Bubbles and Area-Minimizing Currents

Sieon (Sean) Kim
seankim146@gmail.com

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- Differential Geometry + Measure Theory

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- We study "generalized submanifolds" of $\mathbb{R}^n$, standard analysis of smooth surfaces is inadequate due to their lack of compactness properties.
- Currents!

Plateau's Problem

GMT rose to prominence during the mid 1900s with the efforts of early pioneers such as H. Federer and W.H. Fleming to solve *Plateau's problem*, named after the Belgian physicist Joseph Plateau who related the original mathematical problem posed by J. Lagrange in 1760 to the geometry of soap films and soap bubbles.

Statement of Plateau's Problem

Conjecture

For every smooth closed curve Γ *in* $\mathbb{R}^3$*, there is a surface of least area among all surfaces which have* Γ *as their boundary.*

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Currents

Definition

An m-current in $\mathbb{R}^n$ is an element of $(\mathcal{D}^m(U))^*$, the (continuous) dual space of the space of smooth, compactly supported differential m-forms on $\mathbb{R}^n$.

This may seem like (and is) a very technical definition. However, currents may intuitively be understood as a kind of "smooth" surface that satisfies certain extra properties that ensure compactness properties.

Mass of Currents

We now define a notion of "surface-area" for currents, in order to define the idea of area-minimizing currents.

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Those familiar with a bit of analysis will recognize that the mass is analogous to the operator norm. Now, it is clear that we may call a current T area-minimizing if

$$\mathbb{M}(T) \leq \mathbb{M}(R)$$

for any other current R. (Of course, there must be certain restrictions)

Federer-Fleming Compactness Theorem

Theorem (Federer-Fleming Compactness Theorem)

If $\{T_j\}$ is a sequence of integer-rectifiable m-currents with

$$\sup_{j\geq 1} \mathbb{M}(T_j) < \infty, \quad \sup_{j\geq 1} \mathbb{M}(\partial T_j) < \infty,$$

then there is a subsequence $\{T_{j'}\}$ that (weakly) converges to some integer-rectifiable current T in U.

Utility of Compactness

It is a standard fact that the mass $\mathbb{M}$ of currents is lower-semicontinuous. This means that for every sequence of currents $\{T_j\}$ converging weakly to some T, we have

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The proof is almost trivial, yet this result shows the value in considering a sequence of currents in minimization problems such as that of Plateau's problem as guaranteed in the Compactness Theorem: essentially, if one considers a sequence of area-minimizing currents, then the limit will be the global minimizer.

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1. Existence: a simple consequence of the **Banach-Alaoglu Theorem**.
2. Closure: a much more difficult proof, involving a myriad of technical results and machinery.
   - **Slicing**: intersecting a current with the level set of a Lipshitz map to obtain a lower dimensional current.
   - **The Deformation Theorem**: allows one to approximate an integer-multiplicity current with currents with more structure by deforming it onto a grid mesh.

Solution to Plateau's Problem

Theorem

We work in $\mathbb{R}^{m+k}$. Let S be an integer multiplicity $(m-1)$-current with compact support and $\partial S = 0$. Then there is an integer multiplicity m-current T with compact support, $\partial T = S$, and

$$\mathbb{M}(T) \leq \mathbb{M}(R)$$

for every integer multiplicity m-current with compact support and $\partial R = S$.

Proof of the Solution

Proof.

We consider a sequence $\{R_j\}$ of integer multiplicity currents with compact support that span the given boundary such that

$$\lim_{j\to\infty} \mathbb{M}(R_j) = \inf_{R\in\mathcal{I}_S} \mathbb{M}(R),$$

$$\mathcal{I}_S := \{\mathbb{Z}\text{-multiplicity, compactly supported } R \text{ with } \partial R = S\}.$$

We then take a Lipschitz map $f : \mathbb{R}^{m+k} \to B_R(0)$, and construct a new sequence with the R_j. We then apply the **Compactness Theorem** to extract a subsequence that converges to a current T such that

$$\mathbb{M}(T) \leq \inf_{R\in\mathcal{I}_S} \mathbb{M}(R).$$

∎

A Regularity Result

Theorem

A rectifiable, area-minimizing 2-current $T \in \mathbb{R}^3$ is a smooth, embedded manifold on the interior, that is, $\operatorname{supp} T - \operatorname{supp} \partial T$ *is an embedded C^∞ submanifold of $\mathbb{R}^3$.*

The proof was by Fleming in 1962.

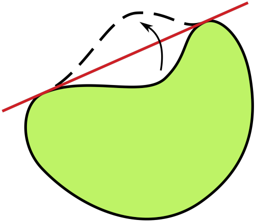
Isoperimetric Inequality

Theorem (Isoperimetric Inequality)

Let $m \geq 2$, and suppose that T is an integer-rectifiable $(m-1)$-current with compact support and $\partial T = 0$. Then there is a compactly supported, integer-rectifiable m-current R such that $\partial R = T$ and

$$(\mathbb{M}(R))^{(m-1)/m} \leq c\mathbb{M}(T),$$

where $c \in \mathbb{R}$ is some constant.



Further Applications of Currents

It is clear from this that currents have a wide variety of applications, owing to their "nice" compactness properties. Some of the notable current uses of currents include their application in the fields of partial differential equations and dynamical systems, calculus of variations and its applications to optimal transport, and the study of analytic varieties in complex (algebraic) geometry.