Folland Measure Theory Sections 2.4-2.6

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Review of Uniform and Pointwise Convergence

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Pointwise Convergence: We say that $\{f_n\}$ converges pointwise to f if for all $\varepsilon > 0$ and $x \in X$, there is an integer $N(x)$ (note the dependency on x) such that

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Uniform Convergence: We say that $\{f_n\}$ converges uniformly to f if for all $\varepsilon > 0$, there exists an integer N (note that it is independent of x) such that

$$n \geq N \implies |f_n(x) - f(x)| < \varepsilon,$$

for all $x \in X$.

Convergence in Measure

- We say a sequence $\{f_n\}$ of measurable functions with $f_n : X \to \mathbb{C}$ is **Cauchy in measure** if for every $\varepsilon > 0$,

$$\mu(\{x \in X : |f_n(x) - f_m(x)| \geq \varepsilon\}) \to 0 \text{ as } m, n \to \infty.$$

- The sequence **converges in measure** to $f : X \to \mathbb{C}$ if for every $\varepsilon > 0$,

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}) \to 0 \text{ as } n \to \infty.$$

Preliminary Lemmas/Theorems

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$\{f_n\}$ Cauchy in measure. Then there is a measurable function f such that $f_n \to f$ in measure, and there is a subsequence f_{n_j} that converges to f a.e. If $f_n \to g$ in measure, then $g = f$ a.e.

Egoroff's Theorem

Theorem (Egoroff's Theorem)

Suppose that $(X, \mathcal{M}, \mu)$ *is a finite measure space, and* $\{f_n\}$ *and* f *are complex-valued measurable functions on* X *such that* $f_n \to f$ *a.e. For every* $\varepsilon > 0$*, there exists* $E \subset X$ *such that* $\mu(E) < \varepsilon$ *and* $f_n \to f$ *uniformly on* E^C*. This is called* **almost uniform convergence***, and it implies p.w. convergence a.e. and convergence in measure.*

Egoroff's Theorem (Proof)

Proof.

WLOG let $f_n \to f$ on X. For $k, n \in \mathbb{N}$, define

$$E_n(k) = \bigcup_{m=n}^{\infty} \{x \in X : |f_m(x) - f(x)| \geq \frac{1}{k}\}.$$

For fixed k, $E_n(k)$ decreases as $n \to \infty$, and $\bigcap_n E_n(k) = \varnothing$, so since $\mu(X) < \infty$, we have that $\mu(E_n(k)) \to 0$ as $n \to \infty$. Given $\varepsilon > 0$ and $k \in \mathbb{N}$, choose n_k so large enough so that

$$\mu(E_{n_k}(k)) < \frac{\varepsilon}{2^k},$$

and let $E = \bigcup_k E_{n_k}(k)$. Then $\mu(E) < \varepsilon$, and we have

$$|f_n(x) - f(x)| < \frac{1}{k}$$

for $n > n_k$ and $x \in E^C$. Thus $f_n \to f$ uniformly on E^C. $\square$

Pointwise, Uniform, and L^1 Convergence

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We saw from **Problem 2** in our last HW that L^1 convergence implies pointwise convergence almost everywhere for a subsequence. Let us summarize the relationships between different modes of convergence.

- Uniform convergence $\implies$ pointwise convergence $\implies$ pointwise convergence a.e.
- L^1 convergence $\implies$ pointwise convergence a.e. for a subsequence
- Uniform convergence $\not\Longleftrightarrow$ L^1 convergence (unless the measure space is finite, then unif $\implies$ L^1)
- Pointwise convergence (a.e.) $\not\Longleftrightarrow$ L^1 convergence

The Product Measure

Recall from last HW that we can construct a product measure on the product measurable space $(X, \mathcal{M})$ with

$$X = \prod_{j=1}^{n} X_j, \quad \mathcal{M} = \bigotimes_{j=1}^{n} \mathcal{M}_j.$$

This measure μ is unique on $\mathcal{M}$ and when the components μ_j are σ-finite, then μ is σ-finite. We want to define integration for this product space $(X, \mathcal{M}, \mu)$ and relate it to integration on the individual component spaces $(X_j, \mathcal{M}_j, \mu_j)$.

The Monotone Class Lemma

Definition

A **monotone class** on a space X is a collection of subsets $\mathcal{C}$ that's closed under countable increasing unions and countable decreasing unions. In other words, if $E_j \in \mathcal{C}$ and

$$E_1 \subset E_2 \subset \cdots, \text{ then } \bigcup_{j=1}^{\infty} E_j \in \mathcal{C}$$

and similarly for intersections. We denote the monotone class generated by a collection of subsets $\mathcal{A}$ by $\mathcal{M}(A)$.

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Theorem (Monotone Class Lemma)

If $\mathcal{A}$ is an algebra of subsets of X, then $\mathcal{M}(\mathcal{A}) = \sigma(\mathcal{A})$.

Proof.

Since $\sigma(\mathcal{A})$ is a monotone class, we have that

$$\mathcal{M}(\mathcal{A}) \subset \sigma(\mathcal{A}),$$

so if we can show that $\mathcal{M}(\mathcal{A})$ is a σ-algebra, we are done. For $E \in \mathcal{M}(\mathcal{A})$, define

$$\mathcal{C}(E) = \{F \in \mathcal{M}(\mathcal{A}) : E \setminus F, F \setminus E, E \cap F \in \mathcal{M}(\mathcal{A})\}.$$

Clearly, $\varnothing, E \in \mathcal{C}(E)$ and $E \in \mathcal{C}(F)$ iff $F \in \mathcal{C}(E)$. $\mathcal{C}(E)$ is a monotone class. If $E \in \mathcal{A}$, then $F \in \mathcal{C}(E)$ for all $F \in \mathcal{A}$ because it is an algebra. Hence, $\mathcal{M}(\mathcal{A}) \subset \mathcal{C}(E)$. This means that $E \in \mathcal{C}(F)$ for all $E \in \mathcal{A}$, so $\mathcal{M}(\mathcal{A}) \subset \mathcal{C}(F)$. Hence, $\mathcal{M}(\mathcal{A})$ is an algebra. But if $\{E_j\} \subset \mathcal{M}(\mathcal{A})$, we have that $\bigcup_1^n E_j \in \mathcal{M}(\mathcal{A})$ for every n, and since it is a monotone class, $\bigcup_1^\infty E_j \in \mathcal{M}(\mathcal{A})$. $\square$

Definitions

Definition

Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are measure spaces, and f a function on $X \times Y$. If $E \subset X \times Y$, then for fixed $x \in X, y \in Y$:

- The x-**section** E_x of E

$$E_x = \{y \in Y : (x, y) \in E\}.$$

- The y-**section** E^y of E is

$$E^y = \{x \in X : (x, y) \in E\}.$$

- The x **and** y **sections** f_x and f^y of f are

$$f_x(y) = f^y(x) = f(x, y).$$

Theorem

The x and y sections are measurable sets in their respective σ-algebra, and if f is $\mathcal{M} \otimes \mathcal{N}$-measurable, then f_x is $\mathcal{N}$-measurable and f^y is $\mathcal{M}$-measurable for all $x \in X$ and $y \in Y$ respectively.

Preliminary Results

Theorem

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Theorem (2.36)

Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces. If $E \in \mathcal{M} \otimes \mathcal{N}$, then $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ are measurable X and Y respectively, and

$$(\mu \times \nu)(E) = \int \nu(E_x) \, \mathrm{d}\mu(x) = \int \mu(E^y) \, \mathrm{d}\nu(y) \, .$$

Preliminary Result Theorem (2.36) (Proof)

Proof.

We shall not present a full proof as it is quite long.

- Suppose the measures are finite. Let $\mathcal{C}$ be the set of all $E \in \mathcal{M} \otimes \mathcal{N}$ such that the conclusions of the thm are true. It can be proved that this collection contains the algebra of finite disjoint unions of rectangles.
- Show that $\mathcal{C}$ is a monotone class, which shows that $\mathcal{C}$ is the σ-algebra generated by the algebra of finite disjoint unions of rectangles by the **MCL**, using the **MCT**, showing that the result is true for finite measure spaces.
- Rewrite $X \times Y$ as a disjoint union of $X_j \times Y_j$ with finite measure. Use the **MCT** to conclude that this works for σ-finite measure spaces.

$\square$

The Fubini-Tonelli Theorem

Theorem (Fubini-Tonelli)

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces.

1. **(Tonelli)** *If $f \in L^+(X \times Y)$, then $g(x) = \int f_x \, \mathrm{d}\nu$ and $h(y) = \int f^y \, \mathrm{d}\mu$ are in $L^+(X)$ and $L^+(Y)$ respectively, and*

$$\begin{aligned}\int f \, \mathrm{d}(\mu \times \nu) &= \int \left[\int f(x,y) \, \mathrm{d}\nu(y) \right] \mathrm{d}\mu(x) \\ &= \int \left[\int f(x,y) \, \mathrm{d}\mu(x) \right] \mathrm{d}\nu(y) \, .\end{aligned}$$

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2. **(Fubini)** *If $f \in L^1(\mu \times \nu)$, then $f_x \in L^1(\nu)$ for a.e. $x \in X$, $f^y \in L^1(\mu)$ for a.e. $y \in Y$, the a.e.-defined functions $g(x) = \int f_x \, \mathrm{d}\nu$ and $h(x) = \int f^y \, \mathrm{d}\mu$ are in $L^1(\mu)$ and $L^1(\nu)$ respectively, and the formulas in 1. are valid.*

The Fubini-Tonelli Theorem (Proof)

Proof.

Tonelli's theorem is **Theorem 2.36** when f is a characteristic function, and since simple functions are linear combinations of char functions, the result holds for nonnegative simple functions by linearity. If $f \in L^+(X \times Y)$, let $\{f_n\}$ be a sequence of simple functions that increase pointwise to f (recall **Theorem 2.10**). The **MCT** implies that the corresponding g_n and h_n increase to g and h and

$$\int g \,\mathrm{d}\mu = \lim \int g_n \,\mathrm{d}\mu = \lim \int f_n \,\mathrm{d}(\mu \times \nu) = \int f \,\mathrm{d}(\mu \times \nu)\,,$$

$$\int h \,\mathrm{d}\nu = \lim \int h_n \,\mathrm{d}\nu = \lim \int f_n \,\mathrm{d}(\mu \times \nu) = \int f \,\mathrm{d}(\mu \times \nu)\,.$$

Hence, $f \in L^+(X \times Y)$ and $\int f \,\mathrm{d}(\mu \times \nu) < \infty$, then $g, h < \infty$ a.e. meaning that $f_x \in L^1(\nu)$ a.e. and $f^y \in L^1(\mu)$ a.e. For Fubini, just take the positive/negative/real/imaginary parts of f. $\square$

Fubini-Tonelli for Complete Measures

Even if μ and ν are complete measures, then $\mu \times \nu$ is usually never complete. Hence, we must take extra care when dealing with the completion of product spaces.

Theorem

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be complete, σ-finite measure spaces, and let $(X \times Y, \mathcal{L}, \lambda)$ be the completion of $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$. If f is $\mathcal{L}$-measurable and either

- *(a)* $f \geq 0$
- *(b)* $f \in L^1(\lambda)$

then f_x and f^y are measurable in their respective σ-algebras for a.e. x and y (if (a) holds), and if (b) holds, then f_x and f^y are also integrable for a.e. x and y. Moreover, $x \mapsto \int f_x \,\mathrm{d}\nu$ and $y \mapsto \int f^y \,\mathrm{d}\mu$ are measurable, and in case (b) also integrable, and

$$\int f \,\mathrm{d}\lambda = \iint f(x, y) \,\mathrm{d}\mu(x) \,\mathrm{d}\nu(y) = \iint f(x, y) \,\mathrm{d}\nu(y) \,\mathrm{d}(\mu(x)\,.$$

Lebesgue Measure on $\mathbb{R}^n$

As described so far, we know how to construct measures on the product of measure spaces. Hence, we are ready to construct the Lebesgue measure on $\mathbb{R}^n$.

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Definition

The **Lebesgue measure on** $\mathbb{R}^n$ is denoted λ^n and is defined as the completion of the product of the one-dimensional Lebesgue measure λ with itself n times on $(\mathbb{R}^n, \mathcal{L}^n)$ where

$$\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}, \ \mathcal{L}^n = \mathcal{L} \otimes \cdots \otimes \mathcal{L}.$$

Properties of the n-dimensional Lebesgue Measure

Theorem (Approximations)

λ^n has the following properties (note the similarities to the 1-dim case):

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  where A_1 is F_σ and A_2 is G_δ and N_1, N_2 are null sets.
- *If $\lambda^n(E) < \infty$, for any $\varepsilon > 0$, there is a finite collection $\{R_j\}_1^N$ of disjoint rectangles whose sides are intervals such that*

$$\lambda^n\left(E \triangle \bigcup_1^N R_j\right) < \varepsilon.$$

Properties of the n-dimensional Lebesgue Measure (cont.)

Theorem (Invariance)

λ^n has the following properties (note the similarities to the 1-dim case):

1. **Translation invariance***: for $a \in \mathbb{R}^n$ define the addition map $\tau_a : \mathbb{R}^n \to \mathbb{R}^n$ by $\tau_a(x) = x + a$. Then*

$$E \in \mathcal{L}^n \implies \tau_a(E) \in \mathcal{L}^n \text{ and } \lambda^n(\tau_a(E)) = \lambda^n(E).$$

Properties of the n-dimensional Lebesgue Measure (cont.)

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2. **Scaling under linear maps**: *if $E \in \mathcal{L}^n$, then $T(E) \in \mathcal{L}^n$ and*

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Properties of the n-dimensional Lebesgue Measure (cont.)

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3. **Rotation invariance**: *if $E \in \mathcal{L}^n$, then rotations are linear maps T such that $TT^T = I$. Let T be one such map. Then*

$$\lambda^n(T(E)) = \lambda^n(E).$$

Change of Variables

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Theorem (CoV for Linear Maps)

Suppose $T \in GL(n, \mathbb{R})$ ($T : \mathbb{R}^n \to \mathbb{R}^n$). If $f : \mathbb{R}^n \to \mathbb{R}$ is a Lebesgue measurable function on $\mathbb{R}^n$, so is $f \circ T : \mathbb{R}^n \to \mathbb{R}$. If $f \geq 0$ or $f \in L^1(\lambda^n)$, then

$$\int f(x)\,\mathrm{d}x = |\det T| \int (f \circ T)(x)\,\mathrm{d}x\,.$$

It can be proved using the Fubini-Tonelli theorem.

Change of Variables (cont.)

The previous theorem applies only to linear maps, a very restrictive condition. We want to generalize this result, and it turns out C^1 diffeomorphisms are functions "nice" enough that our formula holds similarly.

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Definition

Let $G = (g_1, \ldots, g_n)$ be a map from an open set $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^n$, whose components g_j are C^1 (continuous first order partials). Let $D_x(G)$ be the linear map defined by the Jacobian matrix of G at x. G is a C^1 **diffeomorphism** if G is **injective** and $D_x G$ **is invertible** for all $x \in \Omega$.

Change of Variables (cont.)

We are now finally ready to present the change of variables theorem.

Theorem (Change of Variables)

Let $G : \Omega \to \mathbb{R}^n$ be a C^1 diffeomorphism. If f is Lebesgue measurable on $G(\Omega)$, then $f \circ G$ is Lebesgue measurable on Ω. If $f \geq 0$ or $f \in L^1(G(\Omega), \lambda^n)$, then

$$\int_{G(\Omega)} f(x)\,\mathrm{d}x = \int_{\Omega} (f \circ G)(x) |\det D_x G|\,\mathrm{d}x\,.$$

Furthermore, if $\Omega \supset E \in \mathcal{L}^n$, then $G(E) \in \mathcal{L}^n$ and

$$\lambda^n(G(E)) = \int_E |\det D_x G|\,\mathrm{d}x\,.$$

The proof is long and difficult, so it shall be omitted.