Measure Theory HW Sections 7.1-7.2

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1 Problem 1

1.1 Part (a)

Proof.
Let $\mathfrak{X}$ be an infinite dimensional Banach space. Define the unit ball $\mathbb{B}$ as

$$\mathbb{B} := \{x \in \mathfrak{X} : \|x\| \leq 1\}.$$

Suppose the unit ball is compact. Let us cover $\mathbb{B}$ with open balls of radius $\frac{1}{2}$, namely $B_{\frac{1}{2}}(x)$ for each $x \in \mathbb{B}$. Since $\mathbb{B}$ is assumed to be compact, there is a finite subcover

$$B_{\frac{1}{2}}(x_1), \ldots, B_{\frac{1}{2}}(x_n)$$

such that $\mathbb{B} \subseteq \bigcup_{j=1}^{n} B_{\frac{1}{2}}(x_j)$. Clearly, this implies that $x_1, \ldots, x_n$ do not span $\mathfrak{X}$ since if they did, it would imply that $\mathfrak{X}$ is finite dimensional. We now prove a useful lemma.

Lemma 1.1.1. Define $C = \overline{\operatorname{span}(x_1, \ldots, x_n)}$. There is a $y \notin C$ with $\|y\| = 1$ (and hence $y \in \mathbb{B}$) and

$$\|x - y\| > \frac{1}{2}$$

for all $x \in C$.

Proof. Let $y' \notin C$ be fixed. Define $M := \inf \|x - y'\|$ over all $x \in C$. Notice that by the definition of the infimum, there is an $x' \in C$ with

$$\|x' - y'\| < 2M.$$

Setting $y := (x' - y')/\|x' - y'\|$, y is a unit vector and we have

$$\begin{aligned}
\inf_{x \in C} \|x - y\| &= \inf_{x \in C} \left\| x - \frac{y'}{\|x' - y'\|} + \frac{x'}{\|x' - y'\|} \right\| \\
&= \inf_{x \in C} \left\| \frac{x}{\|x' - y'\|} - \frac{y'}{\|x' - y'\|} + \frac{x'}{\|x' - y'\|} \right\| \\
&= \frac{\inf_{x \in C} \|x - y'\|}{\|x' - y'\|} \\
&\geq \frac{M}{2M} \\
&= \frac{1}{2}.
\end{aligned}$$

The second and third equalities are valid as the infimum is taken over the closure of the span (a subspace), and hence scalar multiplication or vector addition by another vector in the span does not affect the infimum. □

By the lemma above, there must be some point $y \in \mathbb{B}$ such that

$$\|y - x\| > \frac{1}{2}$$

for all $x \in \operatorname{span}(x_1, \ldots, x_n)$, but this is a contradiction since the open balls with centers $x_1, \ldots, x_n$ and radius $\frac{1}{2}$ cover $\mathbb{B}$. □

1.2 Part (b)

Proof.
Let X be a set and $\mathcal{F}$ a set of functions $f : X \to Y_f$ where (Y_f, τ_{Y_f}) is a topological space. We define the weak topology, which we denote $\mathcal{T}$, as the coarsest topology on X such that each f is continuous. To show that this topology always exists, notice that we may merely consider the topology

$$\mathcal{T} = \tau(\{f^{-1}(X) : f \in \mathcal{F}, A \in \tau_{Y_f}\}),$$

namely the topology generated by the preimages of open sets in the respective topologies. Then every f is continuous with respect to this topology, and it is clearly the weakest possible such topology by the definition of the topology generated by certain sets. □

1.3 Part (c)

Proof.
Let $\mathfrak{X}$ be a Banach space. The *weak*-topology* on $\mathfrak{X}^*$ is the then the weak topology generated by the functions

$$\operatorname{ev}_x : (\mathfrak{X}^*, \mathcal{T}^*) \to \mathbb{C}, \qquad \operatorname{ev}_x(L) = L(x)$$

for $x \in \mathfrak{X}$, denoting the topology by $\mathcal{T}^*$.

($\Longrightarrow$) Suppose that a net $(L_\lambda)_{\lambda\in\Lambda}$ on $\mathfrak{X}^*$ in the weak*-topology converges to some $L \in \mathfrak{X}^*$, that is, (L_λ) is eventually in each $B \in \mathcal{O}(L)$ where $\mathcal{O}(L)$ is the collection of all open sets in the weak*-topology containing $L \in \mathfrak{X}^*$. We first prove a basic fact about nets.

Lemma 1.3.1. Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a map between topological spaces and let $(x_\lambda)_{\lambda\in\Lambda}$ be a net on X that converges to $x \in X$. Then $f(x_\lambda) \to f(x)$ with respect to τ_Y.

Proof. Notice that if $B \in \mathcal{O}(f(x))$ then $f^{-1}(B) \in \mathcal{O}(x)$ since f is continuous and x is clearly in the preimage. However, we know that there must be a $\lambda_B \in \Lambda$ such that $x_\lambda \in f^{-1}(B)$ for $\lambda \geq \lambda_B$, which clearly implies that $f(x_\lambda) \in B$ for such λ. Since this holds for all such open neighborhoods B of $f(x)$, we clearly have that $(f(x_\lambda))_{\lambda\in\Lambda}$ is eventually in every such open set and we are done. $\square$

Clearly, the lemma above implies that for $x \in \mathfrak{X}$, we have $\mathrm{ev}_x(L_\lambda) \to \mathrm{ev}_x(L)$ in the usual norm topology on $\mathbb{C}$, and by definition this means that

$$L_\lambda(x) \to L(x),$$

or in other words, pointwise convergence of L_λ as functions on $\mathfrak{X}$ since this holds for all $x \in \mathfrak{X}$.

($\Longleftarrow$) Suppose that $L_\lambda \to L$ pointwise on $\mathfrak{X}$. Fix an $x \in X$ and consider some open neighborhood (in the topology of $\mathbb{C}$) V of x. There must be some $\lambda_V \in \Lambda$ such that $L_\lambda(x) \in V$ for all $\lambda \geq \lambda_V$. Since ev_x is continuous, we know that $\mathrm{ev}_x^{-1}(V)$ is open in the weak*-topology on $\mathfrak{X}^*$ and clearly $L \in \mathrm{ev}_x^{-1}(V)$. However, since $\mathrm{ev}_x(L_\lambda) = L_\lambda(x) \in V$, we must also have that $L_\lambda \in \mathrm{ev}_x^{-1}(V)$. Shrinking V adequately proves that this must hold for all such open (in the weak*-topology) neighborhoods of x, and considering all $x \in \mathfrak{X}$ gives the result. $\square$

1.4 Part (d)

Proof.
Let $(L_\lambda)_{\lambda\in\Lambda}$ be a net in the unit ball of $\mathfrak{X}^*$ equipped with the weak topology $\mathcal{T}^*$. Then clearly, for each x,

$$\mathrm{ev}_x(L_\lambda) = L_\lambda(x)$$

and $(L_\lambda(x))_{\lambda\in\Lambda}$ is a net in a compact subset of $\mathbb{C}$, since for each $x \in \mathfrak{X}$

$$|L_\lambda(x)| \leq \|L_\lambda\|_{op} \|x\|_{\mathfrak{X}} \leq \|x\|_{\mathfrak{X}}.$$

Hence, we know that there is a subnet $(L_{h(\mu)}(x))_{\mu\in M}$ in $\mathbb{C}$ that converges to a limit, say $L(x)$. Clearly, since a net is pointwise convergent iff it is convergent in the weak*-topology, we get that $(L_{h(\mu)})_{\mu\in M}$ is a convergent subnet of $(L_\lambda)_{\lambda\in\Lambda}$. However, we know that a set is compact if and only if all nets on the set have a convergent subnet, which means that the unit ball in $\mathfrak{X}^*$ is compact. □

2 Problem 2

2.1 Part (a)

Proof.
Suppose $A \in W_2$. Define the subspace

$$K := \overline{\{\tau^n A \in W_2 : n \in \mathbb{N}\}},$$

and the set

$$M := \{B \in K \subseteq W_2 : x_0 = 1\}.$$

($\Longrightarrow$) Suppose that $\tau^n A \in M$. Then the initial sequence A must have have a 1 in its x_n position since otherwise, the x_0 term of $\tau^n A$ would be 0, a contradiction.

($\Longleftarrow$) Suppose that $n \in A$, that is, A has 1 in its x_n term. Thinking inductively, applying τ n times shifts the first term of A from x_0 to x_n, and obviously $\tau^n A \in M$.

This clearly implies

$$a, \ldots, a + (k-1)n \in A \iff \tau^a A \in M \cup \cdots \cup \tau^{-(k-1)n}(M)$$

where $\tau^{-n}(M) = (\tau^n)^{-1}(M)$, since it means that 1 must be in the x_ath term and each nth term proceeding that, and vice versa. Hence, Szemerédi's will be true if for each k, we can find an n such that

$$M \cup \cdots \cup \tau^{-(k-1)n}(M)$$

is nonempty. □

2.2 Part (b)

Proof.
Let $A \in W_2$ be a binary sequence with positive upper density, that is,

$$d(A) = \limsup_{n\to\infty} \frac{|\{1, \ldots, n\} \cap A|}{n} > 0.$$

We take $(W_2, P(W_2))$ to be our measurable space, identifying W_2 with $P(\mathbb{N})$ in the natural way. Let us define a kind of point mass measure on $(W_2, P(W_2))$, which we shall call δ_i where

$$\delta_i(\mathcal{A}) = \begin{cases} 1 & \exists B \in \mathcal{A} : x_0 = 1 \text{ for } \tau^i B, \\ 0 & \text{else} \end{cases}$$

for any $\mathcal{A} \in P(W_2)$. The intuitive explanation for this measure is that it will return 1 if the collection of binary sequences contains a binary sequence that has a 1 in the ith place, and 0 if it does not. Now, we define the probability measures μ_i on $(W_2, P(W_2))$ by

$$\mu_i(\mathcal{A}) = \frac{1}{i} \sum_{j=1}^{i} \delta_j(\mathcal{A})$$

for all $\mathcal{A} \in P(W_2)$. Since M is merely the space of all left shifts of A such that the first digit in the binary sequence becomes 1, we have

$$\mu_i(M) = \frac{|\{1, \ldots, i\} \cap A|}{i}$$

by Part (a). It is easy to verify that this sends the empty set to 0 and σ-additivity follows from the fact that each δ_j is a measure.

We know that for a measurable space such as $(W_2, P(W_2))$, the space $C(W_2)$ is a Banach space and that its dual space is the space of measures under the total variation norm, i.e. $M(W_2) = (C(W_2))^*$, which is also a Banach space. In this case, our net of measures $\{\mu_i\}$ must lie in the unit ball $\mathbb{B}$, and by the **Banach-Alaoglu theorem** this unit ball of $M(W_2)$ must be compact with respect to the weak*-topology. It is provided that the space of measures on W_2 is metrizable, hence meaning that compactness of the unit ball is equivalent to its sequential compactness, as in our sequence of measures $\{\mu_i\}$ has a convergent subsequence $\{\mu_{i_j}\}$, and we shall call its limit μ.

Let $\varepsilon > 0$ be given. Then, for every $\mathcal{A} \in P(W_2)$, there exists an integer N_1 such that

$$j \geq N_1 \implies |\mu_{i_j}(\mathcal{A}) - \mu(\mathcal{A})| < \varepsilon,$$

since convergence in the weak*-topology is equivalent to pointwise convergence. Verification that μ is indeed a probability measure is easy since each μ_k is a measure, and clearly

$$\mu(M) = \lim_{j \to \infty} \mu_{i_j}(M) = \frac{|\{1, \ldots, i_j\} \cap A|}{i_j} = \limsup_{n \to \infty} \frac{|\{1, \ldots, n\} \cap A|}{n} = d(A).$$

$\square$

2.3 Part (c)

Proof.
Let $(X, \mathcal{M})$ be a measurable space and $T : X \to X$ a measurable function. Let $\{\nu_n\}$ be a sequence of T-invariant measures and let $\nu_n \to \nu$ with respect to the weak*-topology. Since we know that convergence in the weak*-topology is equivalent to pointwise convergence as complex functions, for given $\varepsilon > 0$ and every $A \in \mathcal{M}$, there is an N such that

$$n \geq N \implies |\nu_n(A) - \nu(A)| < \frac{\varepsilon}{2}.$$

Hence, there must be another number N' such that

$$n \geq N' \implies |\nu_n(A) - \nu(T^{-1}A)| = |\nu_n(T^{-1}A) - \nu(T^{-1}A)| < \frac{\varepsilon}{2}.$$

Taking $N_0 = \max(N, N')$, we get that

$$n \geq N \implies |\nu(A) - \nu(T^{-1}A)| \leq |\nu_n(A) - \nu(T^{-1}A)| + |\nu_n(A) - \nu(A)| < \varepsilon,$$

and letting ε shrink arbitrarily, we get that

$$\nu(A) = \nu(T^{-1}A)$$

for all measurable A, meaning that the weak* limit of a sequence of T-invariant measures is also T-invariant. $\square$