Measure Theory HW Sections 3.1-3.2

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1 Problem 1.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. We say a linear operator $T : \mathfrak{X} \to \mathfrak{Y}$ is continuous at a point $a \in X$ if for all $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\|x - a\|_X < \delta \implies \|T(x - a)\| = \|Tx - Ta\|_Y < \varepsilon$$

for $x \in X$.

1.1 Part (a)

Proof.
Suppose we have a linear operator $T : \mathfrak{X} \to \mathfrak{Y}$ where $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces.

($\Longrightarrow$) Suppose T is continuous. Let $\delta > 0$ be the number such that

$$\|x\|_{\mathfrak{X}} \leq \delta \implies \|Tx\|_{\mathfrak{Y}} \leq 1.$$

for $x \in X$. Then we have

$$\|Tx\|_{\mathfrak{Y}} = \left\| \frac{\|x\|_{\mathfrak{X}}}{\delta} T\left(\frac{\delta}{\|x\|_{\mathfrak{X}}} x\right) \right\|_{\mathfrak{Y}} = \frac{\|x\|_{\mathfrak{X}}}{\delta} \left\| T\left(\frac{\delta}{\|x\|_{\mathfrak{X}}} x\right) \right\|_{\mathfrak{Y}}.$$

However,

$$\left\| \frac{\delta}{\|x\|_{\mathfrak{X}}} x \right\|_{\mathfrak{X}} = \delta,$$

so we must have

$$\|Tx\|_{\mathfrak{Y}} \leq \frac{\|x\|_{\mathfrak{X}}}{\delta} \cdot 1 = \frac{1}{\delta} \|x\|_{\mathfrak{X}}.$$

($\Longleftarrow$) Suppose there is some c such that

$$\|Tx\|_{\mathfrak{Y}} \leq c \|x\|_{\mathfrak{X}},$$

i.e. T is bounded. Let $a \in X$ be fixed and $\varepsilon > 0$ be given. Choose $\delta < \frac{\varepsilon}{c}$. Then clearly

$$\|x - a\|_{\mathfrak{X}} < \delta \implies \|T(x - a)\|_{\mathfrak{Y}} \leq c \|x - a\|_{\mathfrak{X}} < \varepsilon$$

and T is continuous. $\square$

1.2 Part (b)

Proof.
We define the operator norm on $\mathfrak{X}^*$ by

$$\|L\|_{op} := \sup_{x \in \mathfrak{X} \setminus \{0\}} \frac{|Lx|}{\|x\|_{\mathfrak{X}}}.$$

Notice that we have an explicit bound on $\|L\|_{op}$, namely it is less than or equal to c where $\|Lx\|_{\mathfrak{Y}} \leq c\|x\|_{\mathfrak{X}}$. To see that it is indeed a norm, notice that we have the triangle inequality by the triangle inequality on $\mathbb{R}$, i.e.

$$\|L+T\|_{op} = \sup_{x \in \mathfrak{X} \setminus \{0\}} \frac{|Lx + Tx|}{\|x\|_{\mathfrak{X}}} \leq \sup_{x \in \mathfrak{X} \setminus \{0\}} \frac{|Lx| + |Tx|}{\|x\|_{\mathfrak{X}}} = \|L\|_{op} + \|T\|_{op}.$$

Absolute homogeneity and positive definiteness are trivial. Basic properties of linear functionals provides the fact that $\mathfrak{X}^*$ is a vector space, so it suffices to show that the space is complete to show that it is Banach.

Let $\varepsilon > 0$ be given and let $\{L_n\}$ be a Cauchy sequence in $\mathfrak{X}^*$ with respect to the operator norm. Clearly, there is an integer N such that

$$n, m \geq N \implies \frac{|L_n x - L_m x|}{\|x\|_{\mathfrak{X}}} < \varepsilon$$

for every $x \in \mathfrak{X}$. Considering fixed $x \in \mathfrak{X}$, we clearly deduce that $\{L_n x\}$ is Cauchy in $\mathbb{R}$ pointwise, since $\|x\|_{\mathfrak{X}}$ is just a constant. By the **Cauchy Criterion** in $\mathbb{R}$, for each $x \in \mathfrak{X}$, we have that $\{L_n x\}$ converges to a limit which we shall call Lx.

We shall now show that $L \in \mathfrak{X}^*$. To do this, we first prove a useful result.

Lemma. If $T \in \mathfrak{X}^*$, then

$$|Tx| \leq \|T\|_{op} \|x\|_{\mathfrak{X}}.$$

Proof.

$$\|T\|_{op} \|x\|_{\mathfrak{X}} = \sup_{x \in X \setminus \{0\}} \frac{|Tx|}{\|x\|_{\mathfrak{X}}} \|x\|_{\mathfrak{X}} = \sup_{x \in X \setminus \{0\}} \geq |Tx|.$$

□

Notice that L must be linear since

$$L(x+y) = \lim_{n \to \infty} L_n(x+y) = \lim_{n \to \infty} (L_n x + L_n y) = Lx + Ly,$$

and

$$L(\lambda x) = \lim_{n \to \infty} L_n(\lambda x) = \lambda \lim_{n \to \infty} L_n x = \lambda Lx.$$

Furthermore, L must be bounded since Cauchy sequences are bounded meaning that there must be some c such that $\|L_n\|_{op} \leq c$, and we have

$$\|L\|_{op} = \sup \frac{|Lx|}{\|x\|_{\mathfrak{X}}} = \sup \left(\lim_{n\to\infty} \frac{|L_n x|}{\|x\|_{\mathfrak{X}}} \right) \leq \sup \left(\limsup_{n\to\infty} \|L_n\|_{op} \right) \leq c$$

Now, all that is left is to prove that $L_n \to L$ in the operator norm. Since $\{L_n\}$ is Cauchy in $\mathfrak{X}^*$, we have an integer N_0 such that

$$n, m \geq N_0 \implies \|L_n - L_m\|_{op} < \varepsilon.$$

Using the lemma proved above, for large enough n, m,

$$|L_n x - L_m x| \leq \|L_n - L_m\| \|x\|_{\mathfrak{X}} < \varepsilon \|x\|_{\mathfrak{X}}$$

for all $x \in \mathfrak{X}$. Taking the limit on the right as $m \to \infty$, we have that

$$|L_n x - Lx| \leq \varepsilon \|x\|_{\mathfrak{X}} \implies \frac{|(L_n - L)x|}{\|x\|_{\mathfrak{X}}} \leq \varepsilon,$$

and taking the supremum over all $x \in X \setminus \{0\}$ gives the result.

□

2 Problem 2.

2.1 Part (a)

Proof.
Let $(X, \mathcal{M})$ be a measurable space and $M(X)$ the space of all (signed) complex measures on X. It is trivial to show the vector space properties of $M(X)$, as since each measure maps into the complex numbers, we may define addition and scalar multiplication in the natural way (we get the properties for free). Hence, all that is left is to show that $M(X)$ is a Banach space.

To do this, we first characterize the total variation of a measure $\mu \in M(X)$ in a different way.

Proposition 2.1.1. If μ is a signed measure on X and $E \in \mathcal{M}$, then

$$|\mu|(E) = \sup \left\{ \sum_{j=1}^{n} |\mu(E_j)| : E_1 \ldots E_n \text{ partition } E, E_j \in \mathcal{M} \right\}.$$

This also holds for an infinite (countable) partition of E.

Proof. Notice that since for any measurable A, we have $|\mu(A)| \leq |\mu|(A)$, we have

$$\sum_{j=1}^{n} |\mu(E_j)| \leq \sum_{j=1}^{n} |\mu|(E_j) = |\mu|(E),$$

and taking the supremum over all partitions gives the inequality. Furthermore, if $P \cup N = X$ is a Hahn-decomposition for μ, then

$$|\mu(E\cap P)| + |\mu(E\cap N)| = \mu^+(E\cap P) + \mu^-(E\cap P) + \mu^+(E\cap N) + \mu^-(E\cap N)$$

where the right side is just the definition of $|\mu|(E)$, so we have

$$\begin{aligned} |\mu|(E) &= |\mu(E\cap P)| + |\mu(E\cap N)| \\ &= \left|\sum_{j=1}^{n} \mu(E_j \cap P)\right| + \left|\sum_{j=1}^{n} \mu(E_j \cap N)\right| \\ &\leq \sum_{j=1}^{n} (|\mu(E_j \cap P)| + |\mu(E_j \cap N)|) \end{aligned}$$

which clearly gives us our result since the last sum must necessarily be less than or equal to the supremum. Hence, we have shown both inequalities and we are done. $\square$

Let $\{\lambda_n\}$ be a Cauchy sequence of measures in $M(X)$. For all $\varepsilon > 0$, there is some integer N such that

$$n, m \geq N \implies \|\lambda_n - \lambda_m\|_{TV} = |\lambda_n - \lambda_m|(X) < \varepsilon.$$

Notice that for any measurable $A \in \mathcal{M}$, we obviously have

$$|\lambda_n(A) - \lambda_m(A)|_{\mathbb{C}} \leq |\lambda_n - \lambda_m|(X) < \varepsilon$$

since $\mu \leq |\mu|$. This means that by the **Cauchy Criterion** on $\mathbb{C}$, there is a measure λ such that $\lambda_n \to \lambda$ uniformly on $\mathcal{M}$. It is easy to see that λ maps the empty set to 0 and finite additivity can also easily be checked. For the latter, consider a finite collection of disjoint sets $\{E_j\}$ with $E = \bigcup_j E_j$ and take n such that

$$|\lambda_n(E) - \lambda(E)| < \frac{\varepsilon}{2}$$

and

$$|\lambda_n(E_j) - \lambda(E_j)| < \frac{\varepsilon}{2} 2^{-j},$$

which gives

$$\left|\lambda_n(E) - \sum_{j=1}^{k} \lambda(E_j)\right| \leq \sum_{j=1}^{k} |\lambda_n(E) - \lambda(E_j)| < \frac{\varepsilon}{2}.$$

This shows that

$$\left|\lambda_n(E) - \sum_{j=1}^{k} \lambda(E_j)\right| \leq |\lambda_n(E) - \lambda(E)| + \left|\lambda_n(E) - \sum_{j=1}^{k} \lambda(E_j)\right| < \varepsilon$$

for all $\varepsilon > 0$, showing finite additivity. Hence, we may use the proposition above since no use of σ-additivity was made in its proof (for the finite partition case).

Clearly, since $\{E_n\}$ is Cauchy in the TV norm, we have that for any partition $\{E_j\}$ of X,

$$\sum_{j=1}^{n} |\lambda_n(E_j) - \lambda_m(E_j)| \leq |\lambda_n - \lambda_m|(X) = \|\lambda_n - \lambda_m\|_{TV} < \varepsilon.$$

Taking the limit as $m \to \infty$, we get that

$$\sum_{j=1}^{n} |\lambda_n(E_j) - \lambda(E_j)| < \varepsilon,$$

for all partitions of X, and taking the supremum over all partitions gives

$$\|\lambda_n - \lambda\|_{TV} = |\lambda_n - \lambda|(X) = \sup \left\{ \sum_{j=1}^{n} |(\lambda_n - \lambda)(E_j)| \right\} \leq \varepsilon,$$

which shows that $\lambda_n \to \lambda$ in the TV norm.

All that is left is to show that λ is indeed a measure by showing σ-additivity. Let $\{E_j\}$ be a sequence of disjoint sets in $\mathcal{M}$ whose union is $E \in \mathcal{M}$. Notice that for each $\varepsilon > 0$, there exists integer N_1 such that

$$n \geq N_1 \implies |\lambda_n(E) - \lambda(E)| < \varepsilon.$$

Furthermore, for every m, there is another integer N_2 such that

$$n \geq N_2 \implies \|\lambda_n - \lambda\|_{TV} < \frac{\varepsilon}{2} 2^{-m},$$

but we know that

$$|\lambda_n(E_j) - \lambda(E_j)| \leq \|\lambda_n - \lambda\|_{TV} < \frac{\varepsilon}{2} 2^{-m}$$

for each E_j, so taking $N := \max(N_1, N_2)$, we get that

$$\left|\lambda_n(E) - \sum_{j=1}^{m} \lambda(E_j)\right| \leq |\lambda_n(E) - \lambda(E)| + \left|\lambda_n(E) - \sum_{j=1}^{m} \lambda(E_j)\right| < \varepsilon,$$

and taking the limit as $m \to \infty$ gives the result. $\square$

2.2 Part (b)

Proof.
Let μ be a fixed positive measure on $(X, \mathcal{M})$. Define the map $\varphi : L^1(\mu) \to M(X)$ by

$$\varphi(f) = f\,\mathrm{d}\mu\,, E \in L^1(\mu).$$

This means that

$$\varphi(f)(E) = \int_E f\,\mathrm{d}\mu\,,$$

and we want to show that φ is a linear isometric embedding, or that φ is linear and

$$\|f\|_1 = \int |f|\,\mathrm{d}\mu = |f\,\mathrm{d}\mu\,|(X) = \|f\,\mathrm{d}\mu\|_{TV}.$$

Linearity follows from the linearity of the integral, i.e.

$$\varphi(af + bg)(E) = a\int_E f\,\mathrm{d}\mu + b\int_E g\,\mathrm{d}\mu\,.$$

To show that φ is isometric, we notice that we have

$$|f\,\mathrm{d}\mu\,|(X) = (f\,\mathrm{d}\mu)^+(P) + (f\,\mathrm{d}\mu)^-(N) = \int_P f\,\mathrm{d}\mu + \int_N f\,\mathrm{d}\mu\,,$$

where $X = P \cup N$ is a Hahn-decomposition for X such that $f \geq 0$ on P μ-a.e. while $f \leq 0$ on N μ-a.e. Clearly, we have that

$$\int_P f\,\mathrm{d}\mu + \int_N f\,\mathrm{d}\mu = \int_P f^+ + f^-\,\mathrm{d}\mu + \int_N f^+ + f^-\,\mathrm{d}\mu = \int_P |f|\,\mathrm{d}\mu + \int_N |f|\,\mathrm{d}\mu$$

since the negative part of f is 0 on P (resp. positive part on N) and $|f| = f^+ + f^-$. Clearly, this shows that

$$\|f\,\mathrm{d}\mu\|_{TV} = \int |f|\,\mathrm{d}\mu = \|f\|_1,$$

and φ is isometric. □

3 Problem 3.

3.1 Part (a)

Proof.
We shall do this using the concavity of the logarithm. Since the logarithm is a concave function, for $\lambda \in [0, 1]$, we have

$$\log(\lambda x + (1-\lambda)y) \geq \lambda \log(x) + (1-\lambda)\log(y).$$

Then since

$$\frac{1}{p} + \frac{1}{q} = 1,$$

we must have that

$$\log\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) \geq \frac{1}{p}\log(a^p) + \frac{1}{q}\log(b^q) = \log\left(a^{\frac{p}{p}}b^{\frac{q}{q}}\right).$$

Taking the exponential on both sides,

$$a^{\frac{p}{p}}b^{\frac{q}{q}} = ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

which gives us our result. □

3.2 Part (b)

Proof.
Let us denote by a and b the following:

$$a = \left(\int |f|^p \, \mathrm{d}\mu\right)^{1/p}, \quad b = \left(\int |g|^q \, \mathrm{d}\mu\right)^{1/q}.$$

Notice that

$$\left(\int \left(\frac{|f|}{a}\right)^p \mathrm{d}\mu\right)^{1/p} = \frac{1}{a}\left(\int |f|^p \, \mathrm{d}\mu\right)^{1/p} = 1 = \left(\int \left(\frac{|g|}{b}\right)^q \mathrm{d}\mu\right)^{1/q}.$$

By **Young's inequality**, we have

$$\int \frac{|fg|}{ab} \, \mathrm{d}\mu \leq \frac{1}{p}\int \left(\frac{|f|}{a}\right)^p \mathrm{d}\mu + \frac{1}{q}\int \left(\frac{|g|}{b}\right)^q \mathrm{d}\mu = 1,$$

which implies

$$\int fg \, \mathrm{d}\mu \leq \int |fg| \leq ab = \left(\int |f|^p \, \mathrm{d}\mu\right)^{1/p}\left(\int |g|^q \, \mathrm{d}\mu\right)^{1/q}.$$

□

3.3 Part (b) Bonus

Proof.
We see that

$$
\begin{aligned}
\|f+g\|_p^p &= \int |f+g|^p \\
&= \int |f+g||f+g|^{p-1} \\
&\le \int |f||f+g|^{p-1} + \int |g||f+g|^{p-1} \\
&\le \left(\int |f+g|^p\right)^{1-1/p} \left(\int |f|^p\right)^{1/p} + \left(\int |f+g|^p\right)^{1-1/p} \left(\int |g|^p\right)^{1/p} \\
&= \left(\int |f+g|^p\right)^{1-1/p} \left(\left(\int |f|^p\right)^{1/p} + \left(\int |g|^p\right)^{1/p}\right) \\
&= \left(\|f+g\|_p^p\right)^{1-1/p} (\|f\|_p + \|g\|_p) \\
&= \|f+g\|_p^{p-1}(\|f\|_p + \|g\|_p),
\end{aligned}
$$

which gives

$$\|f+g\|_p^p \le \|f+g\|_p^{p-1}(\|f\|_p + \|g\|_p) \implies \|f+g\|_p \le \|f\|_p + \|g\|_p.$$

The second inequality was deduced using **Hölder's inequality** on the two integrals. $\square$

3.4 Part (c)

Proof.
Let us define a map $T : L^q(X,\mathbb{R}) \to (L^p(X,\mathbb{R}))^*$ by

$$(T(f))(g) = \int_X fg \, \mathrm{d}\mu \, .$$

When f is 0 the result is trivial, hence we suppose f and g are nonzero functions. Notice that

$$\|g\|_p \|f\|_q \ge \|fg\|_1 \implies \|f\|_q \ge \frac{|(T(f))(g)|}{\|g\|_p}$$

for each $g \in L^p(X)$ by **Hölder's inequality**, and in fact this shows that T is well defined since it implies fg is L^1 and hence integrable, showing that it indeed maps into the dual of $L^p(X)$. To show that

$$\|f\|_q = \|T(f)\|_{op},$$

we must show the opposite direction for the inequality, that is,

$$\|f\|_q \le \|T(f)\|_{op}.$$

Let us consider a sequence of functions g_n defined by

$$g_n(x) = \begin{cases} \operatorname{sgn}(f)|f(x)|^{q-1}, & |f(x)| \leq n, \\ 0, & |f(x)| > n. \end{cases}$$

Clearly, $0 \leq g_n \leq |f|^{q-1}$ and $g_n \uparrow |f|^{q-1}$ and $g_n \in L^p(X)$ since

$$\|g_n\|_p^{q/p} = \left(\int |g_n|^p \right)^{1/q} \leq \left(\int |f|^q \right)^{1/q} < \infty$$

since $f \in L^q(X)$. We have

$$\|g_n\|_p^p = \int |g_n|^p = \int |g_n||g_n|^{\frac{1}{q-1}} \leq \int |g_n f|,$$

and since g_n has a factor of $\operatorname{sgn}(f)$ in its definition, it will necessarily be positive when multiplied by f, so

$$\|g_n\|_p^p = \int |g_n f| = \int g_n f = (T(f))(g_n) \leq \|T(f)\|_{op} \|g_n\|_p.$$

This implies that

$$\left(\int |g_n|^p \right)^{1/q} = \|g_n\|_p^{p-1} \leq \|T(f)\|_{op}.$$

Then, we use **Fatou's lemma** to see that

$$\|f\|_q = \left(\int |f|^{(q-1)p} \right)^{1/q} \leq \liminf_{n \to \infty} \left(\int |g_n|^p \right)^{1/q} \leq \|T(f)\|_{op},$$

since $q = (q-1)p$. □

3.5 Part (d)

Proof.
Let $\Lambda \in (L^p(X, \mathbb{R}))^*$ and define

$$\lambda(A) := \sup \left\{ \Lambda(\chi_E) : E \subseteq A, \mu(E) < \infty \right\}.$$

Clearly, $\lambda(\varnothing) = 0$ since the only subset of the empty set is the empty set itself, and $\Lambda(\mathbf{0}) = 0$ since Λ is a linear functional on $L^p(X)$. For σ-additivity, if $\{A_j\}$ is a sequence of disjoint sets of finite measure whose union is $A \in \mathcal{M}$ (possible since we are working in σ-finite spaces), then

$$\sum_{j=1}^{\infty} \Lambda(\chi_{A_j}) = \Lambda \left(\sum_{j=1}^{\infty} \chi_{A_j} \right) = \Lambda(\chi_{\bigcup_j A_j}) \leq \lambda(A).$$

On the other hand, if $E \subseteq A$ has finite measure and is partitioned by $\{E_j\}$ where every E_j is a subset of or equal to some A_j, then

$$\Lambda(\chi_E) = \Lambda(\chi_{\bigcup_j E_j}) = \Lambda\left(\sum_{j=1}^{\infty} \chi_{E_j}\right) = \sum_{j=1}^{\infty} \Lambda(\chi_{E_j}) \leq \sum_{j=1}^{\infty} \lambda(E_j) \leq \sum_{j=1}^{\infty} \lambda(A_j),$$

and taking the supremum over all $E \subseteq A$ gives the result for partitions of A where each set has finite measure. To extend this to partitions where the sets may not have finite measure, notice that for each such set A_j, we may find some partition of A_j with solely finite sets, and adding these measures together gives the result.

Notice that if $\chi_E \in L^p(X, \mathbb{R})$, then E must be a set of finite measure, meaning that

$$\Lambda(\chi_E) = \lambda(E).$$

Hence, if $\varphi = \sum_{j=1}^{n} a_j \chi_{E_j}$ is a simple function in $L^p(X, \mathbb{R})$, then we have

$$\int \varphi \, \mathrm{d}\lambda = \int \sum_{j=1}^{n} a_j \chi_{E_j} \, \mathrm{d}\lambda = \sum_{j=1}^{n} a_j \lambda(E_j) = \sum_{j=1}^{n} a_j \Lambda(E_j) = \Lambda(\varphi).$$

Hence, if $f \in L^p(X, \mathbb{R}) \cap L^+(X)$, then there is a sequence of simple functions $\{\varphi_n\}$ that increases to f, and the **MCT** provides

$$\int f \, \mathrm{d}\lambda = \lim_{n \to \infty} \int \varphi_n \, \mathrm{d}\lambda = \lim_{n \to \infty} \Lambda(\varphi_n) = \Lambda(f)$$

since the dual is the space of bounded (which implies continuous) linear functionals. Taking $\Lambda(|f|)$ also shows that $f \in L^1(\lambda)$. □

3.6 Part (e)

Proof.
Suppose $A \in \mathcal{M}$ is null, i.e. $\mu(A) = 0$. Then

$$\lambda(A) = \Lambda(\chi_A),$$

but we consider two functions to be equal in $L^p(\mu)$ when they agree μ-a.e., meaning that

$$\Lambda(\chi_A) = \Lambda(\mathbf{0}) = 0 = \lambda(A),$$

implying that $\lambda \ll \mu$. Since we are working on σ-finite spaces and $\lambda \ll \mu$, we apply the **Radon-Nikodym theorem** to deduce the existence of a function g such that

$$\Lambda(f) = \int f \, \mathrm{d}\lambda = \int fg \, \mathrm{d}\mu \,,$$

the Radon-Nikodym derivative of λ wrt μ. □

3.7 Part (f)

Proof.
Let $f \in L^1(\lambda)$. We see that

$$\int f \,\mathrm{d}\lambda = \int fg \,\mathrm{d}\mu < \infty \implies \left|\int fg \,\mathrm{d}\mu\right| < \infty.$$

Furthermore, from the results of part (c), we know that

$$\sup_{f \in L^p(X)} \frac{|(T(g))(f)|}{\|f\|_p} = \|T(g)\|_{op} = \|g\|_q$$

since T is isometric, and since

$$\frac{|(T(g))(f)|}{\|f\|_p} = \frac{\left|\int fg \,\mathrm{d}\mu\right|}{\|f\|_p}$$

and the latter is finite for every $f \in L^p(X)$, we have that $\|g\|_q$ must be finite and $g \in L^q(X, \mathbb{R})$. □

3.8 Part (g)

Proof.
We define the positive part Λ^+ of a functional $\Lambda \in (L^p(X, \mathbb{R}))^*$ by

$$\Lambda^+(f) := \sup\{\Lambda(g) : g \in L^p(X), 0 \leq g \leq f\}.$$

Clearly,

$$\Lambda^+(af) = a\Lambda(f)$$

when a is a positive real number, and both sides will be 0 when it is not. Notice that for two functions $f, h \in L^p(X)$, if $0 \leq g_1 \leq f$ and $0 \leq g_2 \leq h$, then

$$\Lambda(g_1) + \Lambda(g_2) = \Lambda(g_1 + g_2) \leq \Lambda^+(f + h),$$

and taking the supremum over all such functions g_1 and g_2 gives one side of the inequality. On the other hand, for functions $g \in L^p(X)$ satisfying $0 \leq g \leq f + h$, we may write

$$g = g_1 + g_2,$$

where $0 \leq g_1 \leq f$ and $0 \leq g_2 \leq h$. By linearity,

$$\Lambda(g) = \Lambda(g_1) + \Lambda(g_2) \leq \Lambda^+(f) + \Lambda^+(h)$$

and taking the supremum over all such g proves the equality of $\Lambda^+(f + h)$ and $\Lambda^+(f) + \Lambda^+(h)$. Hence, Λ^+ is a linear functional in the dual and clearly is positive. Defining Λ^- by

$$\Lambda^- := \Lambda^+ - \Lambda,$$

we easily see that Λ^- is also positive and linear.

This implies that we may express every linear functional $\Lambda \in (L^p(X, \mathbb{R})^*$ as the difference of two integrals, namely

$$\Lambda(f) = \int f \, \mathrm{d}\lambda_{\Lambda^+} - \int f \, \mathrm{d}\lambda_{\Lambda^-} \, ,$$

where λ_{Λ^+} and λ_{Λ^-} are the corresponding supremum measures for the respective positive linear functionals. Hence, $T : L^q(X, \mathbb{R}) \to (L^p(X, \mathbb{R}))^*$ is surjective. $\square$

3.9 Part (h)

Proof.
Part (c) holds in $L^p(X, \mathbb{C})$ since the polar decomposition is defined for any function $f : X \to \mathbb{C}$. Every other part may be accounted for by merely splitting the functions up into the real, imaginary, positive, and negative components. $\square$