23/5/25

Independent Events: If E and F are two separa ~~separ~~ events and if the occurance of E does not affect and is not affected by the occurance of F then E and F are said to be Independent events.

In other words, two events E and F are said to be independent if

$$\boxed{P(E \cap F) = P(E) \cdot P(F)}$$

Using the def^n of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$P(E|F) = \frac{P(E) \cdot P(F)}{P(F)} = P(E)$$

condition probability becomes an unconditional probability; that's why it becomes ~~indeperte~~ independence events due to the independence of these 2 events.

$$\left.\begin{array}{l} P(E|F) = P(E), \\ P(F|E) = P(F) \end{array}\right\} \rightarrow \text{Independent Events}$$

Problem #1: Two ideal coins are tossed. Let A be the event "head on the first coin" and B be the event "head on the second coin". Check whether the events are independent or not

Sol^n: A: Head on the first coin.

B: " " " second "

$S = \{HH, HT, TH, TT\}$ $\quad P(A) = \frac{2}{4} = 1/2$

$A = \{HH, HT\}$ $\quad P(B) = \frac{2}{4} = 1/2$

$P(A \cap B) = 1/4$

$B = \{HH, TH\}$

$A \cap B = \{HH\}$

$P(A \cap B) = 1/4$

Proven that it is: $P(A) \cdot P(B) = \frac{1}{2} \cdot \frac{1}{2} = 1/4$

Problem #2: There are 3 coins tossed.

A: Head on the 1st coin.
B: Tails on the last 2 coins.
A & B are independent

$S = \{HHH, HTH, HHT, HTT, THH, TTH, THT, TTT\}$

$A = \{HHH, HTH, HHT, HTT\}$ $\quad P(A) = 4/8 = 1/2$

$P(B) = 2/8 = 1/4$

$B = \{HTT, TTT\}$ $\quad P(A \cap B) = 1/8$

$A \cap B = \{HTT\}$

Now, $P(A) * P(B) = \frac{1}{2} \cdot \frac{1}{4} = 1/8 = P(A \cap B)$

So, A and B are independent events.

oblem #3:

Suppose we throw 2 fair dice. Let E_1 denotes the event that the sum of the dice is 6. And F denotes the event that the first dice equals 4.

oln: $P(E_1 \cap F) = 1/36$ $E_1 \cap F = \{(4,2)\}$

$P(E_1) = 5/36$ $E_1 = \{(1,5),(2,4),(4,2),(3,3),(5,1)\}$

$P(F) = 6/36 = 1/6$ $F = \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\}$

$P(E_1) * P(F) = \frac{5}{36} \times \frac{1}{6} \neq \frac{1}{36} = P(E_1 \cap F)$

$\therefore P(E_1) * P(F) \neq P(E_1 \cap F)$

As it's conditional, so not independent.

Modified:

$\rightarrow E_2 = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

$(E_2 \cap F) = \{(4,3)\}$

uld 7.

$P(E_2) = 1/6$

$P(F) = 1/6$

$P(E_2 \cap F) = 1/36$

$P(E_2) \cdot P(F) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} = P(E_2 \cap F)$

t be

em is

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$P(E_2|F) = \frac{P(E_2 \cap F)}{P(F)} = \frac{1/36}{1/6} = 1/6 = P(E_2)$

6

e if

than

qual

So, independent. Unconditional.

(Dunno why tho?)

E_3: (sum = 8) $\qquad P(E_3 | F) \overset{\text{check!}}{=} P(E_3)$

$E_2 = \{(2,6),(6,2),(3,5),(5,3),(4,4)\}$ 5/36

~~$F = \{(4,4)\}$ 1/36~~

$E_2 \cap F = \{(4,4)\} = 1/36$

$F = \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\} = 6/36 = 1/6$

So, sum = 7 is just a special case.

Problem #4: A fire brigade has 2 fire engines operating independently. The probability that a specific fire engine is available when required is 0.99, ~~Find~~ then

(a) what is the probability that an engine is available when nee

(b) " " " " " neither " " " "

Soln: A: The event that the 1^{st} engine is available

B: " " " " second " " "

$P(A) = P(B) = 0.99$

$P(A \cap B) = P(A) \cdot P(B) = (0.99)^2 = 0.9801$

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= 0.99 + 0.99 - 0.9801$
$= 0.9999$

Probability of getting one from 2 is more obvl

(b) $P(A \cup B)^c = 1 - P(A \cup B) = 1 - 0.9999 = 0.0001$

Independence of more than two events:

The multiplication rule for independent events extends very simply to three of more events.

$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$$

$$P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) = P(A_1) \cdot P(A_2) \cdot \ldots P(A_n)$$

absolute independence

pair-wise ": pair-wise does not mean its going to be absolute.

Problem #5 Two coins are tossed. If A is the event "head on the first coin", B is the event "head on the second coin", and C is the event "coins fall alike", then show that the events A, B, C are pair-wise independent but not completely / absolutely ?

Solution: $S = \{HH, HT, TH, TT\}$

$A = \{HH, HT\}$; $A \cap B = \{H, H\}$

$B = \{HH, TH\}$; $B \cap C = \{H, H\}$

$C = \{HH, TT\}$; $C \cap A = \{H, H\}$

$\cancel{P}\ A \cap B \cap C = \{HH\}$

$P(A) = P(B) = P(C) = 2/4 = 1/2$

$P(A \cap B) = P(A \cap C) = P(B \cap C) = 1/4$

$$\left.\begin{aligned} P(A) \cdot P(B) &= \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4} = P(A \cap B) \\ P(B) \cdot P(C) &= \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4} = P(B \cap C) \\ P(A) \cdot P(C) &= \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4} = P(A \cap C) \end{aligned}\right\} \text{Pair-wise independent}$$

Now, $P(A) \cdot P(B) \cdot P(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \cancel{4}\ \frac{1}{8}$

Book eamples, exercises for practice.

$$P(A \cap B \cap C) = \frac{1}{4} \neq P(A) \cdot P(B) \cdot P(C)$$

So, not absolutely independent.

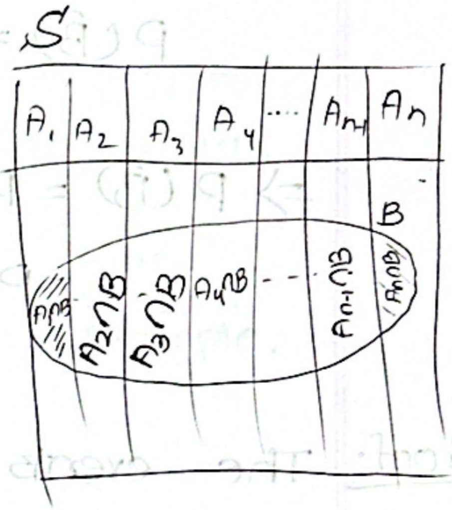
Conditional Probability and Partitions:

Let S denotes the sample space of some random experiment and consider n events $A_1, A_2, A_3, \ldots, A_n$ such that they are mutually exclusive and collectively exhaustive, that is

$$A_i \cap A_j = \emptyset \text{ for all } i \neq j \text{ and}$$

$$S = A_1 \cancel{A} \cup A_2 \cup A_3 \cup \ldots \cup A_n$$

Then it is said that these events form a partition of S.



B is a common event. and it falls over all A_n events. Hence A has A_n ~~partio~~ Partitions and B has some pieces of A.

$$S = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$$

Total Probability Rule:

Suppose that the events $A_1, A_2, A_3, \ldots A_n$ mat partitioning of the sample space, S and $P(A_j) > 0$ for $j = 1, 2, \ldots, n$. Then for any event $B \in S$,

$$P(B) = \sum_{j=1}^{n} P(A_j) \cdot P(B \mid A_j)$$

$$\Rightarrow P(B) = P(A_1) \cdot P(B \mid A_1) + P(A_2) \cdot P(B \mid A_2) + P(A_3) \cdot P(B \mid A_3) + \cdots + P(A_n) \cdot P(B \mid A$$

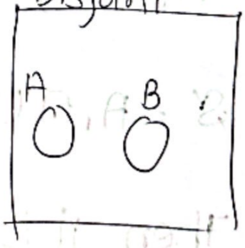
Proof: The events $(A_1 \cap B)$, $(A_2 \cap B)$, $(A_3 \cap B)$, $\ldots$, $(A_n \cap B$ form a partition of the event B:

Hence,

$$B = (A_1 \cap B) \cup (A_2 \cap B) \cup (A_3 \cap B) \cup \cdots \cup (A_n \cap B) \quad \text{— (1)}$$

Since, there n events on the RHS of (1) are mutually exclusive or disjoint then

Disjoint



$$P(A \cup B) = P(A) + P(B)$$

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_n \cap B)$$

$$\Rightarrow P(B) = \sum_{j=1}^{n} P(A_j \cap B)$$

Using the multiplication rule we have

$$P(A_j \cap B) = P(A_j) \cdot P(B \mid A_j)$$

Here,

$$P(B) = \sum_{j=1}^{n} (A_j) \cdot P(B \mid A_j)$$

1·13, 1·14, 1·15, 1·16

Section: ~~5·1~~, ~~5·4~~, 1·4, 1·5, 1·6

Bias-theorem example.

permutation combination

$\boxed{\text{counting Principle}}$