Algebraic Topology Pset 1

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1 Problem 1

1.1 Exercise 3

Proof. ($D^2 \to S^1$): Note that any continuous surjective map $f : D^2 \to S^1$ will be a quotient map since D^2 is compact and S^1 is Hausdorff. Hence, we take the map f defined by

$$f : (x, y) \mapsto (x, \sqrt{1 - x^2}),$$

which gives a quotient map.

($S^2 \to D^2$): Again, S^2 is compact and D^2 is Hausdorff. The continuous map $f : S^2 \to D^2$ defined by

$$f : (x, y, z) \mapsto (x, y, 0)$$

is clearly onto, so it is a quotient map.

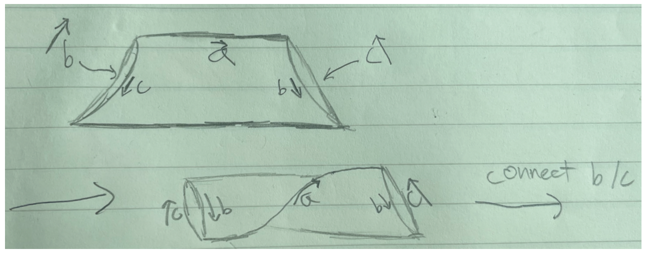
($S^2 \to S^1$): The assumption of compactness and Hausdorffness holds again. We compose the two quotient maps above to obtain the map we desire.

□

1.2 Exercise 9

Proof. We first fold the hexagon in half, aligning the two sides labeled a, resulting in the figure below.

From this, we connect the c sides and b sides with their counterparts respectively, by "twisting" our figure 180 degrees and aligning the orientation of each pair as seen above. It is clear that the figure we get by connecting these sides with matching orientations is homeomorphic to a torus. □



2 Problem 2

If X is a (path)-connected space and $f : X \to Y$ is a homeomorphism, then if for some $x \in X$, $X - \{x\}$ is disconnected, then $Y - \{f(x)\}$ must also be disconnected. We call x a cut point of X and $f(x)$ a cut point of Y. For this problem, we shall use the following fact, which is trivial to prove (just consider the restriction of f onto $X - \{x\}$ to produce a homeomorphism and (dis)connectedness is preserved under said homeomorphism):

Lemma 2.0.1. *Homeomorphic spaces have an equivalent number of cut points and non-cut points.*

The contrapositive of this statement gives a criterion to distinguish whether two spaces are homeomorphic.

2.1 Exercise 1

Proof. These two spaces are not homeomorphic. The spectacle has an infinite number of cut points (anywhere on the line connecting the two circles) while the figure-eight space only has one cut point (the point joining the two circles). □

2.2 Exercise 2

Proof. These two spaces are homeomorphic. Notice that the first space is $S^1 \times \mathbb{R}$. Notice that $\mathbb{R} \approx (0, \infty)$, so

$$S^1 \times \mathbb{R} \approx S^1 \times (0, \infty).$$

Clearly the space on the right is homeomorphic to $\mathbb{R}^2 \setminus \{(0, 0\}$ by conversion between polar coordinates and cartesian coordinates. □

2.3 Exercise 3

Proof. Note that the trefoil knot will have a representation $\gamma : S^1 \to \mathbb{R}^3$ where γ is continuous and injective, and furthermore is a homeomorphism to its image $\gamma(S^1)$. Clearly, γ is itself a homeomorphism between S^1 and the trefoil knot. $\square$

3 Problem 3

3.1 Exercise 1

Proof. We use the contrapositive of Lemma 2.0.1 to deduce the result, as $\mathbb{R}$ has an infinite number of cut points (in fact, every point in $\mathbb{R}$ is a cut point), while $\mathbb{R}^2$ has no cut points. $\square$

3.2 Exercise 2

Proof. Let X be a simply-connected space and Y a topological space, with $f : X \to Y$ a homeomorphism. Clearly, Y is path-connected since path-connectedness is preserved under homeomorphisms. Let $\gamma : [0,1] \to Y$ and $\omega : [0,1] \to Y$ be paths in Y. Then we may compose both paths with the inverse map f^{-1} such that we obtain two paths in X. Since X is simply-connected, there is a (continuous) homotopy $F : [0,1] \times [0,1] \to X$ such that

$$F(x,0) = f^{-1}(\gamma(x)), \quad F(x,1) = f^{-1}(\omega(x)).$$

We then consider the map $\overline{F} := f \circ F : [0,1]^2 \to Y$, and see that

$$\overline{F}(x,0) = \gamma(x), \quad \overline{F}(x,1) = \omega(x),$$

obtaining a homotopy between γ and ω. $\overline{F}$ is clearly continuous since it is a composition of continuous maps. $\square$

3.3 Exercise 3

Proof. ($\Longrightarrow$) Suppose X is simply-connected and let $f : S^1 \to X$ be continuous. For the purposes of interpreting f as a path, we may say that $f : [0,1] \to X$ with

$$f(0) = f(1),$$

that is, f corresponds to a loop in X. We consider the function $g : [0,1] \to X$ that fixes the endpoint of f, that is

$$f(0) = f(1) = g(x),$$

and we have a homotopy $F : [0,1]^2 \to X$ between g and f. Since the homotopy fixes the endpoint, we may interpret the map to be $F : S^1 \times [0,1] \to X$, with

$$F(x,0) = g(x) = f(0), \quad F(x,1) = f(x).$$

We note that there is in fact a quotient map $\varphi : S^1 \times [0,1] \to D^2$ given by

$$\varphi(\mathbf{r}, s) = s\mathbf{r},$$

which is clearly surjective and continuous ($S^1 \times [0,1]$ is compact and D^2 is Hausdorff). The non-injectivity of this map arises at the origin, and forming the quotient space $(S^1 \times [0,1])/\sim$ where

$$(\mathbf{r}, s) \sim (\mathbf{q}, p) \iff s\mathbf{r} = p\mathbf{q},$$

we obviously obtain the natural homeomorphism

$$(S^1 \times [0,1])/\sim \approx D^2.$$

We hence define $\overline{F} : (S^1 \times [0,1])/\sim \to X$ with

$$\overline{F}[(x,t)] = \begin{cases} F(x,t) & \text{if } t \neq 0 \\ f(0) & \text{if } t = 0 \end{cases},$$

where $[(x,t)]$ denotes the equivalence class of (x,t). Clearly, this map is continuous as the homotopy "shrinks" the loop $f(S^1)$ down to the point $f(0)$ as $t \to 0$. The result follows from identifying our domain with D^2. Note that we need not choose the endpoint for our constant function g, we could have chosen any arbitrary point in X.

($\Longleftarrow$) Let the continuous map $f : S^1 \to X$ have an extension $F : D^2 \to X$. □

3.4 Exercise 4

Proof. From the previous exercise, we see that a space is simply-connected iff every loop (a map $f : S^1 \to X$) is homotopic to a point. Clearly, $\mathbb{R}^2 \setminus \{(0,0)\}$ cannot be simply-connected since any loop that contains the origin in its interior cannot be contracted to another point within the interior of the loop. On the other hand, $\mathbb{R}^3 \setminus \{(a,b,c)\}$ is simply-connected since since any path can be deformed to a point by simply "avoiding" (a,b,c) in three-dimensional space. □

3.5 Exercise 5

Proof. First, suppose that $f : \mathbb{R}^3 \to \mathbb{R}^2$ is a homeomorphism. Then clearly $\mathbb{R}^3 \setminus \{(0,0,0)\} \approx \mathbb{R}^2 \setminus \{(0,0)\}$ with the homeomorphism given by the appropriate restriction of f to exclude the origin. However, this restriction cannot be a homeomorphism as $\mathbb{R}^3 \setminus \{(0,0,0)\}$ is simply-connected while $\mathbb{R}^2 \setminus \{(0,0)\}$ is not, imply that our original claim that f is a homeomorphism between $\mathbb{R}^3$ and $\mathbb{R}^2$ must have been false. $\square$

4 Problem 4

4.1 Exercise 1

Proof. Let (X, τ_1) be compact and (Y, τ_2) be Hausdorff. Let $f : X \to Y$ be bijective and continuous. Notice that since continuous maps preserve compactness, for any closed (hence compact) subset $C \subset X$, $f(C)$ is compact in Y. Furthermore, since Y is Hausdorff, each $f(C)$ must be closed. If g is the set theoretic inverse map of f, then for any arbitrary closed $C \subset X$, we have $g^{-1}(C) = f(C)$ to be closed so clearly g is continuous and f is a homeomorphism. $\square$

5 Problem 5

We first answer the question of how the ideas of homeomorphism and homotopy equivalence are related. It turns out that homeomorphism is a stronger notion than homotopy equivalence

Proof. Let X and Y be spaces and $f : X \to Y$ a homeomorphism. Notice that since f is a bijection, we have a two-sided inverse map $g : Y \to X$ such that

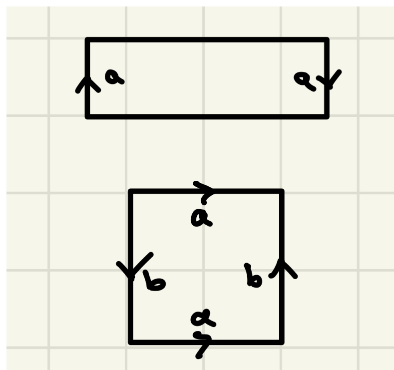
$$f \circ g \equiv \mathbb{1}_Y, \quad g \circ f \equiv \mathbb{1}_X,$$

and clearly each equivalence trivially implies the existence of a homotopy between the pairs of maps. Hence, X and Y are homotopy equivalent. Since the two latter pairs of spaces in Problem 2 were homeomorphic, they are also homeomorphic with respect to their pairs. Note that the spectacles and the figure 8 are also homotopy equivalent as they are both deformation retracts of the Möbius band. This is because deformation retractions are homotopies between the identity map and a projection, we have that the figure 8 and the spectacles are homotopy equivalent to the Möbius band, and transitivity of homotopy equivalence shows the desired result. $\square$

6 Problem 6

6.1 Exercise 1

The following two figures exhibit the fundamental polygons for the Möbius strip and Klein bottle respectively.



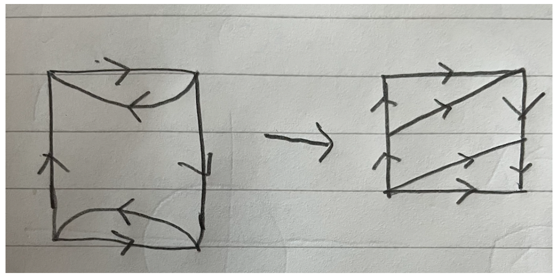
6.2 Exercise 2

Note that by identifying sides a, the top left corner and the bottom left corner are identified, as are the top and bottom right corners. "Twisting" the middle of the cylindrical figure by 180 degrees to correctly match the orientations of the sides b then identifying the two sides to obtain a "torus with a twist", we see that this figure will identify the left corner (the point where the top and bottom left corners were identified) with the right corner (analogous). Hence, our CW-complex will be composed of one 0-cell (the point with all four corners identified), two 1-cells (the two sides which were identified), and one 2-cell (the interior of the fundamental polygon).

6.3 Exercise 3

Proof. We must somehow divide the fundamental polygon for the Klein bottle to obtain some combination of two fundamental polygons for the Möbius strip. A simple cut down the diagonal connecting the bottom left corner of the Klein bottle's fundamental polygon does not suffice since it would result in the two Möbius strips being attached. Hence, we make two cuts, to separate a bit of the top and bottom sides of the Klein bottle to result in two Möbius strips as seen below.

Note that one Möbius band is the middle strip of the divided square



and the other Möbius strip is the remaining part of the figure with the strip "taken out". □

7 Problem 7

7.1 Exercise 2

Proof. The deformation retraction is given by the homotopy between the identity map and the projection of $\mathbb{R}^n \to S^{n-1}$ given by $F : \mathbb{R}^n \times [0,1] \to S^{n-1}$ defined by

$$F(x,t) = t\frac{x}{\|x\|} + (1-t)x.$$

□

7.2 Exercise 5

Proof. Suppose that X deformation retracts to a (fixed) point $x \in X$ via the homotopy $F : X \times [0,1] \to X$ with

$$F(y,0) = \mathbb{1}_X(y), \quad F(y,1) = x$$

for each $y \in X$. Let U be a open neighborhood of x in X. Then clearly

$$F^{-1}(U) = \{(x,t) \in X \times [0,1] : F(x,t) \in U\}$$

is an open neighborhood containing the sets $X \times \{1\}$ and $\{x\} \times [0,1]$. We see that the latter of these sets is compact, and considering every $t \in [0,1]$, we cover $X \times [0,1]$ with V_j open neighborhoods of x and N_j^t open neighborhoods of each t. W see that we may obtain a finite subcover $\{V_1^t \times N_1^t, \cdots, V_n^t \times N_n^t\}$ that covers $\{x\} \times [0,1]$. Intersecting all these sets, we obtain that

$$V = \bigcap_{j=1}^{n} V_j^t$$

is a non-empty open subset of U that contains $\{x\}$. Consider the injection $i : V \hookrightarrow U$ composed with the original homotopy F in the following way:

$$F(i(y), 0) = i, \quad F(i(y), 1) = x,$$

showing that i is nullhomotopic. $\square$

7.3 Exercise 11

Proof. Suppose there is a map $f : X \to Y$ such that there exists maps $g, h : Y \to X$ such that

$$fg \simeq \mathbb{1}_Y, \quad hf \simeq \mathbb{1}_X.$$

Clearly,

$$hfg = (hf)g \simeq \mathbb{1}_X g = g,$$

while

$$hfg = h(fg) \simeq h\mathbb{1}_Y = h,$$

which implies that

$$g \simeq h.$$

Hence, we have that $\mathbb{1}_Y \simeq fg \simeq fh$ and we are done. $\square$

7.4 Exercise 12

Proof. Let $f : X \to Y$ be a homotopy equivalence with $g : Y \to X$ the map such that $fg \simeq \mathbb{1}$ and $gf \simeq \mathbb{1}$. Let $X/\sim$ and $Y/\sim$ be the set of path-components in each space respectively. We induce a map $\bar{f} : X/\sim \to Y/\sim$ defined by

$$f(x) = y \implies \bar{f}([x]) = [y], \quad x \in X, y \in Y$$

where the choice of representatives does not matter by the following argument. Let $x_1, x_2 \in X$ be path-connected by some path $\gamma : [0, 1] \to X$. Then the map $f\gamma : [0, 1] \to Y$ is continuous and

$$f(\gamma(0)) = f(x_1), \quad f(\gamma(1)) = f(x_2),$$

showing that $f(x_1)$ and $f(x_2)$ are also path-connected.

Now suppose that $\bar{f}([x]) = \bar{f}([y])$ for $[x], [y] \in X/\sim$. We see that taking representatives x_0 and y_0 from the two path-components respectively, then

$$f(x_0) = f(y_0),$$

by definition. Applying $g : Y \to X$ on both sides, we have

$$g(f(x_0)) = g(f(y_0)),$$

and we shall denote this common point by $\zeta \in X$.

The two representatives must be in the same equivalence class in $X/\sim$ since the homotopy $F : X \times [0,1] \to X$ such that

$$F(\alpha, 0) = g(f(\alpha)), \quad F(\alpha, 1) = \mathbb{1}_X(\alpha) = \alpha,$$

provides a path between x_0 and ζ. This is done by fixing x_0 in the first argument of F and interpreting the function as a map between $[0,1]$ and X. Specifically, deducing the corresponding fact for y_0 then using the transitivity of path-connectedness with ζ as the mediator finishes the proof of injectivity.

For surjectivity, let $[y] \in Y/\sim$. Let y be a representative for this path-component, and define

$$\zeta := g(y).$$

Again, we use the homotopy $G : Y \times [0,1] \to Y$ such that

$$G(\alpha, 0) = f(g(\alpha)), \quad G(\alpha, 1) = \mathbb{1}_Y(\alpha),$$

and we defining the map $G_y : [0,1] \to Y$ by fixing y in the first argument, we see that

$$G(0) = f(\zeta), \quad G(1) = y,$$

meaning $\bar{f}([\zeta]) = [y]$ and we are done.

It is clear that the inverse map of $\bar{f}$ is $\bar{g} : Y/\sim \to X/\sim$, where the latter function is defined analogously. It can easily be seen that f restricts to a homotopy equivalence between path-components, with the appropriate restriction of g to the image of the path-component under f being the "inverse" map and vice versa. $\square$