Measure Theory HW Sections 2.1-2.3 (Folland)

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March 2023

1 Problem 1

1.1 Part (a)

Proof.
Let $f : (X, \mathcal{M}, \mu) \to ([0,\infty], \mathcal{B}_{[0,\infty]})$ be in L^+. Notice that since f is a measurable function,

$$I := f^{-1}(\{\infty\})$$

is a measurable set in $\mathcal{M}$. Suppose that I is not a null set. Let us define a sequence of simple functions $\{\varphi_n\}$ by

$$\varphi_n = n\chi_I.$$

Clearly $\varphi_n \to f\chi_I$ and $\varphi_1 \leq \varphi_2 \leq \cdots \leq f\chi_I$ so by the **MCT**, we have that

$$\int f\chi_I = \lim_{n\to\infty} \int \varphi_n = \lim_{n\to\infty} \int n\chi_I = \lim_{n\to\infty} n\mu(I) = \infty$$

since $\mu(I) > 0$. However, $\int f \geq \int f\chi_I$, which contradicts the given finiteness.

Suppose that the support of f is not σ-finite. This means that for a countable partition $\{E_j\}$ of $\operatorname{supp} f$ has an index m such that

$$\mu(E_m) = \infty.$$

Let $\{\varphi_n\}$ be an increasing sequence of simple functions greater than 0 converging to f. Let the standard representation of φ_n be

$$\varphi_n = \sum_{j=1}^{k_n} a_j^n \mu(A_j^n),$$

where each A_j^n are pairwise disjoint and the ns are indexes, not exponents. By the **MCT**,

$$\int f = \lim_{n\to\infty} \int \varphi_n \geq \lim_{n\to\infty} \int \varphi_n \chi_{E_m} = \lim_{n\to\infty} \sum_j^{k_n} a_j^n \mu(A_j^n \cap E_m),$$

and since A_j^n are disjoint and their union is X, we apply σ-additivity to obtain

$$\lim_{n\to\infty}\sum_{j}^{k_n} a_j^n \mu(A_j^n \cap E_m) \geq \lim_{n\to\infty} a^n \mu(E_j) = \infty$$

where $a^n := \min(a_1^n, \ldots, a_{k_n}^n)$, implying that

$$\int f = \infty,$$

contradicting the finiteness of $\int f$. Hence, $\operatorname{supp} f$ must be σ-finite. □

1.2 Part (b)

Proof.
As done in the proof of the classical **DCT**, it suffices to prove the result for the case where each function is real valued. Again, we have that

$$g_n + f_n \geq 0 \text{ and } g_n - f_n \geq 0$$

almost everywhere. By **Fatou's Lemma** and the fact that $\int g_n \to \int g$, we have that

$$\int g + f \leq \liminf_{n\to\infty} \int g_n + f_n = \liminf_{n\to\infty} \int g_n + \liminf_{n\to\infty} \int f_n = \int g + \liminf_{n\to\infty} f_n$$

and

$$\int g - f \leq \liminf_{n\to\infty} \int g_n - f_n = \int g + \liminf_{n\to\infty} - \int f_n = \int g - \limsup_{n\to\infty} \int f_n,$$

where the last equality was deduced from the fact that

$$\liminf_n (x_n) = - \limsup_n (-x_n).$$

Combining these two facts, we get

$$\limsup_{n\to\infty} \int f_n \leq \int f \leq \liminf_{n\to\infty} \int f_n,$$

which completes the proof. □

1.3 Part (c)

Proof.
(a) We shall evaluate

$$\int_0^\infty |f_n(x)| \, \mathrm{d}x \, .$$

We notice that f_n is negative until a certain point $\xi \in \mathbb{R}^+$ after which it becomes positive. To find this explicitly, we solve

$$ae^{-na\xi} = be^{nb\xi}$$

to find $\xi = \frac{1}{n(b-a)} \ln \frac{b}{a}$. Hence, we split the integral in the following way:

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\xi} be^{-nbx} - ae^{-nax} \,\mathrm{d}x + \lim_{R \to \infty} \int_{\xi}^{R} ae^{-nax} - be^{nbx} \,\mathrm{d}x\,,$$

which evaluates to

$$\frac{2}{n} \left(e^{-\frac{a}{b-a} \ln(b/a)} - e^{-\frac{b}{b-a} \ln(b/a)} \right).$$

Clearly, the whole part on the right is just a constant, so the sum is just a harmonic series, which obviously diverges.

(b) We evaluate the inner integral as follows:

$$\int_0^\infty ae^{-nax} - be^{-nbx} \,\mathrm{d}x = -\frac{1}{n} e^{-nax}\Big|_0^\infty + \frac{1}{n} e^{-nbx}\Big|_0^\infty = 0.$$

Clearly, $\int_0^\infty f_n = 0$ so the series is equal to zero.

(c) We notice that $\sum f_n$ is merely a difference of geometric series, so we may easily evaluate it to obtain

$$\frac{ae^{-ax}}{1 - e^{-ax}} - \frac{be^{-bx}}{1 - e^{-bx}}.$$

We integrate over the finite interval $[0, R]$ where $R \in [0, \infty)$ to see that

$$\int_0^R \sum_{n=1}^\infty f_n(x) = \int_0^R \frac{ae^{-ax}}{1 - e^{-ax}} \,\mathrm{d}x - \frac{be^{-bx}}{1 - e^{-bx}} \,\mathrm{d}x\,.$$

We evaluate the first integral as the other is the exact same with b instead of a. Making the u-substitution $u = 1 - e^{-ax}$, we get the antiderivative

$$\int \frac{ae^{-ax}}{1 - e^{-ax}} \,\mathrm{d}x = \ln\bigl(1 - e^{-ax}\bigr).$$

Hence, our integral becomes

$$\lim_{\varepsilon \to 0} \ln\bigl(1 - e^{-ax}\bigr) - \ln\bigl(1 - e^{-bx}\bigr)\Big|_{\varepsilon}^{R},$$

and all that is left is to evaluate the limit, since the difference for R is always finite and even approaches 0 for larger and larger R. We see that

$$y = \lim_{\varepsilon \to 0} \frac{1 - e^{-a\varepsilon}}{1 - e^{-b\varepsilon}} \implies e^y = \lim_{\varepsilon \to 0} \frac{1 - e^{-a\varepsilon}}{1 - e^{-b\varepsilon}},$$

and by **L'hopital's rule,** we obtain that this is equal to $\ln \frac{a}{b}$. Hence, the integral is always finite and $\sum f_n \in L^1([0,\infty),\lambda)$. Clearly, since if we take larger and larger R the first difference vanishes, we get

$$\int_0^\infty \sum_{n=1}^\infty f_n(x)\,\mathrm{d}x = 0 - \ln\frac{a}{b} = \ln\frac{b}{a}.$$

□

1.4 Part (d)

Proof.
(a) We see that

$$\lim_{n\to\infty}(1+\frac{x}{n})^{-n}\frac{1}{n}\frac{\sin(x/n)}{1/n} = e^{-x}\times 0\times 1 = 0,$$

so our integrand is a sequence of functions that converges p.w. to 0. We see that since $(1+(x/n))^{-n}$ converges to e^{-x}, the integrand is dominated by the function $g(x) = e^{-x}+1$ for all non-negative x, and by the **DCT**, the integral evaluates to 0.

(b) Using **L'hopital's rule,** the integrand evaluates to 0 on the interval of interest. By **Bernoulli's inequality,** we know that for each positive integer n,

$$(1+x^2)^n \geq 1+nx^2,$$

so we have that

$$\left|\frac{1+nx^2}{(1+x^2)^n}\right| \leq \left|\frac{(1+x^2)^n}{(1+x^2)^n}\right| = 1,$$

so our integrand is dominated by the constant function $g(x) = 1$. By the **DCT**, the integral evaluates to 0.

(c) The limit of the integrand is

$$\frac{1}{1+x^2},$$

since $n\sin(x/n) = \sin(x/n)/(x/n)$, which evaluates to 1 by the classic sine limit. Since $\sin(x/n) \leq x/n$ for positive x, we get that

$$\left|\frac{n\sin\frac{x}{n}}{x(1+x^2)}\right| \leq \frac{1}{1+x^2},$$

so we have found a nonnegative function that dominates our integrand and we may interchange the limit and integral. We now obtain

$$\lim_{n\to\infty}\int_0^\infty \frac{n\sin(x/n)}{x(1+x^2)} = \int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x = \frac{\pi}{2}.$$

(d) Notice that we may just evaluate this integral directly using a u-substitution. Substituting $u = nx$, we get

$$\int_a^\infty \frac{n}{1+(nx)^2}\,\mathrm{d}x = \int_{na}^\infty \frac{1}{1+u^2}\,\mathrm{d}x = \frac{\pi}{2} - \arctan(na).$$

Taking the limit as n goes to infinity, we get three cases:

- if $a > 0$ then $I = 0$,
- if $a = 0$ then $I = \frac{\pi}{2}$,
- if $a < 0$ then $I = \pi$.

We see that if a is negative, there is no function that dominates the integrand when $x = 0$, since the n^2 vanishes. □

1.5 Part (e)

Proof.
In the context of **Theorem 2.27**, our function must have a dominated partial derivative wrt t to interchange the integral and derivative. Furthermore, we are interested in $t \in [1, 1+\varepsilon]$ (some $\varepsilon > 0$) for the first derivative. Notice that

$$\left|\frac{\partial f}{\partial t}(x,t)\right| = \left|\frac{x}{e^{tx}}\right| \le \frac{x}{e^x},$$

so we may interchange the derivative and integral. Evaluating both sides, we obtain

$$\int_0^\infty xe^{-tx}\,\mathrm{d}x = \frac{1}{t^2}.$$

We shall now proceed inductively to show our desired result. Suppose that for some positive integer k, $t \in [1, \infty)$ and we have

$$\int_0^\infty x^k e^{-tx}\,\mathrm{d}x = \frac{k!}{t^{k+1}}.$$

Notice that

$$\left|\frac{\partial f}{\partial t}\right| = \left|\frac{x^{k+1}}{e^{tx}}\right| \le \frac{x^{k+1}}{e^x},$$

so we may interchange the derivative and integral to obtain

$$-\int_0^\infty x^{k+1}e^{-tx}\,\mathrm{d}x = \frac{\partial}{\partial t}\int_0^\infty x^k e^{-tx}\,\mathrm{d}x = \frac{\partial}{\partial t}\frac{k!}{t^{k+1}} = -\frac{(k+1)!}{t^{k+2}},$$

so our result holds for all positive integers n by induction. Taking $t = 1$ gives us our result. The second assertion is proved in a similar way, by considering $t \in [1, \infty)$ and dominating the partial with $x^{2(k+1)}/e^{x^2}$. □

2 Problem 2

2.1 Part (a)

Proof.
Suppose that $f_n \to f$ in L^1, i.e. for every $\varepsilon > 0$ there is a positive integer N such that

$$n \geq N \implies \int |f_n - f| \, \mathrm{d}\mu < \varepsilon.$$

Let us define the sets

$$E_n^{-k} := \left\{ x \in X : |f_n(x) - f(x)| > \frac{1}{k} \right\}.$$

Notice that as n increases, $\mu(E_n)$ must approach 0 since

$$\int |f_n - f| \geq \int_{E_n^k} |f_n - f| > \frac{\mu(E_n^{-k})}{k},$$

so setting $\varepsilon = \frac{1}{k^2}$ and letting n_k be the integer such that $n \geq n_k$ implies

$$\int |f_n - f| \, \mathrm{d}\mu < \frac{1}{k^2},$$

so we have

$$\frac{1}{k^2} > \frac{\mu(E_n^{-k})}{k} \implies \mu(E_n^{-k}) < \frac{1}{k}.$$

Since we want our (sub)sequence to converge to f a.e., our discussion above motivates us to consider the subsequence $\{f_{n_k}\}$. Indeed, for each integer k, we have that

$$|f_{n_k}(x) - f(x)| < \frac{1}{k}$$

for any $x \in (E_n^{-k})^C$. Furthermore, as $k \to \infty$, the set E_n^{-k} becomes measure 0, which gives us our desired p.w. convergence almost everywhere. □

2.2 Part (b)

Proof.
Suppose that $\{f_n\}$ is Cauchy in L^1, i.e. for each $\varepsilon > 0$, there is some N such that

$$n, m \geq N \implies \int |f_n - f_m| < \varepsilon.$$

We define the sets E_n^{-k} in a manner similar to part (a):

$$E_{n,m}^{-k} := \left\{ x \in X : |f_n(x) - f_m(x)| > \frac{1}{2^k} \right\}.$$

In a manner similar to above, we see that for each k, there is an index n_k such that $n, m \geq n_k$ implies

$$\mu(E_{n,m}^{-k}) < \frac{1}{2^k}$$

by considering ε of the form 2^{2^k}. Again, $\mu(E_{n,m}^{-k}) \to 0$ as $k \to \infty$. Hence, it is suffice to now prove that $\{f_{n_k}\}$ is Cauchy in the normal pointwise sense. Let $E = \bigcup_k E_{n,m}^{-k}$ for every combination of n, m. On E^C, we have that there exists some N such that for $m \geq j \geq N$, we have

$$|f_{n_m} - f_{n_j}| \leq \sum_{k=j}^{m-1} |f_{n_{k+1}} - f_{n_k}| \leq \sum_{k=j}^{m-1} \frac{1}{2^k} \leq 2^{1-j},$$

which shows that our subsequence is Cauchy and hence convergent. □

2.3 Part (c)

Proof.
Consider the setting of part (b). We shall rewrite f_{n_k} as follows:

$$f_{n_k} = f_{n_1} + \sum_{j=1}^{k-1} f_{n_{j+1}} - f_{n_j}.$$

We want to show that f is integrable. Taking the absolute value,

$$|f_{n_k}| \leq |f_{n_1}| + \sum_{j=1}^{k-1} |f_{n_{j+1}} - f_{n_j}|$$

by the triangle inequality. By the **MCT**,

$$\int |f| = \lim_{k\to\infty} \int |f_{n_k}| \leq \lim_{k\to\infty} \left(\int |f_{n_1}| + \int \sum_{j=1}^{k-1} |f_{n_{j+1}} - f_{n_j}| \right)$$

which is equivalent to

$$\int |f_{n_1}| + \sum_{j=1}^{\infty} \int |f_{n_{j+1}} - f_{n_j}|,$$

and since there is an N such that

$$j \geq N \implies \int |f_{n_{j+1}} - f_{n_j}| < \frac{1}{2^j},$$

we split the sum and obtain

$$\int |f| < \int |f_{n_1}| + \sum_{j=1}^{N} \int |f_{n_{j+1}} - f_{n_j}| + \sum_{j=N}^{\infty} \frac{1}{2^j} < \infty.$$

Hence, $f \in L^1(X)$.

□

2.4 Part (d)

Proof.
Suppose that $\{f_n\}$ is a Cauchy sequence in $L^1(X)$. Consider the subsequence $\{f_{n_k}\}$ that converges to f found in part (b) and part (c). We shall show that $f_n \to f$. Let $\varepsilon > 0$ be given. We know that there exists an N_0 such that

$$n_k \geq N_0 \implies \int |f_{n_k} - f| < \frac{\varepsilon}{2}$$

and that there is also another integer N_1 such that

$$n, n_k \geq N_1 \implies \int |f_n - f_{n_k}| < \frac{\varepsilon}{2}.$$

Taking $N := \max(N_0, N_1)$, we get that for $n, k \geq N$,

$$\int |f_n - f| \leq \int |f_n - f_{n_k}| + \int |f_{n_k} - f| < \varepsilon,$$

which shows that $L^1(X)$ is complete.

□

3 Problem 3

Proof.
Suppose that $f \in C_c(X)$. Then

$$\operatorname{supp} f = \{x \in X : f(x) \neq 0\}$$

is compact and f is continuous. We must prove that f is integrable. Notice that $\operatorname{supp} f$ has finite measure by assumption, and that the image of the absolute value of f, namely $|f|(\operatorname{supp} f)$, is also compact since continuous maps preserve compactness. Clearly, $|f|$ is bounded by the **Heine-Borel theorem**, and we let $M \in \mathbb{R}$ be the constant such that $|f| \leq M$ for all $x \in X$. Then,

$$\int |f| = \int_{\operatorname{supp} f} |f| \leq M\mu(\operatorname{supp} f) < \infty,$$

so $f \in L^1(X)$. Let $\varepsilon > 0$ be given. Since every integrable function can be approximated arbitrarily well by simple functions, and every simple function can be written as a linear combination of characteristic functions, it suffices to show that there is an $f \in C_c(X)$ such that

$$\int |f - \chi_E| < \varepsilon.$$

for every measurable E. Since μ is regular, we can find compact K and open U such that $K \subseteq E \subseteq U$ and $\mu(U \setminus K) < \varepsilon$. We now apply **Urysohn's Lemma** for locally compact Hausdorff spaces (see lemma 2.12 in Rudin's *Real and Complex Analysis*), to deduce the existence of a function f^* that:

- is continuous,
- $f^* = 0$ on U^C,
- $f^* = 1$ on K,
- f^* is compactly supported,
- $0 \leq f^* \leq 1$.

With this function f^*, we see that

$$\int |f^* - \chi_E| = \int_{U \setminus K} |f^* - \chi_E| \leq \mu(U \setminus K) < \varepsilon,$$

since $|f^* - \chi_E|$ can be at most 1 and the integrand vanishes on K, and its support is contained in U. Clearly, $f^* \in C_c(X)$, so $C_c(X)$ is dense in $L^1(X)$. $\square$