Measure Theory HW Sections 2.4-2.6

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1 Problem 33.

Proof.
Suppose $\{f_n\} \subseteq L^+$ is a sequence of functions and $f_n \geq 0$ for all n and $f_n \to f \in L^+$ in measure. By the definition of the lim inf, there is a subsequence $\{f_{n_k}\}$ such that

$$\liminf_{n\to\infty} \int f_n = \lim_{k\to\infty} \int f_{n_k}.$$

Clearly, $\{f_{n_k}\}$ converges to f in measure since $f_n \to f$ in measure. We choose a subsequence $\{f_{n_{k_j}}\}$ that converges a.e. pointwise to f, which we know exists by the previous homework. We see that

$$\int f = \int \liminf_{j\to\infty} f_{n_{k_j}} \leq \liminf_{j\to\infty} \int f_{n_{k_j}} = \lim_{j\to\infty} \int f_{n_{k_j}} = \lim_{k\to\infty} \int f_{n_k} = \liminf_{n\to\infty} \int f_n.$$

□

2 Problem 34.

2.1 Part (a)

Proof.
Again, choose a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ in measure and

$$\liminf_{n\to\infty} \int f_n = \lim_{k\to\infty} \int f_{n_k}.$$

There is a subsequence $\{f_{n_{k_j}}\}$ such that $f_{n_{k_j}} \to f$ p.w. a.e. Clearly, we have that f is in L^1 and

$$\int f = \int \lim_{j\to\infty} f_{n_{k_j}} = \lim_{j\to\infty} \int f_{n_{k_j}} = \lim_{k\to\infty} \int f_{n_k} = \liminf_{n\to\infty} \int f_n$$

by the **DCT**. Repeating this process for a subsequence $\{f_{n_m}$ that converges to f in measure and satisfies

$$\limsup_{n\to\infty} \int f_n = \lim_{m\to\infty} \int f_{n_m}$$

shows the equality of the lim inf and lim sup with $\int f$ and hence we have the result. $\square$

2.2 Part (b)

Proof.
This result follows directly from part (a) by seeing that

$$\lim_{n\to\infty}\left(\int f_n - \int f\right) = 0 \implies \lim_{n\to\infty}\int |f_n - f| = 0.$$

$\square$

3 Problem 40.

Suppose that f_n, f are measurable complex valued functions for each n such that $f_n \to f$ a.e. and there exists a function $g \in L^1(\mu)$ such that $|f_n| \leq g$ for all n. Let $\varepsilon > 0$ be given. As done in the proof in Folland, we assume that $f_n \to f$ everywhere on X. Define the sets $E_{n,k}$ by

$$E_{n,k} = \bigcup_{m=n}^{\infty} \{x \in X : |f_m(x) - f(x)| \geq \frac{1}{k}\}.$$

For fixed k, we have that

$$\int_{E_{n,k}} f = \lim_{n\to\infty}\int_{E_{n,k}} f_n$$

by the **DCT** and clearly this implies that

$$\lim_{n\to\infty}\int_{E_{n,k}} |f_n - f| = 0.$$

Hence, there is an integer N such that

$$n \geq N \implies \int_{E_{n,k}} |f_n - f| < \varepsilon.$$

However, we know that on $E_{n,k}$ we have $|f_n - f| \geq k^{-1}$ so we have

$$\frac{\mu(E_{n,k})}{k} \leq \int_{E_{n,k}} |f_n - f| < \varepsilon \implies \mu(E_{n,k}) < k\varepsilon.$$

Clearly, this implies that $\mu(E_{n,k}) \to 0$ as $n \to \infty$ so we may just follow Folland's proof to show the rest of the result.

4 Problem 44.

Proof.
Let $\varepsilon > 0$ be given. By **Theorem 2.26**, there is a sequence of simple functions $\{\varphi_n\}$ such that for large enough n,

$$\int |f - \varphi_n| < \frac{\varepsilon}{2}$$

Since each $\varphi_n \in L^1(\lambda)$, there must be a corresponding continuous function g_n that vanishes outside a bounded interval such that

$$\int |\varphi_n - g_n| < \frac{\varepsilon}{2}.$$

Clearly,

$$\int |f - g_n| \leq \int (|\varphi_n - f|) + \int (|\varphi_n - g_n|) < \varepsilon,$$

so $g_n \to f$ in L^1. Hence, we consider the subsequence $\{g_{n_k}\}$ such that $g_{n_k} \to f$ p.w. a.e. on $[a, b]$.

Applying **Egoroff's Theorem**, we know that there is some $E \subseteq [a, b]$ such that $\lambda(E) < \varepsilon$ and $g_{n_k} \to f$ uniformly on E^C. Notice that we may approximate E arbitrarily well with compact subsets since the Lebesgue measure λ is regular. Let E' be a compact set that adequately approximates E such that $\mu(E'^C) < \varepsilon$. Since $g_{n_k} \to f$ uniformly on E' and each g_{n_k} is continuous, we have that $f|_{E'}$ is continuous since uniform convergence preserves continuity and the result is proved. $\square$

5 Problem 52.

Proof.
Suppose $(X, \mathcal{M}, \mu)$ is an arbitrary measure space and Y is countable, $\mathcal{N} = \mathcal{P}(Y)$, and ν is the counting measure on Y.

(*Tonelli*) For convenience, let us enumerate the elements in Y with $1, 2, \ldots$ (since Y is countable, we can do this) such that we are now working with the space $X \times \mathbb{N}$. Let $f \in L^+(X \times \mathbb{N})$. By the fact that

$$\int_{\mathbb{N}} f(x)\, \mathrm{d}\nu(x) = \sum_{n=1}^{\infty} f(n),$$

we have

$$\int \left[\int f(x, y)\, \mathrm{d}\nu(y) \right] \mathrm{d}\mu(x) = \int \left[\int f_x(y)\, \mathrm{d}\nu(y) \right] \mathrm{d}\mu(x) = \int \sum_{n=1}^{\infty} f(x, n)\, \mathrm{d}\mu(x)\,.$$

By the **MCT** on the sequence of functions defined by $f_n(x) = f(x,n)$, we have that

$$\int \sum_{n=1}^{\infty} f(x,n)\,\mathrm{d}\mu(x) = \sum_{n=1}^{\infty} \int f(x,n)\,\mathrm{d}\mu(x)\,,$$

and the value on the right is exactly what

$$\int \left[\int f(x,y)\,\mathrm{d}\mu(x)\right] \mathrm{d}\nu(y)$$

reduces down to. Clearly, $g : x \mapsto \sum_n f_n$ is L^+ and consider a sequence of nonnegative simple functions h_n (resp. g_n) that increases pointwise to $h : x \mapsto \int f_n(x)\,\mathrm{d}\mu(x)$ (resp. g) (hence h is measurable). Then by monotone convergence, we deduce that

$$\int g\,\mathrm{d}\mu = \sum_{n=1}^{\infty} h = \int f\,\mathrm{d}(\mu \times \nu)\,,$$

which shows the result.

(*Fubini*) Again, we just take the different parts (imaginary, real, positive, negative) of $f \in L^1(X \times \mathbb{N})$. $\square$

6 Problem 1.

6.1 Part (a)

Proof.
Clearly ν is nonnegative and is 0 on the empty set since the preimage of the empty set is empty. Let $\{E_n\}$ be a sequence of disjoint sets in M_Y with $E = \bigcup_n E_n$. Then

$$\nu(E) = \mu(T^{-1}(E)) = \mu\left(T^{-1}\left(\bigcup_{n=1}^{\infty} E_n\right)\right) = \mu\left(\bigcup_{n=1}^{\infty} T^{-1}(E_n)\right).$$

Suppose A and B are disjoint sets. Clearly

$$A \cap B = \varnothing \implies T^{-1}(A) \cap T^{-1}(B) = \varnothing,$$

so the preimage of (pairwise) disjoint sets is disjoint and by the σ-additivity of μ, we have

$$\mu\left(\bigcup_{n=1}^{\infty} T^{-1}(E_n)\right) = \sum_{n=1}^{\infty} \mu(T^{-1}(E_n)) = \sum_{n=1}^{\infty} \nu(E_n),$$

which proves that ν is a measure on (Y, M_Y). $\square$

6.2 Part (b)

Proof.
Suppose that $f : (Y, M_Y, \nu) \to (X, M_X, \mu)$ with $f \in L^1(Y, \nu)$. Clearly,

$$f \circ T : (X, M_X, \mu) \to (X, M_X, \mu),$$

We shall first prove this result for when f is a characteristic function. Let $E \in M_Y$ be measurable. Then

$$\int_Y \chi_E \,\mathrm{d}\nu = \nu(E) = \mu(T^{-1}(E)),$$

while on the other hand

$$\int_X \chi_E \circ T \,\mathrm{d}\mu = \int_{T^{-1}(E)} \chi_E \circ T \,\mathrm{d}\mu + \int_{T^{-1}(E^C)} \chi_E \circ T \,\mathrm{d}\mu = \int_E \chi_E \,\mathrm{d}\nu + \int_{E^C} \chi_E \,\mathrm{d}\nu\,,$$

so we have that

$$\int_Y \chi_E \,\mathrm{d}\nu = \int_E \chi_E \,\mathrm{d}\nu = \int_X \chi_E \circ T \,\mathrm{d}\mu\,,$$

and clearly $f \circ T \in L^1(X, \mu)$ when it is a characteristic function. This extends the result to simple functions since the integral is linear, and approximating any L^1 function by an increasing sequence of functions as in **Theorem 2.10 (b)**, we have the result for general $f \in L^1(Y, \nu)$. The converse can be seen easily since if f wasn't L^1, then we necessarily have that $f \circ T$ is not L^1 as well by the argument above. □