Measure Theory
Folland 1.1-1.3

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February 28, 2023

1 1.7

Let us denote by $\bar{\mu} : \mathcal{M} \to \mathbb{R}$ the set function defined by

$$\bar{\mu} : A \mapsto \sum_{k=1}^{n} a_k \mu_k(A), \qquad A \in \mathcal{M}.$$

Verification of non-negativity is trivial since each of the measures are non-negative and each $a_k \in [0, \infty)$. Notice that since μ_k is a measure,

$$\bar{\mu}(\varnothing) = \sum_{k=1}^{n} a_k \mu_k(\varnothing) = \sum_{k=1}^{n} a_k(0) = 0,$$

so only verification of sigma additivity is left for $\bar{\mu}$ to be a measure on $(X, \mathcal{M})$. Suppose that $\{D_i\}$ is a sequence of pairwise disjoint sets where each set is an element of $\mathcal{M}$. By the countable additivity of each μ_k, we have

$$\begin{aligned}
\bar{\mu}(\bigcup_{i=1}^{\infty} D_i) &= \sum_{k=1}^{n} a_k \mu_k(\bigcup_{i=1}^{\infty} D_i) \\
&= \sum_{k=1}^{n} a_k \left(\sum_{i=1}^{\infty} \mu_k(D_i) \right) \\
&= \sum_{i=1}^{\infty} \left(\sum_{k=1}^{n} a_k \mu_k(D_i) \right) \\
&= \sum_{i=1}^{\infty} \bar{\mu}(D_i),
\end{aligned}$$

showing that $\bar{\mu}$ indeed is a measure. ■

2 1.8

Define the sequence of sets $\{H_n\}$ by

$$H_n = \bigcap_{m=n}^{\infty} E_m.$$

This sequence is clearly monotonically increasing, meaning $H_1 \subset H_2 \subset \cdots$. By continuity from below and the definition of the limit inferior of sets,

$$\mu(\liminf_{j\to\infty} E_j) = \mu(\bigcup_{n=1}^{\infty}\left(\bigcap_{m=n}^{\infty} E_m\right)) = \mu(\bigcup_{n=1}^{\infty} H_n) = \lim_{n\to\infty} \mu(H_n).$$

By the monotonicity of a measure, we have that

$$\mu(H_1) \leq \mu(H_2) \leq \cdots,$$

so clearly,

$$\lim_{n\to\infty} \mu(H_n) = \sup_{n\geq 1} \mu(H_n).$$

However, we know that

$$\liminf_{j\to\infty} \mu(E_j) = \sup_{n\geq 1}\left(\inf_{m\geq n} \mu(E_m)\right),$$

so it suffices to show that

$$\mu(H_n) = \mu\left(\bigcap_{m=n}^{\infty} E_m\right) \leq \inf_{m\geq n} \mu(E_m)$$

for every positive integer n. However, it is obvious that

$$\bigcap_{m=n}^{\infty} E_m \subset E_k$$

for each positive integer n and $k \geq n$, implying that by the monotonicity of measure, the inequality must be satisfied. Our explicit chain of (in)equalities is seen below:

$$\mu(\liminf_{j\to\infty} E_j) = \sup_{n\geq 1} \mu(H_n) \leq \sup_{n\geq 1}\left(\inf_{m\geq n} \mu(E_m)\right) = \liminf_{j\to\infty} \mu(E_j).$$

We shall now show

$$\mu(\limsup_{j\to\infty} E_j) \geq \limsup_{j\to\infty} \mu(E_j)$$

provided that $\mu(\bigcup E_j) < \infty$. Again, define a sequence $\{H'_n\}$ by

$$H'_n = \bigcup_{m=n}^{\infty} E_m.$$

This is a decreasing sequence of sets, meaning that

$$H'_1 \supset H'_2 \supset \cdots$$

and that

$$\mu(\bigcap_{n=1}^{\infty} H'_n) = \lim_{n\to\infty} \mu(H'_n)$$

by continuity from above. Furthermore, the monotonicity of a measure implies that

$$\mu(H'_1) \geq \mu(H'_2) \geq \cdots$$

where $\mu(H'_1)$ is necessarily finite by assumption. Hence, we know that

$$\lim_{n\to\infty} \mu(H'_n) = \inf_{n\geq 1} \mu(H'_n).$$

Verification of the inequality

$$\mu(H'_n) \geq \sup_{m\geq n} \mu(E_m)$$

for all n can be done by noticing that

$$\bigcup_{m=n}^{\infty} E_m \supset E_k$$

for any n and $k \geq n$ and using the monotonicity of measure. Hence,

$$\mu(\limsup_{j\to\infty} E_j) = \inf_{n\geq 1} \mu(H_n) \geq \inf_{n\geq 1} \left(\sup_{m\geq n} \mu(E_m)\right) = \limsup_{j\to\infty} \mu(E_j),$$

and the inequality is proved. ■

3 1.9

We see that by splitting E (resp. F) into two disjoint parts (one part in F and one part in F^C), we get the equality

$$\mu(E) + \mu(F) = \mu((E \cup F^C) \cup (E \cap F)) + \mu((F \cap E^C) \cup (E \cap F).$$

By sigma additivity of disjoint sets, this is equivalent to

$$\mu(E \cap F^C) + \mu(E \cap F) + \mu(E^C \cap F) + \mu(E \cap F).$$

It is easy to check that indeed

$$E \cup F = (E \cap F^C) \cup (E \cap F) \cup (E^C \cap F),$$

and since all three of the latter sets are pairwise disjoint, we have that

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F). \blacksquare$$

4 1.10

Clearly, this set function is non-negative since the original measure is non-negative for any input in the sigma algebra, and $A \cap E$ is always in the sigma algebra. Furthermore,

$$\mu_E(\varnothing) = \mu(\varnothing \cap E) = \mu(\varnothing) = 0.$$

To check sigma additivity, let $\{D_n\}$ be a sequence of pairwise disjoint sets in $\mathcal{M}$. Then,

$$\begin{aligned}
\mu_E\left(\bigcup_{n=1}^{\infty} D_n\right) &= \mu(E \cap \left(\bigcup_{n=1}^{\infty} D_n\right)) \\
&= \mu\left(\bigcup_{n=1}^{\infty} (E \cap D_n)\right) \\
&= \sum_{n=1}^{\infty} \mu(E \cap D_n) \\
&= \sum_{n=1}^{\infty} \mu_E(D_n)
\end{aligned}$$

because of the sigma additivity of μ, showing that μ_E is indeed a measure. ■

5 1.11

($\Longrightarrow$) Let μ be a finitely additive set function that is continuous from below. Let $\{D_n\}$ be a sequence of pairwise disjoint sets in the sigma algebra. Define a new sequence $\{H_n\}$ by

$$H_n = \bigcup_{m=1}^{n} D_m.$$

This sequence is clearly monotonically increasing and $\bigcup H_n = \bigcup D_n$. By continuity from below,

$$\begin{aligned}
\mu\left(\bigcup_{n=1}^{\infty} D_n\right) &= \mu\left(\bigcup_{n=1}^{\infty} H_n\right) \\
&= \lim_{n\to\infty} \mu(H_n) \\
&= \lim_{n\to\infty} \mu\left(\bigcup_{m=1}^{n} D_m\right) \\
&= \lim_{n\to\infty} \sum_{m=1}^{n} \mu(D_m) \\
&= \sum_{m=1}^{\infty} \mu(D_m),
\end{aligned}$$

and hence μ is sigma additive.

($\Longleftarrow$) This implication is trivial since a sigma additive (non-negative) set function that sends the empty set to zero is clearly a measure.

The proof for the fact that a finitely additive set function is continuous from above iff it is sigma additive is analogous. To see this, we define the decreasing sequence $\{K_n\}$ by

$$K_n = H_n^C.$$

Obviously, this set decreases down to $X \setminus \bigcup D_n$, and by continuity from above, we have that

$$\begin{aligned}
\mu(X) - \mu\left(\bigcup_{n=1}^{\infty} K_n\right) &= \mu(\bigcap_{n=1}^{\infty} K_n) \\
&= \lim_{n\to\infty} \mu(K_n) \\
&= \mu(X) - \lim_{n\to\infty} \mu\left(\bigcup_{m=1}^{n} D_m\right) \\
&= \mu(X) - \sum_{m=1}^{\infty} \mu(D_m),
\end{aligned}$$

and subtracting $\mu(X)$ (which is finite by assumption) and multiplying by -1 shows that μ is sigma additive. ■

6 1.12

6.1 (a)

Notice that by the definition of the symmetric difference of two sets,

$$0 = \mu(E \triangle F) = \mu\left((E \setminus F) \cup (F \setminus F)\right) = \mu(E \setminus F) + \mu(F \setminus E)$$

since μ is additive and $E \setminus F$ and $F \setminus E$ are disjoint. However, since a measure is always non-negative, this means that for the sum of the last two terms to equal 0, means that both the terms must be zero. Hence, we have

$$\begin{cases} \mu(E \setminus F) = \mu(E \setminus (E \cap F)) = \mu(E) - \mu(E \cap F) = 0, \\ \mu(F \setminus E) = \mu(F \setminus (F \cap E)) = \mu(F) - \mu(E \cap F) = 0, \end{cases}$$

which implies that

$$\mu(E) = \mu(E \cap F) = \mu(F).$$

6.2 (b)

(Reflexive) This is obviously true since the symmetric difference of a set with itself is always empty, and μ is a measure, meaning that this value will be zero.

(Symmetric) Let E and F be sets such that

$$E \sim F, \text{or } \mu(E \triangle F) = 0.$$

Notice that

$$\mu(E \triangle F) = \mu((E \setminus F) \cup (F \setminus E)) = \mu((F \setminus E) \cup (E \setminus F)) = \mu(F \triangle E),$$

so $\sim$ is symmetric. (Transitive) Suppose that $E, F, K \in \mathcal{M}$ such that

$$E \sim F \text{ and } F \sim K.$$

By the well known identity (true because of the associativity of the symmetric difference)

$$(A \triangle B) \triangle (B \triangle C) = A \triangle C$$

for some sets A, B, C, we have that

$$\mu(E \triangle K) = \mu((E \triangle F) \triangle (F \triangle K)).$$

By the definition of the symmetric difference, we have

$$\begin{aligned}
\mu((E \triangle F) \triangle (F \triangle K)) &= \mu(((E \triangle F) \setminus (F \triangle K)) \cup ((F \triangle K) \setminus (E \triangle F))) \\
&= \mu((E \triangle F) \setminus (F \triangle K)) + \mu((F \triangle K) \setminus (E \triangle F)) \\
&= \mu(E \triangle F) - \mu((E \triangle F) \cap (F \triangle K)) + \mu(F \triangle K) - \mu((F \triangle K) \cap (E \triangle F)) \\
&= 0
\end{aligned}$$

because by assumption $E \triangle F$ and $F \triangle K$ have measure 0 and the terms that are subtracted must also be zero as they are subsets of these sets (monotonicity of measure). Hence, $\sim$ is an equivalence relation on $\mathcal{M}$.

6.3 (c)

Notice that we want to show

$$\mu(E \triangle G) = \rho(E, G) \leq \rho(E, F)\rho(F, G) = \mu(E \triangle F) + \mu(F \triangle G).$$

We see that by subadditivity, it suffices to show

$$\mu(E \triangle G) \leq \mu((E \triangle F) \cup (F \triangle G)).$$

Notice that

$$E \triangle G = (E \triangle F) \triangle (F \triangle G) = [(E \triangle F) \cup (F \triangle G)] \setminus [(E \triangle F) \cap (F \triangle G)].$$

Thus, we clearly have

$$E \triangle G \subset (E \triangle F) \cup (F \triangle G),$$

and by the monotonicity of μ, we have

$$\mu(E \triangle G) \leq \mu((E \triangle F) \cup (F \triangle G)) \leq \mu(E \triangle F) + \mu(F \triangle G),$$

showing that the triangle inequality holds for ρ. ρ indeed is a metric on the set of equivalence classes $\mathcal{M}/\sim$ since it satisfies the triangle inequality and is always non-negative by the non-negativeness of μ, with

$$\mu(E \triangle F) = 0,$$

if and only if $E \sim F$, as in both sets belong to the same equivalence class. Symmetry is ensured since the symmetric difference is commutative. ■