MAT102 Course Notes Student Solutions

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Author's Notes

"Great discoveries and improvements invariably involve the cooperation of many minds" - Alexander Graham Bell

If this is your first time (or nth time) taking MAT102, I hope you find the answers provided in this book useful in your studying. Unfortunately, I found that this course' text was terrible. It gives limited information on the discrete mathematics topics it covers and may discourage many students from entering the field of mathematics. I hope you use this resource wisely.

I should also recognize my own human error, parts of this text may be wrong or misleading, if they are I apologize and will try to update this if I can. In the mean time, I applaud you as a student for finding any errors if you happen to.

Before you delve into the contents of this pdf, I will leave you with one piece of advice that many students fail to grasp. You'll notice the proofs look ***backwards****. But, this is not the case; proofs are written in this form; you must start from something you know and* ***build*** *towards what you want to show.*

MANIFESTO - Solutions For All

I personally believe that solutions make most mathematics more accessible. It is a widely debated topic whether there should be any solutions at all to questions in math textbooks. For those who are against it, simply close your eyes and the solutions will vanish, or don't check the solutions manual. Arguments I have seen against this rhetoric are that in math at some point there may not be a solution to a problem you are facing. Having no solutions at all will 'train' the student to tackle problems where no solutions are known. But at this level, it is a ridiculous claim to make, most people who enter this course have never seen a proof in their entire life. On top of that, there is no need to be training students for these types of questions at such a basic and low level of mathematics. I will however bow to the argument that problems in this textbook are used for problem sets, which is fine, however I don't know which ones are and which aren't therefore I have done every single question for the majority that won't show up. The author could have solved the ones that aren't on problem sets and left the ones they will use up to the student (Or just make new questions? In discrete math that is far from difficult).

How can anyone become better at math if they don't see different perspectives or where they may have failed?

To those who have struggled with this course, I leave you with this.

Contents

1 Chapter 1 Exercises Solutions

1. (a) Suppose x_1, x_2 are two solutions of a quadratic equation $ax^2+bx+c = 0$ where $a \neq 0$ it follows that

$$\begin{aligned} x_1 + x_2 &= \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a} \\ &= \frac{-2b}{2a} \\ &= \frac{-b}{a} \end{aligned}$$

Additionally

$$\begin{aligned} x_1 \cdot x_2 &= \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right) \\ &= \frac{\left(-b+\sqrt{b^2-4ac}\right)\left(-b-\sqrt{b^2-4ac}\right)}{4a^2} \\ &= \frac{4ac}{4a^2} \\ &= \frac{c}{a} \end{aligned}$$

Lastly

$$\begin{aligned} x_1^2 + x_2^2 &= (x_1+x_2)^2 - 2x_1x_2 \\ &= \left(\frac{-b}{a}\right)^2 - 2\left(\frac{c}{a}\right) \\ &= \frac{b^2-2ca}{a^2} \end{aligned}$$

(b) Given that the sum of the solutions is 47 and their product is -59.

$$x_1 + x_2 = \frac{-b}{a} = 47 \wedge x_1 \cdot x_2 = \frac{c}{a} = -59 \implies x^2 - 47x - 59$$

Side Note: The symbol $\wedge$ means 'and' in logic. You will unfortunately learn this later in the course depending on who's teaching it.

2. If the dimensions of a rectangle have a perimeter of 12 and a diagonal length of $\sqrt{20}$ then it follows that

$$2x + 2y = 12 \implies x + y = 6 \implies x = 6 - y$$
$$\sqrt{x^2+y^2} = \sqrt{20} \implies \sqrt{(y-6)^2+y^2} = \sqrt{20}$$

Solving for y we get

$$\begin{aligned} y_1 &= 4, y_2 = 2 \\ \implies x_1 &= 2, x_2 = 4 \end{aligned}$$

3. (a) There are no real solutions to $x^2 + x + 1 = 0$ since $b^2 - 4ac < 0$

   (b) Substituting $x = -1 - \frac{1}{x}$ is the issue, we've created a new equation and furthermore added a new solution. Consider the example of $x = x^2$ then $x = 0, 1$, but squaring both sides gives $x = x^4$ by $x = x^2 = x^4$ however we achieve additional roots doing this. We cannot conclude that the solutions of this new equation are the same ones as the solutions of the previous one! Not every solution of $x^2 - \frac{1}{x} = 0$ is a solution of $x^2 + x + 1 = 0$.

4. **Positive numbers** b and perimeter $= 2x+2y$ also, area $= x \cdot y$. It follows that:

$$\begin{aligned} x + y = b \wedge x \cdot y = \frac{b}{2} &\implies y = b - x \\ &\implies x(b - x) = \frac{b}{2} \\ &\implies -2x^2 + 2bx - b = 0 \\ &\implies x = \frac{b \pm \sqrt{b(b-2)}}{2} \end{aligned}$$

   thus $b(b-2) \geq 0$ so $b \geq 2$ since b must be positive as needed.

5. (a) True. See that $b^2 - 4ac > 0 \implies 1 - 4(-c^2) > 0 \implies 1 + 4c^2 > c^2 + 1 \geq 1 > 0$ so clearly it has two distinct real roots.

   (b) True. See that $b^2 - 4ac > 0 \implies b^2 > 4ac \implies 16 > 4a \implies 4 > a$ thus it has a root.

   (c) False. It has at least one solution, not most.

6. **Note:** remember that $x^2 = a \implies x = \pm\sqrt{a}$

   (a) We begin by solving the inequality, we must consider the following two cases.
   Case 1: $x > 0$

$$\begin{aligned} \frac{2}{x} > 3x &\implies 2 > 3x^2 \\ &\implies \frac{2}{3} > x^2 \\ &\implies x < \pm\sqrt{\frac{2}{3}} \\ &\implies 0 < x < \sqrt{\frac{2}{3}} \end{aligned}$$

Case 2: $x < 0$

$$\begin{aligned}\frac{2}{x} > 3x &\implies 2 < 3x^2 \\ &\implies \frac{2}{3} < x^2 \\ &\implies \pm\sqrt{\frac{2}{3}} > x \\ &\implies -\sqrt{\frac{2}{3}} > x\end{aligned}$$

Note: When dividing OR multiplying inequalities by negatives, the direction of the sign changes
Note: Since x is negative in the second case, we flip the sign again when we square root.

(b) We begin by solving the equation

$$x^3 = x$$

Subtracting x from both sides

$$x^3 - x = 0$$

Now, we will factor out an x, and solve for the solutions

$$x(x^2 - 1) = 0 \implies x = 0 \text{ and } x^2 - 1 = 0$$

Solving each solution:

$$x^2 - 1 = 0 \implies x^2 = 1 \implies x = \pm\sqrt{1} \implies x = \pm 1 \vee x = 0$$

Side Note: The symbol $\vee$ means 'or' in logic, you will unfortunately learn this later in this course.

7. Suppose $0 < a \leq b$, it follows that...

$$\begin{aligned}a \leq b &\implies a \cdot a \leq b \cdot a \wedge a \cdot b \leq b \cdot b \\ &\quad \text{multiplying both sides by } a \text{ and then by } b \\ &\implies a \cdot a \leq b \cdot a \leq b \cdot b \implies a^2 \leq b^2\end{aligned}$$

Similarly,

$$a \leq b \implies 0 \leq b - a \implies 0 \leq (\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) \implies \sqrt{a} \leq \sqrt{b}$$

8. Take the case of $a = -2$ and $b = -1$ then clearly $-2 < -1$ but $(-2)^2 < (-1)^2 \implies 4 < 1$ which is obviously false.

9. (a) Suppose $a + b \leq \frac{1}{2}$ where $a, b \in \mathbb{Z}^+$ then

$$\begin{aligned}
\frac{1-a}{a} \cdot \frac{1-b}{b} \geq 1 &\implies (1-a)(1-b) \geq ab \\
&\implies 1 - a - b + ab \geq ab \\
&\implies 1 - a - b \geq 0 \\
&\implies -(a+b) \geq -1 \\
&\implies a + b \leq 1
\end{aligned}$$

Which is clearly true as $a + b \leq \frac{1}{2} < 1$

(b) Suppose $\frac{1-a}{a} \cdot \frac{1-b}{b} \geq 1$ where $a, b \in \mathbb{Z}^+$ then it follows from part a) that $(1-a)(1-b) \geq ab \implies a + b \leq 1$. Take the case where $a = \frac{1}{3} = b$ then clearly $\frac{1}{3} + \frac{1}{3} = \frac{2}{3} \leq 1$ but $\frac{2}{3} > \frac{1}{2}$, so it is false.

Note: $\mathbb{Z}^+$ is all positive integers, which includes 0

10. Suppose $a, b > 0$, from the AGM inequality it follows that

$$\begin{aligned}
\sqrt{ab} \leq \frac{a+b}{2} &\implies \frac{ab}{\sqrt{ab}} \leq \frac{a+b}{2} \\
&\implies 2ab \leq \sqrt{ab}(a+b) \\
&\implies \frac{2ab}{a+b} \leq \sqrt{ab} \\
&\implies \frac{2}{\frac{b+a}{ab}} \leq \sqrt{ab} \\
&\implies \frac{2}{\frac{1}{a} + \frac{1}{b}} \leq \sqrt{ab}
\end{aligned}$$

as needed.

11. The AGM inequality can give us the maximum of two numbers. Using the AGM inequality we can cleverly plug it in.
Note: $(5 + \sqrt{x^4+1}) + (9 - \sqrt{x^4+1}) = 14$
AGM Inequality:

$$\sqrt{ab} \leq \frac{a+b}{2}$$

Let $a = 5 + \sqrt{x^4+1}$ and $b = 9 - \sqrt{x^4+1}$ then...

$$\begin{aligned}
\sqrt{(5 + \sqrt{x^4+1})(9 - \sqrt{x^4+1})} &\leq \frac{(5 + \sqrt{x^4+1}) + (9 - \sqrt{x^4+1})}{2} \\
\sqrt{(5 + \sqrt{x^4+1})(9 - \sqrt{x^4+1})} &\leq \frac{14}{2} \\
(5 + \sqrt{x^4+1})(9 - \sqrt{x^4+1}) &\leq \left(\frac{14}{2}\right)^2 \\
(5 + \sqrt{x^4+1})(9 - \sqrt{x^4+1}) &\leq 49
\end{aligned}$$

The maximum is 49.

12. Suppose that $x, y, u, v \in \mathbb{R}$

    (a) **Note:** Use the basic fact which states that $a^2 \geq 0$

    $$\begin{aligned} a = xy - uv &\implies (xy-uv)^2 \geq 0 \\ &\implies x^2y^2 - 2xyuv + u^2v^2 \geq 0 \\ &\implies x^2y^2 + u^2v^2 \geq 2xyuv \\ &\implies 2(x^2y^2 + u^2v^2) \geq 2(2xyuv) \\ &\implies 2x^2y^2 + 2u^2v^2 \geq 4xyuv \end{aligned}$$

    (b) **Note:** Use the basic fact which states that $a^2 \geq 0$

    $$\begin{aligned} a = xv - yu &\implies 0 \leq (xv - yu)^2 \\ &\implies 0 \leq v^2x^2 - 2xuvy + y^2u^2 \\ &\text{(add } x^2u^2 + v^2y^2 \text{ to both sides)} \\ &\implies x^2u^2 + 2xuvy + v^2y^2 \leq x^2u^2 + v^2x^2 + y^2u^2 + v^2y^2 \\ &\implies (xu + yv)^2 \leq (x^2 + y^2)(u^2 + v^2) \end{aligned}$$

13. Suppose $x, y \in \mathbb{R}$ where $x \neq 0$ it follows from basic facts $(a^2 \geq 0)$

    $$\begin{aligned} a = x^2 - y &\implies 0 \leq (x^2 - y)^2 \\ &\implies 0 \leq x^4 - 2x^2y + y^2 \\ &\implies 2x^2y \leq x^4 + y^2 \end{aligned}$$

    (we can divide by x^2 without changing the inequality since as $x^2 \geq 0$ and $x \neq 0$)

    $$\implies 2y \leq \frac{y^2}{x^2} + x^2$$

    as needed.

14. Suppose we have two real numbers x, y then using the basic fact $(a^2 \geq 0)$ we see that

    $$\begin{aligned} a = (2x - 3y)^2 &\implies (2x - 3y)^2 \geq 0 \\ &\implies 4x^2 - 12xy + 9y^2 \geq 0 \\ &\implies 4x^2 + 9y^2 \geq 12xy \\ &\implies \frac{4x^2 + 9y^2}{6} \geq 2xy \\ &\implies \frac{2}{3} \cdot x^2 + \frac{3}{2} \cdot y^2 \end{aligned}$$

    as needed.

15. Suppose $x > y > z$ then $x > y$ and $y > z$ we want to show that $xy + yz > \frac{(x+y)(y+z)}{2}$, notice that $y - z > 0$ thus $x(y - z) > y(y - z)$ so it follows that $xy - xz > y^2 - yz$. We can add $xy + xz + 2yz$ on both sides to get

$$\begin{aligned}(xy - xz) + (xy + xz + 2yz) > y^2 - yz + xy + xz + 2yz \\ \implies 2xy + 2yz > y^2 + yz + xy + xz \\ \implies xy + yz > \frac{(x+y)(y+z)}{2}\end{aligned}$$

as needed.

16. Suppose x, y, z are nonnegative real numbers such that $x + z \leq 2$, consider that $xz \geq xz - 2y + xy - y^2 + zy$ since replacing $x + z$ with 2 gives $xz \geq xz - y^2$ where $-y^2$ is clearly less than or equal to 0. If $x + z < 2$ then still $2y > y(x + z)$ producing $-2y + y(x + z) - y^2 < 0$. Now, using the AGM inequality with we achieve

$$\frac{(x+z)^2}{4} \geq xz \geq xz - 2y + xy - y^2 + zy$$

where multiplying both sides by 4 and manipulating each element gives

$$x^2 + 2xz + z^2 \geq 4(xz - 2y + xy - y^2 + zy)$$

thus

$$(x - 2y + z)^2 \geq 4xz - 8y$$

Equality occurs in the inequality when $(x-2y+z)^2 = 4xz-8y$ so replacing $x+z$ with 2 gives $xz = y^2+1$ or if $x+z < 2$ we have $xz < y^2+1$. [Uncertain of this proof tbh]

17. Suppose a, b are real numbers and $\epsilon > 0$ then using the basic fact notice that $(a + \epsilon b)^2 \geq 0$ thus

$$\begin{aligned}(a + \epsilon b)(a + \epsilon b) \geq 0 &\implies a^2 - 2a\epsilon b + \epsilon^2 b^2 \geq 0 \\ &\implies a^2 + b^2\epsilon^2 \geq 2a\epsilon b \\ &\implies \frac{a^2 + b^2\epsilon^2}{2\epsilon} \geq ab \\ &\implies \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}\end{aligned}$$

as needed.

18. Suppose a, b are positive real numbers then we can equivalently show that

$$\frac{2a^2 - ab + 2b^2}{2\left(2a^2 + 5ab + 2b^2\right)} \geq 0$$

notice that the denominator is greater than 0 and $2a^2 - ab + 2b^2 = 2(a - b)^2 + 3ab > 0$ thus the entire fraction is greater than or equal to zero so our original inequality holds.

19. (a) Rewrite $\sqrt[4]{xyzw}$ as $\sqrt{\sqrt{xy}\sqrt{zw}}$ then applying the AGM inequality to $\sqrt{xy}$ and $\sqrt{zw}$ we achieve

$$\frac{\sqrt{xy}+\sqrt{zw}}{2} \geq \sqrt{\sqrt{xy}\sqrt{zw}}$$

on the left hand side we can apply AGM again to $\sqrt{xy}$ and $\sqrt{zw}$ to achieve

$$\frac{\frac{x+y}{2}+\frac{z+w}{2}}{2} \geq \frac{\sqrt{xy}+\sqrt{zw}}{2}$$

where the left hand side is equivalent to $\frac{x+w+z+w}{4}$ thus we have the AGM inequality for 4 variables.

(b) Using part a) let $w = \sqrt[3]{xyz}$ then

$$\frac{x+y+z+\sqrt[3]{xyz}}{4} \geq \sqrt[4]{xyz\sqrt[3]{xyz}}$$

notice that the right hand side is equivalent to

$$\sqrt[4]{xyz}\,\sqrt[12]{xyz} = \sqrt[3]{xyz}$$

thus

$$x+y+z+\sqrt[3]{xyz} \geq 4\sqrt[3]{xyz} \implies \frac{x+y+z}{3} \geq \sqrt[3]{xyz}$$

as needed.

20. (I) We consider cases, if $x \geq 0$ then $|x| = x$ and $\sqrt{x^2} = x \geq 0$ thus they are equal. If $x < 0$ then $|x| = -x$ and we have $\sqrt{x^2} = -x$ thus $\sqrt{x^2} = |x|$.
(II) We consider cases, if $x \geq 0$ then $|x| = x$ and $|x|^2 = x^2$, if $x < 0$ then $|x| = -x$ and $|x|^2 = (-x)^2 = x^2$ thus they are equal.
(III) We consider cases, if $x \geq 0$ then $|x| = x$ and clearly since $x = x$ then $x \leq x$ (Note that we have shown one of $<$ or $=$ which is enough to conclude $\leq$). Now, if $x < 0$ then $|x| = -x$ and since $-x > 0$ and $0 > x$ then $-x \geq x$ as a result. We conclude that $x \leq |x|$ as needed.
(IV) We consider cases, if $x, y \geq 0$ then $|x| = x$ and $|y| = y$ and $x \cdot y \geq 0$ thus $|x \cdot y| = x \cdot y = |x| \cdot |y|$. If $x \geq 0$ and $y < 0$ then $|x| = x$ and $|y| = -y$ thus $x \cdot y < 0$ and $|x \cdot y| = -(x \cdot y) = x \cdot (-y) = |x| \cdot |y|$. If $x < 0$ and $y \geq 0$ we arrive at a similar result since $|x| = -x$ and $|y| = y$ then $|x \cdot y| = -(x \cdot y) = (-x) \cdot y = |x| \cdot |y|$. Finally, if $x, y < 0$ then $|x| = -x, |y| = -y$ so $x \cdot y \geq 0$ thus $|x \cdot y| = x \cdot y = (-x)(-y) = |x| \cdot |y|$.

21. The triangle inequality has equality if and only if $x, y \geq 0$ or $x, y < 0$. To see why this is, use properties from the previous exercise and chapter and set $|x+y| = |x| + |y|$. Squaring both sides gives $x^2 + 2xy + y^2 = x^2 + 2|x||y| + y^2$ which creates $xy = |x||y|$. Clearly $|x||y| = |xy|$ so $xy = |xy|$ which can only occur when $x, y \geq 0$ or $x, y < 0$.

22. Consider that $||x| - |y|| \leq ||x|| - ||y|| = |x| - |y|$, all that is left to prove is that $|x| - |y| \leq |x - y|$. Using the fact that $x = (x - y) + y$ we see that $|x| = |(x - y) + y| \leq |x - y| + |y|$ which shows $|x| - |y| \leq |x - y|$ thus $||x|| - ||y|| \leq |x - y|$ as needed.

23. Suppose $x, y \geq 0$ and consider when $x \geq y$ then $x - y \geq 0$ thus $|x - y| = x - y$, similarly $\sqrt{x} - \sqrt{y} \geq 0$ so $|\sqrt{x} - \sqrt{y}| = \sqrt{x} - \sqrt{y}$. Since $x \geq y$ then $xy \geq y^2$ and we can root both sides. (Note that $y \geq 0$ so $|y| = y$)

$$\sqrt{xy} \geq y \implies \sqrt{x}y \geq y$$

multiplying both sides by -2 and adding x gives

$$x - 2\sqrt{x}y \leq x - 2y \implies x - 2\sqrt{x}y + y \leq x - y$$

the left side turns into $(\sqrt{x} - \sqrt{y})^2$ and the right becomes $|x - y|$, rooting both sides gives

$$(\sqrt{x} - \sqrt{y})^2 \leq |x - y| \implies |\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|}$$

(Note that we use $\sqrt{x^2} = |x|$ here) which is what we wanted to show. In the other case consider $y \geq x$ then multiplying both sides by x gives $yx \geq x^2$. It is good to note that $\sqrt{x} - \sqrt{y} < 0$ thus $|\sqrt{x} - \sqrt{y}| = -(\sqrt{x} - \sqrt{y})$. Moreover, $|x - y| = -(x - y)$ since $x - y \leq 0$. We can root both sides of $yx \geq x^2$ to achieve

$$\sqrt{yx} \geq x \implies \sqrt{y}\sqrt{x} \geq x$$

multiplying both sides by -2 and adding y gives

$$-2\sqrt{x}\sqrt{y} + y \leq -2x + y \implies x - 2\sqrt{x}\sqrt{y} + y \leq y - x$$

Since $|x - y| = y - x$ we can replace it, also factoring the lefthand gives $(\sqrt{y} - \sqrt{x})^2$ so

$$(\sqrt{y} - \sqrt{x})^2 \leq |x - y| \implies |\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|}$$

as needed.

24. Using the triangle inequality we can see that

$$\begin{aligned} |x^3 - 4x^2 + x + 1| &\leq |x^3| + |-4x^2 + x + 1| \\ &\leq |x^3| + |-4x^2| + |x + 1| \\ &\leq |x^3| + |-4x^2| + |x| + 1 \end{aligned}$$

since $1 < x < 3$ we see that $-3 < x < 3$ so $|x| < 3$ thus substituting 3 would create an appropriate bound. Therefore let $M = |(3)^3| + |-4(3)^2| + |3| + 1 = 67$.

25. Suppose $1 < x < 2$ then $|x^3+x^2-1| \leq |x^3|+|x^2-1| \leq |x^3|+|x^2|+|-1|$ also $-2 < x < 2$ so $|x| < 2$ and an upper bound is $|(2)^3|+|(2)^2|+1 = 13$ thus $|x^3+x^2-1| < 13$. Also, note that $||x|-|6|| \leq |x-6|$ so a lower bound for this absolute value is $||2|-|6|| = 4$ (by a previous question) thus $4 < |x-6|$. Notice that $x^3+x^2-1 \geq 0$ for $1 < x < 2$ thus $|x^3+x^2-1| = x^3+x^2-1$ and

$$\frac{|x^3+x^2-1|}{|x-6|} < \frac{13}{4} \implies \left|\frac{x^3+x^2-1}{x-6}\right| < \frac{13}{4}$$

choosing $M = \frac{13}{4}$ satisfies the inequality.

26. Suppose a, b, c are three real numbers then by the triangle inequality

$$|a+c| = |(a-b)+(b-c)| \leq |a-b|+|b-c|$$

as needed.

27. (a) **Yes.** Notice that $9 \cdot -13 = -117$.
 (b) **No.** We cannot divide 3 by 9.
 (c) All integers that divide -20 are $\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$
 (d) Suppose k is a nonzero integer, consider that $(k+1)^2-1 = k^2+2k+1-1 = k^2+2k = k(k+2)$ so k can divide the expression since it is a factor (product) of it.

28. The reason that b is a nonzero integer in Definition 1.4.1 is because if $b = 0$ then there cannot exist an integer m such that for some arbitrary a we have $a = m \cdot b$ (other than $a = 0$).

29. (a) **True.** by Definition 1.4.1 we have $0 = 0 \cdot b$ for all $b \neq 0$ with $m = 0$.
 (b) **True.** since 17 is prime, the only integers that divide 17 are itself and 1.
 (c) **False.** let $a = 3$ and $b = -3$ then $a = -1 \cdot b$ and $b = -1 \cdot a$ but $a \neq b$ clearly.
 (d) **True.** Let k be an even number, say $k = 2n$ for some integer n then $k^2+k = (2n)^2+2n = 2(2n^2+n)$ which is divisible by 2. If $k = 2n+1$ then $k^2+k = (2n+1)^2+2n+1 = 4n^2+6n+2 = 2(2n^2+3n+1)$ which is also divisible by 2!

30. (a) The rational solutions to the equation are ± 1 only (since $\pm\sqrt{7}$ are not rational)
 (b) Because it is repeating infinitely. (In fact this number is exactly $\frac{10}{9}$!)
 (c) Consider rationalizing the fraction,

$$\begin{aligned}\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} - 2\sqrt{6} &= \frac{\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}+\sqrt{2}\right)}{\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}+\sqrt{2}\right)} - 2\sqrt{6} \\ &= 5+2\sqrt{6}-2\sqrt{6} \\ &= 5\end{aligned}$$

31. (a) Consider $\sqrt{\frac{4}{16}} = \frac{\sqrt{4}}{\sqrt{16}} = \frac{2}{4}$.

(b) Consider $\sqrt{2} + (-\sqrt{2}) = 0$.

(c) Consider $0 \cdot \sqrt{2} = 0$.

(d) Consider $\frac{\sqrt{2}}{\sqrt{2}} = 1$.

(e) Consider let $a = 1, b = \sqrt{2}$ then $\frac{1}{\sqrt{2}}$ which is not rational.

2 Chapter 2 Exercises Solutions

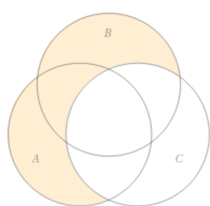
1. Notice that $(2-x)(2-y) < 2(4-x-y)$ when $x \cdot y < 4$ thus since $x, y \in \mathbb{N}$ it follows that only $(1,1), (1,2), (2,1), (1,3), (3,1)$ satisfy the inequality and thus are elements of S so $S = \{(1,1), (1,2), (2,1), (1,3), (3,1)\} = T$ as needed.

2. Notice that $S = \{(2,-2), (2,-1), (2,0), (3,-2), (3,-1), (3,0)\}$ which all satisfy $x + 2y \leq 3$ and $-2 \leq x + 2y$ thus $S \subseteq T$. Consider the point $(0,-1)$ where $-2 \leq -2$ so $(0,-1) \in T$ but $(0,-1) \notin S$ so equality does not hold between these sets.

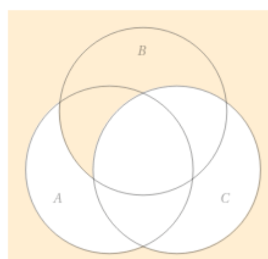
3. $\emptyset \subseteq \mathbb{N}$ since the $\emptyset$ is a subset of every set, $\{\emptyset\} \not\subseteq \mathbb{R}$ since $\mathbb{R}$ is a set containing elements and $\{\emptyset\}$ is a set containing sets it cannot be a subset of $\mathbb{R}$. $\mathbb{Z} \subseteq \mathbb{N} \cup \mathbb{Q}$ is obvious since naturals account for all positive integers and $\mathbb{Q}$ accounts for the negatives so it is a subset. $\mathbb{Z} \not\subseteq \mathbb{N} \cap \mathbb{Q}$ since $-1 \in \mathbb{Z}$ but $-1 \notin \mathbb{N}$.

4. The interval $[a, b]$ is all numbers x such that $a \leq x \leq b$ which obviously $x = a = b$. The interval (a, b) is all numbers x such that $a < x < b$ which empty since $a < b$ contradicts that $a = b$.

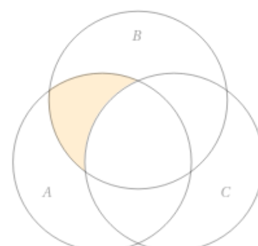
5.



a)

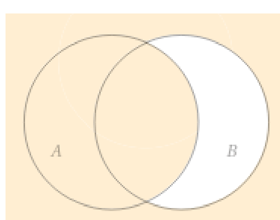


b)

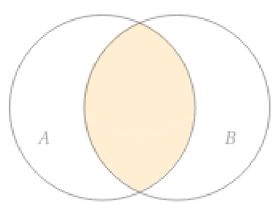


c)

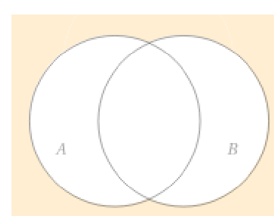
6.



a)



b)



c)

7. (a) It is incorrect because $1.1 \in [1, 7]$ but $1.1 \notin \mathbb{N}$ so it is not in $[0.5, 0.75] \cap \mathbb{N}$

   (b) The cartesian product of two sets is $\{4\} \times \{5\} = \{(4, 5)\} \neq \{20\}$.

(c) Notice that $\{a,b,c\} \cup \emptyset = \{a,b,c\} \cup \{\} = \{a,b,c\}$ since $\{\} \notin \{a,b,c\}$. It is the difference between the union of two sets and a set being an element of a set.

8. $A \cap C = \emptyset$, $A^c \cap B = (-\pi,-1) \cup (1,\pi)$, $(B \cap C) \cap \mathbb{Z} = \{2,3\}$ $B \setminus A = (-\pi,-1) \cup (1,\pi)$.

9. (a) **True.** The first set is $\{1\}$ and the second is $\{0,1\}$.
   (b) **False.** The first set is $\{0, \sqrt{2}, -\sqrt{2}\}$.
   (c) **False.** Clearly $\mathbb{N}$ is not an element of $\mathbb{R}$
   (d) **False.** Since $\mathbb{R} \not\subseteq \mathbb{Z}$ then there are elements in $\mathbb{Z} \times \mathbb{R}$ that are not in $\mathbb{R} \times \mathbb{Z}$.
   (e) **True.** Since $\mathbb{Z} \subseteq \mathbb{Q}$ and $\mathbb{Z} \subseteq \mathbb{R}$ therefore all elements in $\mathbb{N} \times \mathbb{Z}$ will be in $\mathbb{Q} \times \mathbb{R}$.

10. Draw the diagrams yourself.
    (a) True
    (b) False
    (c) False
    (d) True
    (e) False
    (f) False

11. We conclude that $A = B$, we shall also prove this. Suppose $A \setminus B = B \setminus A$ and take $x \in A$ and suppose $x \notin B$ then $x \in A \setminus B$ so $x \in B \setminus A$ but then $x \in B$ and $x \notin A$ which is a contradiction. The other direction is very similar (try it out).

12. **True**, Take $p \in (A \times A) \setminus (B \times B)$ then $p \in A \times A$ and $p \notin B \times B$ then let $p = (x,y)$ so $x, y \in A$ and $x, y \notin B$ thus $x \in A \setminus B$ and $y \in A \setminus B$ so $(x,y) = p \in (A \setminus B) \times (A \setminus B)$. For the other direction, take $p \in (A \setminus B) \times (A \setminus B)$ then if $p = (x,y)$ we have $x \in A \setminus B$ and $y \in A \setminus B$ thus $(x,y) \in A \times A$ and $(x,y) \notin B \times B$ thus $(x,y) = p \in (A \times A) \setminus (B \times B)$ as needed.

13. (a) $C = \{2,4\} \times \{-3,-1,10\} = \{(2,-3),(2,-1),(2,10),(4,-3),(4,-1),(4,10)\}$.
    (b) $B = ([-1,8] \setminus (3,5)) \cap \mathbb{N} = ([-1,3] \cup [5,8]) \cap \mathbb{N} = \{-1,0,1,2,3,5,6,7,8\}$
    (c) $\{x \in \mathbb{R} : 0 < x^2 \leq 25\} = [-5,0) \cup (0,5]$
    (d) $A = (-2,-1) \cup (1,2)$ and $A = (-2,2) \setminus [-1,1]$

14. Let $A = \{\frac{1}{2}\}$ then $A \cap [1,4] = \emptyset = A \cap \mathbb{N}$ and $A \setminus \mathbb{Z} = \{\frac{1}{2}\}$

15. N/A

16. N/A

17. N/A

18. N/A

19. $\{x : a \leq x \leq d\} \setminus \{x : b < x < c\}$

20. Notice that $S = \{x \in \mathbb{R} : -3 < x < 2\}$ so $S = (-3, 2)$ thus $S = U \cap T$.

21. (a) Consider that $x \in (A \cap B)^c$ says $x \notin A \cap B$ or that $x \notin A$ or $x \notin B$ thus $x \in A^c \cup B^c$. For the other direction assume $x \in A^c \cup B^c$ suppose on the contrary that $x \in A \cap B$, but if $x \in A^c$ we arrive at a contradiction and if $x \in B^c$ we do as well thus $x \in (A \cap B)^c$.

    (b) Take $x \in A \setminus (B \cup C)$ then $x \in A$ and $x \notin B \cup C$ or rather that $x \notin B$ and $x \notin C$ but this gives $x \in A$ and $x \notin B$, similarly $x \in A$ and $x \notin C$ so $x \in A \setminus B$ and $x \in A \setminus C$ thus $x \in (A \setminus B) \cap (A \setminus C)$. For the other direction, take $x \in (A \setminus B) \cap (A \setminus C)$ so $x \in A \setminus B$ and $x \in A \setminus C$ thus $x \in A$ and $x \notin B$ and $x \notin C$. By Demorgans law we know that $x \in (B \cup C)^c$ thus $x \in A \setminus (B \cup C)$ as needed.

    (c) Let $x \in A \setminus (B \setminus C)$ then $x \in A$ and $x \notin B \setminus C$ so $x \notin B$ or $x \in C$. If $x \notin B$ then $x \in A \setminus B$ so $x \in (A \setminus B) \cup (A \cap C)$, if $x \in C$ then clearly $x \in A \cap C$ so also $x \in (A \setminus B) \cup (A \cap C)$. For the other direciton suppose $x \in (A \setminus B) \cup (A \cap C)$, if $x \in A \setminus B$ then $x \in A$ and $x \notin B$ so $x \in B^c \cup C$ thus $x \in A \setminus (B \setminus C)$. If $x \in A \cap C$ then $x \in A$ and $x \in C$ so $x \in B^c \cup C$ thus $x \in A \setminus (B \setminus C)$ as needed.

    (d) Take $x \in (A \cap B) \setminus (B \cap C)$ then $x \in A \cap B$ and $x \notin B \cap C$ thus $x \notin B$ or $x \notin C$ and $x \in A$, $x \in B$. Clearly since $x \in B$ then it must be the case that $x \notin C$ so $x \in B \setminus C$ thus $x \in A \cap (B \setminus C)$. For the other direction, suppose $x \in A \cap (B \setminus C)$ then $x \in A$ and $x \in B$ but $x \notin C$. It follows then that $x \in C^c$ so $x \in B^c \cup C^c$ thus $x \notin (B \cap C)$ and $x \in A \cap B$ thus $x \in (A \cap B) \setminus (B \cap C)$.

22. Suppose $A \setminus B \subseteq C$ and let $x \in A \setminus C$ then $x \in A$ and $x \notin C$. Assume by contradiction that $x \notin B$ then $x \in B^c$ thus $x \in A \setminus B$ where $x \in C$ as a result. But, this contradicts that $x \notin C$ so it must be that $x \in B$.

23. Suppose $(A \cup B)^c = A^c \cup B^c$ and take $x \in A$, assume that $x \notin B$ by contradiction then $x \in A^c \cup B^c$ also $x \in A \cup B$ by our assumption. Following our first assumption we see that $x \notin (A \cup B)^c$ but this is impossible so $x \in B$ must hold. For the other direction if $x \in B$ and $x \notin A$ by contradiction then $x \in A \cup B$ and $x \in A^c$ so $x \in A^c \cup B^c$ thus $x \in (A \cup B)^c$ another clearly contradiction thus $x \in A$ must be true aswell. We conclude that $A = B$ as a result.

24. Given $A \cup B \subseteq C \cup D, A \cap B = \emptyset$ and $C \subseteq A$ suppose $x \in B$ and assume the contrary, that $x \notin D$ then $x \in A \cup B$ so $x \in C \cup D$. If $x \notin D$ then it must only hold that $x \in C$ but then $x \in A$ and so $x \in A \cap B$ which is impossible since $A \cap B = \emptyset$. Thus $B \subseteq D$ as needed.

25. (a) $\operatorname{Im}(f) = [0, \infty]$
    (b) $\operatorname{Im}(r) = [\frac{1}{2}, 0)$
    (c) $\operatorname{Im}(g) = \{-1, 1\}$
    (d) $\operatorname{Im}(h) = \mathbb{Z}$

26. (a) $f(3,5) = 3 - 5 = -2$, $f(5,10) = 5 - 10 = -5$.
    (b) $f(13,4) = 13 - 4 = 9$ and $f(10,1) = 10 - 1 = 9$.
    (c) $\operatorname{Im}(f) = \mathbb{Z}$

27. $\operatorname{Im}(f) = \mathbb{Q}$ since a ranges across $\mathbb{Z}$ we achieve any numerator (and any negative fraction) moreover since b ranges $\mathbb{N}$ we avoid dividing by 0.

28. $\operatorname{Im}(f) = \mathbb{Z} \cup \{x \in \mathbb{Z} : \frac{x}{2}\}$ since if both a, b are even we can capture any integer, and any 0.5 points between them such that $\frac{1}{2}, \frac{3}{2} \ldots$ etc...

29. (a) **True.** Suppose f, g are bounded then there exists two positive numbers M, N such that $|f(x)| \leq M$ and $|g(x)| \leq N$ so by the triangle inequality $|f(x) + g(x)| \leq |f(x)| + |g(x)| = M + N$ thus $f + g$ is bounded by $M + N$.
    (b) **True.** Suppose f, g are bounded then there exists two positive numbers M, N such that $|f(x)| \leq M$ and $|g(x)| \leq N$ thus $|f(x)|^2 \leq M^2$ and $|g(x)|^2 \leq N^2$ thus $|f^2 - g^2| \leq |f^2| + |g^2| = M^2 + N^2$ so $f^2 - g^2$ is bounded by $M^2 + N^2$.
    (c) **False.** Let $f(x) = x$ and $g(x) = -x$ then clearly $f(x) + g(x) = x - x = 0$ is bounded but $f(x) - g(x) = 2x$ which is not bounded.
    (d) **True.** Suppose f, g are bounded then there exists two positive numbers M, N such that $|f(x)| \leq M$ and $|g(x)| \leq N$ then $|f(x) \cdot g(x)| = |f(x)||g(x)| \leq M \cdot N$ thus $f \cdot g$ is bounded by $M \cdot N$.
    (e) **False.** Let $f(x) = x$ and $g(x) = \frac{1}{x}$ then $|f(x) \cdot g(x)| = 1$ which is bounded but f, g are not bounded.
    (f) **True.** Suppose $M \geq ||f(x)| + |g(x)|| \geq |f(x)| + |g(x)|$ for some positive M then clearly $|f(x)| \leq M$ and $|g(x)| \leq M$ so both f, g are bounded.

30. **Yes.** notice that $|x|+5 \leq |x|+5$ for all x thus $\frac{1}{|x|+5} \leq \frac{1}{|x|+5}$, also note that $|x+5| \leq |x| + 5$ so multiplying both inequalities gives $\frac{|x+5|}{|x|+5} \leq \frac{|x|+5}{|x|+5} = 1$.

31. (a) $A \cup B = (-3, 4]$
    (b) $A^c \cap B = (-3, 1)$
    (c) $A \setminus B = [3, 4]$
    (d) $D \cap \mathbb{Z} = 1, 5$
    (e) $D \setminus \mathbb{N} = \{\frac{3}{7}, \frac{11}{2}\}$
    (f) $g(D) = \{3, 11, \frac{13}{7}, 12\}$

(g) $g(B \cap D) = \{3, \frac{13}{7}\}$

(h) $g(A) \cap D = [3, 9] \cap D = \{5, \frac{3}{7}, \frac{11}{2}\}$

32. (a) $A \cap B = [1, 2)$

(b) $A \cup B = (-1, 3]$

(c) $A \setminus B = (-1, 1)$

(d) $(A \cap \mathbb{Z}) \times (B \cap \mathbb{Z}) = \{0, 1\} \times \{1, 2, 3\} = \{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3)\}$

(e) $A \cap B \cap \mathbb{Z}^c = [1, 2) \cap \mathbb{Z}^c = (1, 2)$

(f) $f(A) = [0, 2)$

(g) $f(B \cap \mathbb{Z}) = f(\{1, 2, 3\}) = \{1, 2, 3\}$

(h) $f(A \cup B) \cap \mathbb{Z} = [0, 3] \cap \mathbb{Z} = \{0, 1, 2, 3\}$

33. Our goal is to show that $f(\mathbb{R}) = [0, 1)$ thus we show double subset inclusion. Take $y \in f(\mathbb{R})$ then $y = \frac{x^2}{1+x^2}$ for some $x \in \mathbb{R}$ thus $x^2(y-1) = y$ and $x^2 = -\frac{y}{y-1}$. Since $x^2 \geq 0$ and $y \neq 1$ then it must hold that $0 \leq y < 1$ and $y \in [0, 1)$. For the other direction assume $y \in [0, 1)$ then $0 \leq y < 1$, we can construct $x = \sqrt{-\frac{y}{y-1}}$ then $x^2 = -\frac{y}{y-1}$ and $x^2(y-1) = y$ thus $y = \frac{x^2}{1+x^2}$ so $y \in f(\mathbb{R})$.

34. The image of f is $f((0, \infty)) = (-\infty, 4) \cup (4, \infty)$. We prove this using double subset inclusion. Take $y \in f((0, \infty))$ then $y = \frac{4x}{x+1}$ assuming $x \neq -1$ thus $y(x+1) = 4x$ and $x = -\frac{y}{y-4}$ thus $y \neq 4$ and $y \in (-\infty, 4) \cup (4, \infty)$. For the other direction take $y \in (-\infty, 4) \cup (4, \infty)$ then we can construct a $x = -\frac{y}{y-4}$ where $x \in (0, \infty)$ resulting in $y(x+1) = 4x$ and thus $y = \frac{4x}{x+1}$ for some $x \in (0, \infty)$. Thus $y \in f((0, \infty))$ as needed.

35. Our goal is to show that $f((0, \infty)) = [2, \infty)$ thus we show double subset inclusion. Take $y \in f((0, \infty))$ then $y = x + \frac{1}{x}$ for some $x \in (0, \infty)$, manipulating this gives $yx = x^2 + 1$ in which $0 = x^2 - yx + 1$ thus $x = \frac{y \pm \sqrt{y^2-4}}{2}$ so $y^2 - 4 \geq 0$ and thus $y \geq 2$ or $y \leq -2$, however, since $x > 0$ then we can only have $y \geq 0$ thus $y \in [2, \infty)$. For the other direction if $y \in [2, \infty)$ then $y \geq 2$ so $y^2 - 4 \geq 0$ and we can construct $x = \frac{y+\sqrt{y^2-4}}{2}$ where $x > 0$ which translates to $yx = x^2 + 1$ and thus $y = x + \frac{1}{x}$ therefore $y \in f((0, \infty))$ as needed.

36. (a) Suppose $C, D \subseteq A$ and take $y \in f(C \cap D)$ then $y = f(x)$ for some $x \in C \cap D$ thus $x \in C$ and $x \in D$ so $y \in f(C)$ and $y \in f(D)$ therefore $y \in f(C) \cap f(D)$.

(b) Let $f(x) = x^2$ with $A = \mathbb{R} = B$ and $C = \{-1\}$ with $D = \{1\}$ then clearly $f(C \cap D) = f(\emptyset) = \emptyset$ yet $f(C) = \{1\} = f(D)$ so $f(C) \cap f(D) = \{1\}$.

37. Not necessarily, using the above example if $f(x) = x^2$ with $A = \mathbb{R} = B$ and $C = \{-1\}$ with $D = \{1\}$ then clearly $C \cap D = \emptyset$ yet $f(C) = \{1\} = f(D)$ so $f(C) \cap f(D) = \{1\}$

38. (a) Suppose $C, D \subseteq A$ and let $y \in f(C) \setminus f(D)$ then $y \in f(C)$ but $y \notin f(D)$ so $y = f(x)$ for some $x \in C$ but $x \notin D$ thus $x \in C \setminus D$ and $y \in f(C \setminus D)$ as a result.
    (b) Let $f(x) = x^2$ with $A = \mathbb{R} = B$ and $C = \{-1\}$ with $D = \{1\}$ then clearly $f(C) \setminus f(D) = \emptyset$ yet $f(C \setminus D) = \{1\}$.

39. (a) **Yes.** Take $y \in f(A \cap B)$ then $y = f(x)$ for some $x \in A \cap B$ thus $x \in A$ and $x \in B$ so $x \in A \cup B$ and $y \in f(A \cup B)$ as a result.
    (b) **No.** Let $A = B = \mathbb{R}$ and $f(x) = x^2$ with $A = \{1\}B = \{2\}$ then $f(A \cup B) = \{1, 4\}$ but $f(A) \cap f(B) = \{1\} \cap \{4\} = \emptyset$.

40. (a) **Yes.** Suppose $C \subseteq D$ and take $y \in f(C)$ then $y = f(x)$ for some $x \in C$ thus $x \in D$ by our assumption and so $y \in f(D)$.
    (b) **No.** Let $f(x) = x^2$ with $X = \mathbb{R} = Y$ and $C = \{-1\}$ with $D = \{1\}$ then clearly $f(C) \subseteq f(D)$ yet $C \not\subseteq D$.

41. **No.** Because $1 \in \mathbb{N}$ yet there is no negative for 1 in $\mathbb{N}$. Also, **No.** for the same reason as before.

42. The set $\mathbb{Q} \cup [-1, 1]$ is not a field since $\frac{1}{\sqrt{2}} \in [-1, 1]$ and $8 \in \mathbb{Q}$ but $8 \cdot \frac{1}{\sqrt{2}} \notin \mathbb{Q}$ and $8 \cdot \frac{1}{\sqrt{2}} \notin [-1, 1]$ so it is not closed under multiplication.

43. Suppose x is a nonzero element of F and $x \cdot r_1 = x \cdot r_2 = 1$ we have that $x \cdot r_1 = x \cdot r_2$. Since $x \in F$ then $x^{-1} \in F$ by the existence of multiplicative inverses thus we can multiply it on both sides

$$x^{-1}(x \cdot r_1) = x^{-1}(x \cdot r_2) \implies (x^{-1} \cdot x) \cdot r_1 = (x^{-1} \cdot x) \cdot r_2$$

by associativity of F. Since $x^{-1} \cdot x = x \cdot x^{-1}$ by commutativity and by multiplicative inverses we have $x^{-1} \cdot x = 1$ thus

$$1 \cdot r_1 = 1 \cdot r_2 \implies r_1 = r_2$$

by the multiplicative identity of $1 \in F$.

44. (a) It is enough to show that $x + (-1) \cdot x = 0$, note that $x = 1 \cdot x$ by the multiplicative identity of 1 so $1 \cdot x + (-1) \cdot x = 0$ which we can use distributive property to see that $x \cdot (1 + (-1)) = 0$ where $1 + (-1) = 0$ thus we have $x \cdot 0 = 0$ Claim 2.3.2. Thus we conclude that $(-1) \cdot x = -x$ as needed.
    (b) Assume $x, y \in F$ and $x \cdot y = 0$, suppose $x \neq 0$ then we can multiply both sides by $x^{-1} \in F$ since it exists by multiplicative inverses of F thus by associativity and the fact that $x^{-1} \cdot x = 1$ we have that $y = 0$ also that $0 \cdot x^{-1} = x^{-1} \cdot 0 = 0$ by Claim 2.3.2 and commutativity of F.

(c) Assume $x, y, z \in F$ and that $x + z = y + z$ we have that $z \in F$ thus $-z \in F$ so we can add $-z$ to both sides $(x+z)+(-z) = (y+z)+(-z)$ which by associativity of F $x+(z+(-z)) = y+(z+(-z))$ and since $z+(-z) = 0$ we have $x + 0 = y + 0$ where $x + 0 = x$ by the additive identity of $0 \in F$. In conclusion we have that $x = y$ as needed.

45. (a) If $x + 1 \neq 0$ then either $x + 1 = 1$ or $x + 1 = x$ but if either are these are true then $x = 0$ or $1 = 0$ which are impossible by field axioms. Notice that then since $x + 0 = x$ then the only possible case for $x + x$ is $x + x = 1$.

(b) Suppose $x \cdot x \neq 1$ then $x \cdot x = 0$ but then $x = 0$ which is a contradiction or we could have $x \cdot x = x$ but then $x = 1$ as a result which is also impossible thus $x \cdot x = 1$.

(c)

+	0	1	x
0	0	1	x
1	1	x	0
x	x	0	1

·	0	1	x
0	0	0	0
1	0	1	x
x	0	x	1

46. (a) Consider if $a^2 \neq b$ then if $a^2 = 0$ then $a = 0$ which is a contradiction, if $a^2 = a$ then $a = 1$ which is also a contradiction so it must be that $a^2 = 1$. However, this is also impossible since $ab = 1$ as ab cannot be equal to $0, a$ or b (try showing why this is true) but this would mean $a^2 = 1 = ab$ so $a = b$, a contradiction, thus $a^2 = b$. Now, if $b^2 \neq a$ then by similar logic we have a contradiction at $b^2 = 0, b^2 = b$ and $b^2 = 1$ since $ab = 1$ thus $b^2 = a$ must be true.

(b) Since $a^2 = b$ then $a^3 = ab = 1$ similarly $b^3 = ab = 1$ as well.

(c) Assume that $1+1 = a$ then $1+a = b$ or $1+a = 0$ we tackle these cases simultaneosly. If $1 + a = b$ then we have $a + a = 1 + b$ so $a^2 = 1 + b$ and thus by b) we know that $a^2 = b$ which is a contradiction since then $1 = 0$. Thus it must be the case that $1+a = 0$ but then $b+1 = b$ which is also a contradiction since then $1 = 0$. Since both cases lead to a contradiction assume $1+1 = b$ instead, then $1+b = a$ or $1+b = 0$. If $1+b = a$ then similarly $1+(1+b) = 1+a$ so $a+b = 1+a$ meaning $b = 1$ which is impossible. Moreover if $1+b = 0$ then $1+a = 1$ which cannot be the case since then $a = 0$. Clearly if $1 + 1 = b$ we lead to a contradiction in every case thus $1 + 1 = 0$.

47. $a + b = 1, -b = a, b^{-1} = a, a \cdot (1 + b) = a + a \cdot b = a + 1 = b$

48. Yes. Let the identity element be $(1, 1)$ and the zero element be $(0, 0)$ then the following axioms hold:
(Closure) Suppose $(a, b) \in \mathbb{R}^2$ and $(c, d) \in \mathbb{R}^2$ then $(a, b) \cdot (c, d) = (a \cdot c, b \cdot d)$ where clearly $ac \in \mathbb{R}$ and $bd \in \mathbb{R}$ so $(a \cdot c, b \cdot d) \in \mathbb{R}^2$. Consider $(a, b) + (c, d) = (a + c, b + d)$ where also $a + c \in \mathbb{R}$ and $b + d \in \mathbb{R}$ thus $(a + c, b + d) \in \mathbb{R}^2$.

(Associativity) Suppose $(a, b) \in \mathbb{R}^2$, $(c, d) \in \mathbb{R}^2$ and $(e, f) \in \mathbb{R}^2$ then

$$\begin{aligned}(a, b) + [(c, d) + (e, f)] &= (a, b) + (c + e, d + f) \\ &= (a + (c + e), b + (d + f)) \\ &= ((a + c) + e, (b + d) + f) \\ &= [(a, b) + (c, d)] + (e, f)\end{aligned}$$

also

$$\begin{aligned}(a, b) \cdot [(c, d) \cdot (e, f)] &= (a, b) \cdot (c \cdot e, d \cdot f) \\ &= (a \cdot (c \cdot e), b \cdot (d \cdot f)) \\ &= ((a \cdot c) \cdot e, (b \cdot d) \cdot f) \\ &= [(a, b) \cdot (c, d)] \cdot (e, f)\end{aligned}$$

(Commutativity) Suppose $(a, b) \in \mathbb{R}^2$, $(c, d) \in \mathbb{R}^2$ then

$$\begin{aligned}(a, b) + (c, d) &= (a + c, b + d) \\ &= (c + a, d + b) \\ &= (c, d) + (a, b)\end{aligned}$$

moreover,

$$\begin{aligned}(a, b) \cdot (c, d) &= (a \cdot c, b \cdot d) \\ &= (c \cdot a, d \cdot b) \\ &= (c, d) \cdot (a, b)\end{aligned}$$

(Additive/Multiplicative Identity) Suppose $(a, b) \in \mathbb{R}^2$ then with $1 = (1, 1)$ and $0 = (0, 0)$ we have

$$(a, b) + (0, 0) = (a + 0, b + 0) = (a, b)$$

similarly

$$(a, b) \cdot (1, 1) = (a \cdot 1, b \cdot 1) = (a, b)$$

(Existence of negatives/reciprocals) Suppose $(a, b) \in \mathbb{R}^2$ then with $1 = (1, 1)$ and $0 = (0, 0)$ we have

$$(a, b) + (-a, -b) = (a - a, b - b) = (0, 0)$$

where $-a, -b \in \mathbb{R}$ so $(-a, -b) \in \mathbb{R}^2$, also

$$(a, b) \cdot (a^{-1, b^{-1}}) = (a \cdot a^{-1}, b \cdot b^{-1}) = (1, 1)$$

where $a^{-1}, b^{-1} \in \mathbb{R}$ so $(a^{-1}, b^{-1}) \in \mathbb{R}^2$.

(Distributivity) Suppose $(a,b) \in \mathbb{R}^2$, $(c,d) \in \mathbb{R}^2$ and $(e,f) \in \mathbb{R}^2$ then

$$\begin{aligned}(a,b)\cdot[(c,d)+(e,f)] &= (a,b)\cdot[(c+e,d+f)] \\ &= (a(c+e), b(d+f)) \\ &= (ac+ae, bd+bf) \\ &= (a,b)\cdot(c,d)+(a,b)\cdot(e,f)\end{aligned}$$

thus all axioms hold and $\mathbb{R}^2$ with these operations is a field.

49. In the multiplication table it shows that $x^2 = 0$ but this would imply $x = 0$ which is a contradiction.

50. Assume F is an arbitrary field and that $x^2 = 1$ then $x^2 - 1 = 0$ using field axioms which can be factored into $(x+1)(x-1) = 0$. By exercise 2.5.44 b) we know that either $x+1 = 0$ or $x-1 = 0$ thus $x = -1$ or $x = 1$. If F is the field with two elements namely $\{0,1\}$ then $x^2 = 1$ only has one solution which is $x = 1$.

51. Suppose F is a field such that $1+1 = 0$, we can equivalently show that $x+x = 0$ where $x \in F$. Multiplying $1+1 = 0$ by x on both sides achieves $x(1+1) = x+x = x\cdot 0$ so $x+x = 0$ as needed.

52. (a) Let $F = \mathbb{Q}$ then $3, -4, \frac{1}{2}, \frac{2}{5} \in F$, $\mathbb{Q} \subseteq \mathbb{R}$ and $\mathbb{Q}$ is a field but $\sqrt{2} \notin \mathbb{Q}, \pi \notin \mathbb{Q}$
    (b) Consider the smallest field $F = \{0,1\}$ by closure of addition and multiplication we have $1+1 = 2 \in F$ and $1+1+1 = 3 \in F$ so we can capture every element in $\mathbb{N}$, similarly by existence of negatives we achieve all elements of $\mathbb{Z}$ also by existence of multiplicative inverses we have all elements of $\mathbb{Q}$. Thus $\mathbb{Q}$ is the smallest subfield and so $\mathbb{Q} \subseteq F$.

53. Suppose $\sqrt{2}+\sqrt{3} \in F$ where F is a subfield of $\mathbb{R}$ then consider $\frac{1}{\sqrt{2}+\sqrt{3}} = \sqrt{3}-\sqrt{2} \in F$ thus $(\sqrt{2}+\sqrt{3})+(\sqrt{3}-\sqrt{2}) = 2\sqrt{3} \in F$ and so $\frac{1}{2}(2\sqrt{3}) = \sqrt{3} \in F$. Similarly we conclude $(\sqrt{2}+\sqrt{3})-(\sqrt{3}-\sqrt{2}) = 2\sqrt{2} \in F$ since $-(\sqrt{3}-\sqrt{2}) \in F$ thus $\frac{1}{2}(2\sqrt{2}) = \sqrt{2} \in F$.

54. (a) (1) Suppose $x, y, z \in K$ then $x = a+b\sqrt{2}, y = c+d\sqrt{2}$ and $z = e+f\sqrt{2}$ it follows that

$$\begin{aligned}x+(y+z) &= (a+b\sqrt{2})+[(c+d\sqrt{2})+(e+f\sqrt{2})] \\ &= (a+b\sqrt{2})+[(c+e)+(d+f)\sqrt{2}] \\ &= (a+[c+e])+(b+[d+f])\sqrt{2} \\ &= ([a+c]+e)+([b+d]+f)\sqrt{2} \\ &= [(a+c)+(b+d)\sqrt{2}]+(e+f\sqrt{2}) \\ &= [(a+b\sqrt{2})+(b+d\sqrt{2})]+(e+f\sqrt{2}) \\ &= (x+y)+z\end{aligned}$$

and

$$\begin{aligned}
x\cdot(y\cdot z) &= (a+b\sqrt{2})\cdot[(c+d\sqrt{2})\cdot(e+f\sqrt{2})] \\
&= (a+b\sqrt{2})\cdot[(ce+de)+(cf+df)\sqrt{2}] \\
&= (eac+dea+2bcf+2bdf)+(acf+adf+ebc+deb)\sqrt{2} \\
&= [(ac+2bd)+(ad+bc)\sqrt{2}]\cdot(e+f\sqrt{2}) \\
&= [(a+b\sqrt{2})\cdot(c+d\sqrt{2})]\cdot(e+f\sqrt{2}) \\
&= (x\cdot y)\cdot z
\end{aligned}$$

(2) Suppose $x, y \in K$ then $x = a+b\sqrt{2}, y = c+d\sqrt{2}$ so

$$\begin{aligned}
x+y &= (a+b\sqrt{2})+(c+d\sqrt{2}) \\
&= (a+c)+(b+d)\sqrt{2} \\
&= (c+a)+(d+b)\sqrt{2} \\
&= (c+d\sqrt{2})+(a+b\sqrt{2}) \\
&= y+x
\end{aligned}$$

also

$$\begin{aligned}
x\cdot y &= (a+b\sqrt{2})\cdot(c+d\sqrt{2}) \\
&= (ac+2bd)+(ad+bc)\sqrt{2} \\
&= (ca+2db)+(da+cb)\sqrt{2} \\
&= \left(c+d\sqrt{2}\right)\cdot\left(a+b\sqrt{2}\right) \\
&= y\cdot x
\end{aligned}$$

(3) Let $0 = 0+0\sqrt{2}$ and $1 = 1+0\sqrt{2}$ then if $x \in K$ where $x = a+b\sqrt{2}$ we have

$$\begin{aligned}
x+0 &= (a+b\sqrt{2})+(0+0\sqrt{2}) \\
&= (a+0)+(b+0)\sqrt{2} \\
&= a+b\sqrt{2} \\
&= x
\end{aligned}$$

moreover,

$$x\cdot 1 = (a+b\sqrt{2})\cdot(1+0\sqrt{2}) = a+b\sqrt{2} = x$$

thus there is an additive identity and a multiplicative identity.
(5) Finally, suppose $x, y, z \in K$ then $x = a+b\sqrt{2}, y = c+d\sqrt{2}$ and

$z = e + f\sqrt{2}$ it follows that

$$\begin{aligned}
x \cdot (y+z) &= (a+b\sqrt{2}) \cdot [(c+d\sqrt{2}) + (e+f\sqrt{2})] \\
&= (a+b\sqrt{2}) \cdot [(c+e) + (d+f)\sqrt{2}] \\
&= (ac+ea+2bd+2bf) + (ad+af+bc+eb)\sqrt{2} \\
&= [(ac+2bd) + (ea+2bf)] + [(ad+bc) + (af+eb)]\sqrt{2} \\
&= (a+b\sqrt{2}) \cdot (c+d\sqrt{2}) + (a+b\sqrt{2}) \cdot (e+f\sqrt{2}) \\
&= x \cdot y + x \cdot z
\end{aligned}$$

it is very easy to check these axioms since they depend on the fact that $\mathbb{Q}$ is a field.

(b) (0) Let $x, y \in K$ then $x = a + b\sqrt{2}, y = c + d\sqrt{2}$ so

$$x \cdot y = (a+b\sqrt{2}) \cdot (c+d\sqrt{2}) = (ac+2bd) + (ad+bc)\sqrt{2}$$

where $ac + 2bd \in \mathbb{Q}$ and $ad + bc \in \mathbb{Q}$ as well. Similarly for addition

$$x + y = (a+b\sqrt{2}) + (c+d\sqrt{2}) = (a+c) + (b+d)\sqrt{2}$$

since $a + c$ and $b + d$ are both clearly elements of $\mathbb{Q}$ we have closure for K.

(c) (4) I conjecture that the negative of $x = a + b\sqrt{2}$ is $-x = (-a) + (-b)\sqrt{2} \in K$ since

$$\begin{aligned}
x + (-x) &= (a+b\sqrt{2}) + ((-a) + (-b)\sqrt{2}) \\
&= (a+(-a)) + (b+(-b))\sqrt{2} \\
&= 0 + 0\sqrt{2} \\
&= 0
\end{aligned}$$

also $-x \in K$ since $-a, -b \in \mathbb{Q}$. I also conjecture that if $x \neq 0$ then $x^{-1} = \frac{1}{a+b\sqrt{2}} \in K$ we know this to be an element since

$$\frac{1}{a+b\sqrt{2}} \cdot \frac{a-b\sqrt{2}}{a-b\sqrt{2}} = \frac{a-b\sqrt{2}}{a^2-2b^2} = \frac{a}{a^2-2b^2} + \frac{-b}{a^2-2b^2}\sqrt{2}$$

where $\frac{a}{a^2-2b^2}, \frac{-b}{a^2-2b^2} \in \mathbb{Q}$ by properties of the rationals. Multiplying by $a - b\sqrt{2}$ helped us *rationalize* the fraction so that we could break it into rational numbers. Conclusively,

$$x \cdot x^{-1} = (a+b\sqrt{2}) \cdot \frac{1}{a+b\sqrt{2}} = 1 + 0\sqrt{2} = 1$$

so K has multiplicative inverses.

55. (Btw this is a terrible definition since if $P = F = \mathbb{R}$) then the third axiom isn't even satisfied...)

   (a) Since $1 \neq 0$ and $1 \in F$ then either $1 \in P$ or $-1 \in P$, let us assume that $-1 \in P$ however this implies that $(-1)(-1) = 1 \in P$ by (ii). (This relies on the fact that $(-1) \cdot (-1) = 1$ but you can prove this on your own)

   (b) Suppose $x \neq 0$ and $x \in F$ then either $x \in P$ or $-x \in P$. Notice that if $x \in P$ then by (ii) we have $x^2 \in P$, if $-x \in P$ then also by (ii) and field laws we know that $(-x)(-x) = x \cdot x = x^2 \in P$.

   (c) Suppose $a, b, c, d \in F$ and $a < b$ with c, d then $b - a \in P$ and $d - c \in P$, it follows by (i) that $(b-a) + (d-c) = (b+d) - (a+c) \in P$ meaning that $a + c < b + d$.

   (d) Suppose $c > 0$ and $a < b$ then $0 < c$ so $c \in P$ and $b - a \in P$ so by (ii) we have that $c(b-a) = cb - ca \in P$ so $ca < cb$.

   (e) Suppose $c < 0$ and $a < b$ then $-c \in P$ and $b - a \in P$ thus by (ii) $-c(b-a) = ac - bc \in P$ thus $ca > cb$.

56. (a) g is well-defined. Every element of $\mathbb{Z} \times \mathbb{Z}$, g assigns to exactly one element in $\mathbb{Q}$. This is because elements in $\mathbb{Z} \times \mathbb{Z}$ can only be defined one way.

   (b) h is well-defined. Every element of $\mathbb{R}$, h assigns to exactly one element in $\mathbb{R}$.

   (c) r is well-defined. Every element of $\mathbb{R}$, r assigns to exactly one element in $\mathbb{R}$.

   (d) f is well-defined. Every element of $\mathbb{Q} \setminus \{0\}$, f assigns to exactly one element in $\mathbb{Q}$.

   (e) p is well-defined. Every element of $\mathbb{R}$, p assigns to exactly one element in $\mathbb{R}$. This is because $\sin(x)$ and $|x|$ are both well-defined.

   (f) q is well-defined. Every element of $\mathbb{R}$, q assigns to exactly one element in $\mathbb{R}$. This is because $\ln(x)$ and $x^2 - 1, 1 - x^2$ are all well-defined.

57. h can only be well-defined if $f(x) = g(x)$ when $1 \leq x \leq 2$ since if $f(x) \neq g(x)$ we will product two different values for $h(x)$ in the range $1 \leq x \leq 2$.

3 Chapter 3 Exercises Solutions

1. (a) **False.** it translates to $\frac{1}{2} < x < \frac{5}{2} \implies x^2 = 1$ which is clearly false since if $x = 2$ then $\frac{1}{2} < x < \frac{5}{2}$ but $2^2 \neq 1$.
   (b) **False.** it translates to $\frac{1}{2} < x < \frac{5}{2} \wedge x \in \mathbb{Z} \implies x^2 = 1$ which is clearly false since if $x = 2$ then $\frac{1}{2} < x < \frac{5}{2}$ and $x \in \mathbb{Z}$ but $2^2 \neq 1$.
   (c) **False.** it translates to $x^2 = 1 \implies [\frac{1}{2} < x < \frac{5}{2} \wedge x \in \mathbb{Z} \wedge x \neq 2]$ which is false since if $x = -1$ then it is not the case that $\frac{1}{2} < x < \frac{5}{2}$.
2. (a) "For any two real numbers x, y we have that $x \geq y$". This is **False** since if $x = 0, y = 1$ then $0 < 1$ but both are real numbers.
   (b) "There exists two real numbers x, y where $x \geq y$". This is **True** of course since if $x = 1, y = 0$ then $x \geq y$.
   (c) "There exists a real number y such that for every real x, $x \geq y$". This is **False** since any number $x \geq y$ we will have $x - 1 \in \mathbb{R}$ yet $y \geq x - 1$.
   (d) "For every real x there exists a real number y such that $x \geq y$". This is **True** since for every real number x we can choose $y = x - 1 \in \mathbb{R}$ then $x \geq x - 1$ clearly.
   (e) "For every real x there exists a real number y such that $x^2 + y^2 = 1$". This is **False** since if $x = 2$ then $y^2 \geq 0$ so we cannot achieve $x^2 + y^2 = 1$.
   (f) "There exists a real x such that for every y, $x^2 + y^2 = 1$". This is **False** since if $y > 1$ we cannot achieve a negative number for x^2.
3. (a) $\exists y \in \mathbb{R} \forall x \in \mathbb{R}(x \geq y)$ which is **False** by part c) of the question above.
   (b) $\forall x \in \mathbb{Z} \exists y \in \mathbb{Z} \exists c \in \mathbb{Z}(x = y \cdot c)$ which is **True**. If we let $y = x$ and $c = 1$ we are done.
   (c) $\exists x \in \mathbb{N} \exists x \in \mathbb{N}(x^2 + y^2 = 1)$ which is **False** since in this text we consider 0 to not be a natural number.
   (d) $\forall x \in \mathbb{R} \exists y \in \mathbb{R}^+ \exists r \in \mathbb{R}^+(x = y - r)$ which is **True**. If we let $y = x$ and $r = 0$ then we are done.
4. (a) $(\forall x)[((x \in S) \wedge (x \in T)) \wedge x \notin Q]$
   (b) $(\forall x)[(x \in S) \implies ((x \in T) \vee (x \in Q))]$,

$(\exists x)[(x \in S) \wedge ((x \notin T) \wedge (x \notin Q))]$

5. (a) **False.** Let $x = 1 \in \mathbb{R}$ then $P(x)$ clearly but $1^2 > 1$ is not true so $\neg Q(x)$.
   (b) **True.** Because $[(\forall x \in \mathbb{R})P(x)]$ is false in general since not every real number is positive then it is vacuously true.
6. (a) $R = \forall x \in \mathbb{R} \exists y \in \mathbb{R}(x + y < 1)$, $S = \exists y \in \mathbb{R} \forall x \in \mathbb{R}(x + y < 1)$
   (b) $\neg R = \exists x \in \mathbb{R} \forall y \in \mathbb{R}(x + y \geq 1)$, $\neg S = \forall y \in \mathbb{R} \exists x \in \mathbb{R}(x + y \geq 1)$

(c) R is **True** since choosing $y = -x$ gives $x + (-x) = 0 < 1$. S is **False** since letting $x = -y + 1$ will always lead to a contradiction.

7. Let $x = 2$ then $|x - 3| = |2 - 3| = |-1| = 1$ but $|x - 2| = |2 - 2| = 0 \neq 2$ and $x \in \mathbb{R}$.

8. (a)

P	Q	$(P \wedge Q) \vee \neg Q$
F	F	T
F	T	F
T	F	T
T	T	T

(b)

P	Q	$P \implies (P \implies Q)$
F	F	T
F	T	T
T	F	F
T	T	T

(c)

P	Q	$(P \implies Q) \implies P$
F	F	F
F	T	F
T	F	T
T	T	T

(d)

P	Q	$(P \implies Q) \implies (P \wedge Q)$
F	F	F
F	T	F
T	F	T
T	T	T

(e)

P	Q	$(P \wedge Q) \iff (P \vee Q)$
F	F	T
F	T	F
T	F	F
T	T	T

(f)

P	Q	$(P \vee \neg Q) \implies (Q \wedge \neg P)$
F	F	F
F	T	T
T	F	F
T	T	F

9. $R = \neg(P \wedge Q)$

10. (a) It must be that P is false and $Q \implies \neg R$ is false so Q must be true and R also must be true.

    (b) It must be that $(P \wedge Q) \vee R$ is true so R is true or P, Q are true. On the other hand $R \vee S$ must be false so R, S must both be false.

11. (a) Suppose P is false, Q is false and R is true then $P \wedge (\neg Q)$ is false so the implication is true vacuously.

    (b) Suppose P, Q, R are all true then the statement $P \wedge (\neg Q)$ is still false so the implication is true vacuously.

12. (a)

P	Q	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$
F	F	T	T
F	T	T	T
T	F	T	T
T	T	F	F

    (b)

P	Q	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$
F	F	T	T
F	T	F	F
T	F	F	F
T	T	F	F

    (c)

P	Q	$P \iff Q$	$(P \implies Q) \wedge (Q \implies P)$
F	F	T	T
F	T	F	F
T	F	F	F
T	T	T	T

13. (a) $\neg P \vee Q$

    (b) $\neg P \vee Q$

14. We have several cases but to generalize, $P \vee Q$ is false only when P is true or Q is true and R is false. If P is true and R is false then $P \implies R$ is false so both statements are false. If Q is true and R is false then $Q \implies R$ is false so both statements are also false. If both P, Q are true then both $P \implies R$ and $Q \implies R$ are both false so both statements are also equivalent in the manner. If P, Q is false we get vacuously true statements for both making them equal on every truth value of P, Q, R.

15. (a) **No.** If P is true and Q is false then $P \implies Q$ is false but $\neg P \implies \neg Q$ is true vacuously.

    (b) **No.** If R is true and P, Q are false then $(P \wedge Q) \vee R$ is true yet $P \wedge (Q \vee R)$ is false.

(c) **No.** If P is false and Q is false with R being true then $(P \implies Q) \implies R$ is false but then $P \implies (Q \implies R)$ is true vacuously.

(d) **No.** The negation of $P \iff Q$ is $(P \wedge \neg Q) \vee (Q \wedge \neg P)$, in fact $\neg P \iff \neg Q$ is logically equivalent to $P \iff Q$ so it cannot be its negation.

16. (a) If k is even then k is composite or $k = 2$.

    (b) If I get a bad mark in the course then my assignments are not complete.

    (c) If $x > 3$ or $x < -3$ then $x^2 + y^2 \neq 9$.

    (d) If $a \neq 0$ or $b \neq 0$ then $a^2 + b^2 \neq 0$.

    (e) If Anna is graduating then Anna is passing history or psychology.

17. (a) $P \wedge \neg P$ is a contradiction since it is false on every truth value of P.
    $P \iff \neg P$ is a contradiction since it is false on every truth value of P.
    $P \vee \neg P$ is a tautology since it is true on every truth value of P.
    $P \implies \neg P$ is neither a contradiction nor a tautology since if P is true then it is false and if P is false it is true vacuously.
    $(P \wedge Q) \implies Q$ is a tautology since if P is true, we arrive at a true implication for both cases of Q and if P is false the statement is true vacuously.

18. This is a fallacy, the Dog states that all cats have four legs meaning for every arbitrary object, if it is a cat then it has four legs. The dog then says that they have four legs but this cannot conclude whether or not they are a cat.

19. For every set A we have that $A \notin A$.

20. (a) There exists an $x \in A$ such that for all $b \in B$, $b \leq x$

    (b) There exists a positive real number x such that for all natural numbers n we have $\frac{1}{n} \geq x$

21. (a) $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x^2 \leq y^2)$. The original statement is **True** since choosing $y = x - 1$ gives $x^2 > y^2$.

    (b) $(\forall x \in \mathbb{Z})[(x^2 = (x+1)^2) \wedge x^3 \notin \mathbb{Z}]$. The original statement is **False** since $x^2 = (x+1)^2$ implies only $x = -\frac{1}{2}$ where $\left(-\frac{1}{2}\right)^3 \notin \mathbb{Z}$.

    (c) $(\exists n \in \mathbb{N})[(n-1)^3 + n^3 = (n+1)^3]$. The original statement is **True** since there is no natural number such that $(n-1)^3 + n^3 = (n+1)^3$ or rather $n^3 - 6n^2 - 2 = 0$.

    (d) $[(\forall x \in \mathbb{R})(x > 0)] \wedge [(\exists x \in \mathbb{R})(x \neq x + 1)]$. The original statement is **True** vacuously since not every real x is positive.

    (e) $(\exists x \in \mathbb{R})[(x^2 \leq -1) \wedge ((x+1)^2 \neq x^2 + 1)]$. The original statement is **True** since $x^2 \leq -1$ is true vacuously.

(f) $(\exists x \in \mathbb{R})[(x > 0) \wedge (\forall n \in \mathbb{N})(n \cdot x \leq 1)]$. The original statement is **True** since there will always be a natural number we can 'push' x beyond 1.

(g) $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})[(x+y)^2 \neq x^2+y^2]$. The original statement is **True** since if we let $y = 0$ then clearly $x^2 = x^2$.

(h) $(\forall y \in \mathbb{R})(\exists x \in \mathbb{R})(|x+y| \neq |x|+|y|)$. The original statement is **True** since if we let $y = 0$ then $|x| = |x|$ clearly.

(i) $(\exists x \in \mathbb{Q})(\forall n \in \mathbb{N})(n \cdot x \notin \mathbb{Z})$. The original statement is **True** since if $x = \frac{a}{b}$ then we can choose $n = b$.

(j) $(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})[((x + y \leq 7) \wedge (xy = x)) \wedge (x \geq 7)]$. The original statement is **True** since $x + y \leq 7$ thus $x \leq 7 - y$, by our second assumption $x = 0$ or $y = 1$, if $y = 1$ then clearly $x < 7$ and if $x = 0$ then also $x < 7$.

22. (a) $(\exists M \in \mathbb{Z})(\forall x \in \mathbb{R})(x^2 \leq M)$, $(\forall M \in \mathbb{Z})(\exists x \in \mathbb{R})(x^2 > M)$. The original statement is **False** since if there were an M such that $x^2 \leq M$ then we can simply look at $(x+1)^2$.

(b) $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(|x - y| = |x| - |y|)$, $(\forall y \in \mathbb{R})(\exists x \in \mathbb{R})(|x - y| \neq |x| - |y|)$. The original statement is **True** let $y = 0$ then $|x| = |x|$ for all x.

(c) $(\forall x \in \mathbb{R})((x-6)^2 = 4 \implies x = 8)$, $(\exists x \in \mathbb{R})((x-6)^2 = 4 \wedge x \neq 8)$. The original statement is **False** if $x = 4$ then $(4-6)^2 = (-2)^2 = 4$ but $x \neq 8$.

(d) $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})((x^2 - y^2 = 9) \implies |x| \geq 3)$,
$(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})((x^2 - y^2 = 9) \wedge |x| < 3)$. The original statement is **True** since if $x^2 - y^2 = 9$ then clearly only $|x| \geq 3$ since we would not have a solution otherwise.

(e) $(\forall x \in \mathbb{R})((x-1)(x-3) = 3 \implies ((x-1=3) \vee (x-3=3)))$,
$(\exists x \in \mathbb{R})((x-1)(x-3) = 3 \wedge ((x-1 \neq 3) \wedge (x-3 \neq 3)))$. The original statement is **True** clearly.

23. There exists a field F, and an element $a \in F$ where $a^3 = 1$ yet $a \neq 1$. The original statement is **True** since $a^3 - 1 = (a-1)\left(a^2 + a + 1\right) = 0$ thus only $a = 1$.

24. (a) This is wrong because $(\forall x \in \mathbb{Z})[x(x-1) < 0] \equiv \forall x[(x \in \mathbb{Z}) \implies (x(x-1) < 0)]$ so negating this implication should not negate $x \in \mathbb{Z}$. The correct negation is $(\exists x \in \mathbb{Z})[x(x-1) \geq 0]$.

(b) P is true since if $x \in \mathbb{Z}$ then either $x > 0$, $x = 0$ or $x < 0$, if $x = 0$ then the statement is clearly true. If $x > 0$ then $x(x-1) \geq 0$ and if $x < 0$ then $x - 1 < 0$ so $x(x-1) \geq 0$.

25. A function $f : \mathbb{R} \to \mathbb{R}$ is unbounded if for all positive numbers M we have $|f(x)| > M$ for at least one. $x \in \mathbb{R}$.

26. (a_n) is decreasing if $a_n \geq a_{n+1}$ for at least one $n \in \mathbb{N}$

27. (a) Suppose we have a four digit positive integer $k = a_0a_1a_2a_3$ where $a_0 + a_1 + a_2 + a_3$ is divisible by 3. Consider that this number $k = \sum_{i=0}^{3} 10^i \cdot a_i = a_0 + 10a_1 + 100a_2 + 1000a_3$ then we can rewrite this as $k = a_0 + (9a_1 + a_1) + (99a_2 + a_2) + (9999a_3 + a_3)$ so we have

$$k = (a_0 + a_1 + a_2 + a_3) + 3(3a_1 + 33a_2 + 333a_3)$$

since $a_0 + a_1 + a_2 + a_3$ is divisible by 3 then k is also divisible by 3.

(b) Suppose n is a three digit positive number that is divisible by 3, let $n = a_0a_1a_2$. Consider their sum $a_0 + a_1 + a_2$ then $n = (a_0 + a_1 + a_2) + 3(3a_1 + 33a_2)$ since $n = \sum_{i=0}^{2} 10^i \cdot a_i$. Moreover, as n is divisible by 3 then $n = 3k$ for some integer k thus

$$3k - 3(3a_1 + 33a_2) = 3(k - a_1 - 11a_2) = a_0 + a_1 + a_2$$

so their sum is divisible by 3. We can combine this into an if and only if that "n is a three-digit positive integer that is disivible by 3 if and only if its sum of digits is also divisibl by 3"

(c) Using Example 3.6.1 for the forward direction and part b) we conclude that a natural number n is divisible by 3 if and only if its sum of digits is divisible by 3.

28. Suppose n is a natural number divisible by 9 then $n = 9k$ for some integer k. Let $a_0, a_1, \ldots, a_n$ be the digits of n if and only if

$$n = \sum_{i=0}^{n} 10^i \cdot a_i$$

thus $9k = \sum_{i=0}^{n} 10^i \cdot a_i$ so we can rewrite the sum as

$$9k = (a_0 + a_1 + \cdots + a_n) + \sum_{i=0}^{n} (9 \cdot 10^i) \cdot a_{i+1}$$

$$\iff 9k = (a_0 + a_1 + \cdots + a_n) + 9\sum_{i=0}^{n} 10^i \cdot a_{i+1}$$

and we conclude $9k - 9\sum_{i=0}^{n} 10^i \cdot a_{i+1} = a_0 + a_1 + \cdots + a_n$ so the sum of digits is also divisble by 9.

29. The following statement is false, let $n = 2$ then $(2-1)^2 + (2)^3 = 1 + 8 = 9$ and $(2+1)^3 = 27$ but $27 \neq 9$.

30. (a) Suppose x is an odd integer then $x = 2k + 1$ for some integer k, then $x^2 - 1 = (2k+1)^2 - 1 = 4k^2 + 4k = 4k(k+1)$ notice that this number is even and $k(k+1)$ is divisible by 2 since if k is even we can rewrite it as $2j$ for some integer j. If k is odd, then $k+1$ is even so we can also rewrite that as well. Consequently, we arrive at the fact in both cases that $x^2 - 1 = 8p$ for some integer p thus it is divisible by 8.

(b) Suppose m or n are odd integers, take the case where m is odd then $m = 2j + 1$ for some integer j then $m^2 + n^2 = (2j + 1)^2 + n^2 = 4j^2 + 4j + 1 + n^2$. Regardless of whether n is even or odd we are left with a remainder of 1 or 2 when dividing by 4. If n is odd we arrive at a similar case since $m^2 + n^2 = m^2 + (2j+1)^2 = m^2 + 4j^2 + 4j + 1$ and we have a remainder of $1, 2$ depending on whether m is even or odd where $m = 2j + 1$ for some integer j. Since we have a remainder of 1 or 2 in both cases then $m^2 + n^2$ is clearly not divisible by 4 thus the original statement holds.

(c) Suppose $x, y \in \mathbb{R}$ such that $\frac{x}{\sqrt{x^2+1}} = \frac{y}{\sqrt{y^2+1}}$ then

$$x\sqrt{y^2+1} = \sqrt{x^2+1}y \implies x^2 = y^2$$

thus either $x = y$ or $x = -y$ but if $x = -y$ then $y = 0$ so $x = y$ in both cases.

(d) Suppose $x \in \mathbb{R}$ where $x \geq 3$ then $x^3 \geq 27$ and $5x \geq 15$ so adding these inequalities gives $x^3 + 5x \geq 43$ thus $x^3 + 5x > 40$.

31. (a) Assume $x^3 + x^2 = 1$ has a rational solution say $x = \frac{m}{n}$ where m, n are in lowest terms where $m, n \in \mathbb{Z}$ then

$$\left(\frac{m}{n}\right)^3 + \left(\frac{m}{n}\right)^2 = 1$$

which means $m^3 + m^2n + (-n^3) = 0$. If m, n are both even we contradict they are in lowest terms. If m is even and n is odd then m^3 is even, $m^2 \cdot n$ is even and n^3 is odd producing an odd number which is impossible. If m is odd and n is odd then clearly the entire equation is odd. Lastly, if m is odd and n is even then m^3 is odd, $m^2 \cdot n$ is odd and n^3 is odd thus adding three odd numbers which is also odd, an impossible conclusion. It must be the case that this equation has no rational roots.

(b) Assume $x^3 + x + 1 = 0$ has a rational solution say $x = \frac{m}{n}$ where m, n are in lowest terms where $m, n \in \mathbb{Z}$ then

$$\left(\frac{m}{n}\right)^3 + \left(\frac{m}{n}\right) + 1 = 0$$

which simplifies to $m^3 + mn^2 + n^3 = 0$. If both m, n are even then we contradict they were in lowest terms. If m is odd and n is even then m^3 is odd and mn^2 is even but n^3 is even as well thus we will produce an odd number, an impossibility. If m is even and n is odd then m^3 is even, mn^2 is even and n^3 is odd however this also produces an odd number which is impossible. Clearly if m, n are both odd then the entire equation will be odd since we are adding 3 odd numbers. Thus the equation has no rational roots.

(c) Assume $x^5 + 3x^3 + 7 = 0$ has a rational solution say $x = \frac{m}{n}$ where m, n are in lowest terms where $m, n \in \mathbb{Z}$ then

$$\left(\frac{m}{n}\right)^5 + 3\left(\frac{m}{n}\right)^3 + 7 = 0$$

which simplifies to $m^5 + 3mn^4 + 7n^5 = 0$. If both m, n are even then we contradict they were in lowest terms. If m is odd and n is even then m^5 is odd and $7n^5$ is even. Consider that $3mn^4$ is even since mn^4 is even thus this case is impossible since two even numbers added to an odd number added produces an odd number. If m is even and n is odd then m^5 is even and $7n^5$ is odd. Notice that $3mn^4$ is even since mn^4 is even thus this case is impossible by the same reasoning as before. If both m, n are odd then clearly the entire equation produces an odd number since each term will be odd (adding 3 odds together). In conclusion the equation has no rational roots.

(d) Assume $x^5 + x^4 + x^3 + x^2 + 1 = 0$ has a rational solution say $x = \frac{m}{n}$ where m, n are in lowest terms where $m, n \in \mathbb{Z}$ then

$$\left(\frac{m}{n}\right)^5 + \left(\frac{m}{n}\right)^4 + \left(\frac{m}{n}\right)^3 + \left(\frac{m}{n}\right)^2 + 1 = 0$$

which simplifies to $m^5 + m^4n + m^3n^2 + m^2n^3 + n^5 = 0$. If m, n are even then we contradict they were in lowest terms. If m is odd and n is even then m^5 is odd yet every other number will be even thus adding all numbers together will produce an odd number (hanging 1). If m is even and n is odd then all terms except for n^5 will even and thus we will also produce an odd number. If both m, n are odd then adding 5 odd numbers will also still produce an odd number (hanging 1). In conclusion the equation has no rational roots.

32. (a) Assume $x^2 - 4y^2 = 7$ has a natural solution. Then factoring gives $x^2 - 4y^2 = (x - 2y)(x + 2y) = 7$ however this implies that $x - 2y = 7$ and $x + 2y = 1$ or $x - 2y = -7$ and $x + 2y = -1$ where solving both systems reveals the solutions $x = 4, y = -3/2$ or $x = -4, y = 3/2$ both of which are impossible since $y \notin \mathbb{N}$.

(b) Assume $x^2 - y^2 = 10$ has a natural solution. Then factoring gives $(x - y)(x + y) = 10$ however this implies that $x - y = 10, x + y = 1$ or $x - y = -10, x + y = -1$ where the solution to both systems is $x = 11/2, y = -9/2$ or $x = -11/2, y = 9/2$ both of which are impossible since neither x nor y are natural numbers.

33. Assume $x^2 + x + 1 = y^2$ has natural solutions, multiplying the equation by 4 gives $4x^2 + 4x + 4 = 4y^2$ where when we complete the square we arrive at $(2x + 1)^2 + 3 = (2y)^2$ which means $(2x + 1)^2 - (2y)^2 = -3$. This can be changed to $(2x + 1 + 2y)(2x + 1 - 2y) = -3$ where we then have the

systems $2x+1+2y = -3, 2x+1-2y = 1$ or $2x+1+2y = 3, 2x+2-2y = -1$ which give the solutions $x = -5/4, y = -3/4$ or $x = -1/4, y = 5/4$ which is impossible since neither are natural numbers.

34. If two squares of each color are removed from the checkerboard, then the remaining squares cannot be covered exactly by copies of the "T-shape" and its rotations. Since 60 squares remain, 15 T-shapes must be used. Each T-shape covers an odd number of squares of each color. Since the sum of 15 odd numbers is always odd, any board formed from 15 T-shapes has an odd number of squares of each color. Our remaining board has 30 squares of each color, so it can't be covered by 15 T-shapes. Alternatively, since the region has the same number of squares of each color, one can conclude that there must be the number of tiles covering 3 black and 1 white must be the same as the number covering 3 white and 1 black. Thus an even number of tiles must be used, which contradicts the total of 60 squares, since 60 is not 4 times an even number.

35. (a) $M = 2 \cdot 3 \cdot 5 \cdot 7 + 1 = 211$ which is a prime number.
    (b) $M = 7 \cdot 11 \cdot 13 \cdot 19 + 1 = 19020$ which is not a prime number and is divisible by 317 a prime that is not part of the set. $M = 2 \cdot 5 \cdot 11 \cdot 19 \cdot 23 + 1 = 48071$ is not prime and is divisible by 53, a prime that is not part of the set.

4 Chapter 4 Exercises Solutions

1. (a) (Base Case) If $n = 1$ then $1^1 = 1$ and $\frac{4(1)^3-1}{3} = 1$ thus they are equal statements.
(Inductive Hypothesis) Assume for some $k \in \mathbb{N}$ that $1^2 + 3^2 + \cdots + (2k-1)^2 = \frac{4k^3-k}{3}$.
(Inductive Step) We want to show that statement holds for $k+1$ is true aswell, consider the following sum

$$1^2 + 3^2 + \cdots + (2k-1)^2 + (2(k+1))^2 = \frac{4k^3-k}{3} + (2k+2)^2$$

by our inductive hypothesis. Adding these two terms results in

$$\frac{4k^3 + 12k^2 + 23k + 12}{3} = \frac{4(k+1)^3 - (k+1)}{3}$$

thus the statement holds for $k+1$ and by the PMI the statement holds for all $n \in \mathbb{N}$.

(b) (Base Case) If $n = 1$ then $1^1 = 1$ and $\frac{1}{6}(1)(1+1)(1+2) = 1$ thus they are equal statements.
(Inductive Hypothesis) Assume for some $k \in \mathbb{N}$ that $1^2+2^2+\cdots+k^2 = \frac{1}{6}k(k+1)(k+2)$.
(Inductive Step) We want to show that statement holds for $k+1$ is true aswell, consider the following sum

$$1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \frac{1}{6}k(k+1)(k+2) + (k+1)^2$$

by our inductive hypothesis. Adding these two terms results in

$$\frac{k^3}{6} + \frac{k^2}{2} + \frac{k}{3} + k^2 + 2k + 1 = \frac{1}{6}(k+1)(k+2)(k+3)$$

thus the statement holds for $k+1$ and by the PMI the statement holds for all $n \in \mathbb{N}$.

2. (a) (Base Case) If $n = 1$ then $5^1 + 5 = 10$ and $5^{1+1} = 25$ thus $10 < 25$ and our base case holds.
(Inductive Hypothesis) Assume for some $k \in \mathbb{N}$ that $5^k + 5 < 5^{k+1}$ holds.
(Inductive Step) We want to show that statement holds for $k+1$ is true as well, consider the following

$$5^{k+2} = 5 \cdot 5^{k+1} + 5 > 5 \cdot (5^k + 5) + 5 = 5^{k+1} + 30 > 5^{k+1} + 5$$

by our inductive hypothesis thus $k+1$ holds as well and by the PMI the statement holds for all $n \in \mathbb{N}$.

(b) (Base Case) If $n = 1$ then $1 = 1$ and $1^1 = 1$ thus $1 \leq 1$ and our base case holds.
(Inductive Hypothesis) Assume for some $k \in \mathbb{N}$ that $1+2+\cdots+k \leq k^2$ holds.
(Inductive Step) We want to show that the statement holds for $k+1$ as well, consider the sum

$$1 + 2 + \cdots + k + (k+1) \leq k^2 + (k+1)$$

by our inductive hypothesis and since $k > 0$ then $2k > k$ so

$$k^2 + (k+1) \leq k^2 + 2k + 1 = (k+1)^2$$

thus $k+1$ holds as well and by the PMI the statement holds for all $n \in \mathbb{N}$.

(c) (Base Case) If $n = 1$ then $\frac{1}{\sqrt{1}} = 1$ and $\sqrt{1} = 1$ thus $1 \geq 1$ and our base case holds.
(Inductive Hypothesis) Assume for some $k \in \mathbb{N}$ that $\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} \geq \sqrt{k}$.
(Inductive Step) We want to show that the statement holds for $k+1$ as well, consider the sum

$$\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \geq \sqrt{k} + \frac{1}{\sqrt{k+1}}$$

by our inductive hypothesis. Now, adding the last two terms gives

$$\frac{\sqrt{k(k+1)} + 1}{\sqrt{k+1}} \geq \frac{\sqrt{k^2} + 1}{\sqrt{k+1}} \geq \frac{k+1}{\sqrt{k+1}}$$

and the last inequality in the expression is exactly $\sqrt{k+1}$. Thus $k+1$ holds as well and by the PMI the statement holds for all $n \in \mathbb{N}$.

(d) (Base Case) If $n = 1$ then $\frac{1}{\sqrt{1}} = 1$ and $2\sqrt{1} = 2$ thus $2 \geq 1$ and our base case holds.
(Inductive Hypothesis) Assume for some $k \in \mathbb{N}$ that $\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} \leq 2\sqrt{k}$.
(Inductive Step) We want to show that the statement holds for $k+1$ as well, consider the sum

$$\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k} + \frac{1}{\sqrt{k+1}}$$

by our inductive hypothesis. Now, adding the last two terms gives

$$\frac{2\sqrt{k}\sqrt{k+1} + 1}{\sqrt{k+1}} \leq \frac{2\sqrt{k^2 + k + \frac{1}{4}} + 1}{\sqrt{k+1}} \leq \frac{2\left(k + \frac{1}{2}\right) + 1}{\sqrt{k+1}}$$

and the last inequality in the expression is exactly $2\sqrt{k+1}$. Thus $k+1$ holds as well and by the PMI the statement holds for all $n \in \mathbb{N}$.

3. Suppose $0 < a < 1$ then we want to show for any $n \in \mathbb{N}$ that $(1-a)^n < \frac{1}{1+n\cdot a}$, we will use induction for this.
(Base Case) If $0 < a < 1$ and $n = 1$ then we have $a^2 > 0$ thus $\frac{a^2}{1+a} > 0$ so $1 - a < \frac{1}{1+a}$ and our base case holds.
(Inductive Hypothesis) Suppose that if $0 < a < 1$ and k is some natural we have that $(1-a)^k < \frac{1}{1-k\cdot a}$ holds.
(Inductive Step) We want to show that the statement holds for $k+1$ as well, to do this consider that $(1-a)^{k+1} = (1-a)^k(1-a)$ where by our inductive hypothesis it follows that
$$(1-a)^k(1-a) < \frac{1}{1+k\cdot a}(1-a) < \frac{1}{(1+k\cdot a)(1+a)}$$
where we used the base case with $(1-a) < \frac{1}{1+a}$. Following this we achieve
$$\frac{1}{(1+k\cdot a)(1+a)} = \frac{1}{1+a+ak+a^2k} < \frac{1}{1+a+ak}$$
where the last term is equivalent to $\frac{1}{1+(k+1)a}$ and our statement holds for $k+1$ as well. Thus by the PMI the statement holds for all $n \in \mathbb{N}$.

4. We want to show that for every $n \in \mathbb{N}$ that $3^{4n+2} + 1$ is divisible by 10. We will use induction to show this.
(Base Case) If $n = 1$ then $3^{4(1)+2} + 1 = 3^6 + 1 = 730$ and 730 is in fact, divisible by 10 so our base case holds.
(Inductive Hypothesis) Assume for some natural k that $3^{4k+2}+1$ is divisible by 10.
(Inductive Step) We want to show the statement holds for $k+1$ as well, consider the following string of equations
$$3^{4(k+1)+2} + 1 = 3^{4k+2+4} + 1 = 3^4 \cdot 3^{4k+2} + 1 = 81 \cdot 3^{4k+2} + 1$$
we can break up the last equation into $80 \cdot 3^{4k+2} + 1 \cdot 3^{4k+2} + 1$ where 80 is divisible by 10 so $80 \cdot 3^{4k+2}$ is as well and $1 \cdot 3^{4k+2} + 1$ is also divisible by 10 using our inductive hypothesis. Moreover, the entire equation is divisible by 10. Thus $k+1$ holds as well and by the PMI the statement holds for all $n \in \mathbb{N}$.

5. We want to show that for every $n \in \mathbb{N}$ that $n^3 + 2n$ is divisible by 3. We will use induction to show this.
(Base Case) If $n = 1$ then $(1)^3 + 2(1) = 1 + 2 = 3$ where 3 is clearly divisible by 3 and our base case holds.
(Inductive Hypothesis) Assume for some natural k that k^3+2k is divisible by 3.
(Inductive Step) We want to show that the statement holds for $k+1$ as well, consider the following
$$(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 5k + 3 = (k^3 + 2k) + (3k^3 + 3k + 3)$$

where the left hand term is $k^3 + 2k$ our inductive hypothesis which is divisible by 3 and $3(k^3 + k + 1)$ which is also divisible by 3 so our entire equation is divisible by 3. Thus since $k+1$ holds by the PMI the statement holds for all $n \in \mathbb{N}$.

6. We want to show that for every $n \in \mathbb{N}$ that $4^{2n} - 1$ is divisible by 5. We will use induction to show this.
(Base Case) If $n = 1$ then $4^2 - 1 = 15$ where 15 is clearly divisible by 5 and our base case holds.
(Inductive Hypothesis) Assume for some natural k that $4^{2k} - 1$ is divisible by 5.
(Inductive Step) We want to show that the statement holds for $k + 1$ as well, consider the following

$$4^{2(k+1)} - 1 = 16 \cdot 4^{2k} - 1 = 15 \cdot 4^{2k} + (4^{2k} - 1)$$

where the right hand term is $4^{2k} - 1$ our inductive hypothesis which is divisible by 5 and $15 \cdot 4^{2k}$ which is also divisible by 5 so our entire equation is divisible by 5. Thus since $k + 1$ holds by the PMI the statement holds for all $n \in \mathbb{N}$.

7. Assume a is a integer different than 1 we will show by induction that $a^n - 1$ is divisible by $a - 1$ for all natural n.
(Base Case) If $n = 1$ then $a - 1$ is clearly divisible by $a - 1$.
(Inductive Hypothesis) Assume for some natural k that $a^k - 1$ is divisible by $a - 1$
(Inductive Step) We want to show that the statement holds for $k + 1$ or that $a^{k+1} - 1$ is divisible by $a - 1$, to do this consider that $a^{k+1} - 1 = a(a^k - 1) + (a - 1)$ where $a^k - 1$ is divisible by $a - 1$ and $a - 1$ is clearly divisible by $a - 1$ so $a^{k+1} - 1$ is divisible by $a - 1$ as well. Since $k + 1$ holds, by the PMI the statement holds for all $n \in \mathbb{N}$ where $a \neq 1$.

8. Simplifying $\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{2n^3+3n^2+n}{6}$ we can equivalently show that $2n^3 + 3n^2 + n = n(n + 1)(n + 2)$ is divisible by 6. We will show this using induction.
(Base Case) If $n = 1$ then $1 \cdot 2 \cdot 3 = 6$ which is divisible by 6.
(Inductive Hypothesis) Assume $k(k + 1)(k + 2)$ is divisible by 6 for some natural k.
(Inductive Step) We want to show that the statement holds for $k+1$ or that $(k + 1)(k + 2)(k + 3)$ is divisible by 6, to do this we can use our inductive hypothesis and reasoning. Let $6j = k(k+1)(k+2)$ for some integer j then $(k+1)(k+2)(k+3) = k(k+1)(k+2)+3(k+1)(k+2) = 6j+3(k+1)(k+2)$. Since at least one of $k + 1$ or $k + 2$ will be even then we can factor out a 6 making $(k + 1)(k + 2)(k + 3)$ divisible by 6 by definition. Thus since $k + 1$ holds we conclude by the PMI that $2n^3 + 3n^2 + n$ is divisible by 6 for all $n \in \mathbb{N}$. Since this is the case when we divide this expression by 6 we will always return an integer.

9. (a) Consider that $\sum_{k=1}^{100}[k\cdot(-1)^k] = -1+2-3+4-5+6+\cdots+100 = 50$ since we only achieve an addition of 1 every 2 numbers.

   (b) $\sum_{k=2}^{200}\left(\frac{1}{k}-\frac{1}{k+1}\right) = \left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{200}-\frac{1}{201}\right) = \frac{1}{2}-\frac{1}{201} = \frac{199}{402}$ see that terms cancel out in this sum.

   (c) $\prod_{k=1}^{69} 2^{k-35} = 2^{-34}\cdot 2^{-33}\cdots 2^{35} = 1$ since we multiply $2^{-a}\cdot 2^{a}$ at each number in the product.

   (d) $\prod_{i=10}^{99}\frac{i}{i+1} = \frac{10}{11}\cdot\frac{11}{12}\cdots\frac{99}{100} = \frac{1}{10}$ since we cancel out terms through fraction multiplication.

10. (a) **False.** Consider $a_1 = 1, a_2 = 2$ and $b_1 = 1, b_2 = 2$ where $n = 2$ and all terms are real then $\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) = (1+2)(1+2) = 9$ but $\sum_{k=1}^{n}(a_k\cdot b_k) = 1\cdot 1 + 2\cdot 2 = 5$.

   (b) **True.** Expanding the sum gives

$$\begin{aligned}
\sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k &= (a_1+a_2+\cdots+a_n)-(b_1+b_2+\cdots+b_n)\\
&= (a_1+a_2+\cdots+a_n)-(b_1+b_2+\cdots+b_n)\\
&= (a_1+a_2+\cdots+a_n)+(-b_1-b_2-\cdots-b_n)\\
&= \sum_{k=1}^{n}(a_k-b_k)
\end{aligned}$$

   (c) **False.** Consider $a_1 = 1, a_2 = 2$ and $b_1 = 2, b_2 = 1$ where $n = 2$ and all terms are real then $\prod_{k=1}^{n} a_k - \prod_{k=1}^{n} b_k = 2-2 = 0$ but $\prod_{k=1}^{n}(a_k-b_k) = -1$.

   (d) **True.** Expanding the product gives

$$\begin{aligned}
\left(\prod_{k=1}^{n} a_k\right)\Big/\left(\prod_{k=1}^{n} b_k\right) &= \frac{a_1\cdot a_2\cdots a_n}{b_1\cdot b_2\cdots b_n}\\
&= \prod_{k=1}^{n}\frac{a_k}{b_k}
\end{aligned}$$

   where all $b_1, b_2 \ldots b_n$ are nonzero.

11. (a) We will prove the statement using induction.
   (Base Case) If $n = 1$ then $\sum_{i=1}^{1}\frac{i}{2^i} = \frac{1}{2}$ and $2-\frac{1+2}{2^1} = \frac{1}{2}$ thus both expressions are equal and our base case holds.
   (Inductive Hypothesis) Assume for some natural $k \geq 1$ that $\sum_{i=1}^{k}\frac{i}{2^i} = 2-\frac{k+2}{2^k}$.
   (Inductive Step) We want to show that the statement holds for $k+1$, consider the sum in two pieces

$$\sum_{i=1}^{k+1}\frac{i}{2^i} = \sum_{i=1}^{k}\frac{i}{2^i} + \frac{k+1}{2^{k+1}}$$

now using the inductive hypothesis we achieve

$$\begin{aligned}\sum_{i=1}^{k}\frac{i}{2^i}+\frac{k+1}{2^{k+1}} &= 2-\frac{k+2}{2^k}+\frac{k+1}{2^{k+1}}\\ &= 2-\frac{2(k+2)}{2^{k+1}}+\frac{k+1}{2^{k+1}}\\ &= 2-\frac{(k+1)+2}{2^{k+1}}\end{aligned}$$

thus the statement holds for $k+1$ and by the PMI it holds for all $n\in\mathbb{N}$.

(b) We will prove the statement using induction.
(Base Case) If $n=1$ then $\sum_{i=1}^{1}(3i-2)=1$ and $\frac{(1)(3-1)}{2}=1$ thus both expressions are equal and our base case holds.
(Inductive Hypothesis) Assume for some natural $k\geq 1$ that $\sum_{i=1}^{k}(3i-2)=\frac{k(3k-1)}{2}$.
(Inductive Step) We want to show that the statement holds for $k+1$, consider the sum in two pieces

$$\sum_{i=1}^{k+1}(3i-2)=\sum_{i=1}^{k}(3i-2)+(3(k+1)-2)$$

now using the inductive hypothesis we achieve

$$\begin{aligned}\sum_{i=1}^{k}(3i-2)+(3(k+1)-2) &= \frac{k(3k-1)}{2}+(3(k+1)-2)\\ &= \frac{k\left(3k-1\right)}{2}+3k+1\\ &= \frac{3k^2+5k+2}{2}\\ &= \frac{(k+1)(3(k+1)-1)}{2}\end{aligned}$$

thus the statement holds for $k+1$ and by the PMI it holds for all $n\in\mathbb{N}$.

(c) We will prove the statement using induction.
(Base Case) If $n=2$ then $\prod_{i=2}^{2}\left(1-\frac{1}{i^2}\right)=\frac{3}{4}$ and $\frac{2+1}{2\cdot 2}=\frac{3}{4}$ thus both expressions are equal and our base case holds.
(Inductive Hypothesis) Assume for some natural $k\geq 2$ that $\prod_{i=2}^{k}\left(1-\frac{1}{i^2}\right)=\frac{k+1}{2\cdot k}$.
(Inductive Step) We want to show that the statement holds for $k+1$, consider the product in two pieces

$$\prod_{i=2}^{k+1}\left(1-\frac{1}{i^2}\right)=\prod_{i=2}^{k}\left(1-\frac{1}{i^2}\right)\cdot\left(1-\frac{1}{(k+1)^2}\right)$$

now using the inductive hypothesis we achieve

$$\prod_{i=2}^{k}\left(1-\frac{1}{i^2}\right)\cdot\left(1-\frac{1}{(k+1)^2}\right)=\frac{k+1}{2\cdot k}\cdot\left(1-\frac{1}{(k+1)^2}\right)$$
$$=\frac{(k+1)+1}{2\,(k+1)}$$

thus the statement holds for $k+1$ and by the PMI it holds for all $n \geq 2$ where $n \in \mathbb{N}$.

12. (a) Let $P(1), P(2), \ldots$ be a sequence of statements.
If $P(3)$ is true, and $P(k) \implies P(k+3)$ for all $k \in \mathbb{N}$ then $P(n)$ holds for all $n \in \mathbb{N}$ which are a multiples of 3.

    (b) Let $P(1), \ldots$ be a sequence of statements.
If $P(2)$ is true, and $P(k) \implies P(k+3)$ for all $k \in \mathbb{N}$ then $P(n)$ holds for all $n \in \mathbb{N}$ where $n = 2, 5, 8 \ldots$.

    (c) Let $P(1), P(2), \ldots$ be a sequence of statements.
If $P(11)$ is true, and $P(k) \implies P(k+2)$ for all $k \in \mathbb{N}$ then $P(n)$ holds for all $n \in \mathbb{N}$ where $n = 11, 13, 15, \ldots$.

13. Using Proposition 4.2.3 we see that if $x_1, x_2, \ldots x_n$ are real numbers in the interval $[0,1]$ then $\prod_{i=1}^{n}(1-x_i) \geq 1-\sum_{i=1}^{n} x_i$ and Bernoulli's inequality is for any $n \in \mathbb{N}$ and $x \in \mathbb{R}$ with $x \geq -1$ we have that $(1+x)^n \geq 1+n\cdot x$. Using the proposition with $x_1 = x_2 = \cdots = x_n$ where $x_i \geq -1$ we have that $-x_i \leq 1$, assume $0 \leq -x_i$ so that $x_i \in [0,1]$ then

$$(1+x_i)^n = (1-(-x_i))^n \geq 1-\sum_{i=1}^{n} -x_i = 1+n\cdot x_i$$

which is exactly bernoulli's inequality.

14. (Base Case) If $n=2$ then $10^2-1 = 100-1 = 99$ which is divisible by 11.
(Inductive Hypothesis) Assume for some even k that 10^k-1 is divisible by 11.
(Inductive Step) Our goal is to show that the statement is true for the next even number which is then $k+2$, notice that we can break the expression $10^{2k}-1$ into two pieces

$$10^{k+2}-1 = 100\cdot 10^k - 1 = 99\cdot 10^k + 10^k - 1$$

where 10^k-1 is divisible by 11 by the inductive hypothesis and $99\cdot 10^k$ is divisible by 11 since 99 is. Thus $k+2$ is true which is the next even number so the statement holds for all even $n \in \mathbb{N}$ by the PMI.

15. (a) Suppose k is an arbitrary natural number and assume $2^{3k-1}+5\cdot 3^k$ is divisible by 11. We can rewrite this as $2^{3k-1}+5\cdot 3^k = 11m$ for some integer m. Consider $2^{3(k+2)-1}+5\cdot 3^{k+2}$ which is equivalent to

$$2^{3k-1+6}+5\cdot 3^k\cdot 3^2 = 2^6\cdot 2^{3k-1}+45\cdot 3^k = 64\cdot 2^{3k-1}+45\cdot 3^k$$

which can be rewritten as

$$55 \cdot 2^{3k-1} + 9(2^{3k-1}) + 9 \cdot 5 \cdot 3^k = 55 \cdot 2^{3k-1} + 9(2^{3k-1} + 5 \cdot 3^k)$$

where clearly $55 \cdot 2^{3k-1}$ is divisible by 11 and $2^{3k-1} + 5 \cdot 3^k$ is divisible by our assumption thus the entire expression is divisible by 11.

(b) i. **False.** Let $n = 1$ then $2^{3-1} + 5 \cdot 3^1 = 19$ which is clearly not divisible by 11.

ii. **True.** Suppose $n \in \mathbb{N}$ is an even number then $n = 2k$ for some integer k we will use induction to show this is true.
(Base Case) if $n = 2$ then $2^{3n-1} + 5 \cdot 3^n = 2^{3(2)-1} + 5 \cdot 3^2 = 77$ which is divisible by 11.
(Inductive Hypothesis) Assume the statement holds for some even natural number j or that $2^{3j-1} + 5 \cdot 3^j$ is divisible by 11.
(Inductive Step) We actually already proved this! In part a) we showed that for any $k \in \mathbb{N}$ that if $2^{3k-1} + 5 \cdot 3^k$ is divisible by 11 then $2^{3(k+2)-1} + 5 \cdot 3^{k+2}$ is also divisible by 11. Thus replacing k with j we have our desired result and thus by PMI for every positive natural n, $2^{3n-1} + 5 \cdot 3^n$ is divisible by 11.

16. We will use induction to show this.
(Base Case) If $n = 1$ then $a_1 = 1$ and $a_1 = 1^3 - 1 + 1 = 1$ thus our two statements are equivalent and our base case holds.
(Inductive Hypothesis) Assume for some $k \in \mathbb{N}$ that $a_k = k^3 - k + 1$.
(Inductive Step) We want to show that $a_{k+1} = (k+1)^3 - (k+1) + 1$, consider that we already know $a_{k+1} = a_k + 3k(k+1)$ so using our inductive hypothesis we see that $a_{k+1} = k^3 - k + 1 + 3k(k+1) = k^3 + 3k^2 + 2k + 1$ where

$$\begin{aligned} a_{k+1} &= k^3 + 3k^2 + 2k + 1 \\ &= (k+1)^3 - k \\ &= (k+1)^3 - (k+1) + 1 \end{aligned}$$

thus the statement holds for $k + 1$ and by the PMI it holds for all $n \in \mathbb{N}$.

17. We will use induction to show this.
(Base Case) If $n = 1$ then $a_1 = 3$ and $a_1 = 2(1)^3 - 2(1) + 3 = 3$ thus the base case holds.
(Inductive Hypothesis) Assume for some $k \in \mathbb{N}$ that $a_k = 2k^3 - 2k + 3$.
(Inductive Step) We want to show that $a_{k+1} = 2(k+1)^3 - 2(k+1) + 3$ to do this, consider that we already know $a_{k+1} = a_k + 6k(k+1)$ then by

our inductive hypothesis we find that

$$\begin{aligned} a_{k+1} &= 2k^3 - 2k + 3 + 6k(k+1) \\ &= 2k^3 + 6k^2 + 4k + 3 \\ &= 2(k+1)^3 - 2k + 1 \\ &= 2(k+1)^3 - 2(k+1) + 3 \end{aligned}$$

thus the statement holds for $k+1$ and by the PMI $a_n = 2n^3 - 2n + 3$ for any $n \in \mathbb{N}$.

18. (a) Notice that $\left(x - \frac{1}{2}\right)^2 + \frac{1}{4} \geq 0$ where both terms are clearly greater than 0 which can be expanded into $2x^2 - 2x + 1 \geq 0$ and is equivalent to $3x^2 + 3 \geq (x+1)^2 + 1$ thus the inequality holds. The original inequality cannot be proven using induction since we are asked to show it holds for all **real** numbers and not only natural numbers where induction is defined on.

    (b) (Base Case) If $n = 1$ then $a_1 = 2$ and $1^1 + 1 = 2$ so $a_1 \geq 2$ thus the base case holds.
    (Inductive Hypothesis) Assume for some $k \in \mathbb{N}$ that $a_k \geq k^2 + 1$ holds.
    (Inductive Step) We want to show that $a_{k+1} \geq (k+1)^2 + 1$ and we already know that $a_{k+1} = 3 \cdot a_k$ and by our inductive hypothesis we have that $a_{k+1} \geq 3 \cdot (k^2 + 1) = 3k^2 + 3$ so we what truly want to show is $3k^2 + 3 \geq (k+1)^2 + 1$ however this was already proved in part a) thus the statement holds for $k+1$. By the PMI we have that $a_n \geq n^2 + 1$ for all $n \in \mathbb{N}$.

19. (a)

$$\begin{aligned} \sum_{i=m}^{n} (a_i + b_i) &= (a_m + b_m) + (a_{m+1} + b_{m+1}) + \cdots + (a_n + b_n) \\ &= (a_m + a_{m+1} + \cdots + a_n) + (b_m + b_{m+1} + \cdots + b_n) \\ &= \left(\sum_{i=m}^{n} a_i\right) + \left(\sum_{i=m}^{n} b_i\right) \end{aligned}$$

    (b)

$$\begin{aligned} \prod_{i=m}^{n} (c \cdot a_i) &= (c \cdot a_m) \cdot (c \cdot a_{m+1}) \cdots (c \cdot a_n) \\ &= c^{n-(m-1)} (a_m \cdot a_{m+1} \cdots a_n) \\ &= c^{n-m+1} \cdot \prod_{i=m}^{n} a_i \end{aligned}$$

(c)

$$\begin{aligned}\prod_{i=m}^{n}(a_i \cdot b_i) &= (a_m \cdot b_m)(a_{m+1} \cdot b_{m+1}) \cdots (a_n \cdot b_n)\\ &= (a_m \cdot a_{m+1} \cdots a_n)(b_m \cdot b_{m+1} \cdots b_n)\\ &= \left(\prod_{i=m}^{n} a_i\right) \cdot \left(\prod_{i=m}^{n} b_i\right)\end{aligned}$$

20. We will use strong induction to show the statement.
(Base Case) If $n = 1$ then $x_1 = 3$ and $x_1 = 2^1 + 3^{1-1} = 2 + 1 = 3$ so the base case holds.
(Inductive Hypothesis) Assume for some $k \in \mathbb{N}$ that for all natural numbers $1, 2, \ldots, k$ the statement holds.
(Inductive Step) We want to show that the statement holds for $k + 1$ or that $x_{k+1} = 2^{k+1} + 3^k$. We know that $x_{k+1} = 5 \cdot x_k - 6 \cdot x_{k-1}$ where by our inductive hypothesis we can rewrite this as

$$\begin{aligned}x_{k+1} &= 5 \cdot (2^k + 3^{k-1}) - 6 \cdot (2^{k-1} + 3^{k-2})\\ &= 5 \cdot\ 2^k + 5 \cdot\ 3^{k-1} - 3 \cdot\ 2^k - 2 \cdot\ 3^{k-1}\\ &= 2^{k+1} + 3^k\end{aligned}$$

thus the statement holds for $k + 1$ as well and by the PMI it holds for all $n \in \mathbb{N}$.

21. We will use strong induction to show the statement.
(Base Case) If $n = 1$ then $a_1 = 5$ and $a_1 = 1^2 + 4 = 5$ so the base case holds.
(Inductive Hypothesis) Assume for some $k \in \mathbb{N}$ that for all natural numbers $1, 2, \ldots, k$ the statement holds.
(Inductive Step) We want to show that the statement holds for $k + 1$ or that $a_{k+1} = (k + 1)^2 + 4$. We know that $a_{k+1} = 2a_k - a_{k-1} + 2$ so using our inductive hypothesis we can rewrite this as

$$\begin{aligned}a_{k+1} &= 2(k^2 + 4) - ((k - 1)^2 + 4) + 2\\ &= k^2 + 2k + 5\\ &= (k + 1)^2 + 4\end{aligned}$$

thus the statement holds for $k + 1$ and by PMI it holds for all $n \in \mathbb{N}$.

22. We will use strong induction to show the statement.
(Base Case) If $n = 1$ then $a_1 = 6$ and $a_1 = (4 - 1) \cdot 2^1 = 6$ and if $n = 2$ then $a_2 = 8$ and $a_2 = (4 - 2) \cdot 2^2 = 8$ so the base cases holds.
(Inductive Hypothesis) Assume for some $k \in \mathbb{N}$ that for all natural numbers $3, \ldots, k$ the statement holds.

(Inductive Step) We want to show that the statement holds for $k+1$ or that $a_{k+1} = (4-(k+1)) \cdot 2^{k+1}$. We know that $a_{k+1} = 4a_k - 4a_{k-1}$ so using our inductive hypothesis we can rewrite this as

$$\begin{aligned} a_{k+1} &= 2(k^2+4) - ((k-1)^2+4) + 2 \\ &= k^2 + 2k + 5 \\ &= (k+1)^2 + 4 \end{aligned}$$

thus the statement holds for $k+1$ and by PMI it holds for all $n \in \mathbb{N}$.

23. We will use strong induction to show the statement.
(Base Case) If $n = 1$ then $a_1 = 1$ so clearly $1 \leq a_2 \leq 2$. If $n = 2$ then $a_2 = 1$ so also $1 \leq a_2 \leq 2$ and our base cases holds.
(Inductive Hypothesis) Assume for some $k \in \mathbb{N}$ that for all natural numbers $3, \ldots, k$ the statement holds.
(Inductive Step) We want to show that the statement holds for $k+1$ or that $1 \leq a_{k+1} \leq 2$. Notice that $a_{k+1} = \frac{1}{2}\left(a_k + \frac{2}{a_{k-1}}\right)$ where since $1 \leq a_k \leq 2$ and $1 \leq a_{k-1} \leq 2$ then $2 \leq \frac{2}{a_{k-1}} \leq 1$ so $3 \leq a_k + \frac{2}{a_{k-1}} \leq 3$ thus $a_k + \frac{2}{a_{k-1}} = 3$. Dividing this number by 2 gives $\frac{1}{2}\left(a_k + \frac{2}{a_{k-1}}\right) = \frac{3}{2}$ where $1 \leq \frac{3}{2} \leq 2$ thus $1 \leq a_{k+1} \leq 2$. Furthermore the statement holds for $k+1$ and by the PMI we have that it holds for all $n \in \mathbb{N}$.

24. We will use induction to show the statement.
(Base Case) If $n = 1$ then $a_1 = \frac{5}{2}$ and $\frac{1}{2^1} = \frac{1}{2}$ thus $a_1 > \frac{1}{2}$ so our base case holds.
(Inductive Hypothesis) Assume for some $k \in \mathbb{N}$ that the statement holds or that $a_k > \frac{1}{2^k}$.
(Inductive Step) We want to show that the statement holds for $k+1$ as well, consider that $a_{k+1} = \frac{1}{2} \cdot (a_k + 2)$ and $a_k > \frac{1}{2^k}$ by our inductive hypothesis so

$$a_{k+1} > \frac{1}{2} \cdot \left(\frac{1}{2^k} + 2\right) > \frac{1}{2} \cdot \frac{1}{2^k} = \frac{1}{2^{k+1}}$$

thus the statement holds for $k+1$ and by PMI we have that it holds for all $n \in \mathbb{N}$.

25. We will use strong induction to show the statement.
(Base Case) If $n = 3$ then we assume a_1 is odd and a_2 is odd so $a_3 = 2 \cdot a_2 + 3 \cdot a_1$ which is odd since $2 \cdot a_2$ is even and $3 \cdot a_1$ is odd and our base case holds.
(Inductive Hypothesis) Assume for some $k \in \mathbb{N}$ that the statement holds $4, \ldots, k$ on the condition that a_1, a_2 are odd.
(Inductive Step) We want to show that the statement holds for $k+1$ as well, assume a_1, a_2 are odd then using our inductive hypothesis we know that $a_3, \ldots a_k$ are odd. Since $a_{k+1} = 2a_k + 3a_{k-1}$ and a_k, a_{k-1} are odd

then we have $2a_k$ is even and $3a_{k-1}$ is odd so the only possibility is that a_{k+1} is odd as well. Thus the statement holds for $k+1$ as well and by PMI it holds for all $n \in \mathbb{N}$. [Induction feels redundant here, this is clearly true]

26. We will use induction to show the statement holds.
(Base Case) If $n = 1$ then there are $4^1 = 4$ equilateral triangles of side length 1, if we remove one then we can clearly fit exactly one isosceles trapezoid in the triangle. Thus our base case holds.
(Inductive Hypothesis) Assume for some $k \in \mathbb{N}$ that the statement holds or that a board of 2^k length with 4^k triangles
can be tiled into isosceles trapezoids by removing one triangle from the board. (Inductive Step) We want to show that for a triangular board of 4^{k+1} equilateral triangles. Consider the $4^{k+1}th$ triangle we can break this into four 4^k triangles (since $4^{k+1} = 4 \cdot 4^k$) and by the induction hypothesis these can be tiled into isosceles trapezoids however, only on the condition that they all have one removed. To complete this first notice that we remove a tile from the top part of the tile of triangles, then remove 3 pieces on the bottom middle in an isosceles trapezoid form which will take away from the 3 other triangles that don't have a piece removed. By the inductive hypothesis each of the four large triangles can be tiled into isosceles trapezoids and filling in the isosceles triangle that we removed we see that the entire board can be filled with isosceles triangles. Thus the statement holds for 4^{k+1} triangles and by the PMI it must holds for all $n \in \mathbb{N}$.

27. We can't use the argument in Case 1 for the case where k is odd since $k+1$ is even and so we cannot use the trick. of $1 = 2^0$ and adding it to both sides.
We had to use strong induction because to show every arbitrary number holds because in the case where k is odd we had to use the fact that since m is smaller than $k+1$ then it can be written as a sum of distinct nonnegative integer powers of 2. We cannot use usual induction here since we do not know for certain that $m = k$ since in the case that it doesn't or that $m < k$ then the induction hypothesis won't apply.

28. We will use strong induction to show this.
(Base Case) IF $n = 1$ then $1 = 1 \cdot 2^0$ where 1 is odd and 0 is a nonnegative integer power of 2 thus the base case holds.
(Inductive Hypothesis) Assume for some $k \in \mathbb{N}$ that the statement holds for natural numbers $1, 2 \ldots, k$.
(Inductive Step) We want to show that it holds for $k+1$, consider if k is odd, then $k+1$ is even and $k+1 = 2m$ for some integer m. Then since m is smaller then $k+1$ we have that $m = c \cdot 2^j$ for some odd natural c and nonnegative integer j. Multiplying this expression by 2 gives $2m = c \cdot 2^{j+1} = k+1$ where $j+1$ is clearly still a nonnegative integer thus $k+1$ can be written as the product of an odd natural and a

nonnegative integer power 2 when k is odd. If k is even then $k+1$ is odd. Since $k+1$ is odd we can simply write this as $k+1 = (k+1)2^0$ where 0 is a nonnegative integer. Thus in both cases $k+1$ can be written as a product of an odd natural and a nonnegative integer power 2 and by PMI we have that it holds for all $n \in \mathbb{N}$.

29. We will use induction to show this.
(Base Case) If $n = 1$ and $x = 1$ then $x + \frac{1}{x} = 1 + \frac{1}{1}$ also $x^1 + \frac{1}{x^1} = 1$ which is an integer thus the base case holds.
(Inductive Hypothesis) Assume for some $k \in \mathbb{N}$ that the statement holds for natural numbers $1, 2, \ldots, k$ where x is a nonzero real number such that $x + \frac{1}{x}$ is an integer.
(Inductive Step) We want to show the statement holds for $k+1$, let x be a nonzero real number such that $x + \frac{1}{x}$ is an integer. Consider the expression $(x^k + \frac{1}{x^k})(x + \frac{1}{x})$ where the left product is the inductive hypotehsis and the right product is our assumption. Expanding this gives $x^{k+1} + x^{k-1} + x^{-k+1} + \frac{1}{x^{k+1}}$ which is equivalent to $(x^{k+1} + \frac{1}{x^{k+1}}) + (x^{k-1} + \frac{1}{x^{k-1}})$ where the right hand side is an integer by our inductive hypothesis as well. Since this is equivalent to $(x^k + \frac{1}{x^k})(x + \frac{1}{x}) - (x^{k-1} + \frac{1}{x^{k-1}})$ where the left hand side is an integer as is the righthand side thus $x^{k+1} + \frac{1}{x^{k+1}}$ must be an integer as well.

30. Notice that if $k = 1$ the inductive hypothesis fails however we assumed it to be true for all $0, \ldots, k \in N$ which is enough to break the proof.

31. Notice that if $k = 2$ the inductive hypothesis fails since $5^2 = 25 \neq 5$ however we assumed it to be true for all $1, \ldots, k \in \mathbb{N}$ which is enough to break the proof.

32. Notice that if a person has k hairs on their head for some arbitrary number and we set this to 100000 they will clearly not be bald so our assumption is incorrect.

33. (a) Suppose x, y are real numbers such that $0 \leq x \leq 1 \leq y$ then $0 \leq x \leq 1$ and $1 \leq y$ so clearly $x - 1 \leq 0$ and $y - 1 \geq 0$ thus $(x-1)(y-1) = xy - x - y + 1 \leq 0$ thus $x + y \geq xy + 1$.

    (b) Assuming that $a_1, a_2, \ldots, a_n$ are nonnegative real numbers with $n \geq 2$ then each number must be either 1 or have another number with is the multiplicative inverse of that number so that they multiply to 1. Thus for two numbers a_i, a_j it must be that $a_i \leq 1$ and $a_j \geq 1$ or that $a_i \leq 1 \leq a_j$ where $i \neq j$ choosing. one as the multiplicative inverse of the other.

    (c) We will show the statement by induction.
    (Base Case) If $n = 1$ assume a_1 is a non-negative real number with $a_1 = 1$ then clearly $a_1 \geq 1$ so our base case holds.
    (Inductive Hypothesis) Assume for some $k \in \mathbb{N}$ that if $a_1, \ldots, a_k$ are non-negative real numbers with $a_1 \cdots a_k = 1$ then $a_1 + \cdots + a_k \geq k$.

(Inductive Step) We want to show that the statement holds for $k+1$, assume that $a_1, \dots a_{k+1}$ are nonnegative real numbers such that $a_1 \cdots a_{k+1} = 1$. It follows from part b) that $a_i \leq 1 \leq a_j$ for some $i \neq j$, consider that the product of $k+1$ numbers can be written as the product of k numbers since

$$a_1 \cdot a_2 \cdot a_3 \cdots a_{k+1} = (a_1 \cdot a_2) \cdot a_3 \cdots a_{k+1} = 1$$

choosing $a_i = a_1$ and $a_j = a_2$ we know that $0 \leq a_1 \leq 1 \leq a_2$. It follows from part a) that $a_1 + a_2 \geq a_1 \cdot a_2 + 1$
thus using our inductive hypothesis

$$a_1 \cdot a_2 + a_3 + \cdots a_{k+1} \geq k \implies (a_1 \cdot a_2 + 1) + a_3 + \cdots a_{k+1} \geq k+1$$

where $a_1 \cdot a_2 + 1 \leq a_1 + a_2$ then

$$a_1 + a_2 + a_3 + \cdots a_{k+1} \geq k+1$$

which is what we wanted to show. Thus the statement holds for $k+1$ and consequently by the PMI we have that it holds for all $n \in \mathbb{N}$.

(d) If a single $x_i = 0$ in the AGM inequality it results in $\sqrt[n]{0} = 0 \leq \frac{x_1 + \cdots 0 + \cdots x_n}{2}$ which is clearly true since all $x_1, x_2, \dots x_n \geq 0$. Assume this is not the case, then all $x_i's$ are positive, using part c) with

$$a_1 = \frac{x_1}{\sqrt[n]{x_1 \cdot x_2 \cdots x_n}}, a_2 = \frac{x_2}{\sqrt[n]{x_1 \cdot x_2 \cdots x_n}}, \dots$$

we have that $a_1, a_2 \dots a_n$ are nonnegative real numbers and $a_1 \cdot a_2 \cdots a_n = 1$ (you can verify this) it follows that $0 \leq n \leq a_1 + a_2 + \cdots a_n$ which is equivalent to

$$\sqrt[n]{x_1 \cdot x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}$$

as needed.

To be updated...