Measure Theory HW Sections 1.4-1.5

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1 Problem 1

1.1 Part (a)

Proof. Let $(X, \mathcal{M}, \mu)$ be a finite measure space, and let $\mathcal{A} \subseteq \mathcal{M}$ be an algebra of sets such that

$$\sigma(\mathcal{A}) = \mathcal{M},$$

i.e. $\mathcal{M}$ is the σ-algebra generated by $\mathcal{A}$. We must show that for each $M \in \mathcal{M}$ and a given $\varepsilon > 0$, there is a set $A \in \mathcal{A}$ such that

$$\mu(M \triangle A) < \varepsilon.$$

Let us restrict our measure μ to the algebra $\mathcal{A}$. In other words, construct a premeasure $\mu_0 = \mu|_{\mathcal{A}}$. By **Proposition 1.13**, we know that the outer measure $\mu^* : \mathcal{P}(X) \to [0, \infty)$ defined by

$$\mu^*(A) = \inf \left\{ \sum_{j=1}^{\infty} \mu_0(B_j) : B_j \in \mathcal{A}, A \subseteq \bigcup_{j=1}^{\infty} B_j \right\}, \qquad A \subseteq X$$

coincides with the premeasure on $\mathcal{A}$ and that each set in $\mathcal{A}$ is μ^*-measurable. Furthermore, we can apply **Theorem 1.14** to see that in fact our original measure μ is merely the restriction of μ^* to $\mathcal{M}$ since the finiteness of our measure space ensures that the extension of μ_0 to $\mathcal{M}$ is unique (and μ was already one such extension).

Let $\varepsilon > 0$ be given. Let us say that a set $M \in \mathcal{M}$ is *approximated by sets in* $\mathcal{A}$ if there exist sets $A_j \in \mathcal{A}$ such that

- $M \subseteq \bigcup_{j=1}^{\infty} A_j$,
- $\mu\left(\bigcup_{j=1}^{\infty} A_j\right) < \mu(M) + \varepsilon$.

Clearly, if M is approximated by some $\bigcup_{j=1}^{\infty} A_j =: A$ in $\mathcal{A}$, then

$$\mu(M \triangle A) = \mu(M \setminus A) + \mu(A \setminus M) = \mu(A \setminus M) = \mu(A) - \mu(M) < \varepsilon,$$

since the measure of each set is finite. Hence, if we are able to show that every set in $\mathcal{M}$ may be approximated by sets in $\mathcal{A}$, our result follows. To do this, we shall use the first part of exercise 1.18 in Folland, which we shall now state and prove.

Exercise 1.18 Part (a) (Folland). Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra and $\mathcal{A}_\sigma$ the collection of countable unions of sets in $\mathcal{A}$. Let μ_0 be a premeasure on $\mathcal{A}$ and μ^* the induced outer measure on the power set. For any $E \subseteq X$ and $\varepsilon > 0$, there exists $A \in \mathcal{A}_\sigma$ with

$$E \subseteq A \text{ and } \mu^*(A) < \mu^*(E) + \varepsilon.$$

Proof. Let $\varepsilon > 0$ be given. We know that for any $E \subseteq X$,

$$\mu^*(E) = \inf\left\{\sum_{j=1}^{\infty} \mu_0(A_j) : A_j \in \mathcal{A}, E \subseteq \bigcup_{j=1}^{\infty} A_j\right\}.$$

If there was no covering $\bigcup_j A_j \supseteq E$ such that

$$\sum_{j=1}^{\infty} \mu_0(A_j) < \mu^*(E) + \varepsilon,$$

then clearly $\mu^*(E) + \varepsilon$ would be a greater lower bound than $\mu^*(E)$, which contradicts the fact that $\mu^*(E)$ is the infimum of such sums. Let $A := \bigcup_j A_j$. Clearly, $A \in \mathcal{A}_\sigma$ by definition, and

$$\mu^*(A) \le \sum_{j=1}^{\infty} \mu_0(A_j) < \mu^*(E) + \varepsilon,$$

by the definition of the infimum proving the result. □

Using this result, we deduce that for every $M \in \mathcal{M}$, there exists some covering $A := \bigcup_j A_j \supseteq M$ such that

$$M \subseteq A \text{ and } \mu^*(A) < \mu^*(M) + \varepsilon,$$

where μ^* is the outer measure induced by our measure μ as described above. Since both $A, M \in \mathcal{M}$ and the measure μ is merely the restriction of the outer measure to $\mathcal{M}$, we can write

$$\mu(A) < \mu(M) + \varepsilon \implies \mu(M \triangle A) = \mu(A \setminus M) < \varepsilon,$$

and we have shown our result. □

1.2 Part (b)

We propose that if $(X, \mathcal{M}, \mu)$ is a σ-finite measure space, then the statement in part (a) is true if M is μ^*-measurable, where μ^* is the outer measure defined in the proof of part (a).

Proof. We begin by stating and proving a rendition of part (b) of the aforementioned exercise 18.

Exercise 1.18 Part (b) (Folland). Let us consider the same assumptions an definitions in our statement for part (a) of exercise 1.18 above. Suppose that μ_0 is σ-finite. If E is μ^*-measurable, there exists $B \in \mathcal{A}_{\sigma\delta}$ (the set of all countable intersections of sets in $\mathcal{A}_\sigma$) with

$$E \subseteq B \text{ and } \mu^*(B \setminus E) = 0.$$

Proof. Suppose that $E \subseteq X$ is μ^*-measurable. Since μ_0 is σ-finite, we may write

$$X = \bigcup_{j=1}^{\infty} X_j,$$

where $\mu_0(X_j) < \infty$ for each j. Let us define sets E_j by

$$E_j = E \cap X_j,$$

which clearly satisfies $\bigcup_j E_j = E$. By part (a) of exercise 1.18, for every positive integer j and some fixed positive integer i, there are sets $K_j \in \mathcal{A}_\sigma$ such that $E_j \subseteq K_j$ and

$$\mu^*(K_j) < \mu^*(E_j) + \frac{1}{i}2^{-j}.$$

Furthermore, we see that for any $Y \subseteq X$,

$$\mu^*(Y \cap (E \cap X_j)) + \mu^*(Y \cap (E \cap X_j)^C)$$

is equivalent to

$$\mu^*(Y \cap (E \cap X_j)) + \mu^*(Y) - \mu^*(Y \cap (E \cap X_j)) = \mu^*(Y)$$

since X_j has finite measure, meaning that each E_j is measurable. Hence,

$$\mu * (K_j) = \mu^*(K_j \cap E_j) + \mu^*(K_j \setminus E_j) = \mu^*(E_j) + \mu^*(K_j \setminus E_j).$$

However, since E_j is finite,

$$\mu^*(K_j \setminus E_j) = \mu^*(K_j) - \mu^*(E_j) < \frac{1}{i}2^{-j}.$$

Defining $D_i = \bigcup_j K_j \in \mathcal{A}_\sigma$, we see that

$$\begin{aligned}
\mu^*(D_i \setminus E) &= \mu^*\left(\bigcup_{j=1}^{\infty}(K_j \cap E^C)\right) \\
&\leq \mu^*\left(\bigcup_{j=1}^{\infty}(K_j \cap E_j^C)\right) \\
&\leq \sum_{j=1}^{\infty} \mu^*(K_j) - \mu^*(E_j) < \frac{1}{i}
\end{aligned}$$

where the first inequality is derived from the fact that $E_j \subseteq E$ implies $E_j^C \supseteq E^C$. Hence, we see that the set $B = \bigcap_i D_i \in \mathcal{A}_{\sigma\delta}$ will surely contain E and

$$\mu^*(B \setminus E) \leq \mu^*(D_i \setminus E) < \frac{1}{i}$$

for every positive integer i, showing that $\mu^*(B \setminus E) = 0$ and B is the set with our desired properties.

$\square$

From the result from above, we see that since μ_0 is merely the restriction of μ to $\mathcal{A}$, it is surely σ-finite, implying that if $M \in \mathcal{M}$ is μ^*-measurable, then there exists some set $B \supseteq M$ that satisfies

$$\mu(M \triangle B) = 0 + \mu(B \setminus M) = 0,$$

as the outer measure μ^* is derived from our given measure. $\square$

1.3 Part (c)

Proof. To do this, we shall use **Theorem 1.18** from Folland.

Theorem 1.18. If $E \in \mathcal{L}$, the set of Lebesgue measurable sets, then

$$\lambda(E) = \inf\{\lambda(U) : E \subseteq U \text{ and } U \text{ is open}\},$$

where $\lambda : \mathcal{L} \to [0, \infty]$ is the Lebesgue measure.

Let $\varepsilon > 0$ be given. There must be an open set $U \supseteq E$ such that

$$\lambda(U) < \lambda(E) + \varepsilon$$

since if such a set did not exist, then $\lambda(E) + \varepsilon$ would be a larger lower bound, contradicting the fact that $\lambda(E)$ is defined to be the greatest such lower bound for the set. Since E is a (finite) Lebesgue measurable set,

$$\lambda(U) = \lambda(U \cap E) + \lambda(U \cap E^C) = \lambda(E) + \lambda(U \setminus E)$$

which implies
$$\lambda(U \setminus E) = \lambda(U) - \lambda(E) < \varepsilon,$$
where we could subtract $\lambda(E)$ from both sides since E has finite measure. However,
$$\lambda(E \triangle U) = \lambda(E \setminus U) + \lambda(U \setminus E) = 0 + \lambda(U \setminus E) < \varepsilon,$$
so we are done. ☐

1.4 Part (d)

Proof. Let $\alpha \in (0,1)$ be given. Define $\varepsilon = \frac{1}{\alpha} - 1 > 0$. Notice that by the definition of the Lebesgue measure (as an infimum of the countable sum of measures of open intervals by **Lemma 1.17**), we may find open intervals I_j such that $A := \bigcup_j I_j \supseteq E$ and
$$\lambda(E \triangle A) < \varepsilon \lambda(I).$$
Let us fix one of the intervals I_k, which we shall now call I. By monotonicity,
$$\lambda(I \cap (E \triangle A)) \leq \lambda(E \triangle A) < \varepsilon \lambda(I).$$
However, it is clear that
$$I \cap \bigcup_{j=1}^{\infty} I_j = I \cap A = I,$$
so we have
$$\lambda(I \cap (E \triangle A)) = \lambda((I \cap E) \triangle (I \cap A)) = \lambda((I \cap E) \triangle I)$$
by the distributivity of intersection over symmetric difference. Evaluating this,
$$\lambda((I \cap E) \triangle I) = \lambda((I \cap E) \setminus I) + \lambda(I \setminus (E \cap I)) = \lambda(I \setminus E)$$
and finally
$$\lambda(I \cap (E \triangle A)) = \lambda(I \setminus E) = \lambda(I) - \lambda(E \cap I)$$
since E has finite measure. Hence,
$$\lambda(I) < \lambda(E \cap I) + \varepsilon \lambda(I) \leq (1 + \varepsilon) \lambda(E \cap I) \implies \alpha \lambda(I) < \lambda(E \cap I)$$
by monotonicity and the definition of ε.

☐

2 Problem 2

2.1 Part (a)

Proof. Let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be an outer measure on X. By **Carathéodory's theorem**, we know that the set

$$\mathcal{M} := \{A \subseteq X : A \text{ is } \mu^* \text{ measurable}\}$$

is a σ-algebra and that the set function $\mu : \mathcal{M} \to [0, \infty]$ defined by

$$\mu = \mu^*|_{\mathcal{M}}$$

is a measure on $(X, \mathcal{M})$. Let $N \in \mathcal{M}$ be a null set and let B be a subset of N. To prove that $(X, \mathcal{M}, \mu)$ is complete, we must show that $B \in \mathcal{M}$. For any $E \in \mathcal{P}(X)$, we have that

$$\mu^*(E) \leq \mu^*(E \cap B) + \mu^*(E \cap B^C) = \mu^*(E \cap B^C) \leq \mu^*(E),$$

where we used the monotonicity and subadditivity of an outer measure. Clearly each inequality in the chain turns into an equality, so

$$\mu^*(E) = \mu^*(E \cap B) + \mu^*(E \cap B^C),$$

and $B \in \mathcal{M}$, showing that $(X, \mathcal{M}, \mu)$ is complete. $\square$

2.2 Part (b)

Proof. Let us denote by $\mathcal{L}$ the collection of Lebesgue measurable sets. We must show that the Lebesgue measure restricted to the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$ is not complete, as in there is a subset of some null set $N \in \mathcal{B}_{\mathbb{R}}$ that is not in $\mathcal{B}_{\mathbb{R}}$.

To do this, we shall consider the classical Cantor set, which we denote by C. Notice that by the second part of **Proposition** 1.22 in Folland, the Cantor set has measure zero, meaning that $\mathcal{L}$ contains $\mathcal{P}(C)$. Furthermore, by part (c) of the same proposition,

$$|C| = \mathfrak{c},$$

as in the Cantor set has the cardinality of the continuum. Hence, we must have that

$$|\mathcal{P}(C)| > \mathfrak{c},$$

a strict inequality given by **Cantor's theorem**.

On the other hand, consider **Proposition 1.23** in Folland, which we state below.

> **Proposition 1.23.** Let X be a set, $\mathcal{E} \subseteq \mathcal{P}(X)$, and let us denote by $\sigma(\mathcal{E})$

the σ-algebra generated by $\mathcal{E}$. Then

$$\sigma(\mathcal{E}) = \bigcup_{\alpha \in \Omega} \mathcal{E}_\alpha,$$

where Ω is the set of countable ordinals (refer to Folland for the definition of each $\mathcal{E}_\alpha$).

Clearly, for

$$X = \mathbb{R}, \text{ and } \mathcal{E} = \tau$$

where τ is the usual topology (family of open sets according to the Euclidean metric on $\mathbb{R}$), we have that

$$\mathcal{B}_\mathbb{R} = \bigcup_{\alpha \in \Omega} \mathcal{E}_\alpha.$$

However, we know that

$$\tau \subseteq \mathcal{P}(\mathbb{R}) \implies |\mathbb{N}| \leq |\tau| \leq \mathfrak{c},$$

so we use part (b) of **Proposition 0.14** and deduce that

$$|\mathcal{B}_\mathbb{R}| = \left| \bigcup_{\alpha \in \Omega} \mathcal{E}_\alpha \right| \leq \mathfrak{c}.$$

Hence,

$$|\mathcal{B}_\mathbb{R}| \leq \mathfrak{c} < |\mathcal{P}(C)|,$$

which shows that there must be at least one subset of C that is not in the real Borel σ-algebra, meaning that $\mathcal{B}_\mathbb{R}$ does not contain all subsets of any null set. Hence, $(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mu)$ is not complete. $\square$

Note. One can show that $|\mathcal{B}_\mathbb{R}| = \mathfrak{c}$ by proving that any infinite σ-algebra has cardinality of at least $\mathfrak{c}$, but for this problem the inequality suffices.

2.3 Part (c)

Proof. By a "complete σ-algebra", we mean a σ-algebra that contains all subsets of any null set with respect to a given measure, which is the Lebesgue (outer) measure in this case. To begin, we shall cite **Theorem 1.9** in Folland.

Theorem 1.9. If $(X, \mathcal{M}, \mu)$ is a measure space, then the family $\overline{\mathcal{M}}$ defined by

$$\overline{\mathcal{M}} := \{E \cup F : E \in \mathcal{M} \wedge F \subseteq N, N \in \mathcal{N}\}$$

where

$$\mathcal{N} := \{N \in \mathcal{M} : \mu(N) = 0\}$$

is a σ-algebra, and there is a unique extension $\overline{\mu}$ of μ to a complete measure on $\overline{\mathcal{M}}$.

We shall show that in fact $\overline{\mathcal{M}}$ is the smallest σ-algebra containing $\mathcal{M}$ that gives rise to a complete measure space. Let C denote the collection of all complete σ-algebras on X that contain $\mathcal{M}$. Then clearly

$$\bigcap_{S \in C} S \subseteq \overline{\mathcal{M}},$$

since $\overline{\mathcal{M}} \in C$. To show the reverse inclusion, consider some set $E \cup F \in \overline{\mathcal{M}}$, where

$$E, N \in \mathcal{M} \text{ and } F \subseteq N, \mu(N) = 0.$$

Since $\bigcap S$ is a σ-algebra (intersection of σ-algebras is a σ-algebra) and $\mathcal{M} \subseteq \bigcap S$, it is clear that $E \cup F \in \bigcap S$ as each subset of any null set $N \in \mathcal{M}$ must be in $\bigcap S$ since each $S \in C$ is complete. Hence,

$$\overline{\mathcal{M}} \subseteq \bigcap_{S \in C} S \implies \overline{\mathcal{M}} = \bigcap_{S \in C} S,$$

and $\overline{\mathcal{M}}$ must be the smallest σ-algebra containing $\mathcal{M}$.

Now all that is left is to show that

$$\mathcal{L} = \overline{\mathcal{B}_{\mathbb{R}}},$$

as this would imply that the set of Lebesgue measurable sets is the smallest complete σ-algebra containing the real Borel sets. The inclusion

$$\overline{\mathcal{B}_{\mathbb{R}}} \subseteq \mathcal{L}$$

is trivial since $\mathcal{L}$ is a complete σ-algebra containing $\mathcal{B}_{\mathbb{R}}$. Conversely, suppose that $E \in \mathcal{L}$. By part (b) of **Theorem 1.19**, we may write

$$E = \left(\bigcup_{j=1}^{\infty} K_j \right) \cup N,$$

where $N \in \mathcal{L}$ is a null set and each K_j is compact (and hence in $\mathcal{L}$ as it contains every closed set). However,

$$\overline{\mathcal{B}_{\mathbb{R}}} = \{A \cup B : A \in \mathcal{B}_{\mathbb{R}} \text{ and } B \subseteq M, \lambda(M) = 0\},$$

meaning that $E \in \overline{\mathcal{B}_{\mathbb{R}}}$ since $\mathcal{B}_{\mathbb{R}}$ is closed under countable unions. Hence,

$$\mathcal{L} \subseteq \overline{\mathcal{B}_{\mathbb{R}}} \implies \mathcal{L} = \overline{\mathcal{B}_{\mathbb{R}}},$$

and this shows that $\mathcal{L}$ is the smallest complete σ-algebra that contains $\mathcal{B}_{\mathbb{R}}$ and $(\mathbb{R}, \mathcal{L}, \lambda)$ is the smallest complete measure space containing $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu)$. □

3 Problem 3

3.1 Part (a)

Proof. Suppose $(X, \mathcal{M})$ is a product of n measurable spaces, where

$$X = \prod_{i=1}^{n} X_i \text{ and } \mathcal{M} = \bigotimes_{i=1}^{n} \mathcal{M}_i.$$

We shall define the set $\mathcal{A}$ by

$$\mathcal{A} = \left\{ \prod_{j=1}^{n} A_j : A_j \in \mathcal{M}_j \right\}.$$

This set is in fact an elementary family on X since it contains the empty set, is closed under finite intersection, and every member's complement may be written as a disjoint union of elements in the set. Closure under intersection follows from the identity

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D).$$

The verification of the last claim can (probably) be verified by induction once closure under intersections is established.

By **Proposition 1.7**, we may extend this elementary family to an algebra on X, which we shall denote $\overline{\mathcal{A}}$. By the statement of the proposition, we may explicitly write

$$\overline{\mathcal{A}} = \left\{ \bigcup_{k=1}^{m} M_k : M_k \in \mathcal{A} \text{ and } M_k \cap M_i = \varnothing \text{ for } k \neq i \right\}$$

Now, let us define a premeasure $\mu_0 : \overline{\mathcal{A}} \to [0, \infty]$ defined by

$$\mu_0 \left(\prod_{j=1}^{n} A_j \right) = \prod_{j=1}^{n} \mu_j(A_j).$$

This function is clearly monotonic and sends the empty set to 0, so just need to check σ-additivity. Suppose that $\{E^k\} \subseteq \mathcal{P}(\overline{\mathcal{A}})$ is a sequence of disjoint sets with

$$E^k = \prod_{j=1}^{\infty} A_j^k.$$

Then

$$\begin{aligned}
\mu_0\left(\bigcup_{k=1}^{n} E^k\right) &= \mu_0\left(\bigcup_{k=1}^{\infty}\prod_{j=1}^{n} A_j^k\right), \\
&= \mu_0\left(\prod_{j=1}^{n}\bigcup_{k=1}^{\infty} A_j^k\right), \\
&= \prod_{j=1}^{n}\mu_j\left(\bigcup_{k=1}^{\infty} A_j^k\right), \\
&= \prod_{j=1}^{n}\sum_{k=1}^{\infty}\mu_j(A_j^k), \\
&= \sum_{k=1}^{\infty}\prod_{j=1}^{n}\mu_j(A_j^k), \\
&= \sum_{k=1}^{\infty}\mu_0(E^k)
\end{aligned}$$

(turns out the product does not commute with unions... dunno how to fix so i'll leave it for now) by the compatibility of unions with cartesian products and the fact that each μ_j is a measure. Since μ_0 is a premeasure on the algebra $\overline{\mathcal{A}}$, we may now extend it to an outer measure $\mu^* : \mathcal{P}(X) \to [0,\infty]$ such that for every set $E \in \mathcal{P}(X)$, we have

$$\mu^*(E) = \inf\left\{\sum_{j=1}^{\infty}\mu_0(A_j) : A_j \in \overline{\mathcal{A}} \text{ and } E \subseteq \bigcup_{j=1}^{\infty} A_j\right\}$$

By **Theorem 1.14**, we know that since $\sigma(\overline{\mathcal{A}}) = \mathcal{M}$, there is a measure μ whose restriction to $\mathcal{A}$ is μ_0. Furthermore, we have an explicit characterization of this measure, namely

$$\mu = \mu^*|_{\mathcal{M}},$$

where μ^* is the outer measure that we described above. We know that this measure is unique when μ_0 is σ-finite, and at least know that it is the maximal measure that extends μ_0 on $\mathcal{M}$ in general.

□

3.2 Part (b)

Proof. We notice that if we desire a measure similar to the one constructed in part (a), we must necessarily have that

$$\mu_i(X_i) = 1$$

for all indices $i \in \mathbb{Z}^+$ such that $E_i = X_i$, as otherwise, the product with either diverge to infinity or give rise to the trivial zero measure. This makes it clear that in fact any such product will reduce down to the finite case.

Let $j_1, \ldots, j_k$ be the indices for which E_i is not just X_i, and let $\mathcal{M}_{j_k}$ denote their corresponding σ-algebras. As done above, one can define

$$\mathcal{A} = \left\{ \prod_{j=1}^{\infty} A_j : A_{j_k} \in \mathcal{M}_{j_k}, A_j = X_j \text{ for other indices} \right\}$$

The verification that this set is an elementary family on $X := \prod_j X_j$, is (hopefully) entirely analogous to the finite case. We extend this elementary family to an algebra $\overline{\mathcal{A}}$ by adding all finite disjoint unions of sets within the family.

Clearly, we may define a set function $\mu_0 : \overline{\mathcal{A}} \to [0, \infty]$ defined by

$$\mu_0 \left(\prod_{j=1}^{\infty} A_j \right) = \prod_{j=1}^{\infty} \mu_j(A_j) = \prod_{n=1}^{k} \mu_{j_n}(A_{j_n}).$$

Proceeding in a similar manner from part (a), we may then construct an outer measure by extending μ_0, then restrict it to $\sigma(\overline{\mathcal{A}})$ to obtain a measure space. $\square$