *marginal risk contribution* and RC for *risk contribution.* Given that $\nabla\sigma(w) = \frac{\Sigma_n w}{(w^T \Sigma_n w)^{1/2}}$, we have that

$$\mathrm{MRC}_i = \frac{(\Sigma_n w)_i}{(w^T \Sigma_n w)^{1/2}} \tag{16}$$

$$\mathrm{RC}_i = w_i \frac{(\Sigma_n w)_i}{(w^T \Sigma_n w)^{1/2}} \tag{17}$$

$$\mathrm{PRC}_i = \frac{\mathrm{RC}}{\sigma(w)} = w_i \frac{(\Sigma_n w)_i}{(w^T \Sigma_n w)} \tag{18}$$

Having this in mind, we can break down portfolio volatility by asset and sectors.

In Figure (2) and (3) we report *percentage risk contributions* (PRC) and portfolio weights by asset and sectors respectively. From Figure (2) we see that tech giants like Apple, Microsoft, Alphabet and Amazon contribute to more than half (51.50%) of the total risk: for this reason the portfolio cannot be described as well-diversified. The by-sector analysis confirms this finding: tech-heavy sectors like IT and Communication Services take a big share of the total risk. This is usually the case for cap-weighted index portfolios. It's interesting to note that we can draw the same conclusion just by looking at portfolio weights. This is not true in general (it would not be the case for a multi-asset portfolio) and it is due to the fact that large-cap US stocks share a similar volatility profile.

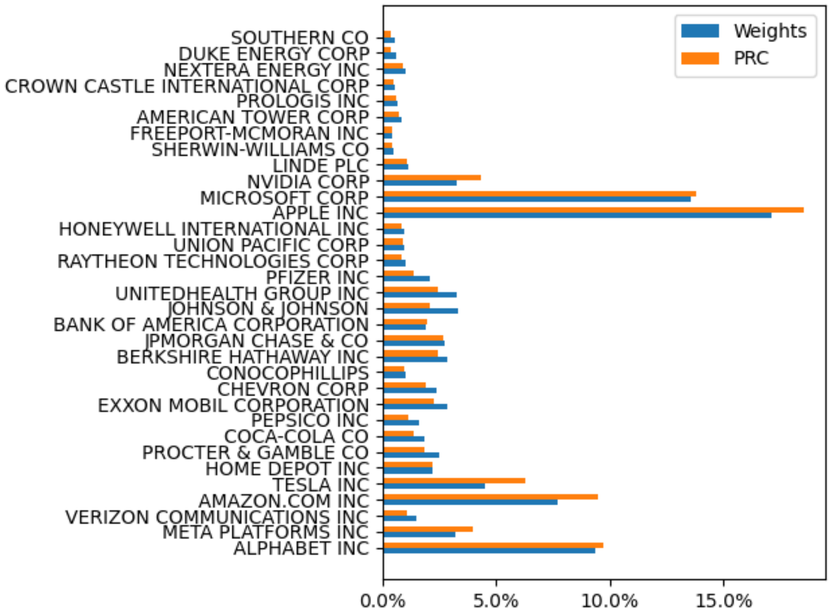

Figure 2: Weights and percentage risk contributions (PRC) of a portfolio of US stocks

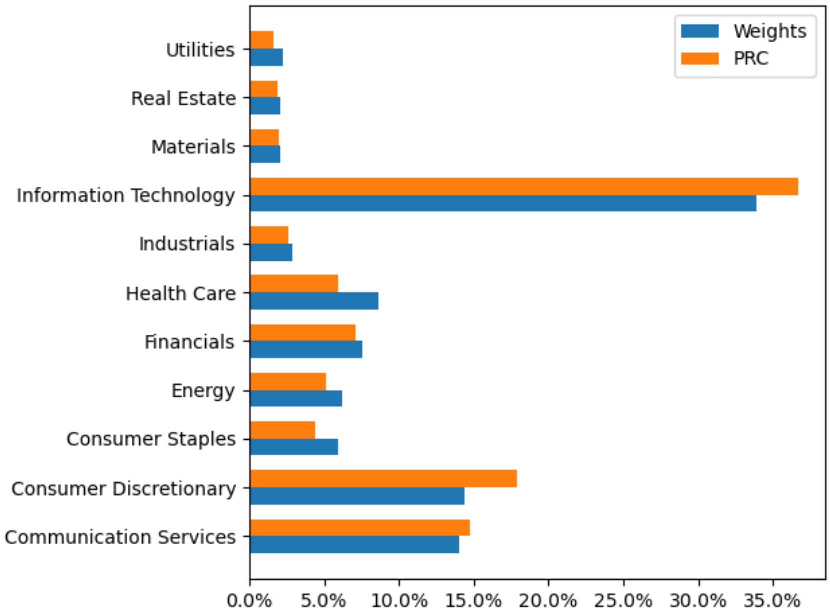

Figure 3: Sectors weights and percentage risk contributions (PRC) of a portfolio of US stocks

Table 1: Risk decomposition by asset

ISIN	Name	Weight	PRC
US0378331005	APPLE INC	17.09%	18.54%
US5949181045	MICROSOFT CORP	13.57%	13.81%
US02079K1079	ALPHABET INC	9.36%	9.7%
US0231351067	AMAZON.COM INC	7.68%	9.45%
US88160R1014	TESLA INC	4.53%	6.27%
US67066G1040	NVIDIA CORP	3.27%	4.31%
US30303M1027	META PLATFORMS INC	3.21%	3.99%
US46625H1005	JPMORGAN CHASE & CO	2.74%	2.67%
US0846707026	BERKSHIRE HATHAWAY INC	2.88%	2.46%
US91324P1021	UNITEDHEALTH GROUP INC	3.27%	2.45%
US30231G1022	EXXON MOBIL CORPORATION	2.84%	2.28%
US4370761029	HOME DEPOT INC	2.21%	2.19%
US4781601046	JOHNSON & JOHNSON	3.31%	2.06%
US0605051046	BANK OF AMERICA CORPORATION	1.92%	1.96%
US1667641005	CHEVRON CORP	2.36%	1.9%
US7427181091	PROCTER & GAMBLE CO	2.5%	1.83%
US1912161007	COCA-COLA CO	1.82%	1.4%
US7170811035	PFIZER INC	2.08%	1.39%
US7134481081	PEPSICO INC	1.62%	1.17%
IE00BZ12WP82	LINDE PLC	1.16%	1.1%
US92343V1044	VERIZON COMMUNICATIONS INC	1.49%	1.07%
US20825C1045	CONOCOPHILLIPS	1.04%	0.95%
US9078181081	UNION PACIFIC CORP	0.99%	0.93%
US65339F1012	NEXTERA ENERGY INC	1.04%	0.89%
US4385161066	HONEYWELL INTERNATIONAL INC	0.93%	0.87%
US75513E1010	RAYTHEON TECHNOLOGIES CORP	1.0%	0.86%
US03027X1000	AMERICAN TOWER CORP	0.82%	0.75%
US74340W1036	PROLOGIS INC	0.66%	0.62%
US22822V1017	CROWN CASTLE INTERNATIONAL CORP	0.57%	0.5%
US8243481061	SHERWIN-WILLIAMS CO	0.47%	0.45%
US35671D8570	FREEPORT-MCMORAN INC	0.4%	0.44%
US8425871071	SOUTHERN CO	0.56%	0.38%
US26441C2044	DUKE ENERGY CORP	0.61%	0.37%

Table 2: Risk decomposition by asset

Sector	Weight	PRC
Information Technology	33.93%	36.66%
Consumer Discretionary	14.42%	17.92%
Communication Services	14.05%	14.76%
Financials	7.54%	7.09%
Health Care	8.67%	5.9%
Energy	6.24%	5.12%
Consumer Staples	5.95%	4.39%
Industrials	2.92%	2.65%
Materials	2.03%	1.99%
Real Estate	2.05%	1.87%
Utilities	2.21%	1.64%

2.3 Question 2.3

By risk parity portfolio we mean the equal risk contribution (ERC) portfolio, that is a portfolio whose weights satisfy the following relations

$$\mathrm{RC}_i = \mathrm{RC}_j, \quad \forall i = 1, \ldots, n. \tag{19}$$

There is no close formed solution to this problem and one has to solve the following optimization problem:

$$w^\star = \arg\min \sum_{i=1}^{n} \left(\frac{w_i(\Sigma_n w)_i}{w^T \Sigma_n w} - \frac{1}{n} \right)^2 \tag{20}$$

$$\text{s.t.} \quad 1^T w = 1 \tag{21}$$

$$0 \leq w \leq 1 \tag{22}$$

The results are reported in Figure (4) and in the following table.

Figure 4: Asset weights and percentage risk contributions (PRC) of the equal risk contribution portfolio (ERC).

Table 3: Risk decomposition by asset for the ERC portfolio

ISIN	Name	w_ERC	PRC_ERC
US7170811035	PFIZER INC	3.76%	3.04%
US0605051046	BANK OF AMERICA CORPORATION	2.72%	3.04%
US20825C1045	CONOCOPHILLIPS	2.78%	3.04%
US7427181091	PROCTER & GAMBLE CO	3.65%	3.04%
US91324P1021	UNITEDHEALTH GROUP INC	3.5%	3.03%
US7134481081	PEPSICO INC	3.55%	3.03%
US35671D8570	FREEPORT-MCMORAN INC	2.48%	3.03%
US92343V1044	VERIZON COMMUNICATIONS INC	3.51%	3.03%
US26441C2044	DUKE ENERGY CORP	3.73%	3.03%
US8425871071	SOUTHERN CO	3.5%	3.03%
US88160R1014	TESLA INC	2.14%	3.03%
US46625H1005	JPMORGAN CHASE & CO	2.83%	3.03%
US4370761029	HOME DEPOT INC	2.81%	3.03%
US0378331005	APPLE INC	2.78%	3.03%
US1912161007	COCA-COLA CO	3.35%	3.03%
US8243481061	SHERWIN-WILLIAMS CO	2.84%	3.03%
US0231351067	AMAZON.COM INC	2.45%	3.03%
US74340W1036	PROLOGIS INC	2.85%	3.03%
US02079K1079	ALPHABET INC	2.89%	3.03%
US03027X1000	AMERICAN TOWER CORP	2.86%	3.03%
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US30303M1027	META PLATFORMS INC	2.43%	3.03%
US1667641005	CHEVRON CORP	3.07%	3.03%
US30231G1022	EXXON MOBIL CORPORATION	3.1%	3.03%
US0846707026	BERKSHIRE HATHAWAY INC	3.16%	3.03%
US4385161066	HONEYWELL INTERNATIONAL INC	2.99%	3.03%
US5949181045	MICROSOFT CORP	2.99%	3.03%
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US65339F1012	NEXTERA ENERGY INC	3.03%	3.02%
US67066G1040	NVIDIA CORP	2.35%	3.02%
US4781601046	JOHNSON & JOHNSON	4.04%	3.02%

3 Further development

Possible ways to extend this work are:

- In addition to analysing portfolio diversification in terms of assets and sectors, it would be interesting to compute factors diversification through risk decomposition [2]
- Exploring new numerical algorithms for solving efficiently the risk parity problem at scale [1]

References

[1] Choi1, J., and Chen, R. Improved iterative methods for solving risk parity portfolio.

[2] Roncalli, T., and Weisang, G. Risk parity portfolios with risk factors.