Optimization of a Bullet

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Abstract

We optimize the shape of a bullet given by the rotation of a polynomial curve under certain physical constraints with respect to air-resistance, using tools from calculus of variations in the general case. We produce a complete solution of the simpler case where the curve determining the radially generated bullet is quadratic. A potential method for generating air-resistance minimizers in the general case of smooth curves is provided, and a discussion of the physical feasibility of the analysis is then conducted.

Contents

1 Introduction

One field in the intersection of engineering and mathematics is optimization, a powerful tool used in both professions alike. In engineering, optimization is often utilized to minimize or maximize certain (un)desirable factors involved in real world applications. In mathematics, we study the existence and identity of such optimal solutions, providing the rigorous basis upon which we implement the techniques used in engineering.

In this paper, we shall study the shape of bullets and the optimization of their aerodynamic qualities. Since we know that the shape of the solid is what regulates the extent to which it is affected by air resistance, the main factor influencing a bullet's ability to fly effectively, this ultimately comes down to optimizing the bullet's figure to experience minimal air resistance. As a result, we shall conduct a mathematical analysis of the matter, first solving a certain simplified problem with standard optimization techniques, then investigate the general case using tools from calculus of variations.

While we shall present the methodology of the mathematical tools utilized in the bullet optimization problem, it is necessary to rewrite the given practical problem into a mathematical optimization problem and impose the necessary conditions before a thorough analysis can be conducted. For simplicity, we assume that the bullet's shape is given by a solid of rotation about a fixed axis, the rotational axis about which the bullet has rotational symmetry.

We then proceed to assume the aforementioned conditions on the graph modeling the bullet. We will provide these conditions in Section 3 with justifications to their presence.

To rewrite the problem, we must parametrize the curve that will determine the shape of the bullet via revolution about its rotational axis. Before tackling the more difficult problem of considering arbitrary curves on the xy-plane, we solve a simpler problem, in which we assume that the curve is the graph of a certain at most quadratic polynomial. Since linear polynomials are uniquely determined by two variables and quadratic polynomials are determined by three, we will be working with a system involving a finite number of variables in the simpler case. Such parametrizations of given curves by finite variables are important since they are computationally economic and the most likely to have a closed form optimal solution.

After parametrizing and expressing the shape of the bullet using a quadratic polynomial, we shall calculate the air resistance by using the principle of conservation of momentum. As it is known that the air molecules collide with the surface of a bullet, each air molecule would naturally experience a change in its velocity (recall that velocity is a vector) following the collision. We assume that the air molecules are small and ideal, and hence we suppose that the speed (magnitude of velocity) of the air molecule is not affected by the collision. In other words, the only change to the air molecules' velocities is the direction of the vectors. For simplicity, we also assume that the air conditions are ideal, i.e. the air molecules are fixed in the space. In reality, this assumption would not be true, but the large number of air molecules makes it reasonable.

Since the air molecules are ideal, we assume that the changes in velocity following the collision are in accordance with the law of the reflection. Although this law is usually applied to the path of a ray of light, we may also assume that small particles also adhere to this law. Given a plane and a point on the plane, there are normal vectors which are perpendicular to this plane at the point. Suppose one of the air particles collides with the plane and the angle between the velocity vector (before collision) and the normal vector at the point of collision is θ. The law of reflection states that the angle between the normal and the velocity vector of the air particle following the collision is also θ. This fact is utilized to derive an expression for the air resistance experienced by the bullet in terms of the coefficients of the polynomial determining its shape. We then find the quadratic polynomial (under certain additional constraints) which yields the least air resistance by using the tools of mathematical optimization.

We now discuss the logical flow of this paper and its structure. In the Preliminaries section, certain necessary concepts from mathematics and physics are introduced and reviewed. In the Main Theory section, we interpret the physical problem and derive a corresponding mathematical optimization problem for which we find the desired optimal solution. We conclude by summarizing the findings and suggesting methods of extending the techniques used in this paper to more general, sophisticated situations.

2 Preliminaries

In this section, we prepare the necessary tools necessary to conduct our mathematical analysis of the problem. Since the bullet's shape is determined by a solid of revolution, it is natural to study the formula that provides its volume in order to ascertain one of the constraints of the optimization problem. Furthermore, it is necessary to discuss the idea of momentum and the conservation thereof, as we use the conservation of momentum to calculate the effect of air resistance on our bullet. To solve the mathematical optimization problem itself, we shall utilize techniques from the calculus of variations.

2.1 Volume of a Solid of Revolution

We study the volume of solid of revolution in this subsection. First, note that we must rotate the curve of interest around perpendicular to the axis of rotation, indicating that the shell method (which provides the volume of a solid of revolution found by rotating the given graph perpendicular to the axis of rotation) must be used to derive our desired volume. Let $R \in \mathbb{R}$ be a fixed constant representing the chosen radius of our bullet. Given a function $f : \mathbb{R} \to \mathbb{R}$, recall the solid of revolution of f on a closed interval $[0, R]$ is determined by rotating the graph Γ_f about some axis for $x \in [0, R]$, where

$$\Gamma_f := \{(x, f(x)) \in \mathbb{R}^2\} \subset \mathbb{R}^2.$$

Let S be a subset of $\mathbb{R}$. We use the notation $\Gamma_f(S) := \Gamma_{f|_S}$ to represent the graph of f restricted to the set $S \subset \mathbb{R}$. For example, $\Gamma_f([0, R])$ represents the graph of f restricted to the domain $[0, R]$. The following diagram from Wikipedia visually demonstrates what rotating such a graph using the shell method may look like.

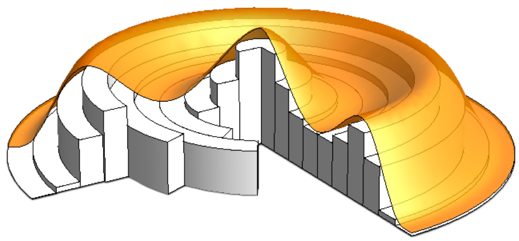

Figure 1: The shell method.

We then know the volume of the solid of revolution given by rotating f about y-axis (i.e. y-axis is taken to be the axis of rotation) is given by

$$V_f = \int_0^R 2\pi x f(x)\, \mathrm{d}x\,.$$

We refer to [Ste16] for the formula above.

2.2 Conservation of Momentum

To provide this section we refer to [HRW13]. The momentum of a particle with mass m and velocity v is given by

$$\rho = mv.$$

If there exist two or more particles with constant mass in an isolated system with no external force acting on the particles, then total momentum is preserved after collisions. In other words, in an isolated system of n particles, if $m_1, \ldots, m_n$ represent the mass of the particles, and $v_1, \ldots, v_n$ their respective velocities prior to collision, and $v_1', \ldots, v_n'$ the velocities following collision, then these variables satisfy

$$\sum_{i=1}^{n} m_i v_i = \sum_{i=1}^{n} m_i v_i'$$

This property is known as the conservation of momentum, and will be used to calculate the air resistance. It is easy to deduce from this formula that if the velocity of an air molecule is perturbed, the velocity of the bullet will also experience variation. Specifically, let m_b be the mass of the bullet and m_a the masses of the air particles that collide with it (assuming that air particles have constant mass), with v_b and v_b' representing the velocity of the bullet, and

$v_{a_1}, \ldots, v_{a_n}$ and $v'_{a_1}, \ldots, v'_{a_n}$ the velocities of the individual air particles before and after collision respectively. By conservation of momentum, we have that

$$\begin{aligned} m_b v_b + \sum_{i=1}^{n} m_a v_{a_i}, &= m_b v'_b + \sum_{i=1}^{n} m_a v'_{a_i}, \\ -m_a \sum_{i=1}^{n} (v'_{a_i} - v_{a_i}) &= m_b (v'_b - v_b), \\ \Delta v_b &\propto \sum_{i=1}^{n} \Delta v_{a_i}, \end{aligned}$$

where Δv_{a_i} and Δv_b are the change in velocity of the ith air molecule and the bullet after collision respectively. Hence, the change in the bullet's velocity is proportional to the total change in the air molecules' velocities. This fact will be utilized to derive a formula for the air resistance experienced by the bullet as it collides with a blanket of air molecules.

2.3 Optimization of a Functional

The subject of calculus of variations is used to study functionals (which are, for our purposes, a mapping from the set of functions on $\mathbb{R}$ to $\mathbb{R}$ itself) and their behavior. One such property is a functional's extrema, or specific functions $f : \mathbb{R} \to \mathbb{R}$ that maximize or minimize its value (we assume that any function discussed in this paper is "nice" enough, i.e. it is differentiable as many times as necessary). Hence, we shall derive a functional representing the air resistance in terms of the integral of x, a function f and its derivatives, or in other words,

$$F[f] = \int_{x_1}^{x_2} J(x, f, f') \,\mathrm{d}x$$

then attempt to minimize it, subject to certain predetermined constraints (where F is the air resistance and J is some function of x, f, and f').

We now introduce two different types of constraints, and methods to deal with them. The problem we shall be studying will be of the form

$$\begin{aligned} \min_{f(x)} F[f] &= \int_{x_1}^{x_2} J(x, f, f') \,\mathrm{d}x \,, \\ G[f] &= \int_{x_1}^{x_2} K(x, f, f') \,\mathrm{d}x = C, \\ f(x) &\geq 0, \\ f(x_1) &= A, \qquad\qquad f(x_2) = B, \end{aligned}$$

where $A, B, C \in \mathbb{R}$ are fixed, i.e. those with one integral constraint, one pointwise inequality (non-integral) constraint, and specified boundary conditions. We handle the inequality separately, by introducing the **slack function** s defined by

$$f(x) = s^2(x),$$

which ensures the non-negativity of the function. By the chain rule, we see that

$$f'(x) = 2s(x)s'(x),$$

providing the information necessary to replace each instance of f (resp. f') in F and G with s^2 (resp. $2ss'$), creating a new problem without an inequality constraint below

$$\begin{aligned}
\min_{s(x)} F[s] &= \int_{x_1}^{x_2} J(x, s^2, 2ss')\,\mathrm{d}x\,, \\
G[s] &= \int_{x_1}^{x_2} K(x, s^2, 2ss')\,\mathrm{d}x = C, \\
f(x_1) &= A, \qquad\qquad f(x_2) = B.
\end{aligned}$$

These problems are known in literature as "isoperimetric problems", and can be solved by forming a function classically labelled the Lagrangian, defined by

$$\mathcal{L}(x, s, s') = J(x, s^2, 2ss') + \lambda K(x, s^2, 2ss'),$$

where λ is a constant known as a **Lagrange multiplier**, then finding solutions to the **Euler-Lagrange equation**, the second order ODE below:

$$\frac{\partial \mathcal{L}}{\partial s} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial s'} = 0.$$

The crux of this method is realizing that finding a function extremizing the integral of the Lagrangian is equivalent to finding a function extremizing F satisfying the isoperimetric constraint G. Solving the Euler-Lagrange equation does exactly the former, providing the extremal function for the integral of the Lagrangian, hence giving the extremal function for F that satisfies the constraints (though technically, it is only a necessary condition for extrema, we shall assume that it is also sufficient as it often produces the correct solution). The exact rigorous details are unnecessary and shall be omitted for our purposes, but can be found in the second chapter of [GF64].

Constrained optimization of a functional is often a difficult task, which requires the solution to a complicated differential equation. The problem we shall study is no exception, and the differential equation that results in fact is too difficult to solve without numerical methods. As it is impossible to verify the validity of the solution and check whether it truly minimizes the functional F, we shall merely present the resulting differential equation to provide a potential method of finding a solution.

3 Main Theory

We shall now finally discuss the physical problem itself. In the first section, we interpret the problem mathematically and create an optimization problem with inequality and equality constraints by setting appropriate parameters. We then provide a potential method of solving the general problem and a complete analysis of the quadratic case, finding the second degree polynomial which minimizes the air resistance experienced by the bullet.

3.1 Mathematically Modeling the Problem

Let us first account for all of the constants and variables within this system. Suppose that the fixed radius and volume of the bullet are represented by $R \in \mathbb{R}$ and $V \in \mathbb{R}$ respectively. Furthermore, let the function $f : \mathbb{R} \to \mathbb{R}$ determine the solid of rotation (i.e. by rotation about the y-axis). It is clear that we are only concerned with the behavior of the function on the closed interval $[0, R]$. Hence, the shape of the bullet will be determined by rotating the region bound by the following four curves about the y-axis:

$$\begin{cases} \Gamma_f([0,R]), \\ x = 0, \\ y = 0, \\ x = R. \end{cases}$$

Since we set the volume of the bullet to be equal to some fixed constant V, it is clear that one of the constraints to our problem must be that the volume of the aforementioned solid of revolution must be equivalent to this value. That is, by the formula given by the shell method in Section 2.1, we must have

$$\begin{aligned} V &= \int_0^R 2\pi x f(x)\,\mathrm{d}x \\ &= 2\pi \int_0^R x f(x)\,\mathrm{d}x \end{aligned}$$

We now calculate the change of velocity of the bullet when an air molecule collides with the bullet at a point with x-coordinate between (and including) 0 and R. Suppose that the bullet is traveling in the positive y direction with some velocity v_b. Then, we can consider any air molecule (assumed to be stationary) to be moving in the negative y direction relatively. Suppose that the air molecule collides with the bullet at point $(x, f(x))$ with $x \in [0, R]$. Since the collision is local to this single point, the situation is unchanged if we instead consider the affine function tangent to the curve $\Gamma_f([0, R])$ at point x (the tangent line to the curve at $(x, f(x))$). Let θ be the angle between the x-axis and the tangent line. It is easy to verify that the angle between the normal vector of the tangent line and the positive y-axis is also θ.

Simple geometry then verifies that the directional unit vector of the air molecule following the collision is determined by $(\sin 2\theta, -\cos 2\theta)$. That is, the unit direction of the air molecules changes from $(0, -1)$ to $(\sin 2\theta, -\cos 2\theta)$ after the collision.

We suppose that a blanket of air molecules collides with the bullet from the positive y direction, at large enough quantities such that it is suffice to approximate the situation by assuming that a solid of revolution represents the total change in the y direction (air resistance) experienced by the bullet after collision. We shall now determine this quantity explicitly.

Since the bullet is rotationally symmetric about the y-axis, there will be no horizontal shift in the bullet's direction. We see that when θ satisfies

$$\tan\theta = -f'(x),$$

the change in the y-component of the bullet will be proportional to $1 + \cos 2\theta$ since change in the y-component of the bullet is proportional to that of the air molecules'. From the property mentioned above, we have

$$\begin{aligned} 1 + \cos 2\theta &= 2 - 2\sin^2\theta \\ &= 2 - \frac{2\tan^2\theta}{1+\tan^2\theta} \\ &= 2 - 2\left(1 - \frac{1}{\tan^2\theta + 1}\right) \\ &= \frac{2}{\tan^2\theta + 1}. \end{aligned}$$

We now find the total change in the y-component of the bullet by integrating these individual changes in the y-component of the air molecules and using the proportionality of the two quantities below:

$$\Delta = C\int_0^R 2\pi x \frac{2}{1+(f'(x))^2}\,\mathrm{d}x = 4\pi C\int_0^R \frac{x}{1+(f'(x))^2}\,\mathrm{d}x\,,$$

where Δ is the change in the bullet's y-component.

Since $4\pi C$ is merely a constant and we are only worried about the magnitude of Δ, it is suffice to find

$$\min F[f] = \int_0^R \frac{x}{1+[f'(x)]^2}\,\mathrm{d}x\,.$$

We now finally impose the inequality constraint $f(x) \geq 0$ for $x \in [0, R]$ on the bullet to prevent any notions of "negative" volume arising.

3.2 Statement of The Optimization Problem

From the results of the last subsection, we have arrived at the mathematical problem we desire to solve:

$$\begin{aligned} \min_{f(x)} F[f] &= \int_0^R \frac{x}{1+(f'(x))^2}\,\mathrm{d}x \\ G_1[f] &= 2\pi\int_0^R xf(x)\,\mathrm{d}x = V, \\ G_2(x) &= f(x) \geq 0, \end{aligned}$$

one clearly of the form described in Section 2.3. Since such isoperimetric problems also require boundary conditions, we assume that $f(0) = H$ and $f(R) = 0$, where $H \in \mathbb{R}$ is a constant representing the fixed height of the bullet.

3.3 A Simper Problem

Before tackling the more general variational optimization problem, we study a simpler function (not functional) minimization problem which does not rely on much of the heavy machinery of optimization and calculus of variations. We suppose that the function f is a concave quadratic polynomial and that its derivative vanishes at $x = 0$, i.e. if f is uniquely determined by real coefficients $a, b, c \in \mathbb{R}$ such that

$$f(x) = ax^2 + bx + c,$$

then we have $a \leq 0$ and $b = 0$ by assumption. Despite restricting the problem, these assumptions are reasonable since bullets often have a smooth tip and concave exterior body in the real world.

As derived in the previous section, we now plug in our explicit formula for f' given by $f'(x) = 2ax + b$ into the function we want to optimize, and find the objective function

$$F(a, b) = \int_0^R \frac{x}{1 + (2ax + b)^2} \, \mathrm{d}x \, .$$

It is important to note that F is a real valued function, but not a functional. A closed form of this integral can be evaluated using the following two identities derived via elementary antiderivatives, substitution, and partial fraction decomposition:

$$\begin{aligned} \int_0^R \frac{2ax + b}{1 + (2ax + b)^2} \, \mathrm{d}x &= \frac{1}{4a} \left[\log\bigl(1 + (2ax + b)^2\bigr)\right]_0^R \\ \int_0^R \frac{1}{1 + (2ax + b)^2} \, \mathrm{d}x &= \frac{1}{2a} \left[\arctan(2ax + b)\right]_0^R . \end{aligned}$$

Evaluating the integral results in

$$F(a, b) = \frac{\log\left(\frac{(2aR+b)^2+1}{b^2+1}\right) - 2b(\arctan(2aR + b) - \arctan(b))}{8a^2},$$

and minimizing the air resistance on the bullet is clearly a non-linear constrained optimization problem.

Furthermore, we also find an explicit formula for the volume constraint, namely g_1. Merely plugging in our f and integrating with the shell method, we find the formula

$$V = \int_0^R 2\pi x f(x) \, \mathrm{d}x = 2\pi \int_0^R x(ax^2 + bx + c) \, \mathrm{d}x = 2\pi \left[\frac{aR^4}{4} + \frac{bR^3}{3} + \frac{cR^2}{2}\right].$$

For clarity and organization, we now restate the optimization problem for this specific easier problem below:

$$\begin{aligned}
\min F(a,b,c,x) &= \frac{\log\left(\frac{(2aR+b)^2+1}{b^2+1}\right) - 2b(\arctan(2aR+b) - \arctan(b))}{8a^2}, \\
g_1(a,b,c,x) &= 2\pi\left[\frac{aR^4}{4} + \frac{bR^3}{3} + \frac{cR^2}{2}\right] - V = 0, \\
g_2(a,b,c,x) &= -(ax^2+bx+c) \le 0, \qquad (x \in [0,R]), \\
g_3(a,b,c,x) &= a \le 0,
\end{aligned}$$

However, by the property of f' vanishing at 0, b was equal to 0, so the constraint g_1 can be simplified down to

$$g_1(a,b,c,x) = 2\pi\left(\frac{aR^4}{4} + \frac{cR^2}{2}\right) - V = \frac{\pi}{2}R^2(aR^2+2c) - V.$$

Hence, we choose the problem degenerates to finding

$$\begin{aligned}
\min F(a,c,x) &= \frac{1}{8a^2}\log\left(1+4a^2R^2\right) \\
g_1(a,c,x) &= \frac{\pi}{2}R^2(aR^2+2c) - V = 0, \\
g_2(a,c,x) &= -(ax^2+c) \le 0, \qquad (0 \le x \le R), \\
g_3(a,c,x) &= a \le 0.
\end{aligned}$$

Since F is now independent of all but one variable, we may treat it as a function from $\mathbb{R}$ to $\mathbb{R}$ and take its derivative. Taking the derivative of F with respect to a, we have

$$F'(a) = \frac{R^2}{a+4R^2a^3} - \frac{\log\left(1+4R^2a^2\right)}{4a^3}$$

for which we have $F'(a) \ge 0$ for all $a \le 0$. This inequality holds because

$$\begin{aligned}
\frac{R^2}{a+4R^2a^3} &\ge \frac{\log\left(1+4R^2a^2\right)}{4a^3} \\
\frac{4R^2a^3}{a+4R^2a^3} &\le \log\left(1+4R^2a^2\right) \\
\frac{4R^2a^2}{1+4R^2a^2} &\le \log\left(1+4R^2a^2\right) \\
\frac{y-1}{y} &\le \log y,
\end{aligned}$$

and the last inequality is valid by the mean value theorem on $\log x$ on the interval $[1,y]$ (the substitution $y = 1+4R^2a^2$ was made). Hence, this implies that F is increasing on the negative real axis, and we must minimize the new objective function $F^\star(a,c) = a$ to find $\min F$. To do this, we first note that since the quadratic polynomial is decreasing, we have the inequalities

$$aR^2 + c \geq ax^2 + c \geq 0$$

for any $x \in [0, R]$. Hence, our g_2 constraint is encompassed by a new constraint which we shall denote

$$g_2^\star(a,c,x) = -(aR^2 + c) \leq 0.$$

The final problem we are left with is

$$\begin{aligned}
\min F^\star(a,c) &= a,\\
g_1(a,c) &= \frac{\pi}{2}R^2(aR^2 + 2c) - V = 0,\\
g_2^\star(a,c) &= -(aR^2 + c) \leq 0,\\
g_3(a,c) &= a \leq 0.
\end{aligned}$$

Combining constraint g_1 and $g_2^\star$, we derive the inequality

$$V = \frac{\pi}{2}R^2(aR^2 + 2c) \geq \frac{\pi}{2}R^2(aR^2 - 2aR^2) = -\frac{\pi R^4 a}{2},$$

which gives us

$$a \geq -\frac{2V}{\pi R^4},$$

and our desired minimum $a^\star$ has been found, where

$$a^\star = -\frac{2V}{\pi R^4}.$$

We finally calculate the c term to find the optimal triplet

$$(a,b,c) = \left(-\frac{2V}{\pi R^4}, 0, \frac{2V}{\pi R^2}\right),$$

which gives us our optimal quadratic polynomial $f^\star$ minimizing the air resistance factor, defined by

$$f^\star(x) = -\frac{2V}{\pi R^4}x^2 + \frac{2V}{\pi R^2} = \frac{2V}{\pi R^4}(R^2 - x^2).$$

Hence, given the volume and radius of a bullet, a specific quadratic polynomial that minimizes the air resistance experienced by the bullet can be explicitly determined when certain suppositions and constraints are assumed to be satisfied. To visualize this result, we refer to the following graphs of the curve restricted to the first quadrant, with fixed $V = 4$ and radii $R = 0.8$ and $R = 1.3$ respectively.

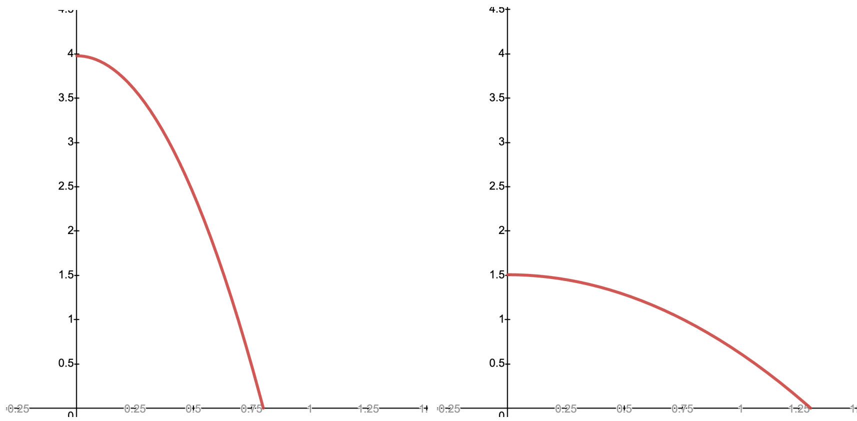

Figure 2: Optimal quadratic curves for $V = 4$ with $R = 0.8$ and $R = 1.3$ respectively. Graphs drawn in Desmos.

3.4 The General Problem

We now provide a possible approach to solving the general problem. As described in Section 2.3, we introduce the slack function s defined by

$$f(x) = s^2(x)$$

and rewrite the isoperimetric problem of 3.2 in terms of s and s'. To do this, we substitute in s^2 for all instances of f and $2ss'$ for all instances of f'. Hence, we form the Lagrangian $\mathcal{L}$ in terms of s and s', and minimize

$$\int_0^R \mathcal{L}^\star(x, s, s')\,\mathrm{d}x = \int_0^R \frac{x}{1 + [2s(x)s'(x)]^2} + \lambda(2\pi x s^2(x))\,\mathrm{d}x$$

for our desired function. Explicitly writing out this Lagrangian's Euler-Lagrange equation, an optimal s must be a solution to

$$\frac{\partial \mathcal{L}^\star}{\partial s} - \frac{d}{dx}\frac{\partial \mathcal{L}^\star}{\partial s'} = 0.$$

Though the differential equation that results is most likely impossible to solve analytically, if a solution $s^\star(x, \lambda)$ is obtained, then one may use the boundary conditions $f(0) = A$ and $f(R) = 0$ as well as the other isoperimetric conditions to deduce the explicit value of λ. This optimal $s^\star$ value may then be squared to obtain an optimal $f^\star$. The actual computations and the subsequent analysis and verification of the minimality of the functional is beyond the scope of this paper, and could be the subject of further research.

4 Conclusion

Throughout this paper, we studied potential methods of deducing an optimal shape for a bullet with respect to air resistance. We found an explicit minimizer in the quadratic case, and provided a method that could be followed to find a general function that minimizes the air resistance experienced by the bullet during flight. This was done through application of the principle of conservation of momentum and the law of reflection to obtain the factor proportional to the air resistance force. Future work on this problem may generalize and extend our results to account for more physical phenomena to more accurately reflect real life, such as the smooth flow of air around the bullet, or a case where collisions are not considered elastic.

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