Set

The cardinality of a set is the number of distinct elements in the set. $n(A)$ or $|A|$

$\{1,2,3\}$ $\{1,2,2,3\}$

same but elements increased ; repeated

Practice: $n(A)=5$, $n(B)=50$, $n(C)=\infty$

$n(D)=3$, $n(E)=6$

Venn Diagram: Universal set denoted as U / Ω.

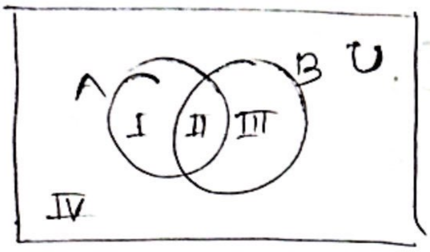
Sample Space: Ω

proper subset : cannot be equal eg. $A \subset B$

B will have at least on element.

Subset: ~~$A \subseteq B$~~ ; can be equal ; $A \subseteq B$

Disjoint, Proper, Equal, Overlapping



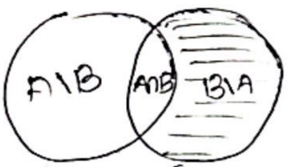
$\mathrm{I} + \mathrm{II} + \mathrm{III} = A \cup B$

$$\left.\begin{array}{l} \mathrm{I} : A \setminus B \\ \mathrm{II} : A \cap B \\ \mathrm{III} : B \setminus A \end{array}\right\} A \cup B$$

$\mathrm{IV} : (A \cup B)^c$

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$B \setminus A = \{x \mid x \in B \text{ and } x \notin A\}$



$n(A \setminus B) = n(A) - n(A \cap B)$

$n(B \setminus A) = - - -$

2^n = region number , n = number of sets

e.g. for 3 sets : $2^3 = 8$

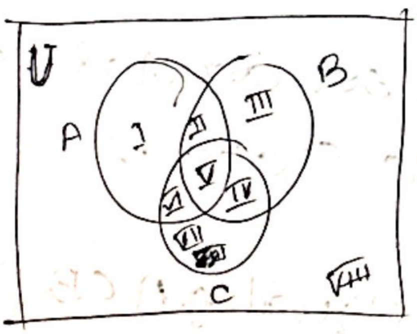


Fig: a

$\mathrm{V} : A \cap B \cap C$

$$\left\{\begin{array}{l} \mathrm{II} : (A \cap B) \cap C' \\ \mathrm{VI} : (A \cap C) \cap B' \\ \mathrm{IV} : A' \cap (B \cap C) \end{array}\right.$$

$$\left\{\begin{array}{l} \mathrm{I} : A \cap (B' \cap C') \\ \mathrm{III} : B \cap (A' \cap C') \\ \mathrm{VII} : C \cap (A' \cap B') \end{array}\right.$$

$\mathrm{VIII} : (A \cup B \cup C)^c = U \setminus (A \cup B \cup C)$

$= A^c \cap B^c \cap C^c$

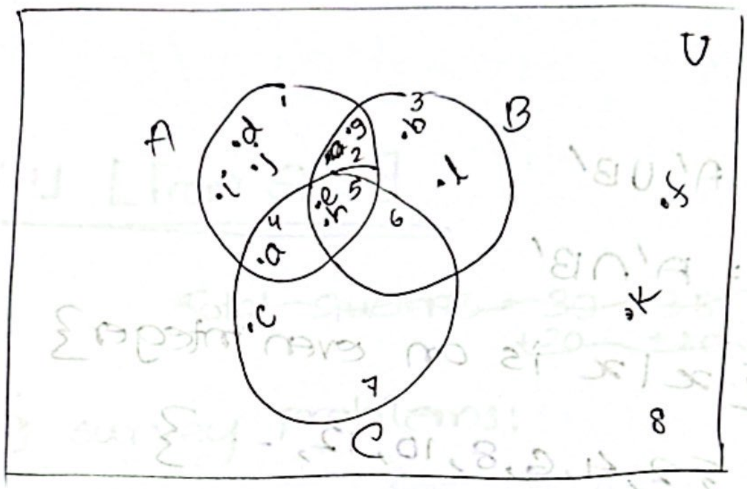
Practice:

$A = \{a, d, e, g, h, i, j\}$

$B = \{b, e, g, h, l\}$

$C = \{a, c, e, h\}$

$U = \{a, b, c, d, e, f, g, h, i, j, k, l\}$



If A subset of B, B superset of A.

Power Set: All the possibles subsets from a set.

$n(\wp(A)) = 2^{n(A)}$

$A \{1, 2, 3\} \quad 2^3 = 8$

$\wp(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, A\}$

Tree: $A = \{\emptyset\}$

$A = \{a\}$

$A = \{a, b\}$

$A = \{a, b, c\}$

Symmetric Difference:

$$A \oplus B = (A \setminus B) \cup (B \setminus A)$$
$$= (A \cup B) \setminus (A \cap B)$$
$$= \mathrm{I} + \mathrm{III} \quad [\text{Fig: a}]$$

$$(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$$

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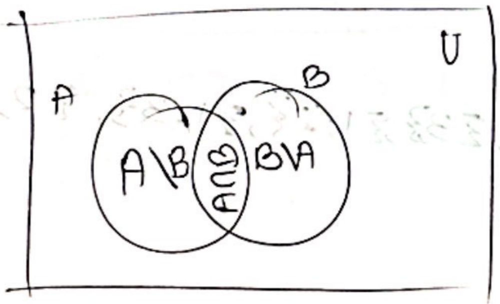
$$(A \cap B)' = A' \cup B'$$
$$(A \cup B)' = A' \cap B'$$

$A = \{x \mid x \text{ is an even integer}\}$
$= \{2, 4, 6, 8, 10, 12, \ldots\}$

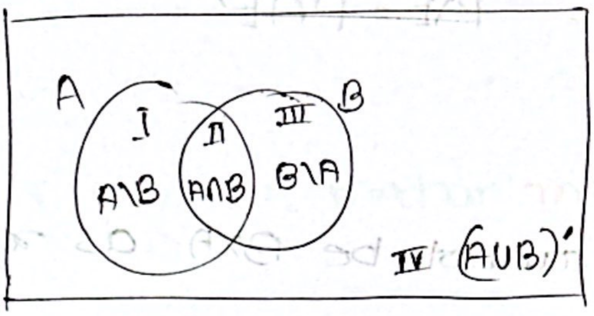
$A = \{\} = \phi$

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$



$$n(A \setminus B) = n(A) - n(A \cap B)$$

Inclusion-Exclusion Principle



$n(A) = \mathrm{I} + \mathrm{II}$

$n(B) = \mathrm{II} + \mathrm{III}$

$n(A \cap B) = \mathrm{II}$

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

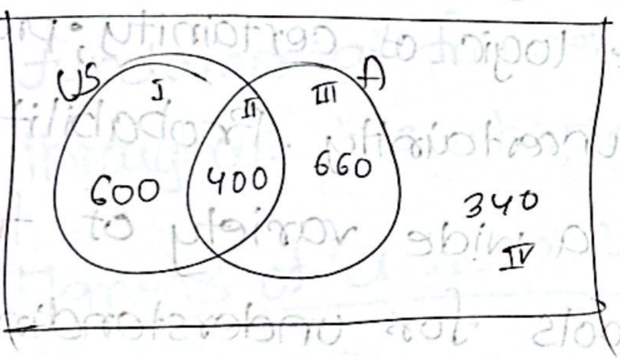
$= \mathrm{I} + \mathrm{II} + \mathrm{II} + \mathrm{III} - \mathrm{II}$

$= \mathrm{I} + \mathrm{II} + \mathrm{III}$

Example 1 [From Slide]

~~Total Students = 30 + 35 = 60 + 55~~
~~+20 +20 = 115~~

Solving survey problems:

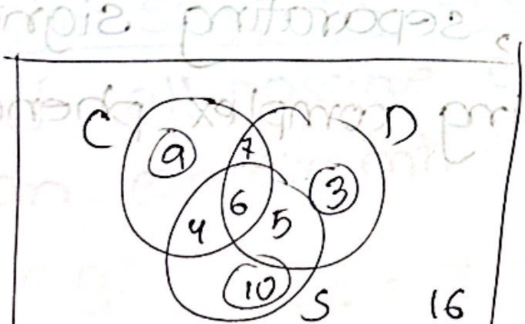


I : US + ~~D~~ Disagreed

II : US + Agreed

III : Mexican + Agreed

IV : ~~US~~ Mexican + Disagreed.



Total: 16

a) 9

b) 5

c) 3+5+10 = 18

d) 9+3+10 = 22

e) 4+6+7+5 = 22

f) 16

Exactly 2 movie style: 4+7+5 = 16

Exactly 3: 6

$E \setminus P = E \cap P'$ $\quad$ $P \setminus E = P \cap E'$

~~AB~~ $A \times B$ cannot be $B \times A$ as x, y will get interchanged so Cartesian product is asymmetric. Special case / set or some set is symmetric.

Probability

~~II~~ Why we need to study probability?

⇒ Mathematics is the logic of certainity, probability is the logic of uncertainity. Probability is extremely useful in a wide variety of fields since it provides tools for understanding and explaining variations, separating signal from noise, and modelling complex phenomena.

Some areas where we can apply the knowledge of Probability: [applications]

① Statistics:

② Physics : Quantam mechanisms, Statistical mechanism

③ Biology: genetic

④ CSE: AI, networking, randomized algorithm, neural

more: medicine, life science. etc.

Probabilistics Models: A probabilistic model is a mathematical description of an uncertain situation. It must be in accordance with a fundamental framework that we discuss throughout this chapter.

Elements of a Probabilistics Model:

- The Sample Space: Ω, which is the set of all possible ~~outcomes~~ of outcomes of an experiment.

e.g. $\Omega = \{TT, TH, HT, HH\}$

flips 2 coins.

• <u>The Probability Law</u>: which assigns a set A of possible outcomes (also called event) a nonnegative number $P(A)$, called the probability of A that encodes our knowledge or belief about the collective "likelihood" of the elements of A.

<u>Probability Axioms</u>:

① <u>Non-negativity</u>: $P(A) \geq 0$, for any event A.

② <u>Additivity</u>: If A and B are two disjoint events, then the probability of their union statisfies $P(A \cup B) = P(A) + P(B)$

③ <u>Normalization</u>: The probability of entire sample space Ω is equal to 1, i.e. $P(\Omega) = 1$.

e.g.

$$\Omega = \{TT, TH, HT, TT\}$$

Individual probability = 1/4 e.g. $P(HH) = 1/4$

Sum of all probability $= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$.

Sample Spaces and Events:

Every probability model involves an underlying process called the <u>experiment</u>, that will provide/produce exactly one out of several possible outcomes. e.g. dice.

The set of all possible outcomes is called the sample space. e.g. dice

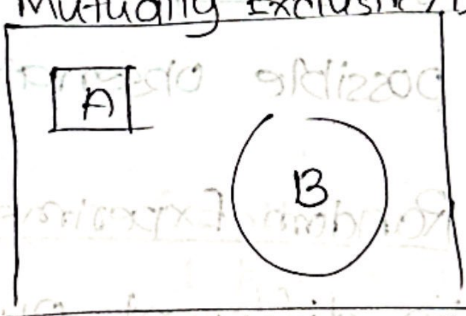
-tesian duct $\rightarrow 6\times 6 = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$
$(2,1), (2,2), \ldots$
$(6,5), (6,6)\}$

Appropriate Sample Space ensures 2 things.
→ mutually exclusive
→ collective exhaustive

16/5/25

Mutually Exclusive
Collective exhaustive

Mutually Exclusive/Disjoint



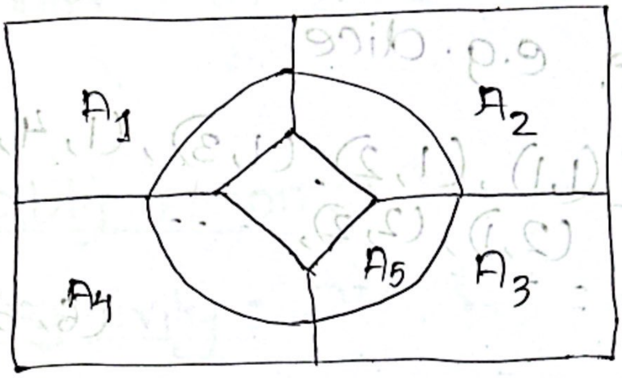
A collection of sets $A_1, A_2, \ldots, A_n$ is mutually exclusive if and only if

$A_i \cap A_j = \emptyset \; ; \; i \neq j$

e.g. $A_1 \cap A_2 = \emptyset$

$A_2 \cap A_5 = \emptyset$

Collectively Exhaustive



$\Omega = S$

A collection of sets $A_1, A_2, A_3, \ldots, A_n$ is collectively exhaustive iff, $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n = S = \Omega$

Outcome: An outcome of an experiment is any possible observation of that experiment.

↳ Random, Definite

Random Experiment: An experiment that can result in different outcomes, even though it is repeated in the same manner every time is called a random experiment. e.g. rolling a dice.

: Discrete, continuous

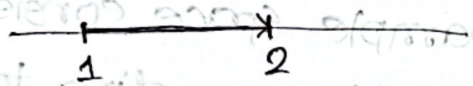
Sample Space: The set of all possible outcomes of a random experiment, denoted by S or Ω.

Discrete Sample Space: A sample space is discrete if it consists of a finite number of outcomes.

e.g. whole numbers, prime numbers

continuous → real number between 2 numbers.
Infinite numbers.

Continuous Sample Space: If a sample space contains an infinite number of possibilities equal to the number of points on a line segment.



In otherwise, a sample space is continuous if it contain an interval of real numbers $(1,2)$ or $[1,2]$

Event: An event is a subset of the sample space, ~~of~~ of a random experiment.

e.g. $S = \{HH, HT, TH, TT\}$

$S_1 = \{HH, TH, HT\}$ ← ~~Greet~~ "Getting at least 1 head" event.

$S_1 \subset S$ ← proper subset

Equally Likely Events: 2 or more events are said to be equally likely if they have the same chance of occurrence. e.g. dice ; events of getting an even or odd is same : 3/6 .

Joint Probability: 2 or more events from a joint event if all of them occur simultaneously and the ~~prob~~ probability of these joint event is calle the 'Joint Probability'. e.g. 2 or even number indic

A: Heart Disease
B: Fast Food $\Big\}$ $P(A \cap B)$

Problem: Suppose a sample space consists of 500 ~~perso~~ people and are distributed according to the gender and employement status as shown below:

$n(M) = 275$
$n(F) = 225$ } 500

$n(E) = 335$
$n(U) = 165$ } 500

Gender \ Status	Employed (E)	UnEmployed (U)	Total
Male(M)	255	20	275
Female(F)	80	145	225
Total	335	165	500

M: The selected person is male

F: " " " " female

E: " " " " employed

U: " " " " Unemployed

$M \cap E$:

$M \cap U$:

$F \cap E$:

$F \cap U$:

$$P(M) = \frac{n(M)}{n(S)} = \frac{275}{500}$$

$$P(F) = \frac{n(F)}{n(S)} = \frac{225}{500}$$

$$P(M \cap E) = \frac{n(M \cap E)}{n(S)} = \frac{255}{500}$$

$$P(M \cap U) = \frac{n(M \cap U)}{n(S)} = \frac{20}{500}$$

$$P(F \cap E) = \frac{80}{500}$$

$$P(F \cap U) = \frac{145}{500}$$

mutually exclusive

$$P(M) = P(M \cap E) + P(M \cap U)$$
$$= \frac{255}{500} + \frac{20}{500} = \frac{275}{500}$$

- No common points
- Sum to calculate the probability of 2 events.

[1.3] Book

Conditional Probability: It provides us a way to reason about the outcome of an experiment, based on partial information.

Def^n: Conditional probability corresponds to a modified probability model that reflects partial information about the outcome of an experiment. This modified model has a smaller ~~st~~ sample space than the original model

Problem #1

e.g. First ~~die →~~ dice → 4 Total: 6

$6 \times 6 = 36$

So (second dice) $\neq$ 1/6 = probability

amples
reased $\Rightarrow (4,1)\ (4,2)\ (4,3)\ (4,4)\ (4,5)\ (4,6)$

Defn: The probability of an event A when it is known that some other event B has occured (already) is called the conditional Probability of the event A and is denoted by $P(A|B)$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(AB)}{P(B)}$$

Similarly,

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)} = \frac{P(AB)}{P(A)}$$

$$P(A \cap B) = P(A|B) \cdot P(B) \quad \text{①}$$

$$P(B \cap A) = P(B|A) \cdot P(A) \quad \text{①}$$

Intersection,

$$P(A \cap B) = P(B \cap A)$$

$$\Rightarrow \boxed{P(A|B) \cdot P(B) = P(B|A) \cdot P(A)}$$

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(probability of A as B already occured) $P(A|B) = \frac{P(A\cap B)}{P(B)}$ ← already occured

Examples
1.6
1.7
1.8
1.9

$P(B|A) = \frac{P(A\cap B)}{P(A)}$

$$\boxed{P(A\cap B) = \overbrace{P(A|B)\cdot P(B)}^{\text{multiplication block}} = P(B|A)\cdot P(A)}$$ ⇒ Simple multiplication law

$P(A|B)$ ↓ Conditional Probability

$P(B)$ ↓ Unconditional Probability

Multiplication Law:

Assuming that all of the conditioning events have positive probability, we have

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1\cap A_2\cap A_3\cap \dots \cap A_n)$$

$$= P(A_1)\,P(A_2|A_1)\,P(A_3|A_1\cap A_2)\,P(A_4|A_1\cap A_2\cap A_3)\dots P\left(A_n \middle| \bigcap_{i=1}^{n-1} A_i\right)$$

Problem #2: A family has 2 children. What is the conditional probability that both of them are boys, given that at least one of them is a boy.

condition

condition

Solution: $S = \{(b,b), (b,g), (g,b), (g,g)\}$

$A = \{(b,b)\}$

$B = \{(b,b), (b,g), (g,b)\} \Rightarrow$ condition

$A \cap B = \{(b,b)\}$

$n(S) = 4$

$n(A) = 1$

$n(B) = 3$

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{1}{3} \quad \left[\frac{\frac{n(A\cap B)}{n(S)}}{\frac{n(B)}{n(S)}} \right.$$

Problem #3: The probability that a married man watches a certain TV show is 0.4 and that of his wife is 0.5. The probability that a man watches the show given that ~~*~~ his wife does is 0.7. Now, find the following probabilities:

(a) The probability that the married couple watches the TV show.

(b) The probability that his wife watches the show given that her husband does.

(c) the probability that $\underbrace{\text{at least}}_{\text{union}}$ one of them watches the show.

Soln: H: W:

$P(H) = 0.4$

$P(W) = 0.5$

$P(H|W) = 0.7$

H = event husband watches the show

W = event wife watches the show.

(a) $P(W \cap H) = P(H \cap W) = P(W) \cdot P(H|W) = 0.5 \times 0.7$
$= 0.35$ (Ans)

(b) $P(W|H) = \frac{P(W \cap H)}{P(H)} = \frac{0.35}{0.4} = 0.875$

(c) $P(H \cup W) = P(H) + P(W) - P(H \cap W)$
$= 0.4 + 0.5 - 0.35$
$= 0.55$ (Ans)

Problem #4: A box contains 7 red balls and 3 black balls. The balls are drawn from the box one after another. Find the probability that the first two balls are red and the third one is black if

(a) The balls are replaced before the next draw. [Place the picked ball into the box again] [Balls constant]

(b) The balls are not replaced. [Balls not constant]

$R_1 R_2 B_3$ } sequence

Solution: (a) $P(R_1 \cap R_2 \cap B_3)$

$n(R) = 7$
$n(B) = 3$
$\overline{\quad 10 \quad}$

$= P(R_1) \cdot P(R_2) \cdot P(B_3)$

$= \left(\frac{7}{10}\right)\left(\frac{7}{10}\right)\left(\frac{3}{10}\right) = \frac{147}{1000}$

(b) $P(R_1 \cap R_2 \cap B_3) = P(R_1) \cdot P(R_2 | R_1) \cdot P(B_3 | R_1 \cap R_2)$

$= \left(\frac{7}{10}\right)\left(\frac{6}{9}\right)\left(\frac{3}{8}\right)$

$= \frac{126}{720}$

$= \frac{7}{40}$

#5: A coin is tossed until a head appeared or it has been ~~toos~~ tossed three times. Given that the head does not appear on the first toss, what is the probability that the coin is tossed 3 times?

~~H, HT,~~

$S = \{H, TH, TTH, TTT\}$

If unbaised, or fair coin,

ty $P(H) = P(T) = 1/2$

$P(TH) = 1/4$

$P(TTH) = 1/8$

$P(TTT) = 1/8$

	HH	HT	~~H~~TH	TT
H	HHH	HHT	HTH	HTT
T	THH	THT	~~THT~~ TTH	TTT

A : The event that the coin is tossed three times

B : No head appeared on the first toss.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\# \left[\frac{\frac{n(A \cap B)}{n(S)}}{\dots} \right]$$

$A = \{TTH, TTT\}$

$B = \{TH, TTH, TTT\}$

$(A \cap B) = \{TTH, TTT\}$

$P(B) = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = 1/2$

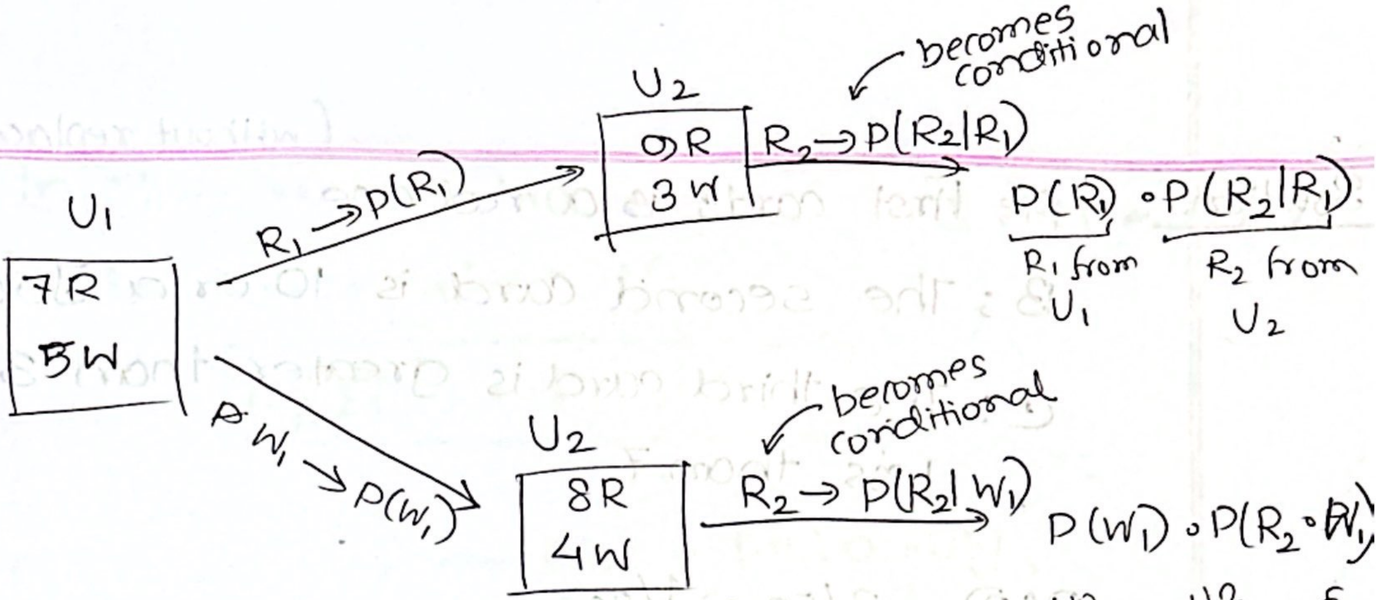
$P(A \cap B) = \frac{1}{8} + \frac{1}{8} = 1/4$

$P(A|B) = \frac{1/4}{1/2} = \frac{1}{4} \times 2 = 1/2$

Need to do individual events to calculate.

one's appearance won't hamper another's occurence

Problem #6: Suppose we have 2 Urns, U_1 and U_3. The first urn contains 5 white balls and 7 red balls, while the second urn contains 3 white balls and 8 red balls. One ball is transferred from the first urn to the second urn unseen and then a ball is drawn from the second urn. What is the probability that the picked ball is red?



placed red in U2 — picked red from U2 — or — placed white in U2 — picked white from U2

$$P(\text{Red Ball from Urn 2}) = P(R_1 \cap R_2) + W(W_1 \cap R_2)$$

$$= P(R_1) \cdot P(R_2|R_1) + P(W_1) \cdot P(R_2|W_1)$$

$$= \left(\frac{7}{12}\right) \times \left(\frac{9}{12}\right) + \left(\frac{5}{12}\right) \times \left(\frac{8}{12}\right)$$

$$= \frac{103}{144}$$

Problem #7: Three cards are drawn in succession without replacement from an ordinary deck of playing cards. Find the probability that the first card is a red ace, the second card is a 10 or a Jack and the third card is greater than 3 but less than 7.

(without replacement)

Solution: A: First cards is a red ace

B: The second card is 10 or a Jack

C: The third card is greater than 3 but less than 7.

$P(A) = 2/52 = 1/26$

$P(B) = 8/51$

$P(C) = 12/50 = 6/25$

nested condition loop

$P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot [P(C|\cancel{B \cap A})(A \cap B)]$

$= \left(\frac{1}{26}\right) \cdot \left(\frac{8}{51}\right) \cdot \left(\frac{6}{25}\right)$

23/5/25

Independent Events: If E and F are two separate ~~Sepen~~ events and if the occurance of E does not affect and is not affected by the occurance of F then E and F are said to be Independent events.

In other words, two events E and F are said to be independent if

$$\boxed{P(E \cap F) = P(E) \cdot P(F)}$$

Using the defn of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$P(E|F) = \frac{P(E) \cdot P(F)}{P(F)} = P(E)$$

condition probability becomes an unconditional probability, that's why it becomes ~~indepente~~ independence events due to the independence of these 2 events.

$$\left.\begin{array}{l} \underline{P(E|F) = P(E)} \\ \underline{P(F|E) = P(F)} \end{array}\right\} \rightarrow \text{Independent Events}$$

Problem #1: Two ideal coins are tossed. Let A be the event "head on the first coin" and B be the event "head on the second coin". Check whether the events are independent or not.

Sol^n: A: Head on the first coin.

B: " " " second "

$S = \{HH, HT, TH, TT\}$ $P(A) = \frac{2}{4} = 1/2$

$A = \{HH, HT\}$ $P(B) = \frac{2}{4} = 1/2$

$B = \{HH, TH\}$ $P(A \cap B) = 1/4$

$A \cap B = \{HH\}$

$P(A \cap B) = 1/4$

Proven that it is! $P(A) \cdot P(B) = \frac{1}{2} \cdot \frac{1}{2} = 1/4$

Problem #2: There are 3 coins tossed.

A: Head on the 1^{st} coin.

B: Tails on the last 2 coins.

} A & B are independent

$S = \{HHH, HTH, HHT, HTT, THH, TTH, THT, TTT\}$

$A = \{HHH, HTH, HHT, HTT\}$ $P(A) = 4/8 = 1/2$

$B = \{HTT, TTT\}$ $P(B) = 2/8 = 1/4$

$A \cap B = \{HTT\}$ $P(A \cap B) = 1/8$

Now, $P(A) * P(B) = \frac{1}{2} \cdot \frac{1}{4} = 1/8 = P(A \cap B)$

So, A and B are independent events.

n #3:

Suppose we throw 2 fair dice. Let E_1 denotes the event that the sum of the dice is 6. And F denotes the event that the first dice equals 4.

$P(E_1 \cap F) = 1/36$ $\qquad$ $E_1 \cap F = \{(4,2)\}$

$P(E_1) = 5/36$ $\qquad$ $E_1 = \{(1,5), (2,4), (4,2), (3,3), (5,1)\}$

$P(F) = 6/36 = 1/6$ $\qquad$ $F = \{(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)\}$

$P(E_1) * P(F) = \frac{5}{36} \times \frac{1}{6} \neq \frac{1}{36} = P(E_1 \cap F)$

$\therefore P(E_1) * P(F) \neq P(E_1 \cap F)$

As it's conditional, so not independent.

Modified:

$\rightarrow E_2 = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

$P(E_2) = 1/6$ $\qquad$ $(E_2 \cap F) = \{(4,3)\}$

$P(F) = 1/6$

$P(E_2 \cap F) = 1/36$

$P(E_2) \cdot P(F) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} = P(E_2 \cap F)$

$P(E_2 | F) = \frac{P(E_2 \cap F)}{P(F)} = \frac{1/36}{1/6} = 1/6 = P(E_2)$

E_3: (sum = 8) $P(E_3|F) = P(E_3)$

$E_2 = \{(2,6),(6,2),(3,5),(5,3),(4,4)\}$ 5/36

~~$F = \{(4,4)\}$ 1/36~~

$E_2 \cap F = \{(4,4)\} = 1/36$

$F = \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\} = 6$

$= 6/36 = 1/6$

So, sum = 7 is just a special case.

Problem #4: A fire brigade has 2 fire engines operating independently. The probability that a specific fire engines is available when required is 0.99, ~~Find~~ then

(a) What is the probability that an engine is available when need

(b) " " " " " " neither " " " "

Sol^n: A: The event that the 1^{st} engine is available

B: " " " " second " " "

$P(A) = P(B) = 0.99$

$P(A\cap B) = P(A)\cdot P(B) = (0.99)^2 = 0.9801$

(a) $P(A\cup B) = P(A) + P(B) - P(A\cap B)$
$= 0.99 + 0.99 - 0.9801$
$= 0.9999$

Probability of getting one from 2 is more obv!

(b) $P(A\cup B)^c = 1 - P(A\cup B) = 1 - 0.9999 = 0.0001$

Independence of more than two events:

The multiplication rule for independent events extends very simply to three of more events.

$$P(A\cap B\cap C) = P(A)\cdot P(B)\cdot P(C)$$

$$P(A_1\cap A_2\cap A_3\cap \dots \cap A_n) = P(A_1)\cdot P(A_2)\cdot \dots P(A_n)$$

absolute independence

pair-wise " : pair-wise does not mean its going to be absolute.

Problem #5 Two coins are tossed. If A is the event "head on the first coin", B is the event "head on the second coin", and C is the event "coins fall alike", then show that the events A, B, C are pair-wise independent but not completely/absolutely"?

Solution: $S = \{HH, HT, TH, TT\}$

$A = \{HH, HT\}$; $A \cap B = \{H, H\}$

$B = \{HH, TH\}$; $B \cap C = \{H, H\}$

$C = \{HH, TT\}$; $C \cap A = \{H, H\}$

$A \cap B \cap C = \{HH\}$

$P(A) = P(B) = P(C) = 2/4 = 1/2$

$P(A \cap B) = P(A \cap C) = P(B \cap C) = 1/4$

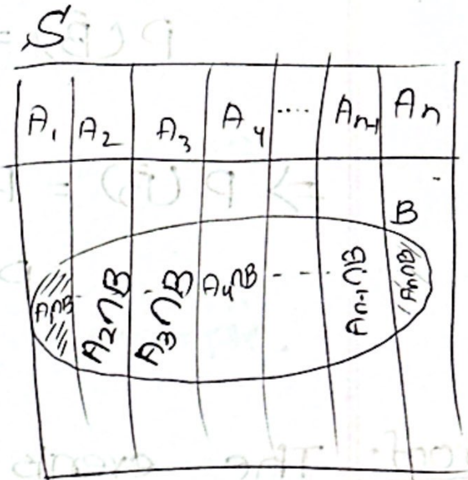
$$\left.\begin{array}{l} P(A) \cdot P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = P(A \cap B) \\ P(B) \cdot P(C) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = P(B \cap C) \\ P(A) \cdot P(C) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = P(A \cap C) \end{array}\right\} \text{Pair-wise independent}$$

Now, $P(A) \cdot P(B) \cdot P(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \cancel{4}\ \frac{1}{8}$

$$P(A \cap B \cap C) = \frac{1}{4} \neq P(A) \cdot P(B) \cdot P(C)$$

So, not absolutely independent.

Conditional Probability and Partitions:



B is a common event and it falls over all A_n events. A has A_n ~~partio~~ Partitions and B has some pieces of A.

$$S = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n$$

Let S denotes the sample space of some random experiment and consider n events $A_1, A_2, A_3, \dots, A_n$ such that they are mutually exclusive and collectively exhaustive, that is

$$A_i \cap A_j = \emptyset \text{ for all } i \neq j \text{ and}$$

$$S = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n$$

Then it is said that these events form a partition of S.

Total Probability Rule:

Suppose that the events $A_1, A_2, A_3, \ldots A_n$ make partitioning of the sample space, S and $P(A_j) > 0$ for $j = 1, 2, \ldots, n$. Then for any event $B \in S$,

$$P(B) = \sum_{j=1}^{n} P(A_j) \cdot P(B|A_j)$$

$$\Rightarrow P(B) = P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) + P(A_3) \cdot P(B|A_3) + \cdots + P(A_n) \cdot P(B|A_n)$$

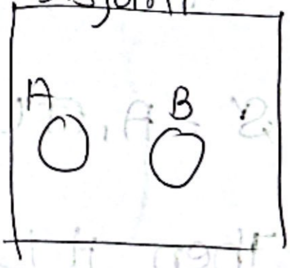
Proof: The events $(A_1 \cap B), (A_2 \cap B), (A_3 \cap B), \ldots, (A_n \cap B)$ form a partition of the event B.

Hence,

$$B = (A_1 \cap B) \cup (A_2 \cap B) \cup (A_3 \cap B) \cup \ldots \cup (A_n \cap B) \quad \text{— (1)}$$

Since, the n events on the RHS of (1) are mutually exclusive or disjoint then

Disjoint



$P(A \cup B) = P(A) + P(B)$

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_n \cap B)$$

$$\Rightarrow P(B) = \sum_{j=1}^{n} P(A_j \cap B)$$

Using the multiplication rule we have

$$P(A_j \cap B) = P(A_j) \cdot P(B \mid A_j)$$

Here,

$$P(B) = \sum_{j=1}^{n} (A_j) \cdot P(B \mid A_j)$$