# Currents in Geometric Measure Theory

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**Abstract**

This expository paper will explore the theory of currents and their properties in view of Plateau's problem, while providing a brief introduction to geometric measure theory. We present the Deformation Theorem and the theory of slicing to culminate in a proof for the Federer-Fleming Compactness Theorem for integer-rectifiable currents. The Compactness Theorem will then be applied to provide a solution to Plateau's problem.

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# 1 Introduction

Geometric Measure Theory (henceforth abbreviated GMT) is a field of mathematics drawing from the tools of measure theory to investigate the geometry of subsets of Euclidean space. It rose to prominence during the mid 1900s with the efforts of early pioneers such as H. Federer and W.H. Fleming to solve *Plateau's problem*, named after the Belgian physicist Joseph Plateau who related the original mathematical problem posed by J. Lagrange in 1760 to the geometry of soap films and soap bubbles. It states:

**Conjecture 1.0.1** (Plateau's Problem)**.** *For every smooth closed curve $\Gamma$ in $\mathbb{R}^3$, there is a surface of least area among all surfaces which have $\Gamma$ as their boundary.*

Plateau showed that via physical arguments about the surface tension of soap films, they mimic area-minimizing surfaces spanning a given boundary, namely, a piece of wire to act as the perimeter, as seen in Figure 1. Standard differential geometric arguments considering submanifolds of Euclidean space failed to solve this problem due to their lack of compactness properties, which led to the development of currents, which indeed possess the desirable properties yet still resemble such submanifolds.

Figure 1: Soap film is the surface of minimal surface area given a wire boundary.
https://commons.wikimedia.org/wiki/File:Minimal_surfaces._Plateau%27s_problem_07.jpg

This expository paper aims to present the highlights of GMT and develop an introduction to currents that requires only a basic real analysis and point-set topology background. Our efforts shall culminate in the Compactness Theorem for integer-rectifiable currents, first proven by Federer and Fleming and published in [FF60]. Then, we shall solve Plateau's problem and state a regularity result attributed to Fleming, first proved in 1962 in [Fle62]. The main texts of reference for this paper are [KP08] and [Sim14], and those desiring a rigorous and detailed development of GMT without omissions should consult Federer's excellent reference text, [Fed69].

# 2 Preliminaries

In this section, we introduce the preliminary measure theory necessary for geometric measure theory and currents in particular. Those interested in a more detailed development of the theory of measures and integration may consult [Fol94].

## 2.1 Measures and $\sigma$-Algebras

The theory of measures arises from one wanting to develop a rigorous method of assigning a "volume" or "mass" to subsets of a set $X$ (take $\mathbb{R}^3$ for example). We call a map that performs such designations a *measure*, which is canonically denoted $\mu$. Unfortunately, the intuitive path of considering the measure as a map $\mu : \mathcal{P}(X) \to [0, \infty]$ (where $\mathcal{P}(X)$ is the set of all subsets of $X$), as in attempting to assign a volume to every single subset of $X$ while retaining fundamental properties such as being able to decompose a set into a finite number of subsets and evaluate an equivalent total volume sometimes turns out problematic. Supposing that $\mu$ satisfies certain physically expected properties such as invariance under rigid transformations, even in the most applicable case of $\mathbb{R}^3$, considering the domain of $\mu$ to be the power set $\mathcal{P}(\mathbb{R}^3)$ leads to impossibilities, such as the Banach-Tarski Paradox, due to the collection of subsets simply being too complicated or fine. We are hence led to the notion of a $\sigma$-algebra, a collection of subsets of $X$ which shall serve as an appropriate domain for the measure $\mu$.

**Definition 2.1.1** ($\sigma$-Algebras)**.** A $\sigma$-algebra $\mathcal{A}$ on a given set $X$ is a collection of subsets of $X$ that satisfies

1. $\varnothing$ and $X$ are elements of $\mathcal{A}$;
2. (Closure under complements) If $A \in \mathcal{A}$, then $A^C \in \mathcal{A}$;
3. (Closure under countable unions) If $\{E_j\}$ is a countable collection of subsets such that $E_j \in \mathcal{A}$ for each positive integer $j$, then $\bigcup_j E_j$ is also in $\mathcal{A}$.

It is worth noting that $\mathcal{A}$ may be the set of all subsets of $X$, and it is easy to verify that the power set $\mathcal{P}(X)$ is indeed a $\sigma$-algebra on $X$. The most common type of $\sigma$-algebra is the *Borel $\sigma$-algebra* on a topological space, the smallest $\sigma$-algebra containing all open sets.

We call the pair $(X, \mathcal{A})$ a *measurable space*, and we are now ready to define measures.

**Definition 2.1.2** (Measures)**.** A measure on a measurable space $(X, \mathcal{A})$ is a map $\mu : \mathcal{A} \to [0, \infty]$ that satisfies

1. $\mu(\varnothing) = 0$;

2. ($\sigma$-Additivity) For a countable pairwise-disjoint collection of sets $\{S_j\} \subset \mathcal{A}$, $\mu$ satisfies
$$\mu\left(\bigcup_{j=1}^{\infty} S_j\right) = \sum_{j=1}^{\infty} \mu(S_j).$$

A triple $(X, \mathcal{A}, \mu)$ is called a *measure space*. On a measure space $(X, \mathcal{A}, \mu)$, we say that a statement about $x \in X$ true except for on a set $N$ such that $\mu(N) = 0$, then we say that the statement is satisfied $\mu$-almost everywhere, or more concisely, $\mu$-a.e. The $\mu$ may be dropped when the context is clear.

We call $\mu$ *finite* if $\mu(X) < \infty$. Of course, measures may take on non-finite values, but in practice mostly measures that are "reasonably finite" encountered. This notion of reasonable finiteness is quantified as the property of $\sigma$-finiteness.

**Definition 2.1.3** ($\sigma$-Finiteness)**.** Given a measure space $(X, \mathcal{A}, \mu)$, we say that $\mu$ is a $\sigma$-finite measure on $X$ (or just $\sigma$-finite) if there are sets $E_j \in \mathcal{A}$ with

$$X = \bigcup_{j=1}^{\infty} E_j$$

and $\mu(E_j) < \infty$ for each $j$.

It turns out that relaxing the $\sigma$-additivity condition in the definition of measures to a condition known as $\sigma$-subadditivity defined below gives rise to the notion of an *outer measure*, which allows us to consider the domain as the power set with some slight changes to the definition.

**Definition 2.1.4** (Outer Measures)**.** An outer measure on $X$ is a map $\mu^* : \mathcal{P}(X) \to [0, \infty]$ that satisfies

1. $\mu^*(\varnothing) = 0$;
2. (Monotonicity) For subsets $A, B \subset X$ with $A \subset B$, we have $\mu^*(A) \leq \mu^*(B)$;
3. ($\sigma$-Subadditivity) For $\{S_j\} \subset \mathcal{P}(X)$, $\mu^*$ satisfies
$$\mu^*\left(\bigcup_{j=1}^{\infty} S_j\right) \leq \sum_{j=1}^{\infty} \mu^*(S_j).$$

Outer measures may always be restricted to a smaller domain to become a proper measure on this restriction, and it turns out this domain is the collection of sets $S \subset X$ such that
$$\mu^*(A) = \mu^*(A \cap S) + \mu^*(A \cap S^C)$$

for every $A \in X$, and this collection turns out to be a $\sigma$-algebra. This result is known as Caratheódory's Theorem and such sets in the $\sigma$-algebra are called $\mu^*$-measurable sets.

We shall restrict our attention to the case where $X = \mathbb{R}^n$ in this paper. Though there exist a multitude of distinct (outer) measures on $\mathbb{R}^n$, the two most significant to our development of currents and GMT are the *Lebesgue measure* and *Hausdorff measure*, with the former being a measure induced from restricting an outer measure, while the latter is an outer measure.

**Definition 2.1.5** (Lebesgue Measure)**.** We define the $n$-dimensional Lebesgue outer measure $(\mathcal{L}^n)^* : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$ first, then restrict the domain to the appropriate $\sigma$-algebra to obtain a measure. For each interval $I = (a, b)$ (it may or may not contain its endpoints), define its length to be $\ell(I) := b - a$. For any $n$-dimensional rectangle $R = \prod_1^n I_j$ that is the product of $n$ open intervals, define

$$\operatorname{vol}(R) = \prod_{j=1}^{n} \ell(I_j).$$

Then for any $S \subset \mathbb{R}^n$, the $n$-dimensional Lebesgue outer measure is defined to be

$$(\mathcal{L}^n)^*(S) = \inf \left\{ \sum_{j=1}^{\infty} \operatorname{vol}(R_j) : \text{each } R_j \text{ is an } n\text{-rectangle and } S \subset \bigcup_{j=1}^{\infty} R_j \right\}.$$

We now restrict $(\mathcal{L}^n)^*$ to

$$\mathcal{M}^n := \{S \subset \mathbb{R}^n : (\mathcal{L}^n)^*(A) = (\mathcal{L}^n)^*(A \cap S) + (\mathcal{L}^n)^*(A \cap S^C), \quad A \in \mathbb{R}^n\},$$

and define $(\mathcal{L}^n)^*|_{\mathcal{M}^n} := \mathcal{L}^n$, the $n$-dimensional Lebesgue measure.

A basic measure theoretic result is that $\mathcal{M}^n$ is strictly larger than the Borel $\sigma$-algebra on $\mathbb{R}^n$, meaning that most, if not all, the sets that one encounters naturally (open, closed, and unions/intersections thereof) will be $(\mathcal{L}^n)^*$-measurable.

**Definition 2.1.6** (Hausdorff Measure)**.** For a set $U \subset \mathbb{R}^n$, we define its diameter by

$$\operatorname{diam} U := \sup\{\|x - y\| : x, y \in U\},$$

where the empty set has 0 diameter and $\|\cdot\|$ denotes the usual Euclidean norm. Furthermore, let

$$\Omega_m = \frac{\pi^{m/2}}{\Gamma(\frac{m}{2} + 1)},$$

the volume of the $m$-dimensional unit ball in $\mathbb{R}^m$, except we allow $m$ to take on non-integer values. Let $S \in \mathcal{P}(\mathbb{R}^n)$ and let $\delta > 0$. We define the auxiliary measure

$$\mathcal{H}_\delta^m(S) = \inf \left\{ \Omega_m \sum_{j=1}^{\infty} (\frac{\operatorname{diam} U_i}{2})^m : S \subset \bigcup_{j=1}^{\infty} U_j, \operatorname{diam} U_j < \delta \right\},$$

which is clearly monotonically decreasing as we take smaller $\delta$. Hence, $\lim_{\delta \to 0} H_\delta^m(S)$ exists (though may not be finite), and we define the $m$-dimensional Hausdorff measure on $\mathbb{R}^n$ as

$$\mathcal{H}^m(S) := \sup_{\delta > 0} \mathcal{H}_\delta^m(S) = \lim_{\delta \to 0} \mathcal{H}_\delta^m(S).$$

Note that $m$ does not need to equal $n$, yet when this equality does occur, the $n$-dimensional Hausdorff measure and Lebesgue measure coincide on an appropriate domain.

We may want to "restrict" measures to certain sets, which motivates the following notation. If $(X, \mathcal{A}, \mu)$ is a measure space and $A \in \mathcal{A}$, then we define the *restriction of $\mu$ to $A$* to be the measure $\mu \llcorner A$ on $\mathcal{A}$ such that

$$(\mu \llcorner A)(S) = \mu(A \cap S)$$

for all $S \in \mathcal{A}$.

## 2.2 Density of Measures

We briefly discuss the density of measures in this subsection.

**Definition 2.2.1** (Borel Regular Measures)**.** Let $\mu$ be an outer measure on $\mathbb{R}^m$. We say that $\mu$ is *Borel regular* if every Borel set is $\mu$-measurable and for every $S \subset \mathbb{R}^m$, there exists a Borel $B \subset S$ such that $\mu(B) = \mu(S)$.

**Definition 2.2.2** (Upper Density of Measures)**.** Let $\mu$ be a Borel regular measure on $\mathbb{R}^{m+k}$, and let $S \subset \mathbb{R}^{m+k}$, $x \in \mathbb{R}^{m+k}$. The *upper density* of $\mu$ is defined to be

$$\Theta^{m*}(\mu, S, x) := \limsup_{r \to 0} \frac{\mu(S \cap B(x, r))}{\Omega_m r^m},$$

where $B(x, r)$ is the open ball of radius $r$ centered at $x$, and $\Omega_m$ is as Definition 2.1.6.

The lower density $\Theta_*^m$ is defined similarly with lim inf instead of lim sup. We present the two following facts about densities without proof.

**Proposition 2.2.1.** *If $\mu$ is a Borel regular measure on $\mathbb{R}^{m+k}$, then the following are satisfied:*

1. $\mu(S) \geq \delta \mathcal{H}^m(\{x \in S : \Theta^*(\mu, \mathbb{R}^{m+k}, x) \geq \delta\})$ *if* $S$ *is open;*
2. $\Theta^{m*}(\mathcal{H}^m, S, x) \leq 1$ *for* $\mathcal{H}^m$*-almost every* $x \in S$ *if* $\mathcal{H}^m(S) < \infty$.

## 2.3 Integration

We shall now discuss the integration of measurable functions with respect to measures. For a function $f : X \to Y$ and a collection of subsets $\mathcal{B} \subset \mathcal{P}(Y)$, we define

$$f^{-1}(\mathcal{B}) := \{A \subset X : \text{there is a } B \in \mathcal{B} \text{ with } f(A) = B\}$$

**Definition 2.3.1** (Measurable Functions)**.** For two given measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, a map $f : X \to Y$ is $(\mathcal{A}; \mathcal{B})$*-measurable* (or just *measurable* if the context is understood) if

$$f^{-1}(\mathcal{B}) \subset \mathcal{A}.$$

*Remark* 2.3.1. Note that most measurable functions that one will encounter will map into $\mathbb{R}$ with its Borel $\sigma$-algebra $\mathcal{B}_{\mathbb{R}}$. For such functions $f : (X, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}}$, we write that they are $\mathcal{A}$-measurable.

Readers familiar with topology will notice the resemblance of this definition with that of a continuous map between topological spaces (the latter replaces the $\sigma$-algebras with the respective topologies on the (co)domain), yet measurability is a significantly weaker condition. This can be seen by the following basic result, which says that the pointwise limit of a sequence of real measurable functions is measurable, contrasted to the falsehood of the statement with measurability replaced with continuity.

**Theorem 2.3.1.** *If* $\{f_n\}$ *is a sequence of measurable functions with* $f_n : X \to \mathbb{R}$ *for each* $n$*, (*$\mathbb{R}$ *is equipped with its Borel* $\sigma$*-algebra) and there is a function* $f : X \to \mathbb{R}$ *such that*

$$f_n(x) = f(x)$$

*for all* $x \in X$*, i.e.* $f_n \to f$ *pointwise, then* $f$ *is measurable.*

Before defining integration for general measurable functions with real codomain, we define simple functions and their integrals, then extend to the general case.

**Definition 2.3.2** (Characteristic Functions)**.** Given a set $X$ and a subset $S \subset X$, the characteristic function of $S$ is $\chi_S : X \to \{0, 1\}$ defined by

$$\chi_S(x) := \begin{cases} 0 & \text{when } x \notin S, \\ 1 & \text{when } x \in S. \end{cases}$$

**Definition 2.3.3** (Simple Functions)**.** A function is *simple* if its range is a finite set.

Clearly, we may always represent a simple function $\varphi : X \to \mathbb{R}$ with a linear combination of characteristic functions, namely

$$\varphi = \sum_{j=1}^{n} a_j \chi_{E_j},$$

where $a_j$ are the distinct values that $\varphi$ takes on and each $E_j = \varphi^{-1}(\{a_j\}) \subset X$. This is called the *standard representation* of $\varphi$. Conversely, any such linear combination of characteristic functions also defines a simple function. Let $\mathbb{R}^+$ denote the set of non-negative real numbers. Note that simple functions with codomain $\mathbb{R}^+$ (and also those with any general measurable space as the codomain) are always measurable with respect to the *trace $\sigma$-algebra* of $\mathbb{R}^+$ induced by the Borel $\sigma$-algebra on $\mathbb{R}$, the collection of all Borel sets intersected with $\mathbb{R}^+$. We first define the integral of non-negative real simple functions.

**Definition 2.3.4.** Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\varphi : X \to \mathbb{R}^+$ be simple with $\varphi = \sum_1^n a_j \chi_{E_j}$. We define the integral of $\varphi$ with respect to $\mu$ as

$$\int_X \varphi \, \mathrm{d}\mu = \sum_{j=1}^{n} a_j \mu(E_j).$$

When there is no confusion, we may omit the $X$ and/or $\mathrm{d}\mu$ and write $\int \varphi$, or specify a "variable" and write $\int \varphi(x) \, \mathrm{d}\mu$ to mean the same thing. We may also integrate over certain measurable subsets $A \subset X$ in $\mathcal{A}$ by defining

$$\int_A \varphi \, \mathrm{d}\mu = \int \varphi \chi_A \, \mathrm{d}\mu = \sum_{j=1}^{n} a_j \mu(E_j \cap A).$$

We are now equipped to extend the integral to all $(X; \mathbb{R}^+)$-measurable functions.

**Definition 2.3.5.** For a measurable $f : X \to \mathbb{R}^+$, we define

$$\int f \, \mathrm{d}\mu := \sup \left\{ \int \varphi \, \mathrm{d}\mu : 0 \leq \varphi \leq f, \varphi \text{ simple} \right\}.$$

We now finally arrive at the definition of the integral for general measurable functions.

**Definition 2.3.6** (Integration of Measurable Functions)**.** For a measurable $f : X \to \mathbb{R}$, we define

$$\int f \, \mathrm{d}\mu := \int f^+ \, \mathrm{d}\mu - \int f^- \, \mathrm{d}\mu \,,$$

where

$$f^+(x) = \max(f(x), 0) \text{ and } f^-(x) = \max(-f(x), 0).$$

We call $f^+$ and $f^-$ the positive and negative parts of $f$ respectively.

The measurability of the positive and negative parts of $f$ is easy to verify due to the measurability of functions composed with the max function, and we call functions with both $\int f^+$ and $\int f^-$ finite to be *$\mu$-integrable* (we may sometimes omit the $\mu$). Since $f^+ + f^- = |f|$, $f$ is integrable iff $\int |f| < \infty$, and we write $f \in L^1(\mu)$, recalling that $\mu$ is the measure on $(X, \mathcal{A})$. Furthermore, we say that $f$ is *locally $\mu$-integrable* if

$$\int_K |f| \, \mathrm{d}\mu < \infty$$

for all compact $K \subset X$.

This integral satisfies all of the properties that is expected from acquaintance with the *Riemann integral*, namely, that of linearity and monotonicity, and is called the *Lebesgue integral* for functions with domain $(\mathbb{R}, \mathcal{M}, \mathcal{L}^1)$. Furthermore, a theorem due to Lebesgue states that a bounded function $f : \mathbb{R} \to \mathbb{R}$ is *Riemann integrable* on a closed, bounded interval $[a, b]$ if and only if it is continuous $\mathcal{L}^1$-a.e. on $[a, b]$, where $\mathcal{L}^1$ is the one-dimensional Lebesgue measure.

## 2.4 Measures and Representations

We shall now discuss two theorems that demonstrate the utility of measures in representing certain operations: the Radon-Nikodym Theorem and the Riesz Representation Theorem (also known as the Riesz-Markov-Kakutani Theorem).

The former is a foundational result that allows one measure to be represented as integration with respect to another measure, given that they satisfy the following condition (and are $\sigma$-finite.

**Definition 2.4.1** (Absolute Continuity)**.** Let $(X, \mathcal{A})$ be a measurable space equipped with measures $\mu$ and $\nu$. We say that a measure $\nu$ is absolutely continuous with respect to $\mu$ and write $\nu \ll \mu$ if $\nu$ is 0 on every set $S$ such that $\mu(S) = 0$.

**Theorem 2.4.1** (Radon-Nikodym)**.** *Suppose $(X, \mathcal{A})$ is a measurable space on which the $\sigma$-finite measures $\mu$ and $\nu$ are defined. If $\nu \ll \mu$ then there exists an $\mathcal{A}$-measurable function $f : X \to \mathbb{R}^+$ such that for each $S \in \mathcal{A}$,*

$$\nu(S) = \int_S f \, \mathrm{d}\mu \, .$$

This function $f$ is called the *Radon-Nikodym derivative* of $\nu$ with respect to $\mu$, and is sometimes denoted $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$. The proof may be found in any introductory text on measure theory, such as [Fol94].

We now discuss a version of the Riesz Representation Theorem that will later be utilized in representing currents through integration.

*Remark* 2.4.1. When talking about vector spaces and elements thereof, we may sometimes use a bold font such as in $\mathbf{v}$ to emphasize that these objects are in fact vectors.

**Definition 2.4.2** (Support)**.** Let $(X, \tau)$ be a topological space and $V$ a vector space over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. For a function $f : X \to V$, we define its *support* to be

$$\operatorname{supp} f := \overline{\{x \in X : f(x) \neq \mathbf{0}\}}.$$

We say that $f$ has compact support (or is compactly supported) if $\operatorname{supp} f$ is compact in $X$. The most common case is when $f$ maps into $\mathbb{R}$ as its codomain.

We now introduce the idea of locally-compact Hausdorff spaces, which are topological spaces with enough structure to allow "nice" regularity properties. We adopt the convention of defining a neighborhood about a point to be any open set containing the point.

**Definition 2.4.3** (LCH Spaces)**.** Let $(X, \tau)$ be a topological space. $X$ is called *locally-compact (and) Hausdorff* (abbreviated LCH) if

1. (Locally-compact) Each point $x \in X$ has a neighborhood $N_x$ such that $\overline{N_x}$ is compact;
2. (Hausdorff) For every two distinct points $x, y \in X$, there are neighborhoods $N_x$ and $N_y$ about $x$ and $y$ respectively such that $N_x \cap N_y = \varnothing$.

Many of the measures of interest on LCH spaces satisfy certain regularity properties, two of which are *outer regularity* and *inner regularity*. The former states that for a Borel measure $\mu$ on an LCH $(X, \tau)$, if

$$\mu(S) = \inf\{\mu(U) : U \supset S, U \in \tau\}$$

for some Borel measurable $S$, then $\mu$ is inner regular on $S$, while inner regularity on $S$ means

$$\mu(S) = \sup\{\mu(K) : K \subset S, K \text{ compact}\}.$$

Radon measures are a certain family of measures which are not overly restrictive, yet "nice enough" such that one may derive representation theorems such as Theorem 2.3.2

**Definition 2.4.4** (Radon Measures)**.** A measure $\mu$ on an LCH space $X$ with its Borel $\sigma$-algbera is a Borel measure that is finite on all compact sets, outer regular on all Borel sets, and inner regular on all open sets.

We are now ready to present the Riesz Representation Theorem in the form stated in [Sim14], which says that a continuous linear functional may be represented through integration.

**Theorem 2.4.2** (Riesz-Markov-Kakutani)**.** *Let $H$ be a given finite-dimensional real complete inner product space (with inner product $\langle\cdot,\cdot\rangle_H$ and induced norm $\|\cdot\|_H$) and let $C_c(X,H)$ denote the space of continuous functions $f : X \to H$ with compact support. Suppose $X$ is a locally-compact Hausdorff space equipped with its Borel $\sigma$-algebra $\mathcal{B}_X$, and let $L : C_c(X,H) \to \mathbb{R}$ be linear with*

$$\sup_{\|f\|\leq 1, \operatorname{supp} f \subset K} L(f) < \infty$$

*for every compact $K \subset X$. Then there is a Radon measure $\mu$ on $X$ and $\mu$-measurable $\nu : X \to H$ with $\|\nu(x)\|_H = 1$ $\mu$-a.e. and*

$$L(f) = \int_X \langle f(x), \nu(x)\rangle_H \, \mathrm{d}\mu(x)$$

*for any $f \in C_c(X,H)$.*

The proof is technical and omitted, yet may be found following [Sim14] Theorem 1.4.14.

# 3 Differential Forms, Distributions, and Currents

The purpose of this section is to introduce the material necessary to define and understand currents. A current is essentially merely a differential form with distribution coefficients, a notion which shall be formalized below.

## 3.1 Differential Forms

*Remark* 3.1.1. Throughout the section, $U$ will be understood to be an open subset of $\mathbb{R}^n$.

We shall define two different types of spaces on $\mathbb{R}^n$, the first being the space $\bigwedge_m(\mathbb{R}^n)$ of *$m$-vectors* on $\mathbb{R}^n$, and its dual $\bigwedge^m(\mathbb{R}^n)$ consisting of the *$m$-covectors* on $\mathbb{R}^n$. Our approach to defining these spaces will be analogous to that in [KP08], with little to no manifold theory being a pre-requisite to our development.

Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ denote the standard basis for $\mathbb{R}^n$.

**Definition 3.1.1.** Let $\sim$ be an equivalence relation on $(\mathbb{R}^n)^m$ defined by

1. $(u_1, \dots, \alpha u_i, \dots, u_j, \dots, u_m) \sim (u_1, \dots, u_j, \dots, \alpha u_i, \dots, u_m)$;
2. $(u_1, \dots, u_i, \dots, u_j, \dots, u_m) \sim (u_1, \dots, u_i + \alpha u_j, \dots, u_j, \dots, u_m)$;
3. $(u_1, \dots, u_i, \dots, u_j, \dots, u_m) \sim (u_1, \dots, -u_j, \dots, u_i, \dots, u_m)$,

for all $\alpha \in \mathbb{R}$ and $1 \leq i < j \leq m$, and extending the relation to be symmetric and transitive.

We denote by $u_1 \wedge \cdots \wedge u_m$ the equivalence class of $(u_1, \ldots, u_m)$ under $\sim$, and call the former a *simple $m$-vector*. We now consider the vector space of linear combinations of these simple $m$-vectors under the equivalence relation $\approx$ defined by

1. $\alpha(u_1 \wedge \cdots \wedge u_m) \approx (\alpha u_1) \wedge \cdots \wedge u_m$;
2. $(u_1 \wedge u_2 \wedge \cdots \wedge u_m) + (v_1 \wedge u_2 \wedge \cdots \wedge u_m) \approx (u_1 + v_1) \wedge u_2 \wedge \cdots \wedge u_m$.

Finally, we consider the equivalence classes of linear combinations of simple $m$-vectors under $\approx$, which is called the vector space of $m$-vectors in $\mathbb{R}^n$ and denoted $\bigwedge_m(\mathbb{R}^n)$. Clearly $\bigwedge_1(\mathbb{R}^n) = \mathbb{R}^n$ and when $m > n$, $\bigwedge_m(\mathbb{R}^n)$ is the trivial vector space containing only the zero vector.

We may define the *wedge product* (also referred to as the *exterior product*), which is a map

$$\wedge : \bigwedge_\ell(\mathbb{R}^n). \times \bigwedge_m(\mathbb{R}^n) \to \bigwedge_{\ell+m}(\mathbb{R}^n)$$

and is an anticommutative and associative multiplication-type operation. One may notice that since each $m$-vector can be represented as a linear combination of simple $m$-vectors, we may form a basis for $\bigwedge_m(\mathbb{R}^n)$ by considering $m$-vectors of the form

$$\mathbf{e}_{j_1} \wedge \mathbf{e}_{j_2} \wedge \cdots \wedge \mathbf{e}_{j_m},\ 1 \leq j_1 < \cdots < j_m \leq n.$$

Before discussing $\bigwedge^m(\mathbb{R}^n)$, we introduce some convenient notation.

**Definition 3.1.2** (Multi-index)**.** A *multi-index* $\alpha$ is an $n$-tuple of non-negative integers, i.e. an element of $\mathbb{N}^n$. We adopt the convection that $\mathbb{N}$ contains 0.

We are now ready to discuss the space $\bigwedge^m(\mathbb{R}^n)$ of alternating $m$-linear functions on $(\mathbb{R}^n)^m$.

**Definition 3.1.3** (Multilinear Maps)**.** We say that a function $f : (\mathbb{R}^n)^m \to \mathbb{R}$ is *multilinear* or $m$-linear if it is linear in each of its $m$ arguments.

**Definition 3.1.4** (Alternating Maps)**.** An $m$-linear function $f$ is said to be *alternating* if

$$f(u_1, \ldots, u_i, \ldots, u_j, \ldots, u_m) = -f(u_1, \ldots, u_j, \ldots, u_i, \ldots u_m)$$

for all $1 \leq i < j \leq m$.

We call the alternating $m$-linear functions $\omega \in \bigwedge^m(\mathbb{R}^n)$ to be the $m$-covectors on $\mathbb{R}^n$.

Clearly, $\bigwedge^1(\mathbb{R}^n)$ is merely the dual space of $\mathbb{R}^n$ since all functions of one variable are trivially alternating. Let us denote by $dx^1, \ldots, dx^n \in \bigwedge^1(\mathbb{R}^n)$ the dual basis with respect to $\mathbf{e}_1, \ldots, \mathbf{e}_n \in \mathbb{R}^n$ defined by

$$dx^i(\mathbf{e}_j) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases},$$

where $\delta_{ij}$ is the *Kronecker delta.*

Similar to the wedge product we defined on $\bigwedge_m(\mathbb{R}^n)$, a notion of a "multiplication" of covectors may be explored.

**Definition 3.1.5** (Wedge Product of Covectors)**.** We define the wedge product of $\omega_1, \ldots, \omega_m \in \bigwedge^1(\mathbb{R}^n)$ by

$$(\omega_1 \wedge \cdots \omega_m)(u_1, \ldots, u_m) := \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{m} \omega_{\sigma(j)}(u_j) = \det([\omega_i(u_j)]),$$

where $S_n$ is the set of all permutations of $\{1, \ldots, n\}$ (bijections onto itself).

Analogous to how the wedge product of standard basis vectors $\mathbf{e}_{j_i}$ formed the basis for $\bigwedge_m(\mathbb{R}^n)$, the set of $m$-covectors of the form

$$dx^{j_1} \wedge \cdots \wedge dx^{j_m}$$

form a basis for $\bigwedge^m(\mathbb{R}^n)$ since any $\omega \in \bigwedge^m(\mathbb{R}^n)$ may be represented as

$$\begin{aligned} \omega &= \sum_{1 \leq j_1 < \cdots < j_m \leq n} \omega_{j_1, \ldots, j_m} dx^{j_1} \wedge \cdots \wedge dx^{j_m}, \\ &= \sum_{\alpha \in I_{m,n}} \omega_\alpha dx^\alpha \end{aligned}$$

where

$$I_{m,n} := \{\alpha = (j_1, \ldots, j_m) \in \mathbb{N}^m : 1 \leq j_1 < \cdots < j_m \leq n\}.$$

The utility of the multi-indices $\alpha$ is seen here, where we understand

$$\omega_\alpha = \omega_{j_1, \ldots, j_m} = \omega(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_m})$$

and

$$dx^\alpha = dx^{j_1} \wedge \cdots \wedge dx^{j_m}.$$

Likewise, the $m$-vectors $\xi$ may be represented in the form

$$\xi = \sum_{\alpha \in I_{m,n}} \xi^\alpha \mathbf{e}_\alpha,$$

where each component is defined analogously.

We are interested in the interactions between the spaces $\bigwedge_m(\mathbb{R}^n)$ and $\bigwedge^m(\mathbb{R}^n)$. The latter turns out to be the dual of the former, meaning that we may introduce a *dual pairing* between an $m$-covector $\omega$ and an $m$-vector $\xi$.

**Definition 3.1.6** (Dual Pairing)**.** For $\omega \in \bigwedge^m(\mathbb{R}^n)$ and $\xi \in \bigwedge_m(\mathbb{R}^n)$, the dual pairing $\langle \cdot, \cdot \rangle : \bigwedge^m(\mathbb{R}^n) \times \bigwedge_m(\mathbb{R}^n) \to \mathbb{R}$ (not to be confused with the inner product) is defined by

$$\langle \omega, \xi \rangle = \left\langle \sum_{\alpha \in I_{m,n}} \omega_\alpha dx^\alpha, \sum_{\alpha \in I_{m,n}} \xi^\alpha \mathbf{e}_\alpha \right\rangle = \sum_{\alpha \in I_{m,n}} \omega_\alpha \xi^\alpha.$$

Note that the dual pairing is a more general concept that may be applied to a $\mathbb{F}$-vector space $V$ and its dual $V^*$, where $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{F}$ defined by

$$\langle L, u \rangle = L(u)$$

for all $L \in V^*$ and $u \in V$.

We are now ready to define differential forms on $\mathbb{R}^n$.

**Definition 3.1.7** (Differential Forms)**.** Recalling that $U$ is an open subset of $\mathbb{R}^n$, a map $\varphi : U \to \bigwedge^m(\mathbb{R}^n)$ is a *differential $m$-form* on $\mathbb{R}^n$.

We may represent any differential form $\varphi$ in terms of a linear combination of basis elements of $\bigwedge^m(\mathbb{R}^n)$ by defining the functions

$$\varphi_\alpha(p) = \langle \varphi(p), \mathbf{e}_\alpha \rangle,$$

noticing this is a dual pairing between $\varphi(p) \in \bigwedge^m(\mathbb{R}^n)$ and the basis $m$-vector $\mathbf{e}_\alpha$, where $\alpha$ is a multi-index. Then, we may write

$$\varphi = \sum_{\alpha \in I_{m,n}} \varphi_\alpha dx^\alpha.$$

Let us define $\mathcal{E}^m(U) := C^\infty(U, \bigwedge^m(\mathbb{R}^n))$. The elements $\varphi$ of $\mathcal{E}^m(U)$ are called the *smooth $m$-forms on $U$*, and are differential forms with smooth coefficient functions $\varphi_\alpha$. We may define a derivative operator known as the *exterior derivative* $d : \mathcal{E}^m(U) \to \mathcal{E}^{m+1}(U)$ by

$$d\varphi := \sum_{j=1}^{n} \sum_{\alpha \in I_{m,n}} \frac{\partial \varphi_\alpha}{\partial x^j} dx^j \wedge dx^\alpha,$$

given that $\varphi = \sum \varphi_\alpha dx^\alpha$. A surprising property of the exterior derivative is that

$$d^2\varphi = 0$$

for all $\varphi \in \mathcal{E}^m(U)$, which may be verified through tedious calculation.

The most fundamental theorem to calculus on manifolds is a generalization of the Fundamental Theorem of Calculus to manifolds, known as the Generalized Stokes's Theorem. We state it without proof, but the interested reader may explore its background in [Tu11]

**Theorem 3.1.1** (Generalized Stokes's Theorem)**.** *Let $\varphi$ be a smooth $(m-1)$-form with compact support on an oriented, $m$-dimensional manifold with boundary $M$, where the boundary $\partial M$ is given the induced orientation. Then, we have*

$$\int_M d\varphi = \int_{\partial M} \varphi.$$

We now consider the space $\mathcal{D}^m(U) \subset \mathcal{E}^m(U)$ consisting of all differential forms $\varphi$ such that its component functions $\varphi_\alpha$ have compact support contained in $U$. We equip $\mathcal{D}^m(U)$ with the locally-convex Hausdorff topology described in Appendix A. We include the following lemma without proof.

**Lemma 3.1.2.** *The space $\mathcal{D}^m(U)$ is separable, meaning that there is a countable sequence $\{\varphi_j\} \subset \mathcal{D}^m(U)$ that is dense in $\mathcal{D}^m(U)$.*

Finally, we define a norm on $\mathcal{D}^m(U)$.

**Definition 3.1.8.** Let $\varphi \in \mathcal{D}^m(U)$. We define the norm of $\varphi$ as

$$\|\varphi\| := \sup_{x \in U} \sqrt{\varphi(x) \cdot \varphi(x)}.$$

Note that the $\varphi(x) \cdot \varphi(x)$ term in the definition refers to the induced inner product on $\bigwedge^m(\mathbb{R}^n)$ by the Euclidean dot product. More precisely, for $\omega, \eta \in \bigwedge^m(\mathbb{R}^n)$,

$$\omega \cdot \eta = \left( \sum_{\alpha \in I_{m,n}} \omega_\alpha dx^\alpha \right) \cdot \left( \sum_{\alpha \in I_{m,n}} \eta_\alpha dx^\alpha \right) = \sum_{\alpha \in I_{m,n}} \omega_\alpha \eta_\alpha,$$

so

$$\sqrt{\varphi(x) \cdot \varphi(x)} = \sqrt{\sum_{\alpha \in I_{m,n}} \varphi_\alpha^2},$$

clearly resembling the Euclidean norm.

## 3.2 Currents

We are finally equipped with sufficient machinery to define and understand currents.

First, we introduce some notation.

**Definition 3.2.1.** We write $W \subset\subset U$ for $W \subset \mathbb{R}^n$ if $W \subset U$ and the closure of $W$, namely $\overline{W}$, is compact and contained within $U$.

**Definition 3.2.2** (Currents)**.** We define

$$\mathcal{D}_m(U) = (\mathcal{D}^m(U))^*,$$

where the right is the space of continuous maps $T : \mathcal{D}^m(U) \to \mathbb{R}$, i.e. the dual space of $\mathcal{D}^m(U)$. We call an element $T \in \mathcal{D}_m(U)$ an *$m$-current*.

Much like in Definition 2.3.2, we want to speak of the support of a current $T \in \mathcal{D}_m(U)$.

**Definition 3.2.3** (Support of a Current)**.** Support $T \in \mathcal{D}_m(U)$. We define the *support* of $T$ to be the (relatively) closed subset of $U$ defined by

$$\operatorname{supp} T := U \setminus \left(\bigcup W\right),$$

where we apply the union over all open $W \subset\subset U$ with

$$\varphi \in \mathcal{D}^m(U),\ \operatorname{supp}\varphi \subset W \implies T(\varphi) = 0.$$

To speak of the "finiteness" of a current, we introduce a (family of) seminorms on the space of currents, known as the *mass* of a current.

**Definition 3.2.4** (Mass)**.** We define the *mass* of a current $T \in \mathcal{D}_m(U)$, denoted $\mathbb{M}(T)$, by

$$\mathbb{M}(T) := \sup_{\|\omega\| \leq 1} T(\omega)$$

where $\omega \in \mathcal{D}^m(U)$. Furthermore, for any open set $W \subset U$, we define

$$\mathbb{M}_W(T) := \sup_{\|\omega\| \leq 1, \operatorname{supp}\omega \subset W} T(\omega).$$

Currents are of interest due to their compactness and convergence properties, which leads us to define the weak convergence of currents.

**Definition 3.2.5** (Weak Convergence)**.** A sequence $\{T_j\} \subset \mathcal{D}_m(U)$ is said to converge weakly to $T \in \mathcal{D}_m(U)$ if

$$\lim_{j \to \infty} T_j(\omega) = T(\omega)$$

for all $\omega \in \mathcal{D}^m(U)$.

The following lemma follows easily from standard analysis arguments, yet shows the value in considering a sequence of currents in minimization problems such as that of Plateau's problem: essentially, if one considers a sequence of area-minimizing currents, then the limit will be the global minimizer.

**Lemma 3.2.1** (Lower-semicontinuity of Mass). *The mass function is lower-semicontinuous on the space of currents, meaning that for a sequence $\{T_j\} \subset \mathcal{D}_m(U)$ weakly converging to some $T \in \mathcal{D}_m(U)$, we have*

$$\mathbb{M}_W(T) \leq \liminf_{j\to\infty} \mathbb{M}_W(T_j)$$

*for all open $W \subset U$.*

*Proof.* Since

$$T_j(\varphi) \leq \sup_{\|\varphi\|\leq 1} T_j(\varphi),$$

we take the limit on the left side to obtain

$$T(\varphi) \leq \sup_{\|\varphi\|\leq 1} T_j(\varphi).$$

Since this holds for all $\varphi \in \mathcal{D}^m(U)$, we take the supremum over all $\varphi$ with norm less than or equal to 1 on the left and the lim inf on the right to see that by the definition of the mass seminorm, we have the result. □

Often, it is useful to represent currents through the integration of a Radon measure, which is the content of Lemma 3.2.2.

**Lemma 3.2.2** (Representation of Currents). *Suppose $T \in \mathcal{D}_m(U)$ has finite mass $\mathbb{M}_W(T)$ for every open $W \subset\subset U$. There exists a Radon measure $\mu_T$ on $U \subset \mathbb{R}^n$ and $\mu_T$-measurable map $\widetilde{T} : U \to \bigwedge_m(\mathbb{R}^n)$ satisfying $\|T(\varphi)\| = 1$ $\mu_T$-a.e., such that*

$$T(\varphi) = \int_U \langle \varphi(x), \widetilde{T}(x)\rangle \, \mathrm{d}\mu_T(x)\,.$$

*Proof.* This is a direct application of Theorem 2.3.2. □

Furthermore, with this integral representation, we may define a new current $T \llcorner A \in \mathcal{D}_m(U)$ where $A \subset U$ is some $\mu_T$-measurable set defined by

$$(T \llcorner A)(\varphi) := \int_A \langle \varphi, \widetilde{T}\rangle \, \mathrm{d}\mu_T\,.$$

Likewise, if $\zeta$ is some locally $\mu_T$-integrable function on $U$, then we define $T \llcorner \zeta \in \mathcal{D}_m(U)$ by

$$(T \llcorner \zeta)(\varphi) := \int \langle \varphi, \widetilde{T}\rangle \zeta \, \mathrm{d}\mu_T\,.$$

Knowing that currents with finite mass may be represented through integration, we are motivated by the Generalized Stokes's Theorem (Theorem 3.1.1) to define the *boundary* of a current according to its exterior derivative.

**Definition 3.2.6** (Boundary of a Current)**.** Let $T \in \mathcal{D}_m(U)$. We define the *boundary* of $T$ to be the $(m-1)$-current $\partial T$ according to

$$\partial T(\varphi) = T(d\varphi)$$

for all $\varphi \in \mathcal{D}^m(U)$.

By the fact that $d^2 = 0$, we have that $\partial(\partial T) = \partial^2 T = 0$, meaning that the boundary of a boundary vanishes.

We now introduce the family of currents that will be of most interest throughout the rest of the paper. The definition will require the knowledge of *rectifiable sets* and *approximate tangent spaces.*

**Definition 3.2.7** (Countably Rectifiable Sets)**.** Let $1 \leq m \leq n$. A set $S \subset \mathbb{R}^n$ is *countably $m$-rectifiable* if

$$S \subset S_0 \cup \left( \bigcup_{j=1}^{\infty} F_j(\mathbb{R}^m) \right),$$

where

1. $\mathcal{H}^m(S_0) = 0$;
2. $F_j : \mathbb{R}^m \to \mathbb{R}^n$ are *Lipschitz* functions, meaning that for each $j = 1, 2, \ldots$, there exists a constant $\mathrm{Lip}(F_j) \in \mathbb{R}^+$ such that

$$\|F_j(x) - F_j(y)\| \leq \mathrm{Lip}(F_j)\|x - y\|.$$

**Definition 3.2.8** (Approximate Tangent Space)**.** Let $S \subset \mathbb{R}^n$ be $\mathcal{H}^m$-measurable with $\mathcal{H}^m(S \cap K) < \infty$ for each compact $K$ and $\theta : S \to \mathbb{R}^+$ be locally $\mathcal{H}^m$-integrable. An $m$-dimensional linear subspace $W \subset \mathbb{R}^n$ is the *approximate tangent space* to $S$ at $x \in \mathbb{R}^n$ with multiplicity $\theta$ if

$$\lim_{\lambda \to 0^+} \int_{\lambda^{-1}(S-x)} f(y)\, \mathrm{d}\mathcal{H}^m(y) = \theta(x) \int_W f(y)\, \mathrm{d}\mathcal{H}^m(y)$$

for all continuous $f : \mathbb{R}^n \to \mathbb{R}$ with compact support, where $\lambda^{-1}(S - x)$ consists of $y = \lambda^{-1}(z - x)$ for some $z \in S$. We shall write $T_x S := W$ when $W$ exists.

*Remark* 3.2.1. If $S$ is an $m$-dimensional $C^1$ submanifold of $\mathbb{R}^n$ (see definition 4.1.1), then the tangent space arising from the smooth structure and the approximate tangent space are equivalent notions. Hence, we may use the notation $T_x S$ for both without confusion when the context is understood.

*Remark* 3.2.2. It turns out that for every $\mathcal{H}^m$-measurable and countably rectifiable $S$, the approximate tangent space $T_xS$ exists $\mathcal{H}^m$-a.e. In fact, the converse is also true, given some locally $\mathcal{H}^m$-integrable $\theta : S \to \mathbb{R}^+$ such that the tangent plane $T_xS$ exists $\mathcal{H}^m$-a.e. with respect to *theta*.

**Definition 3.2.9** (Integer-Multiplicity Rectifiable Currents)**.** Let $T \in \mathcal{D}_m(U)$ and $1 \leq m \leq n$. We say that $T$ is an *integer-rectifiable rectifiable $m$-current* (or just an *integer-rectifiable current*) if there exist $S, \theta$, and $\xi$ such that:

1. $S$ is $\mathcal{H}^m$-measurable and a countably $m$-rectifiable subset of $U$ such that $\mathcal{H}^m(S \cap K)$ is finite for each compact $K \subset U$;
2. $\theta : U \to \mathbb{N}$ is locally $\mathcal{H}^m$-integrable;
3. $\xi : S \to \bigwedge^m(\mathbb{R}^n)$ is $\mathcal{H}^m$-measurable and $\xi(x)$ is a simple unit $m$-vector in $T_xS$ $\mathcal{H}^m$-a.e.;
4. $T$ may be written as

$$T(\varphi) = \int_S \langle \varphi(x), \xi(x) \rangle \theta(x) \, \mathrm{d}\mathcal{H}^m(x)$$

for each $\varphi \in \mathcal{D}^m(U)$.

We call $\theta$ and $\xi$ the *multiplicity* and *orientation* of $T$ respectively.

# 4 The Theory of Currents

We present Brian White's 1989 proof of the Compactness Theorem in [Whi89], introducing a series of preliminary theorems and lemmas to finally culminate in the result. Due to the highly technical nature of the full proof (including all the intermediary lemmas), certain proofs will be omitted or merely sketched and the curious reader will be directed to an appropriate reference when necessary. We now state the Compactness Theorem.

**Theorem 4.0.1** (Federer-Fleming Compactness Theorem)**.** *If $\{T_j\}$ is a sequence of integer-rectifiable $m$-currents with*

$$\sup_{j \geq 1} (\mathbb{M}_W(T_j) + \mathbb{M}_W(\partial T_j)) < \infty$$

*for each $W \subset\subset U$, then there is a subsequence $\{T_{j'}\}$ that weakly converges to some integer-rectifiable $T \in \mathcal{D}_m(U)$ in $U$.*

## 4.1 Slicing

The first tool that will be used to prove 4.0.1 is the *slicing* of currents. Slicing theory arises from the difficulty that it is impossible to define the "intersection" of two currents that satisfies the usual expected properties of submanifolds of $\mathbb{R}^n$. Instead, given an $m$-current $T \in \mathcal{D}_m(U)$, we may intersect $T$ with the level set $f^{-1}(y)$ of a desired Lipschitz function $f$ to obtain a lower dimensional current.

The theory of slicing is very closely related to the *coarea formula*, which requires the notion of the *approximate gradient* of a function.

We first handle the case where we want to differentiate a function $f$ relative to $S$ where $S$ is an $m$-dimensional $C^1$ submanifold of $\mathbb{R}^n$.

**Definition 4.1.1.** We call $S$ an $m$-dimensional $C^1$ submanifold of $\mathbb{R}^{m+k}$ if $S \subset \mathbb{R}^{m+k}$ is a set where each $x \in S$ has an open neighborhood $W \subset \mathbb{R}^{m+k}$ such that there exists an bijective, $C^1$ map $\phi : V \subset \mathbb{R}^m \to W$, where $V$ is open with

1. $D\phi$ is of (maximal) rank $m$ for all $x \in V$;
2. $\phi$ is proper, meaning that for every compact $K \subset W$, $\phi^{-1}(K)$ is compact in $V$;
3. $\phi(V) = W \cap S$.

The idea behind $\phi$ above is to provide an appropriate parameterization between $\mathbb{R}^m$ and the $m$-dimensional $S$ embedded in $\mathbb{R}^{m+k}$. $\phi$ is not only a homeomorphism but is in fact a $C^1$ diffeomorphism, meaning that it has $C^1$ inverse. This can be deduced by the full rank assumption and the Inverse Function Theorem.

**Definition 4.1.2.** We define the *tangent space to $S$ at $x = \phi(u)$* to be $D\phi(V)$. We denote this space by $T_xS$.

**Definition 4.1.3.** Suppose $x \in S$ and $f : W \to \mathbb{R}^\ell$ where $W$ contains a neighborhood of $x$ in $S$. $f$ is *differentiable relative to $S$ at $x$* if there is $\tilde{f} : \widetilde{W} \to \mathbb{R}^\ell$ such that

1. $\widetilde{W}$ is a neighborhood of $x$ in $\mathbb{R}^n$;
2. $f|_{S\cap\widetilde{W}} = \tilde{f}|_{S\cap\widetilde{W}}$;
3. $\tilde{f}$ is differentiable at $x$.

When these conditions are satisfied, then we have the map $D_Sf : T_xS \to \mathbb{R}^\ell$ defined by

$$D_Sf(x) := D\tilde{f}(x)|_{T_xS}$$

and call this the *differential of $f$ relative to $S$ at $x$*.

**Definition 4.1.4.** When $\ell = 1$ in Definition 4.1.3, the *gradient of $f$ relative to $S$ at $x$* is defined to be $\nabla^S f(x) \in T_xS$ such that

$$\langle D_Sf, u\rangle = \nabla^S f(x) \cdot u$$

for all $u \in T_xS$, where the left side is a dual pairing.

For the following discussion, let $S$ be $\mathcal{H}^m$-measurable and countably rectifiable with $S \subset \mathbb{R}^{m+k}$. Recalling 3.2.7, we present a preliminary result on the representation of $S$.

**Lemma 4.1.1.** *We may write $S = \bigcup_{j=0}^{\infty} S_j$ where*

1. $\mathcal{H}^m(S_0) = 0$;
2. $S_i \cap S_j = \varnothing$ *if* $i \neq j$;
3. *for* $j \geq 1$, $S_j \subset T_j$ *where* $T_j$ *is an* $m$*-dimensional, embedded* $C^1$ *submanifold of* $\mathbb{R}^n$.

*Proof.* See Proposition 5.4.3 of [KP08]. □

Using this lemma, we are now equipped to define the approximate gradient.

**Definition 4.1.5** (Approximate Gradient)**.** Suppose that $S$ is as defined above with $\mathcal{H}^m(S \cap K) < \infty$ for each compact $k \subset \mathbb{R}^n$. Representing $S$ as $\bigcup_{j=0}^{\infty} S_j$ as in Lemma 4.1.1, we define the *approximate gradient of $f$ relative to $S$* to be the map $\nabla^S f : \bigcup_{j=1}^{\infty} S_j \to T_x S$ defined by

$$\nabla^S f(x) := \nabla^{S_j} f(x), \quad j \geq 1$$

for $x \in S_j$, whenever the right side makes sense.

The utility of having $f : \mathbb{R}^{m+k} \to \mathbb{R}$ be Lipschitz is revealed in Rademacher's Theorem, which is stated without proof.

**Theorem 4.1.2** (Rademacher's Theorem)**.** *If $f : \mathbb{R}^n \to \mathbb{R}^{n'}$ is Lipschitz, then $f$ is differentiable $\mathcal{L}^n$-a.e. and the differential of $f$ is measurable.*

**Lemma 4.1.3.** *Let $S \subset \mathbb{R}^{m+k}$ be $\mathcal{H}^m$ measurable and countably rectifiable and let $f : \mathbb{R}^{m+k} \to \mathbb{R}$ be Lipschitz. Then the approximate gradient $\nabla^S f : \bigcup_1^{\infty} S_j \to T_x S$ exists $\mathcal{H}^m$-a.e.*

*Proof.* Let $x \in S$ and consider the $C^1$ parameterization $\phi : V \subset \mathbb{R}^m \to \mathbb{R}^{m+k}$ with the properties described in Definition 4.1.1, namely $\phi(V) = S \cap W$. By Theorem 4.1.2, $f \circ \phi : \mathbb{R}^m \to \mathbb{R}$ is differentiable $\mathcal{L}^m$-a.e., yet $\mathcal{L}^m = \mathcal{H}^m$ on $\mathbb{R}^m$, so it is also true $\mathcal{H}^m$-a.e. Composing with the $C^1$ inverse $\phi^{-1}$, we see that $(f \circ \phi) \circ \phi^{-1} = f$, yet the composition of two differentiable a.e. maps is differentiable so $f$ must be differentiable $\mathcal{H}^m$-a.e. We then merely take $\tilde{f} = f$ to obtain the existence of the approximate gradient $\mathcal{H}^m$-a.e. □

We may now define the *slice* of $T$ by some Lipschitz $f$.

**Definition 4.1.6** (Slice of a Current)**.** Let $T \in \mathcal{D}_m(U)$ and

$$\mathbb{M}_W(T) + \mathbb{M}_W(\partial T) < \infty$$

for every open $W \subset\subset U$. If $f : U \to \mathbb{R}$ is Lipschitz, the *slice* of $T$ by $f$ at $r$ is defined to be the current $\langle T, f, r\rangle \in \mathcal{D}_{m-1}(U)$

$$\langle T, f, r\rangle := \partial(T \llcorner \{f < r\}) - (\partial T) \llcorner \{f < r\}.$$

*Remark* 4.1.1. This definition is defined almost everywhere, except for at the countable set at which

$$(\mu_T + \mu_{\partial T})\{f = r\} = 0,$$

where the measures are as in Lemma 3.2.2.

The relationship between slicing and the approximate gradient is provided by a special case of the *coarea formula*, and is demonstrated below.

**Proposition 4.1.1.** *Let $T$ be an integer-rectifiable current in $\mathcal{D}_m(U)$ with multiplicity $\theta$. For all open $W \subset U$ and if $\mathcal{H}^m$-measurable $S$ is countably $m$-rectifiable and $f : \mathbb{R}^{m+k} \to \mathbb{R}$ is Lipschitz, then*

$$\int_{-\infty}^{\infty} \mathbb{M}_W(\langle T, f, r\rangle)\,\mathrm{d}r = \int_{S\cap W} |\nabla^S f(x)|\theta(x)\,\mathrm{d}\mathcal{H}^m(x) \leq \left(\operatorname*{ess\,sup}_{x\in S\cap W} |\nabla^S f(x)|\right)\mathbb{M}_W(T).$$

We shall use the two following results about slices of currents in our proof of the Compactness Theorem.

**Lemma 4.1.4.** *If $T$ is an integer-rectifiable current, then so is $\langle T, f, r\rangle$ for $\mathcal{L}^1$-almost every $r \in \mathbb{R}$.*

**Theorem 4.1.5** (Slicing Lemma)**.** *Suppose $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is Lipschitz and $\{T_j\}$ is a sequence of $m$-currents that weakly converge to $T \in \mathcal{D}_m(U)$ with*

$$\sup_{j\geq 1}(\mathbb{M}_W(T_j) + \mathbb{M}_W(\partial T_j)) < \infty$$

*for each $W \subset\subset U$. Then,*

1. *For $\mathcal{L}^1$-almost every $r$, there is a subsequence $\{T_{j'}\}$ such that*

$$\langle T_{j'}, f, r\rangle \to \langle T, f, r\rangle$$

*and*

$$\sup_{j\geq 1}(\mathbb{M}_W(\langle T_{j'}, f, r\rangle) + \mathbb{M}_W(\partial\langle T_{j'}, f, r\rangle) < \infty;$$

2. *if for some $W_0 \subset\subset U$, we have*

$$\lim_{j\to\infty}(\mathbb{M}_{W_0}(T_j) + \mathbb{M}_{W_0}(\partial T_j)) = 0,$$

*then there is some subsequence such that*

$$\lim(\mathbb{M}_{W_0}(\langle T_{j'}, f, r\rangle) + \mathbb{M}_{W_0}(\partial\langle T_{j'}, f, r\rangle) = 0.$$

*Proof.* To prove (1), we consider a subsequence $\{T_{j'}\}$ such that the sum of the associated Radon measures $\mu_{T_{j'}} + \mu_{\partial T_{j'}}$ converges to another Radon measure $\mu$. One then proves that this Radon measure is associated with the slice $\langle T, f, r\rangle$ to show weak convergence a.e. Considering further subsequences and using the facts that

$$\overline{\int_a^b} \mathbb{M}_W(\langle T_j, f, r\rangle)\,\mathrm{d}r \leq \mathrm{Lip}(f)\mathbb{M}_W(T_j \llcorner \{a < r < b\}),$$

where

$$\overline{\int} f\,\mathrm{d}\mu := \inf\left\{\int \psi\,\mathrm{d}\mu : 0 \leq f \leq \psi,\ \psi \text{ is } \mu\text{-measurable}\right\},$$

and

$$\partial\langle T_j, f, r\rangle = -\langle \partial T_j, f, r\rangle.$$

$\square$

For a more detailed proof, consult Lemma 8.1.16 of [KP08].

## 4.2 The Deformation Theorem

Let us consider the $(m+k)$-dimensional integer lattice $\mathbb{Z}^{m+k}$ where $m$ and $k$ are positive integers. Let $C = [0,1]^{m+k}$ denote the $(m+k)$-dimensional unit cube. For $j$ a positive integer such that $1 \leq j \leq m+k$, we denote by $\mathcal{L}_j$ the set of all $j$-dimensional faces of the $m+k$ dimensional cubes

$$\mathbf{t}_z(C) := \prod_{i=1}^{m+k} [z_i, z_i + 1]$$

with $z = (z_1, \ldots, z_{m+k} \in \mathbb{Z}^{m+k}$ ranging over the $(m+k)$-dimensional integer lattice. For each $m$-dimensional face $F \in \mathcal{L}_m$, there is a corresponding integer-rectifiable $m$-current $[\![F]\!] \in \mathcal{D}^m(\mathbb{R}^{m+k})$ defined by

$$[\![F]\!](\varphi) = \int_F \langle \varphi(x), \xi(x)\rangle\,\mathrm{d}\mathcal{H}^m(x)\,,$$

given that we have made a choice of orientation $\xi(x) = \pm\tau_1 \wedge \cdots \wedge \tau_{m+k}$ where $\tau_1, \cdots, \tau_{m+k}$ is an orthonormal basis for $T_x\mathbb{R}^{m+k}$.

The Deformation Theorem allows one to approximate a current with finite mass and boundary with finite mass with a linear combination of currents $[\![F]\!], F \in \mathcal{L}_m$. There are two versions of the Deformation Theorem: the scaled version and the unscaled version. The former is a direct consequence of the latter and is presented below, while the unscaled version is not necessary for our purposes.

We denote by $\eta_t : \mathbb{R}^{m+k} \to \mathbb{R}^{m+k}$ the map defined by

$$\eta_t = tx,$$

a *homothety.*

**Definition 4.2.1** (Pullback)**.** Given a linear $f : \mathbb{R}^n \to \mathbb{R}^{n'}$, the *pullback* $f^\# : \bigwedge^m(\mathbb{R}^{n'}) \to \bigwedge^m(\mathbb{R}^n)$ is defined for $\varphi \in \bigwedge^m(\mathbb{R}^{n'})$

$$f^\#\varphi(u_1, \ldots, u_m) = \varphi(f(u_1), \ldots, f(u_m)), \quad u_1, \ldots, u_m \in \mathbb{R}^n.$$

**Definition 4.2.2** (Push-forward of a Current)**.** Let $f$ be a smooth map between open sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^{n'}$. Let $\varphi \in \mathcal{D}^m(V)$, $T \in \mathcal{D}_m(U)$ and $f|_{\operatorname{supp} T}$ be *proper*, meaning that $f^{-1}(K) \cap \operatorname{supp} T$ is compact for all compact $K \subset V$. The *push-forward* $f_\# T$ under $f$ of $T$ is defined by

$$f_\# T(\varphi) = T(\zeta f^\# \varphi),$$

where $\zeta$ is any function in $C_c^\infty(U)$ that is equal to one in a neighborhood of the compact $\operatorname{supp} T \cap \operatorname{supp} f^\# \varphi$.

Note that the choice of $\zeta$ is independent of $f_\# T$. A notable property of the push-forward is that it commutes with the boundary of a current. That is,

$$\partial f_\# T = f_\# \partial T$$

for $T \in \mathcal{D}_m(U)$ and $\partial T \in \mathcal{D}_{m-1}(U)$.

We may now state the Scaled Deformation Theorem.

**Theorem 4.2.1** (Scaled Deformation Theorem)**.** *Suppose* $T \in \mathcal{D}_m(\mathbb{R}^{m+k})$ *with*

$$\mathbb{M}(T) + \mathbb{M}(\partial T) < \infty$$

*and* $\rho > 0$ *is fixed. Then*

$$T - P = \partial R + S$$

*where* $P \in \mathcal{D}_m(\mathbb{R}^{m+k}), R \in \mathcal{D}_{m+1}(\mathbb{R}^{m+k}), S \in \mathcal{D}_m(\mathbb{R}^{m+k})$ *and satisfy*

1. $$P = \sum_{F \in \mathcal{L}_m} p_F \eta_{\rho\#} [\![F]\!]$$

   *for* $F \in \mathcal{L}_M$ *where* $p_F \in \mathbb{R}$;

2. $$\mathbb{M}(P) \leq c\mathbb{M}(T), \quad \mathbb{M}(\partial P) \leq c\mathbb{M}(\partial T),$$
   $$\mathbb{M}(R) \leq c\rho\mathbb{M}(T), \quad \mathbb{M}(S) \leq c\rho\mathbb{M}(\partial T),$$

   *where the constant* $c \in \mathbb{R}$ *only depends on* $m$ *and* $k$;

*3.*
$$\operatorname{supp} P \cup \operatorname{supp} R \subset \left\{x \in \mathbb{R} : \operatorname{dist}(x, \operatorname{supp} T) < 2\rho\sqrt{m+k}\right\}$$
$$\operatorname{supp} \partial P \cup \operatorname{supp} S \subset \left\{x \in \mathbb{R} : \operatorname{dist}(x, \operatorname{supp} \partial T) < 2\rho\sqrt{m+k}\right\}.$$

*When $T$ is an integer-rectifiable current, then $P$ and $R$ are also, while if $\partial T$ is of integer-rectifiable, then so is $S$.*

The theorem is named after its proof, which deforms the given current $T$ onto a square grid by projecting the surface onto the square faces then estimating the error by analyzing the area swept out by the boundary $\partial T$ during the deformation. A detailed, rigorous proof may be found in [Sim14] or [KP08], yet must be relegated in this paper to put our focus on the Compactness Theorem.

The most important corollary of 4.1.1 for our purposes is the Weak Polyhedral Approximation Theorem.

**Theorem 4.2.2** (Weak Polyhedral Approximation Theorem)**.** *Let $T \in \mathcal{D}_m(U)$ be an integer-rectifiable current with $\mathbb{M}_W(\partial T) < \infty$ for all $W \subset\subset U$. There is a sequence of currents $\{P_j\}$ of the form*
$$P_j = \sum_{F \in \mathcal{L}_m} p_F^j \eta_{\rho\#}[\![F]\!]$$
*for $p_F^j \in \mathbb{Z}$ and $\rho_j \downarrow 0$ with $P_j \to T$ weakly in $U$.*

*Proof.* Directly applying 4.1.1. onto $T$ and sequence $\rho_j \downarrow 0$ with $\rho = \rho_j$ for each positive integer $j$, we obtain currents $P_j, R_j, S_j$ such that
$$T - P_j = \partial R_j + S_j$$
and
$$\mathbb{M}(R_j) \leq c\rho_j \mathbb{M}(T) \to 0 \text{ as } j \to \infty,$$
$$\mathbb{M}(S_j) \leq c\rho_j \mathbb{M}(\partial T) \to 0 \text{ as } j \to \infty.$$
On the other hand, we also have
$$\mathbb{M}(P_j) \leq c\mathbb{M}(T) \text{ and } \mathbb{M}(\partial P_j) \leq c\mathbb{M}(\partial T).$$
From the inequalities above applied onto the top equation, we see that the right side vanishes (as $\partial R_j(\varphi) = R_j(d\varphi)$ implies that $\partial R_j$ vanishes), and hence we obtain that $T - P_j \to 0$, meaning that $P_j \to T$ weakly. Hence, we have proved the case for $U = \mathbb{R}^{m+k}$ and finite mass for $T$ and $\partial T$.

When $U$ is not the whole space, we take a Lipschitz $\phi : \mathbb{R}^{m+k} \to \mathbb{R}$ such that $\phi > 0$ on $U$ while $\phi = 0$ on $U^C$. Assuming that $\{x \in \mathbb{R}^{m+k} : \phi(x) > \lambda\} \subset\subset U$ for all $\lambda > 0$.

For $\mathcal{L}^1$-almost every $\lambda > 0$, by Proposition 4.1.1, the currents $T_\lambda := T \llcorner \{x : \phi(x) > \lambda\}$ satisfy $\mathbb{M}(\partial T_\lambda) < \infty$. Note that $\operatorname{supp} T_\lambda \subset\subset U$, and we use the argument in the previous paragraph to approximate $T_\lambda$ with a sequence of currents $P_j$ for any $\lambda$. We then take a sequence $\lambda_i \downarrow 0$ to conclude that we may approximate $T$ on $U$. □

The Weak Polyhedral Approximation Theorem sees immediate use in conjunction with the $m-1$ case of the Compactness Theorem. Since our proof of 4.0.1 is inductive, we may assume this case and deduce the $m$-dimensional case (proving the base case as well of course).

**Theorem 4.2.3** (Boundary Rectifiability Theorem)**.** *Let $T \in \mathcal{D}_m(U)$ be an integer-rectifiable current with $\mathbb{M}(\partial T) < \infty$ for all $W \subset\subset U$. Then $\partial T \in \mathcal{D}_{m-1}(U)$ is an integer-rectifiable current.*

*Proof.* We apply Theorem 4.2.2 to $\partial T$ to find a sequence of integer-rectifiable currents $\{P_j\}$ that weakly converge to $\partial T$ and since this limit must necessarily be a current of integer multiplicity by the $m-1$ case of Theorem 4.0.1. □

Another important result that follows from the Deformation Theorem is the Isoperimetric Inequality.

**Theorem 4.2.4** (Isoperimetric Inequality)**.** *Let $m \geq 2$, and suppose that $T \in \mathcal{D}_{m-1}(\mathbb{R}^{m+k})$ is integer-rectifiable. Assume that $\operatorname{supp} T$ is compact and $\partial T = 0$. Then there is a compactly supported, integer-rectifiable current $R \in \mathcal{D}_M(\mathbb{R}^{m+k})$ such that $\partial R = T$ and*

$$(\mathbb{M}(R))^{(m-1)/m} \leq c\mathbb{M}(T),$$

*where $c \in \mathbb{R}$ depends solely on $m$ and $k$.*

*Proof.* See Theorem 7.9.1 of [KP08]. □

## 4.3 Proving the Compactness Theorem

The existence portion of the Compactness Theorem follows immediately from the Banach-Alaoglu Theorem, a standard result in functional analysis which we state below.

**Theorem 4.3.1** (Banach-Alaoglu)**.** *Let $X$ be a complete normed vector space. Then the closed unit ball $\overline{\mathbb{B}}$ of the continuous dual space $X^*$ is compact with respect to the weak*-topology, where*

$$\overline{\mathbb{B}} := \{\Lambda \in V^* : |\Lambda \boldsymbol{x}| \leq 1, \boldsymbol{x} \in V\}.$$

**Lemma 4.3.2.** *Suppose $\{T_j\}$ is a sequence of $m$-currents and*

$$\sup_{j \geq 1} \mathbb{M}_W(T_j) < \infty$$

*for every $W \subset\subset U$. There is a subsequence $\{T_{j'}\}$ and a $T \in \mathcal{D}_m(U)$ such that*

$$\int_U \langle \omega, \widetilde{T}_{j'} \rangle \, \mathrm{d}\mu_{T_{j'}} \to \int_U \langle \omega, \widetilde{T} \rangle \, \mathrm{d}\mu_T$$

*for every $\omega \in \mathcal{D}^m(U)$. That is,*

$$T_{j'} \to T$$

*weakly.*

*Proof.* By Theorem 4.3.1, the unit ball of $\mathcal{D}_m(U)$ $\overline{\mathbb{B}}$ is weakly compact. However, by Lemma 3.1.2, we know that $\mathcal{D}^m(U)$ is separable, and a standard functional analytic argument deduces that $\overline{\mathbb{B}}$ is hence metrizable and *sequentially compact* (since compactness $\iff$ sequential compactness in metric spaces). We then consider the dilation map $L : \mathcal{D}_m(U) \to \mathcal{D}_m(U)$ defined by

$$x \mapsto \lambda x$$

for some adequate $\lambda \in \mathbb{R}$ such that $\{T_j\} \subset L(\overline{\mathbb{B}})$. Since $\mathcal{L}$ is clearly continuous and continuous maps preserve compactness, our sequence is contained in a (sequentially) compact ball and the existence of a subsequence that converges to a limit $T \in L(\overline{\mathbb{B}}) \subset \mathcal{D}_m(U)$ is immediate. □

The lemma above clearly implies the existence of the limit in Theorem 4.0.1. The difficulty in proving the theorem lies in verifying that the limit $T$ is indeed an integer-rectifiable current, so much so that it has been dubbed the Closure Theorem. We now present a few intermediary results to begin our proof.

**Lemma 4.3.3.** *If $\mu$ is a Radon measure on $\mathbb{R}^{m+k}$ and $f$ is locally $\mu$-integrable and defined on $\mathbb{R}^{m+k}$, then for $\mu$-almost every $x$,*

$$\lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} \|f(y) - f(x)\| \, \mathrm{d}\mu(y) = 0.$$

**Lemma 4.3.4** (Lower Density Lemma)**.** *If $T \in \mathcal{D}_m(U)$ and*

$$\mathbb{M}_W(T) + \mathbb{M}_W(\partial T) < \infty$$

*for every $W \subset\subset U$, then:*

1. *for $\mu_T$-almost every $x \in \mathbb{R}^{m+k}$,*

$$\lim_{r \to 0} \frac{\lambda(x,r)}{\mu_T(B(x,r))} = 1,$$

*where*

$$\lambda(x,r) := \inf\{\mathbb{M}(S) : \partial S = \partial(T \llcorner B(x,r)), S \in \mathcal{D}_m(U)\};$$

2. *if $\partial T = 0$ and $\partial(T \llcorner B(x,r))$ is integer-rectifiable for all $x \in \mathbb{R}^{m+k}$ and almost every $r \in \mathbb{R}$, there is a $\delta > 0$ such that*

$$\Theta_*^m(\mu_T, \mathbb{R}^{m+k}, x) > \delta$$

*for $\mu_T$-almost every $x \in U$.*

**Lemma 4.3.5** (Constant Vectorfield Lemma)**.** *Let $T \in \mathcal{D}_m(\mathbb{R}^{m+k})$ satisfy $\mathbb{M}_W(T) < \infty$ for all $W \subset\subset U$, $\partial T = 0$, and $\widetilde{T}(x) = \vec{\omega} \in \bigwedge_m(\mathbb{R}^{m+k})$ for every $x \in \mathbb{R}^{m+k}$. Let $V$ be the vector subspace of $\mathbb{R}^{m+k}$ consisting of the vectors in the directions in which $T$ is translation invariant. That is, the subspace defined by*

$$V := \{v \in \mathcal{D}^m(\mathbb{R}^{m+k}) : T(v) = T(rv) \text{ for all } r \in \mathbb{R}\}.$$

*Then $\vec{\omega} \in \bigwedge_m(V)$.*

We may now present the proof of the closure portion of the Compactness Theorem, beginning with a weaker version then generalizing the result to the full theorem.

**Theorem 4.3.6** (Weak Closure Theorem)**.** *Let $\{T_j\}$ be a sequence of integer-rectifiable rectifiable $m$-currents in $\mathcal{D}_m(\mathbb{R}^{m+k})$ with*

$$\sup_{j \geq 1}(\mathbb{M}(T_j) + \mathbb{M}(\partial T_j)) < \infty,$$

*with $\partial T = 0$. If $T_j \to T$, then $T \in \mathcal{D}_m(U)$ is also integer-rectifiable.*

*Proof.* The proof will proceed by induction. The base case is trivial for 0-currents since every 0-current is integer-rectifiable. Hence, we suppose the result holds for $(m-1)$-currents.

Recalling the Slicing Lemma (Theorem 4.1.5), for every Lipschitz $f : \mathbb{R}^{m+k} \to \mathbb{R}$, there is a subsequence $j'$ such that

$$\sup_{j' \geq 1}(\mathbb{M}(\langle T_{j'}, f, r\rangle + \mathbb{M}(\partial\langle T_{j'}, f, r\rangle) < \infty$$

and $\langle T_{j'}, f, r\rangle \to \langle T, f, r\rangle$ for $\mathcal{L}^1$-almost every $r \in \mathbb{R}$. Furthermore, each $\langle T_j, f, r\rangle$ is integer-rectifiable by Lemma 4.1.4, and the limit $\langle T, f, r\rangle$ is also an integer-rectifiable $(m-1)$-current by the induction hypothesis. We also have that

$$\langle T, f, r\rangle = \partial(T \llcorner \{f < r\}),$$

since $\partial T = 0$.

$T \in \mathcal{D}_m(\mathbb{R}^{m+k})$ must be of finite mass since

$$\sup_{j \geq 1}(\mathbb{M}(T_j) + \mathbb{M}(\partial T_j)) < \infty,$$

and we use (2) of Lemma 4.2.4 to deduce the existence of $\delta > 0$ such that if

$$M := \{x \in \mathbb{R}^{m+k} : \Theta^m_*(\mu_T, \mathbb{R}^{m+k}, x) > \delta\},$$

then $\mu_T(M^C) = 0$. This implies that $\mu_T \ll \mathcal{H}^m \llcorner M$ and that $\mathcal{H}^m(M) < \infty$. By a simple functional analytic argument, $\mathcal{H}^m \llcorner M$ is a Radon measure.

We apply the Radon-Nikodym Theorem (Theorem 2.4.1), and by setting $\theta(x) = d\mu_T/d(\mathcal{H}^m \llcorner M)$, we may write

$$T(\varphi) = \int \langle \varphi(x), \widetilde{T}(x) \rangle \theta(x) \, \mathrm{d}\mathcal{H}^m \llcorner M(x) = \int_M \langle \varphi(x), \tau(x) \rangle \, \mathrm{d}\mathcal{H}^m(x)\,,$$

where $\tau(x) = \widetilde{T}(x)\theta(x) \in \bigwedge^m(\mathbb{R}^{m+k})$. To prove that $T$ is integer-rectifiable according to Definition 3.2.9, we must show that $M$ is countably $m$-rectifiable, $\theta(x)$ is an integer, and that for $\mathcal{H}^m$-almost every $x$, $\widetilde{T}(x)$ is a simple $m$-vector in $T_xM$. (The local integrability of $\theta$ is easy to see as the integral becomes equivalent to $\mathbb{M}(T) < \infty$)

Since $\mathcal{H}^m(M) < \infty$, we apply (2) of Proposition 2.2.1 to see

$$\Theta^{m*}(\mathcal{H}^m, M, x) \leq 1 \tag{1}$$

for $\mathcal{H}^m$-almost every $x \in M$. Applying Lemma 4.3.3, we see that for $\mathcal{H}^m$-almost every $a \in M$,

$$\lim_{r \to 0} \frac{1}{\mathcal{H}^m(M \cap B(0,r))} \int_{M \cap B(a,r)} \|\tau(a) - \tau(x)\| \, \mathrm{d}\mathcal{H}^m(x) = 0, \tag{2}$$

where $B(a, r)$ is the open ball with radius $r$ centered at $a$. Fix a point $\overline{a} \in M$ where both of these statements are satisfied (the set where both statements are false has measure 0).

By the definition of $M$ and (2), we obtain

$$\Theta^m_*(\mathcal{H}^m, M, \overline{a}) = \frac{1}{\|\tau(\overline{a})\|} \Theta^m_*(\mu_T, \mathbb{R}^{m+k}, \overline{a}) > 0.$$

Let $\Lambda$ be a sequence of positive numbers converging to 0. We define the maps $\eta_{\lambda,\overline{a}} : \mathbb{R}^{m+k} \to \mathbb{R}^{m+k}$ by

$$\eta_{\lambda,\overline{a}} := \frac{x - \overline{a}}{\lambda}$$

for $\lambda \in \mathbb{R}$. For every $\lambda \in \Lambda$,

$$\mathcal{H}^m \llcorner \eta_{\lambda,\overline{a}}(M) = \frac{1}{\lambda^m} \mathcal{H}^m \llcorner (x - \overline{a})(M),$$

implying that the left side is a Radon measure (due to the translation invariance of $\mathcal{H}^m$). Fix some constant $0 < r < \infty$. We then have

$$\begin{aligned}\limsup_{\lambda\in\Lambda}\,((\mathcal{H}^m \llcorner \eta_{\lambda,\overline{a}}(M))(B(0,r))) &= \limsup_{\lambda\in\Lambda} \frac{1}{\lambda^m}\mathcal{H}^m(M\cap B(\overline{a},\lambda r),\\ &= \Omega_m r^m \limsup_{\lambda\in\Lambda} \frac{\mathcal{H}^m(M\cap B(\overline{a},\lambda r)}{(\lambda r)^m\Omega_m},\\ &= \Omega_m r^m \Theta^{m*}(\mathcal{H}^m, M, \overline{a}),\\ &< \infty,\end{aligned}$$

where again the first equality is deduced from the translation invariance of $\mathcal{H}^m$. By a certain result on the convergence of measures, there is a subsequence $\Lambda' \subset \Lambda$ and a Radon measure $\mu$ such that for $\lambda \in \Lambda'$,

$$\mathcal{H}^m \llcorner \eta_{\lambda,\overline{a}}(M) \to \mu \tag{3}$$

weakly, and this pointwise convergence implies that for $\lambda \in \Lambda'$,

$$(\mathcal{H}^m \llcorner \eta_{\lambda,\overline{a}}(M)) \wedge \tau(\overline{a}) \to (\mu \wedge \tau(\overline{a})),$$

weakly, where

$$((\mathcal{H}^m \llcorner \eta_{\lambda,\overline{a}}(M)) \wedge \tau(\overline{a}))(\varphi) := \int \langle \varphi, \tau(\overline{a})\rangle \,\mathrm{d}(\mathcal{H}^m \llcorner \eta_{\lambda,\overline{a}}(M))$$

and

$$(\mu \wedge \tau(\overline{a}))(\varphi) := \int \langle \varphi, \tau(\overline{a})\rangle \,\mathrm{d}\mu$$

for all $\varphi \in \mathcal{D}^m(\mathbb{R}^{m+k})$. Let us define

$$T_\lambda := \eta_{\lambda,\overline{a}\#}T,$$

where the right side denotes the push-forward under $\eta_{\lambda,\overline{a}}$ of $T$ defined in Definition 4.2.2. We also have that for every $\varphi \in \mathcal{D}^m(\mathbb{R}^{m+k})$,

$$T_\lambda \to \mu \wedge \tau(\overline{a}), \quad \lambda \in \Lambda' \tag{4}$$

and

$$\mu_{T_\lambda} \to \|\tau(\overline{a})\|\mu, \quad \lambda \in \Lambda' \tag{5}$$

weakly. These two assertions may be verified by noting that for every $0 < R < \infty$,

$$\begin{aligned}\mathbb{M}_{B(0,R)}(T_\lambda - ((\mathcal{H}^m \llcorner \eta_{\lambda,\overline{a}}(M)) \wedge \tau(\overline{a})) &= \frac{1}{\lambda^m}\mathbb{M}_{B(a,\lambda R)}(T - (\mathcal{H}^m \llcorner M) \wedge \tau(\overline{a}))\\ &= \frac{1}{\lambda^m}\int_{M\cap B(a,\lambda R)} \|\tau(x) - \tau(\overline{a})\| \,\mathrm{d}\mathcal{H}^m(x) \to 0\end{aligned}$$

by (1) and (2).

It is easy to deduce from our previous discussion that $\langle T_\lambda, f, r\rangle$ are integer-rectifiable $(m-1)$-currents for every Lipschitz $f$ and $\mathcal{L}^1$-almost every $r$, and clearly

$$\mathbb{M}(T_\lambda) + \mathbb{M}(\partial T_\lambda)$$

by the finiteness of the sum of masses of $T$ and $\partial T$. By (3) and the Slicing Lemma, there is a subsequence $\Lambda'' \subset \Lambda'$ such that

$$\langle T_\lambda, f, r\rangle \to \langle \mu \wedge \tau(\overline{a}), f, r\rangle, \quad \lambda \in \Lambda''$$

$\mathcal{L}^1$-a.e. By the inductive hypothesis, the limit slice is rectifiable, and by Lemma 4.3.4, there is a $\delta > 0$ such that

$$\Theta^m_*(\mu, \mathbb{R}^{m+k}, x) > \frac{\delta}{\|\tau(\overline{a})\|} > 0 \tag{6}$$

for $\mu$-almost every $x \in M$.

By (4) and $\partial T_\lambda = 0$, we also have $\partial(\mu \wedge \tau(\overline{a})) = 0$. Hence, we apply the Constant Vectorfield Lemma (Lemma 4.3.5) to see that $\mu \wedge \tau(\overline{a})$ must be translation invariant in at least $m$ directions. We also use (5) to deduce that $\mu \wedge \tau(\overline{a})$ must be translation invariant in at most $m$ dimensions. Hence, our current in translation invariant in exactly $m$ dimensions. Thus, we apply Lemma 4.3.5 again to deduce that $\tau(\overline{a}) \in \bigwedge_m(V)$ is a simple $m$-vector, and that there is a collection $P_1, \dots, P_p$ of $m$-planes parallel to the $m$-dimensional subspace determined by $\tau(\overline{a})$ such that

$$\mu = \sum_{j=1}^{p} \alpha_j \mathcal{H}^m \llcorner P_j$$

where $\alpha_j > 0$. We note that by (1) and (3), we have

$$\sum_{j=1}^{p} \alpha_j \leq \Theta^{m*}(\mathcal{H}^m, M, a) \leq 1.$$

However, by (6), we know that each $\alpha_j \leq \frac{\delta}{\|\tau(\overline{a})\|}$, so $p$ must be finite.

By a long and technical argument using the Slicing Lemma and the Poincare inequality, it is concluded that in fact $p = 1$ and $\alpha_1 = 1$, and $P_1$ passes through the origin. Furthermore, $P_1$ must be independent of the subsequences of $\Lambda$ since it was completely determined by $\tau(\overline{a})$.

Let us fix some continuous, compactly supported function $f : \mathbb{R}^{m+k}$. We see that

$$\lim_{\lambda \to 0} \int_{\eta_{\lambda,\overline{a}}(M)} f \, \mathrm{d}\mathcal{H}^m = \int_{\mathbb{R}^{m+k}} f \, \mathrm{d}\mu = \int_{P_1} f \, \mathrm{d}\mathcal{H}^m \,,$$

since $\mathcal{H}^m \llcorner \eta_{\lambda,\overline{a}}(M) \to \mu$ weakly by (3). However, this clearly means that $P = T_{\overline{a}}M$, the approximate tangent space to $M$ at $\overline{a}$ by Definition 3.2.8, and letting $\overline{a}$ vary, this clearly holds for almost every $\overline{a} \in M$ and $M$ has an approximate tangent plane $\mathcal{H}^m$-a.e. By Remark 3.2.1, $M$ is countably $m$-rectifiable.

Now all that is left is to verify that $\theta$ is integer valued. Notice that

$$\begin{aligned}\langle \mu \wedge \tau(\overline{a}), f, r\rangle &= \partial(\mu \wedge \tau(x) \llcorner \{f < r\})(\varphi)\\ &= \int_M \langle d\varphi, \tau(x)\rangle \,\mathrm{d}\mathcal{H}^m \llcorner P\\ &= \int_M \langle d\varphi, \widetilde{T}(x)\rangle \theta(x) \,\mathrm{d}\mathcal{H}^m \llcorner P\end{aligned}$$

is an integer-rectifiable $(m-1)$-current by the induction hypothesis. Hence, $\theta(x) \in \mathbb{Z}$ and we are done. □

Now that we have established the weak version of the theorem, we are ready to prove the Closure Theorem in full generality.

**Theorem 4.3.7** (Closure Theorem)**.** *Let $U \subset \mathbb{R}^{m+k}$ be open and let $\{T_j\} \subset \mathcal{D}_m(U)$ be a sequence of integer-rectifiable $m$-currents with*

$$\sup_{j\geq 1}(\mathbb{M}_W(T_j) + \mathbb{M}_W(\partial T_j)) < \infty.$$

*Suppose $T_j \to T$ weakly. Then $T$ is an integer-rectifiable $m$-current in $\mathcal{D}_m(U)$.*

*Proof.* First consider the case where $U = \mathbb{R}^{m+k}$. We use the Boundary Rectifiability Theorem (Theorem 4.2.3) to deduce that each $\partial T_j$ is an integer-rectifiable $(m-1)$-current. Since $\partial(\partial T) = 0$ and $\partial T_j \to \partial T$, we apply Theorem 4.3.6 to see that $\partial T$ is also an integer-rectifiable $(m-1)$-current. By the Isoperimetric inequality (Theorem 4.2.4), there is an integer-rectifiable $m$-current $R \in \mathcal{D}_m(\mathbb{R}^{m+k})$ such that $\partial T = \partial R$. Hence, $\mathbb{M}(\partial R) = \mathbb{M}(\partial T) < \infty$, so $\partial R$ is an integer-rectifiable $(m-1)$-current by the Weak Closure Theorem above.

Clearly, we have $T_j - R \to T - R$ weakly, and by the definition of $R$, we have $\partial(T - R) = 0$. Hence, by the $m-1$ case of the Compactness Theorem (Theorem 4.0.1), $T - R$ is an integer-multiplicity $m$-current, showing that we need not have the $\partial T = 0$ condition in Theorem 4.3.6.

To prove the result for open $U \subset \mathbb{R}^{m+k}$ where $U$ is not the whole space, we fix $W \subset\subset U$ and let

$$\mathscr{C} := \{B(x, r) \subset\subset U\}$$

be an open covering for $\overline{W}$. Fixing $x \in U$ such that $B(x,r) \subset\subset U$ for some $r > 0$, we have

$$\sup_{j\geq 1}(\mathbb{M}_{B(x,r)}(T_j)+\mathbb{M}_{B(x,r)}(\partial T_j)) < \infty \iff \sup_{j\geq 1}(\mathbb{M}(T_j \llcorner B(x,r))+\mathbb{M}((\partial T_j)\llcorner B(x,r))) < \infty.$$

By the Slicing Lemma, we have a subsequence $j'$ such that

1. $T_{j'} \llcorner B(x,r) \to T \llcorner B(x,r)$ weakly;
2. $\sup\langle T, f, r\rangle = \sup(\mathbb{M}(\partial(T_{j'} \llcorner B(x,r)) - (\partial T_j) \llcorner B(x,r))) < \infty$, where

$$f(y) = \operatorname{dist}(x,y).$$

Combining these two facts, we clearly have

$$\sup(T_{j'} \llcorner B(x,r)) + \mathbb{M}(\partial(T_{j'} \llcorner B_(x,r))) < \infty.$$

By putting $U = \mathbb{R}^{m+k}$,we get that $T_{j'} \llcorner B(x,r) \to T \llcorner B(x,r)$ weakly with the latter being an integer-rectifiable $m$-current. Since $x$ was arbitrary, we are done. □

# 5 Plateau's Problem

With the Compactness Theorem finally established, we are now equipped to prove the existence of solutions to Plateau's Problem, as provided in [Sim14].

**Theorem 5.0.1** (Existence of Solutions)**.** *Let $S$ be an $(m-1)$-dimensional integer-rectifiable current with compact support in $\mathbb{R}^{m+k}$ with $\partial S = 0$. Then there is a compactly supported $m$-dimensional rectifiable current $T \in \mathcal{D}_m(\mathbb{R}^{m+k})$ with $\partial T = S$ and $\mathbb{M}(T) \leq \mathbb{M}(R)$ for every integer-rectifiable $m$-current $R$ with compact support and $\partial R = S$.*

*Proof.* Let us define the family of currents

$$\mathcal{I}_S := \{R \in \mathcal{D}_m(\mathbb{R}^{m+k}) : R \text{ integer-rectifiable, compactly supported, } \partial R = S\}.$$

Taking a sequence $\{R_j\} \subset \mathcal{I}_S$ satisfying

$$\lim_{j\to\infty} \mathbb{M}(R_j) = \inf_{R\in\mathcal{I}_S} \mathbb{M}(R),$$

and letting $B(0,R)$ be an open ball in $\mathbb{R}^{m+k}$ containing $\operatorname{supp} S$, we define $f : \mathbb{R}^{m+k} \to B(0,R)$ to be the retraction of $\mathbb{R}^{m+k}$ onto $B(0,R)$. Clearly, $\operatorname{Lip} f = 1$ and hence, we have

$$\mathbb{M}(f_{\#}R_j) = \mathbb{M}(R_j).$$

However, $\partial f_{\#} R_j = f_{\#} \partial R_j = f_{\#} S = S$, because $f$ restricted to $B(0, R)$ is identical to the identity function. Thus, $f_{\#} R_j \subset \mathcal{I}_S$ and we have

$$\lim_{j \to \infty} \mathbb{M}(f_{\#} R_j) = \inf_{R \in \mathcal{I}_S} \mathbb{M}(R).$$

Applying the Compactness Theorem 4.0.1, there is a subsequence $\{T_{j'}\}$ and integer-rectifiable limit $T \in \mathcal{D}_m(\mathbb{R}^{m+k})$ such that $f_{\#} R_{j'} \to T$, and by the lower-semicontinuity of mass (Lemma 3.2.1), we have

$$\mathbb{M}(T) \leq \inf_{R \in \mathcal{I}_S} \mathbb{M}(R).$$

Furthermore, $\operatorname{supp} T \subset B(0, R)$ and

$$\partial T = \lim \partial f_{\#} R_{j'} = \lim f_{\#} \partial R_{j'} = S,$$

so we have $T \in \mathcal{I}_S$, and we are done. $\square$

The complete solution of Plateau's Problem in $\mathbb{R}^3$ is provided by the regularity result below.

**Theorem 5.0.2** (Regularity of Solutions)**.** *A rectifiable, area-minimizing* 2*-current* $T \in \mathbb{R}^3$ *is a smooth, embedded manifold on the interior, that is,* $\operatorname{supp} T - \operatorname{supp} \partial T$ *is an embedded* $C^\infty$ *submanifold of* $\mathbb{R}^3$*.*

This result was shown by Fleming in 1962. It is clear from this that currents have a wide variety of applications, owing to their "nice" compactness properties. Some of the notable current uses of currents include their application in the fields of partial differential equations and dynamical systems, calculus of variations and its applications to optimal transport, and the study of analytic varieties.

# A The Topology on $\mathcal{D}^m(U)$

We define the topology on $\mathcal{E}^m(U)$ by considering some arbitrary $\omega \in \mathcal{E}^m(U)$ with the representation

$$\omega = \sum_{\alpha \in I_{m,n}} \omega_\alpha dx^\alpha,$$

where each $\omega_\alpha : U \to \mathbb{R}$ is smooth and we use the notation established in Section 3.1. We define the seminorms

$$\nu_K^\beta(\omega) = \sup_{\alpha \in I_{m,n}} \left\| D^\beta \omega_\alpha \right\|$$

where $K \subset U$ is compact and the *length of* $\beta = (\beta_1, \ldots, \beta_m)$, defined

$$|\beta| = \beta_1 + \cdots + \beta_m,$$

satisfies $|p| \leq k$ where $k \in \mathbb{Z}$. The family of these seminorms for all $\beta$ and compact $K \subset U$ induces a locally-convex, translation invariant Hausdorff topology on $\mathcal{D}^m(U)$.

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