Signal Processing and Speech Communication Lab.
Graz University of Technology

AS

Adaptive Systems—Homework Assignment 1

Name(s) Matr.No(s).

The analytical part of your homework (your calculation sheets) **as well as** the MATLAB simulation protocol have to be delivered as **hard copy** to our mailbox at Inffeldgasse 16c, ground floor, no later than **2011/11/18**. Use a printed version of **this entire document** as the title pages and fill in your **name(s) and matriculation number(s)**. Submitting your homework as a LaTeX document can earn you **up to 3 points!**

Your MATLAB programs (`*.m` files) and the simulation protocol (in pdf format!) have to be submitted via **e-mail** to the address `hw2.spsc@tugraz.at` no later than **2011/11/18**. The subject of the e-mail consists of the assignment number and your matriculation number(s) **"Assignment1, MatrNo1, MatrNo2"**. You have to zip (or tar) all your homework files to one single file with the name `Assignment1_MatrNo1_MatrNo2.zip`, e.g., `Assignment1_9833280_9933281.zip`, which has to be attached to the e-mail.

Please make sure that your approaches, procedures, and results are clearly presented. Justify your answers!

Analytical Problem 1.1 (6 Points)—Facts about Wiener Filters

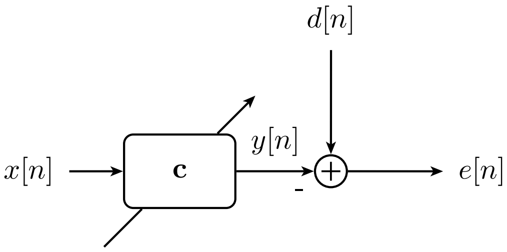

Figure 1: Wiener Filter

Consider the linear filtering problem depicted in Fig. 1. Note that $x[n]$ and $d[n]$ are jointly stationary stochastic processes. We want to optimize the filter coefficients according to the mean squared error (MSE) criterion:

$$\mathbf{c}_{MSE} = \underset{\mathbf{c}}{\operatorname{argmin}}\, J(\mathbf{c}) = \underset{\mathbf{c}}{\operatorname{argmin}}\, E\left\{e^2[n]\right\}.$$

The *Principle of Orthogonality* states:

> The estimate $y[n]$ of the desired signal $d[n]$ is optimal in the MSE-sense if and only if the error $e[n]$ is orthogonal to the input $x[n-m]$ for all m.

(a) Show that this principle is true.

(b) Show that whenever the filter operates in its optimum (i.e., if $\mathbf{c} = \mathbf{c}_{MSE}$), also the error $e[n]$ and the estimate $y[n]$ are orthogonal to each other.

(c) Derive the minimum MSE (MMSE) achievable with a Wiener filter, i.e., compute

$$\min_{\mathbf{c}} J(\mathbf{c}) = J(\mathbf{c}_{MSE}).$$

Analytical Problem 1.2 (15 Points)—System Identification

Consider again the linear filter problem depicted in Fig. 1. We will now use this structure to determine the optimum coefficient vector $\mathbf{c}$ for a desired signal

$$d[n] = \mathbf{h}^T \mathbf{x}[n] + \nu[n]$$

where $\mathbf{h}$ is a vector of length N_h collected the values of a system impulse response and $\mathbf{x}[n]$ is the tap-input vector of according length. $\nu[n]$ is a noise process jointly stationary with $x[n]$.

(a) Assuming no further knowledge is available, determine the length $N = N_h$ coefficient vector optimal in the MSE sense ($\mathbf{c}_{MSE}$). Having done so, identify conditions for perfect identification, i.e, such that $\mathbf{c}_{MSE} = \mathbf{h}$.

(b) Assume now that the conditions from **(a)** are fulfilled. Determine the MMSE!

(c) Consider again the noisy system identification problem. The system to be identified is $\mathbf{h} = [1, -1, 1]^T$ and the input signal is a zero-mean, white process with unit variance. The noise process $\nu[n]$ is also zero-mean, white, and uncorrelated to the input process. Its variance is given as $\sigma_\nu^2 = 0.5$. Determine $\mathbf{c}_{MSE}$ and the corresponding MMSE for $N = 2$, $N = 3$, and $N = 4$. What happens in these cases?

(d) Instead of assuming that $\nu[n]$ is white and uncorrelated with $x[n]$, we now assume that

$$\nu[n] = \frac{1}{2}\left(x[n] + x[n-1]\right).$$

Determine the cross-correlation sequence $r_{\nu x}[k]$ and the autocorrelation sequence $r_{\nu\nu}[k]$ for $k = 0, \ldots, 4$. Compute the variance σ_ν^2.

(e) Taking into account the dependence between the noise process and the input process from task **(d)**, compute the optimal filter coefficients $\mathbf{c}_{MSE}$ for $N = 3$. Does $\mathbf{c}_{MSE} = \mathbf{h}$ still hold? If yes, why, if no, why not? Compare the MMSE from the optimum solution to the MSE obtained by letting $\mathbf{c} = \mathbf{h}$. What difference can you observe?

MATLAB Problem 1.3 (12 Points)—Least-Squares Tracking of a Time-Varying System

Consider again the system identification problem for a time-varying impulse response. In other words, we again focus on the linear filtering problem depicted in Fig. 1 and a desired signal

$$d[n] = \mathbf{h}^T[n]\mathbf{x}[n] + \nu[n]$$

where $\mathbf{h}[n]$ indicates that the impulse response now depends on the actual time n.

(a) Write a MATLAB function that computes the optimum filter coefficients in the sense of *least squares* according to the following specifications:

```
function c = ls_filter( x, d, N)
% x ... input signal
% d ... desired output signal (of same length as x)
% N ... number of filter coefficients
```

(b) For the 'unknown' system, implement a filter with the following time-varying 3-sample impulse response:

$$\mathbf{h}[n] = \begin{bmatrix} 1 \\ 0.001n \\ 2 - 0.003n \end{bmatrix}.$$

Visualize this time-varying impulse response using a waterfall plot (MATLAB function `waterfall`).

(c) Assume now, that $\nu[n] = 0$. Generate 1000 samples (for $n = 0 \dots 999$) of an input signal drawn from a stationary white noise process with zero mean and variance $\sigma_x^2 = 1$. Compute the output $d[n]$ of the system, under the condition that all delay elements are initialized with zeros (i.e., $x[n] = 0$ for $n < 0$).

The adaptive filter has 3 coefficients ($N = 3$). By calling the MATLAB function `ls_filter` with length-M segments of both $x[n]$ and $d[n]$, the coefficients of the adaptive filter $\mathbf{c}[n]$ for $n = 0 \dots 999$ can be computed. Note, we then obtain $\mathbf{c}[n] = \operatorname{argmin}_{\mathbf{c}} J(\mathbf{c}, n)$ where we may rewrite the cost function as $J(\mathbf{c}, n) = \sum_{k=n-M+1}^{n} |e[k]|^2$. For a segment lengths of $M \in \{20, 50\}$, create plots comparing the elements of the coefficient vector $c_j[n]$ to the elements of the impuls response vector $h_j[n]$. Compare and discuss your results and explain the effects of M.

(d) Repeat task **(c)** for $\nu[n]$ being a zero-mean white noise process with variance $\sigma_\nu^2 = 0.5$.

(e) Repeat task **(c)** for $\nu[n]$ being generated by

$$\nu[n] = \frac{1}{2}\left(x[n] + x[n-1]\right).$$

Analytical Problem 1.4 (5 Points)—Bonus Exponentially Weighted Least Squares

We stick to the linear filtering problem depicted in Fig. 1. Instead of assuming (jointly) stationary stochastic processes, consider that observations of $x[n]$ and $d[n]$ for $n = 0 \dots n_0$ are given.

(a) Derive the optimum filter coefficients $\mathbf{c}$ in the sense of exponentially weighted least squares, i.e., find $\mathbf{c}_{opt} = \operatorname{argmin}_{\mathbf{c}} J(\mathbf{c})$ where the cost function is

$$J(\mathbf{c}) = \sum_{n=0}^{n_0} \lambda^{n_0 - n} \cdot |e[n]|^2,$$

with the so-called 'forgetting factor' $0 < \lambda \leq 1$. Use vector/matrix notation!

(b) Explain the effect of the weighting and answer for what scenario(s) such an exponential weighting may be meaningful. Discuss how to select λ for a stationary, a slowly time-varying, or a quickly time-varying scenario.