Algebraic Topology Pset 2

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1 Problem 1

1.1 Exercise 1

This statement is incorrect due to the assumption that n may take on the value 2. The correct statement would be:

Theorem 1.1.1. $\mathbb{R}^n$ *with* $n \geq 3$ *and finitely many points missing is simply-connected.*

1.2 Exercise 2

Proof. We saw from the previous problem set that $\mathbb{R}^2 \setminus \{(0,0)\}$ is *not* simply-connected since a loop with a hole in its interior cannot be contracted to a point, which shows that the $n \geq 3$ assumption must be fulfilled. We may put each of the finite number of points in disjoint open balls since $\mathbb{R}^3$ is Hausdorff, and then with any loop with a fixed endpoint, we may contract it to a point by adjusting the homotopy whenever it hits one of these balls to merely go around its boundary, avoiding the points. Hence, $\mathbb{R}^3$ with finitely many points missing must be simply-connected, and the also proof works for $n \geq 4$. $\square$

2 Problem 2

2.1 Exercise 1

Proof. Suppose A is a 3×3 matrix with entries in $\mathbb{R}^+$. Let us define the map $f : S^{2+} \to S^{2+}$ where

$$S^{2+} := \{x \in S^2 : x \text{ has non-negative entries}\} \subset \mathbb{R}^3,$$

defined by

$$f(x) = \frac{Ax}{\|Ax\|}.$$

We notice that S^{2+} is homeomorphic to the unit disk D^2 by the map $g : S^{2+} \to D^2$ which projects each point on the upper sphere S^{2+} down onto the disk (sends the z component to 0). Forming the map $gfg^{-1} : D^2 \to D^2$, we see that we may apply the Brouwer fixed point theorem to see that there exists a point $\bar{x} \in D^2$ (interpreting D^2 to be in $\mathbb{R}^3$) such that

$$g(f(g^{-1}(\bar{x}))) = \bar{x},$$

and applying g^{-1} to both sides,

$$\frac{Ag^{-1}(\bar{x})}{\|Ag^{-1}(\bar{x})\|} = g^{-1}(\bar{x}).$$

Setting $\xi = g^{-1}(\bar{x})$, we get

$$A\xi = \|A\xi\|\xi,$$

and we are done since the norm is obviously positive and never zero since the entries of our matrix and vector are always positive. □

2.2 Exercise 2

Proof. Suppose that A is invertible and has non-negative entries. Clearly, A is injective and hence has trivial kernel, meaning that we may use the same argument as above to deduce that

$$A\xi = \|A\xi\|\xi$$

for all $\xi \in S^{2+}$, since the zero vector is not in S^{2+}, implying that $\|A\xi\|$ will never be zero and f is well-defined. □

3 Problem 3

3.1 Exercise 1

Proof. (See proofs of Exercise 2 and Exercise 3 before this one). Let M be a Möbius strip embedded in $\mathbb{R}^3$. Suppose there is a continuous retraction $r : M \to \partial M$. Since ∂M is a loop in $\mathbb{R}^3$, we know that it is homeomorphic to S^1 by definition. Since homeomorphisms preserve the fundamental group of spaces, we know that $\pi_1(\partial M) \cong \mathbb{Z}$. We also know that $\pi_1(M) \cong \mathbb{Z}$

by Exercises 2 and 3. Notice that if $i : \partial M \to M$ is the inclusion map, then $r \circ i : \partial M \to \partial M$ is equal to the identity map $\mathrm{id}_{\partial M}$ on the boundary. Similar to the maps f_* and g_* in Exercise 3, we obtain the induced maps $r_* : \pi_1(M) \to \pi_1(\partial M)$ and $i_* : \pi_1(\partial M) \to \pi_1(M)$ such that

$$r_* \circ i_* = \mathrm{id}_{\pi_1(\partial M)},$$

and this implies that r_* must be surjective while i_* is injective. Notice that the retraction to the boundary of the Möbius strip must be even. Specifically, $r_*(n) = 2n$ (when $\pi_1(M)$ and $\pi_1(\partial M)$ are interpreted as $\mathbb{Z}$) since the boundary "winds" around the strip twice. However, this contradicts the fact that r_* is surjective, since $2n$ is not a generator of $\mathbb{Z}$. Hence, r cannot be a retraction of the Möbius strip onto its boundary.

□

3.2 Exercise 2

Proof. We consider the fundamental polygon of the Möbius strip, namely, the unit square $[0,1] \times [0,1]$ (we have not identified the top and bottom yet). The "core" line of the Möbius strip induced from this construction is then the line with endpoints $(\frac{1}{2}, 0)$ and $(\frac{1}{2}, 1)$. The deformation retraction from the unit square to this core line is then given by the homotopy $F : [0,1]^2 \times [0,1] \to [0,1]^2$ defined by

$$F((x,y),t) = (x,y)(1-t) + (\frac{1}{2}, y)t.$$

Identifying the endpoints by quotienting by $(\frac{1}{2}, 0) \sim (\frac{1}{2}, 1)$ gives the desired core loop of the Möbius strip, and the deformation retraction is given as above acting appropriately along the points we have glued. □

3.3 Exercise 3

Proof. We shall prove that for path-connected spaces, the fundamental group is preserved up to isomorphism under homotopy equivalence, as this would imply that the Möbius band would have fundamental group isomorphic to $\mathbb{Z}$. The path-connectedness is necessary to remove any dependency on the basepoint of the fundamental group. Suppose X and Y are path-connected with maps $f : X \to Y$, $g : Y \to X$ such that

$$fg \simeq \mathrm{id}_Y, \quad gf \simeq \mathrm{id}_X.$$

We claim that f provides the isomorphism between $\pi_1(X)$ and $\pi_1(Y)$ when considering the naturally induced map between homotopy classes $f_* : \pi_1(X) \to \pi_1(Y)$. Clearly, f_* must be bijective up to homotopy since it has a two-sided inverse (up to homotopy) given by the induced map $g_* : \pi_1(Y) \to \pi_1(X)$. Note that for any two homotopy classes $[\varphi], [\vartheta] \in \pi_1(X)$,

$$f_*([\varphi][\vartheta]) = f_*([\varphi \cdot \vartheta]) = [f(\varphi \cdot \vartheta)] = [f(\varphi) \cdot f(\vartheta)] = [f(\varphi)][f(\vartheta)],$$

where the third equality may be easily seen by unravelling the definition of compositions of paths. Hence, we have an isomorphism of groups, and the Möbius band M must have the fundamental group $\pi_1(M) \cong \mathbb{Z}$. $\square$

4 Problem 4

4.1 Exercise 1

Proof. Suppose that f_0, g_0, f_1, g_1 are all paths such that

$$f_0 \cdot g_0 \simeq f_1 \cdot g_1,$$

and

$$g_0 \simeq g_1.$$

Notice that we must have $\bar{g_0} \simeq \bar{g_1}$ by merely composing the homotopy between g_0 and g_1 with the continuous map that sends $(x,t) \mapsto (1-x,t)$. Then, we see that

$$f_0 \simeq f_0 \cdot (g_0 \cdot \bar{g_0}) \simeq (f_0 \cdot g_0) \cdot \bar{g_0} \simeq (f_1 \cdot g_1) \cdot \bar{g_0} \simeq f_1 \cdot (g_1 \cdot \bar{g_1}) \simeq f_1.$$

$\square$

4.2 Exercise 2

Proof. Let $x_0, X_1 \in X$ and consider the two homotopic paths h, h' that fix the endpoints x_0 and x_1. We already know that β_h and $\beta_{h'}$ are isomorphisms between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$. To show that they are in fact equivalent, notice that

$$h \cdot f \cdot \bar{h} \simeq h' \cdot f \cdot \bar{h}'$$

since each component is homotopic to each other. Hence,

$$\beta_h[f] \simeq \beta_{h'}[f]$$

for all $[f] \in \pi_1(X, x_0)$ and since the fundamental group only concerns homotopy classes, $\beta_h = \beta_{h'}$ and we are done. $\square$

4.3 Exercise 3

Proof. Suppose that $\pi_1(X)$ is abelian. For any two change of basepoint maps β_h and β_g connecting points $x_0, x_1 \in X$, we have

$$\beta_h[f] = [h \cdot f \cdot \bar{h}] = [g \cdot \bar{g} \cdot h \cdot f \cdot \bar{h} \cdot g \cdot \bar{g}].$$

Notice that

$$\overline{\bar{g} \cdot h} = \bar{h} \cdot g,$$

so we have

$$[g \cdot (\bar{g} \cdot h) \cdot f \cdot (\bar{h} \cdot g) \cdot \bar{g}] = \beta_g([\bar{g} \cdot h][f][\bar{g} \cdot h]^{-1}) = \beta_g[f]$$

by commutativity, so $\beta_h = \beta_g$.

Conversely, suppose that $\pi_1(X)$ is not abelian and that basepoint-change map are unique given the endpoints. Consider the constant map c that fixes the endpoint x_0, and let $[f]$ and $[g]$ be any two loops at x_0. Notice that we must have

$$\beta_g[f] = [g \cdot f \cdot \bar{g}] = [g][f][\bar{g}] = [f] = \beta_c[f]$$

since $\beta_g = \beta_c$. However, this implies that $\pi_1(X, x_0)$ must be abelian, a contradiction. Hence, there cannot be a unique basepoint-change map given the endpoints. □

4.4 Exercise 6

Proof. Suppose that X is path-connected. Suppose that $f : S^1 \to X$ is a map with $[f] \in [S^1, X]$ and $x_1 \in X$ with

$$f(0) = x_1 = f(1)$$

when interpreted as a path. There is some path $h : I \to X$ joining x_0 and x_1 since X is path-connected, and clearly the homotopy class $[h \cdot f \cdot \bar{h}]$ will map to $[f] \in [S^1, X]$, meaning that Φ is surjective.

Suppose that $[f]$ and $[g]$ are conjugate elements of $\pi_1(X, x_0)$, meaning that there is some $[h] \in \pi_1(X, x_0)$ such that

$$[f] = [h][g][h]^{-1}.$$

□

4.5 Exercise 16

Proof. (i) Suppose there is a retraction $r : \mathbb{R}^3 \to M$, where $M \approx S^1$. Let f be any loop in M with basepoint $x_0 \in M$. In $\mathbb{R}^3$, we may take the linear homotopy $F : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ between f and the constant map at x_0 (denoted c), defined by

$$F(x,t) = (t-1)f(x) + tc.$$

By definition $r|_M = \mathrm{id}_M$, so we see that by considering the appropriate restriction of $rF : \mathbb{R}^3 \times [0,1] \to M$, we obtain a homotopy between $rf = f$ and the constant loop at x_0. This clearly contradicts the fact that $\pi_1(M) \cong \pi_1(S^1)$ is nontrivial.

(ii) We notice that

$$\pi_1(S^1 \times D^2) \cong \pi_1(S^1) \times \pi_1(D^2) \cong \mathbb{Z},$$

and that

$$\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$$

since all the involved spaces are path-connected. If $r : S^1 \times D^2 \to S^1 \times S^1$ is a retraction, then the functorality of the fundamental group induces maps $r_* : (S^1 \times D^2) \to \pi_1(S^1 \times S^1)$ and $i_* : \pi_1(S^1 \times S^1) \to \pi_1(S^1 \times D^2)$, where i_* is an injection. However, this is impossible since there is no injection from $\mathbb{Z}^2$ to $\mathbb{Z}$. Hence, such a retraction r cannot exist.

(iii) Note that $\pi_1(S^1 \times D^2) \cong \mathbb{Z}$ and $\pi_1(A) \cong \mathbb{Z}$, where the latter is since $A \approx S^1$. Furthermore, A must be contractible to a point on X since it does not go around the hole completely. Again, suppose $r : S^1 \times D^2 \to A$ is a retraction and obtain the induced maps r_* and i_*. Note that r_* must map all homotopy classes to the constant loop at the point to which we contracted A, and hence i_* must also be the trivial injection into $\pi_1(X) \cong \mathbb{Z}$. However, since i_* maps to only one element of $\pi_1(X)$, it cannot be injective, hence implying that such a retraction cannot exist.

(iv) Note that $\pi_1(D^2 \vee D^2)$ is clearly trivial since every loop may be contracted to a point. On the other hand, its boundary $S^1 \vee S^1$ cannot have trivial fundamental group since the loop that circumnavigates one of the two circles is not contractible. Clearly a retraction cannot exist as it would imply an injection from a nontrivial group into the trivial group.

(v) Again, $\pi_1(X)$ is trivial, yet the boundary has nontrivial fundamental group. $\square$

4.6 Exercise 18

Proof. Let B be the boundary along with A and e^n are attached. Clearly these two components are independently path-connected and their intersection is B, which is also path-connected. Let $i : A \hookrightarrow X$ be the injection of A into X and $i_* : \pi_1(A) \to \pi_1(X)$ be the induced map. Since the space is path-connected, it is suffice to consider one fixed basepoint $x_0 \in B$. Let f be an arbitrary loop with basepoint x_0. By Lemma 1.15, $[f]$ may be represented as

$$[f] = [g_1 \cdot h_2 \cdots],$$

where each g_i and h_i is a loop contained A or e^n. Without loss of generality, let each g_i be in A and h_i be in e^n. Notice that e^n is simply-connected so that h_i may be contracted x_0 and we obtain that

$$[f] = [g],$$

where g is completely contained in A. Hence, $i_*([g]) = [f]$ and i_* is a surjection.

(i) We see that $X = S^1 \vee S^2$ is obtained by attaching the boundary of a 2-cell to a specified point x_0 on S^1. The inclusion then induces a surjection $i_* : \pi_1(S^1) \hookrightarrow \pi_1(X)$. However, this map is also an isomorphism since it is obviously injective by definition. Hence, $\pi_1(X) \cong \pi_1(S^1) \cong \mathbb{Z}$.

(ii) We show the case where X is a CW-complex with finitely many cells. Denoting by X^1 the 1-skeleton of X, we see that by considering $A = X^1$ and attaching the next appropriate cells individually e^n with $n \geq 2$ (namely, the 2-cells if it has any), then proceeding inductively with the new quotiented space being the new A, we will eventually exhaust the finite number of cells and create the whole cell complex X with this construction. Throughout this process, we obtain the following sequence of inclusions

$$X^1 \overset{i_1}{\hookrightarrow} (X^1 \cup e^n)/\sim \overset{i_2}{\hookrightarrow} \cdots \overset{i_k}{\hookrightarrow} X,$$

and merely composing every map to create $i : X^1 \hookrightarrow X$ defined by

$$i = i_k \cdots i_2 i_1,$$

we see that $i_* : \pi_1(X^1) \to \pi_1(X)$ must be surjective as

$$i_* = i_{k*} \cdots i_{2*} i_{1*}$$

and each map on the right is surjective by the lemma we proved.

$\square$

5 Problem 5

5.1 Exercise 1

Proof. (i) We have

$$1_\times = 1_\times \times 1_\times = (1_\circ \circ 1_\times) \times (1_\times \circ 1_\circ) = (1_\circ \times 1_\times) \circ (1_\times \circ 1_\circ) = 1_\circ \circ 1_\circ = 1_\circ.$$

(ii) Let us denote the common value $1_\circ = 1_\times$ by e. Notice that

$$a \times b = (a \circ e) \times (e \circ b) = (a \times e) \circ (e \times b) = a \circ b.$$

Commutativity is ensured by the calculation

$$a \times b = a \circ b = (e \times a) \circ (b \times e) = (e \circ b) \times (a \circ e) = b \times a = b \circ a.$$

(iii) Associativity of $\circ$ is ensured by

$$(a \circ b) \circ c = (a \circ b) \circ (e \circ c) = (a \circ e) \circ (b \circ c) = a \circ (b \circ c),$$

where the interchange property was used in the second equality. The result for $\times$ follows since $a \circ b = a \times b$. $\square$

5.2 Exercise 2

Proof. (i) The identity element of $\Omega(G, x_0)$ is the constant loop at x_0, which we shall denote by c. Indeed, we see that

$$(c \times f)(t) = x_0 f(t) = f(t)$$

and

$$(f \times c)(t) = f(t)x_0 = f(t).$$

Furthermore, the inverse of each loop f is the loop g defined by

$$g(t) = [f(t)]^{-1},$$

which we shall denote by f^{-1}. The binary operation $\times$ is clearly associative since the group operation on G is associative.

(ii) To prove that $\times$ is well-defined on $\pi_1(G, x_0)$, consider any two homotopy classes $[\varphi], [\vartheta] \in \pi_1(G, x_0)$ and let $\varphi, \varphi' \in [\varphi]$ and $\vartheta, \vartheta' \in [\vartheta]$ be pairs of representatives of the two respective homotopy classes. Let

$F : G \times I \to G$ be the homotopy between φ and φ' and G the one between ϑ and ϑ' respectively. Notice that $\varphi \times \vartheta$ and $\varphi' \times \vartheta'$ must be homotopic by the map $(F \times G) : G \times I \to G$, (note that $\times$ is commutative) since

$$(F \times G)(x, 0) = \varphi(x)\vartheta(x), \quad (F \times G)(x, 1) = \varphi'(x)\vartheta'(x).$$

$F \times G$ must be continuous since the multiplication in G is continuous and the individual homotopies are continuous. The identity element is the homotopy class of the constant loop c at x_0, and the inverse homotopy class of $[\varphi] \in \pi_(G, x_0)$ is the homotopy class of φ^{-1} defined by

$$\varphi^{-1}(t) = [\varphi(t)]^{-1}$$

where φ is some representative. This is well defined since if φ and ϑ are in the same homotopy class with the homotopy F between them, then the composition between F and the map $x \mapsto x^{-1}$ (which is continuous in G) provides the homotopy between φ^{-1} and ϑ^{-1}. Associativity is obvious from the group structure of G.

(iii) Let $f, g, \varphi, \vartheta \in \pi_1(G, x_0)$ (consider the representatives for convenience). We see that

$$((f \times g) \circ (\varphi \times \vartheta))(t) = \begin{cases} f(2t)g(2t) & 0 \leq t \leq \frac{1}{2} \\ \varphi(2t-1)\vartheta(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases},$$

and

$$(f \circ \varphi) \times (g \circ \vartheta))(t) = \begin{cases} f(2t)g(2t) & 0 \leq t \leq \frac{1}{2} \\ \varphi(2t-1)\vartheta(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases},$$

showing the desired property.

(iv) By part (ii) of Exercise 1, we know that both $\circ$ and $\times$ are commutative. Hence, $\pi_1(G, x_0)$ must be abelian. $\square$

5.3 Exercise 3

Proof. (i) Convinced.

(ii) We see that for $t \in [0,1]^n$,

$$((f \circ g) \times (\varphi \circ \vartheta))(t) = \begin{cases} (f \circ g)(t_1, 2t_2, \ldots, t_n) & t_2 \in [0, \frac{1}{2}] \\ (\varphi \circ \vartheta)(t_1, 2t_2 - 1, \ldots, t_n) & t_2 \in [\frac{1}{2}, 1] \end{cases},$$

$$= \begin{cases} f(2t_1, 2t_2, \ldots, t_n) & t \in [0, \frac{1}{2}] \times [0, \frac{1}{2}] \times [0,1]^{n-2} \\ g(2t_1 - 1, 2t_2, \ldots, t_n) & t \in [\frac{1}{2}, 1] \times [0, \frac{1}{2}] \times [0,1]^{n-2} \\ \varphi(2t_1, 2t_2 - 1, \ldots, t_n) & t \in [0, \frac{1}{2}] \times [\frac{1}{2}, 1] \times [0,1]^{n-2} \\ \vartheta(2t_1 - 1, 2t_2 - 1, \ldots, t_n) & t \in [\frac{1}{2}, 1] \times [\frac{1}{2}, 1] \times [0,1]^{n-2} \end{cases},$$

while

$$((f \times \varphi) \circ (g \times \vartheta))(t) = \begin{cases} (f \times \varphi)(2t_1, \ldots, t_n) & t_1 \in [0, \frac{1}{2}] \\ (g \times \vartheta)(2t_1 - 1, \ldots, t_n) & t_1 \in [\frac{1}{2}, 1] \end{cases},$$

$$= \begin{cases} f(2t_1, 2t_2, \ldots, t_n) & t \in [0, \frac{1}{2}] \times [0, \frac{1}{2}] \times [0,1]^{n-2} \\ g(2t_1 - 1, 2t_2, \ldots, t_n) & t \in [\frac{1}{2}, 1] \times [0, \frac{1}{2}] \times [0,1]^{n-2} \\ \varphi(2t_1, 2t_2 - 1, \ldots, t_n) & t \in [0, \frac{1}{2}] \times [\frac{1}{2}, 1] \times [0,1]^{n-2} \\ \vartheta(2t_1 - 1, 2t_2 - 1, \ldots, t_n) & t \in [\frac{1}{2}, 1] \times [\frac{1}{2}, 1] \times [0,1]^{n-2} \end{cases},$$

so the interchange property must hold.

(iii) This is easy to see by the commutativity of $\times$ and $\circ$. $\square$