Optimization of the Aerodynamic Qualities of a Bullet

Sieon (Sean) Kim
Seoul International School

The Problem

Consider a bullet shot horizontally without the influence of any external forces. Suppose that the bullet experiences a collision with a sheet of N air molecules, where N is large. We wish to model this situation and specifically minimize the effect of the collision on the bullet's velocity following the interaction, i.e. minimizing the air resistance experienced by the bullet.

Developing the Model

To simplify our analysis, we make the following physical assumptions:

1. The bullet is radially symmetric about a vertical axis and is given by rotating a “nice enough” curve (sufficiently differentiable and continuous) about said axis.
2. The air “molecules” the bullet encounters travel parallel to the axis straight toward the bullet at unit speed, and follow the law of reflection. We also assume that collisions are elastic.
3. The bullet has a fixed volume and radius and is not deformed by its collision with the air molecules.

Based on the discussion above, we set the constants and variables as follows:

Symbol	Quantity	Type
R	Radius of bullet	Constant
V	Volume of bullet	Constant
f	Rotated curve	Function

Suppose that the graph $\Gamma_f \subset \mathbb{R}^2$ of f in the first quadrant of the xy-plane is rotated about the y-axis to give a solid of revolution, the bullet. We imagine that the bullet then encounters a sheet of air molecules as described above. A particular example describing the trajectory of a single air molecule hitting the bullet (or a cross section thereof) is shown below.

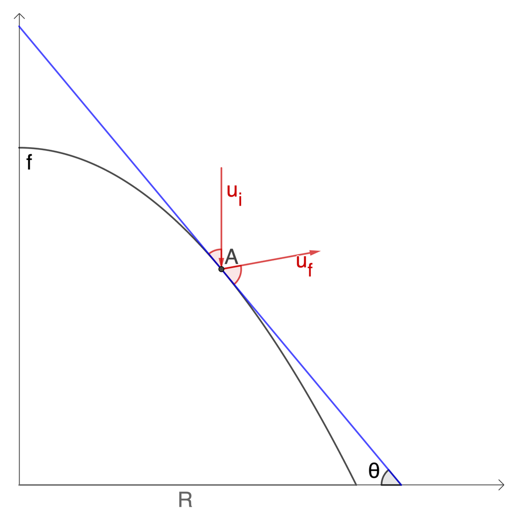

Figure 1. A single air molecule hits the bullet

By our second assumption, the air molecule bounces with equivalent angles of incidence and reflection with respect to the tangent line to Γ_f. From the figure and basic trigonometry, we deduce that the initial unit direction vector is given by $u_i = \langle 0, -1\rangle$ which becomes $u_f = \langle \sin(2\theta), -\cos(2\theta)\rangle$ after collision. By conservation of momentum, we know that if v_b and v_b' represent the velocity of the bullet, and $v_{a_1}, \ldots, v_{a_n}$ and $v_{a_1}', \ldots, v_{a_n}'$ the velocities of the individual air particles before and after collision, we have

$$m_b v_b + \sum_{i=1}^{n} m_a v_{a_i} = m_b v_b' + \sum_{i=1}^{n} m_a v_{a_i}', \implies \Delta v_b \propto \sum_{i=1}^{n} \Delta v_{a_i}.$$

This implies that the change in the vertical velocity of the bullet following collision with a single air particle is proportional to $1+\cos(2\theta)$, and radial symmetry ensures that any horizontal contribution cancels out after all air molecules are considered.

Formulating the Optimization Problem

We are now equipped to convert our physical situation to a mathematical optimization problem. First, notice that

$$1+\cos(2\theta) = \frac{2}{\tan^2(\theta)+1},$$

and according to Figure 1. we have $\tan(\theta) = -f'(x)$ for $x \in [0, R]$. Recalling that the volume of a solid of revolution given by rotating f about the y-axis is the integral of $2\pi x f(x)\,\mathrm{d}x$, we know that the net change in velocity of the bullet is hence given by

$$\int_0^R 2\pi x\,[1+\cos(2\theta(x))]\,\mathrm{d}x = \int_0^R 2\pi x \frac{2}{1+[f'(x)]^2}\,\mathrm{d}x\,,$$

and since constants do not affect the extrema of a functional, we are left with the problem of finding

$$\min F[f] = \int_0^R \frac{x}{1+[f'(x)]^2}\,\mathrm{d}x\,.$$

We also add in the constant volume (of the bullet) and the positive volume constraint.

The Optimization Problem

We have arrived at the mathematical problem we desire to solve on the interval $[0, R]$:

$$\begin{aligned}
\min_{f(x)} F[f] &= \int_0^R \frac{x}{1+[f'(x)]^2}\,\mathrm{d}x\\
G_1[f] &= \int_0^R 2\pi x f(x)\,\mathrm{d}x = V,\\
G_2(x) &= f(x) \geq 0,
\end{aligned}$$

Where G_1 is the constant volume constraint and G_2 ensures that no notion of “negative volume” arises. This optimization problem is known as an isoperimetric problem, one with an integral constraint and the goal of extremizing a functional rather than a function. We also require the boundary conditions $f(0) = H$ and $f(R) = 0$, where $H \in \mathbb{R}$ is a constant representing the fixed height of the bullet.

The General Case

We first take care of the inequality constraint by ensuring that f is non-negative by introducing a slack function $s : [0, R] \to \mathbb{R}$ defined as follows:

$$f(x) = s^2(x), \quad f'(x) = 2s(x)s'(x).$$

Our optimization problem is then reformulated according to the rule above, with s^2 and $2ss'$ replacing each instance of f and f' respectively. We hence form the Lagrangian

$$\mathcal{L}(x, s, s') = \frac{x}{1+[2s(x)s'(x)]^2} + \lambda(2\pi x s^2(x)),$$

such that finding an s solving the dual problem of minimizing $\int \mathcal{L}\,\mathrm{d}x$ is equivalent to solving the initial isoperimetric problem. Solving the Euler-Lagrange equation below provides a necessary condition for finding an extremum satisfying the constraints, yet we may assume it to be sufficient for the physical purposes of our analysis:

$$\frac{\partial \mathcal{L}}{\partial s} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial s'} = 0.$$

This provides a potential method of approaching the general problem, yet there are limitations due to issues such as constraint qualifications and existence/uniqueness of solutions that are beyond the scope of this poster.

The Quadratic Case

Although the constrained optimization problem in the general case is very difficult, the simpler case where f is a quadratic polynomial is much more tractable. Letting

$$f(x) = ax^2 + bx + c,$$

we make two further assumptions:

1. f' vanishes at 0 to ensure the bullet has a smooth tip.
2. f is concave.

The first assumption implies $b = 0$ and the second implies $a \leq 0$. By plugging in the polynomial $ax^2 + c$ for each instance of f and evaluating each integral, the problem degenerates down to solving the following:

$$\begin{aligned}
\min F(a, c, x) &= \frac{1}{8a^2}\log\left(1+4a^2R^2\right)\\
g_1(a, c, x) &= \frac{\pi}{2}R^2(aR^2+2c) - V = 0,\\
g_2(a, c, x) &= ax^2 + c \geq 0, \quad (0 \leq x \leq R),\\
g_3(a, c, x) &= a \leq 0.
\end{aligned}$$

We now treat F as a map $\mathbb{R} \to \mathbb{R}$ with $F'(a) \geq 0$ for all $a \leq 0$ by the mean value theorem. Hence, F is increasing on the negative real axis, and we must minimize the new objective function $F^\star(a, c) = a$ to find $\min F$. Further, we may replace x with R in our g_2 constraint and retain the same information, so the final problem we are left with is

$$\begin{aligned}
\min F^\star(a, c) &= a,\\
g_1(a, c) &= \frac{\pi}{2}R^2(aR^2+2c) - V = 0,\\
g_2(a, c) &= aR^2 + c \geq 0,\\
g_3(a, c) &= a \leq 0.
\end{aligned}$$

Combining constraint g_1 and g_2, we derive the minimum for a:

$$V = \frac{\pi}{2}R^2(aR^2+2c) \geq \frac{\pi}{2}R^2(aR^2-2aR^2) = -\frac{\pi R^4 a}{2} \implies a \geq -\frac{2V}{\pi R^4}.$$

We finally calculate the c term to find the optimal quadratic polynomial $f^\star$ minimizing the air resistance factor, defined by

$$f^\star(x) = -\frac{2V}{\pi R^4}x^2 + \frac{2V}{\pi R^2} = \frac{2V}{\pi R^4}(R^2 - x^2).$$

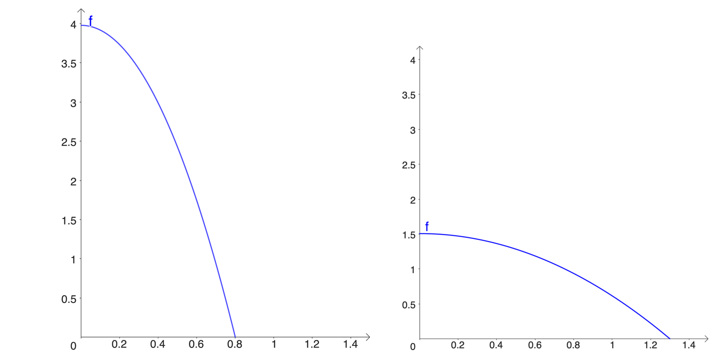

Figure 2. Optimal quadratic curves for $V = 4$ with $R = 0.8$ and $R = 1.3$ respectively

References

[1] Gel′fand, I. M. and Fomin, S. V. Calculus of variations. Prentice-Hall, 1964.

[2] Halliday, David and Resnick, Robert and Walker, Jearl. Fundamentals of physics. Wiley, 2013.

seankim146@gmail.com