Path Detection:
A Quantum Computing Primitive

Shelby Kimmel

Middlebury College

Based on work with
Stacey Jeffery: arXiv: 1704.00765 (Quantum vol 1 p 26)
Michael Jarret, Stacey Jeffery, Alvaro Piedrafita, arXiv:1804.10591 (ESA 2018)
Kai DeLorenzo, Teal Witter, arXiv:1904.05995 (TQC 2019)

Middlebury

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- New primitive: ***st*-connectivity**

Outline:

A. Introduction to st-connectivity
B. st-connectivity makes a good algorithmic primitive
   1. Widely applicable
   2. Easy to analyze (without knowing quantum mechanics)
C. Examples

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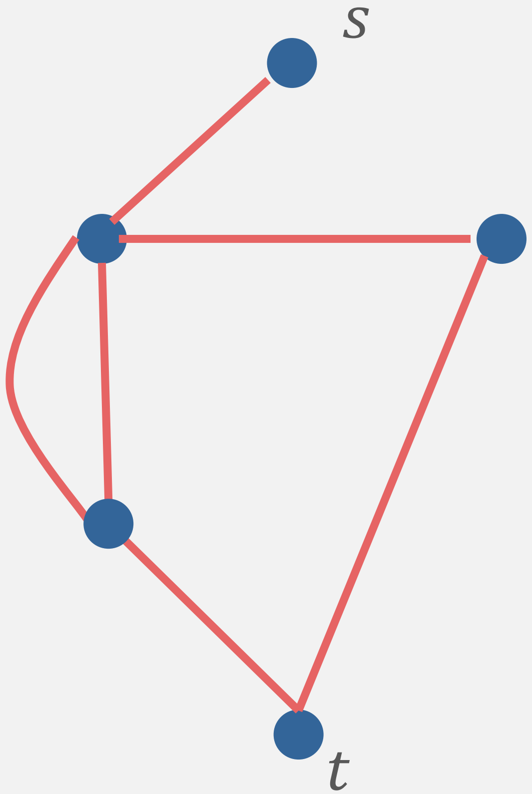
Applications:

- Read-once Boolean formulas (query optimal)
- Total connectivity (query optimal)
- Cycle detection (query optimal)
- Bipartiteness (query optimal)

st-connectivity

$\boldsymbol{st - connectivity}$:
is there a path from s to t?



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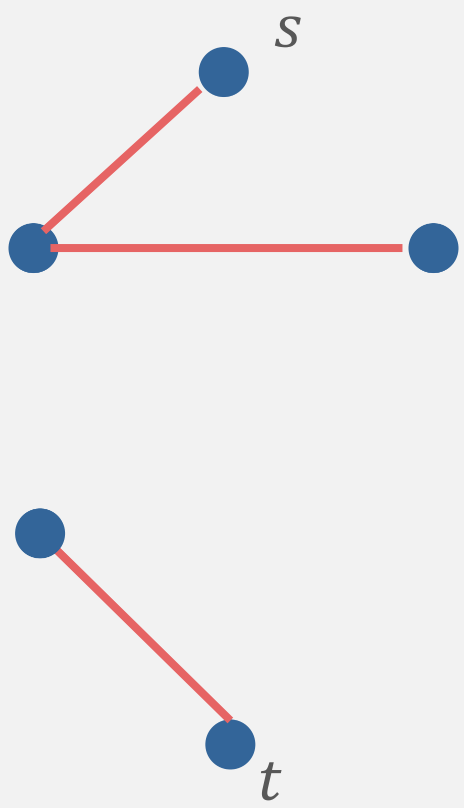


Input to Algorithm

Bit String:

$\boxed{x_1}\boxed{x_2} \ldots \boxed{x_n}$

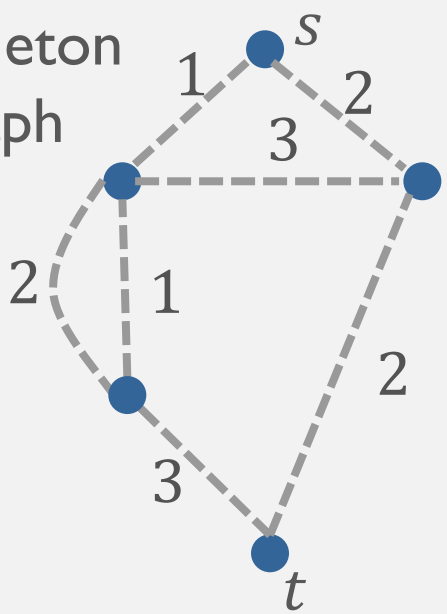
Input to Algorithm

Bit String:

x_1	x_2	…	x_n
1	0		1

Input to Algorithm

Skeleton Graph



Bit String:

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Input to Algorithm

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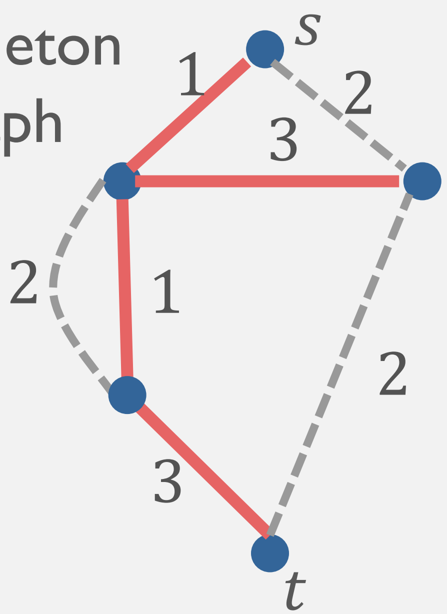

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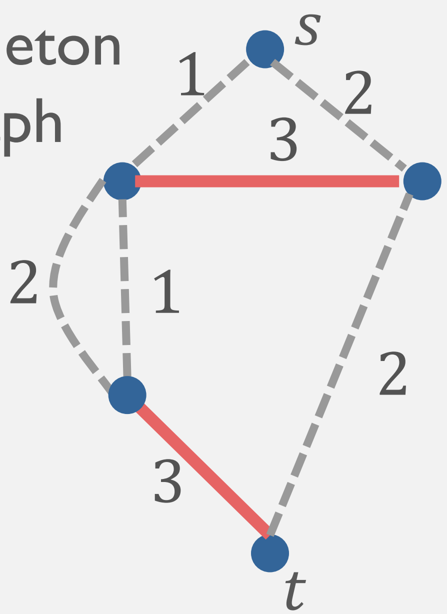
Input to Algorithm

Skeleton Graph



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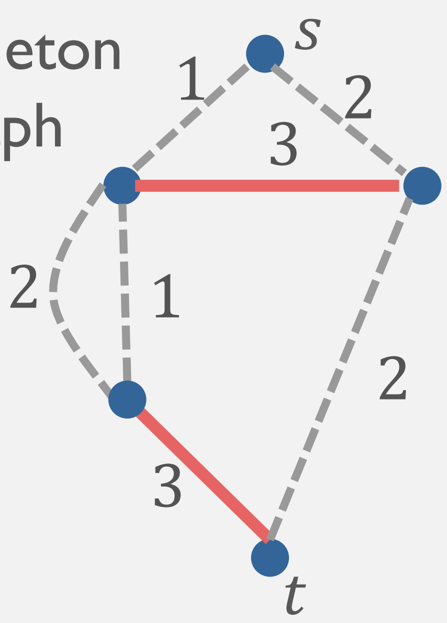
Skeleton Graph



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Catch:

- Bit string initially hidden
- Goal: solve while revealing as few bits as possible → minimize **Query Complexity**

Input to Algorithm

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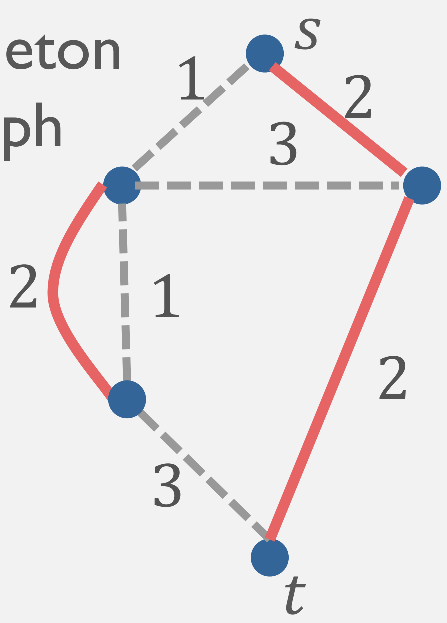

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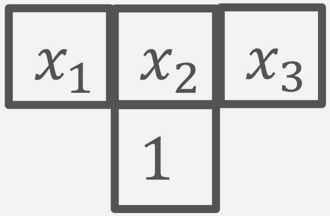

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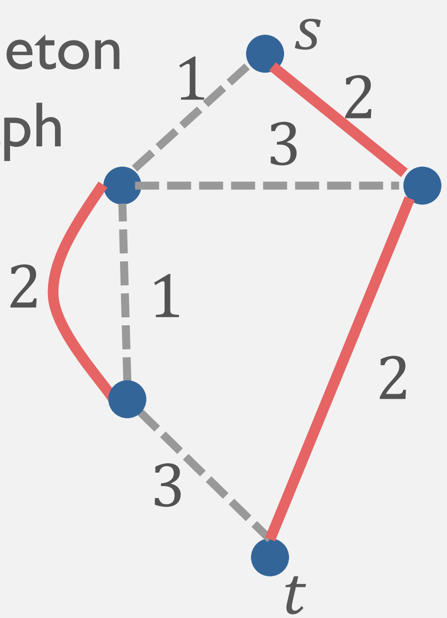

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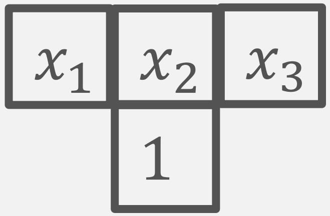

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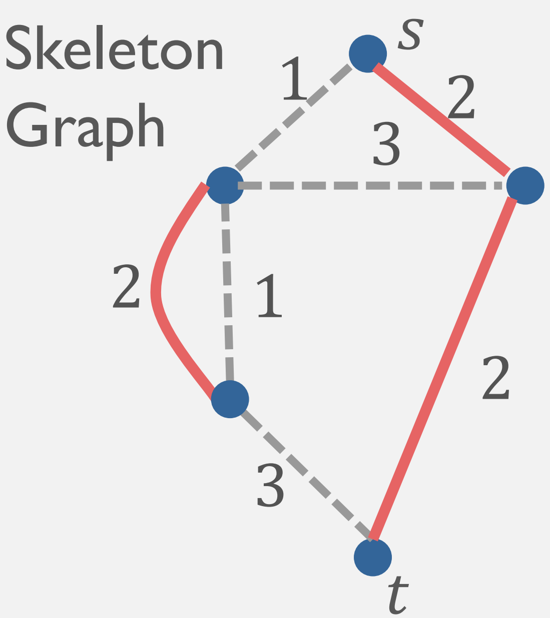
$\approx ?$

Time Complexity

Input to Algorithm



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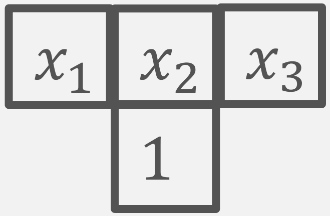

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Skeleton
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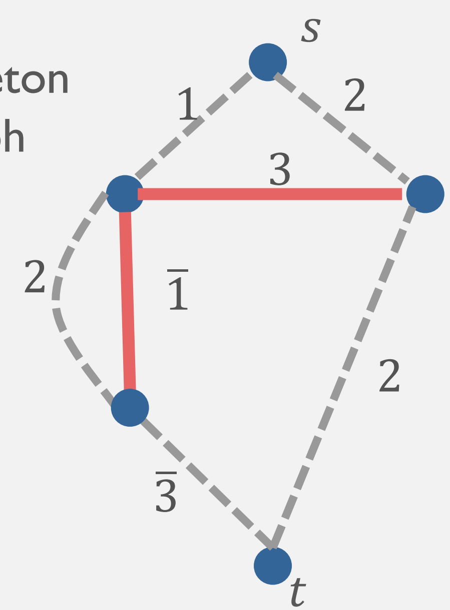

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Outline:

A. Introduction to st-connectivity
B. st-connectivity makes a good algorithmic primitive
    1. Widely applicable
        - Boolean Formulas
        - Cycle Detection
    2. Easy to analyze (without knowing quantum mechanics)
C. Example

Boolean Formulas

$\wedge$ AND: outputs 1 if all input subformulas have value 1

$\vee$ OR: outputs 1 if any input subformulas have value 1

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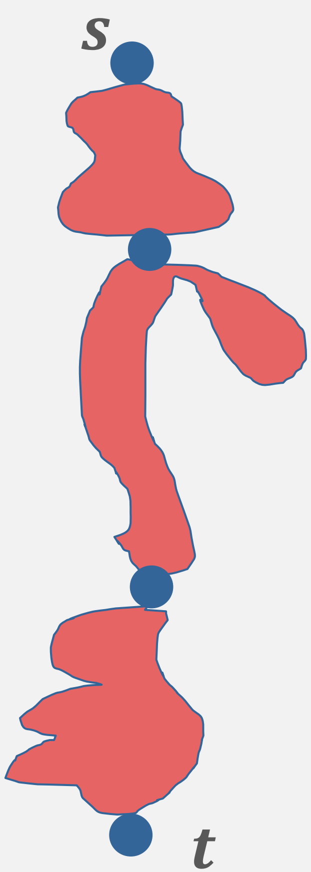

s and t connected if all subgraphs connected

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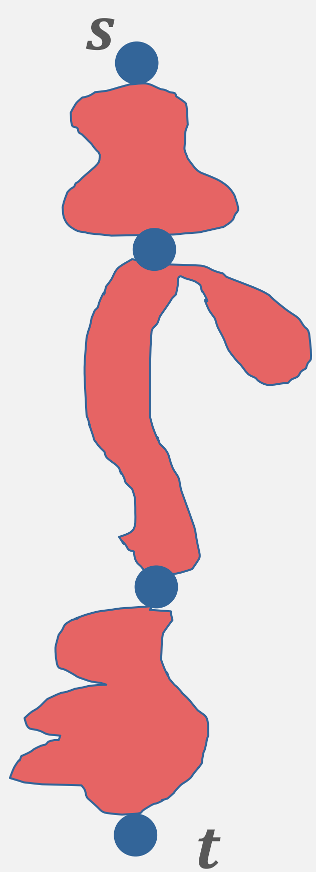

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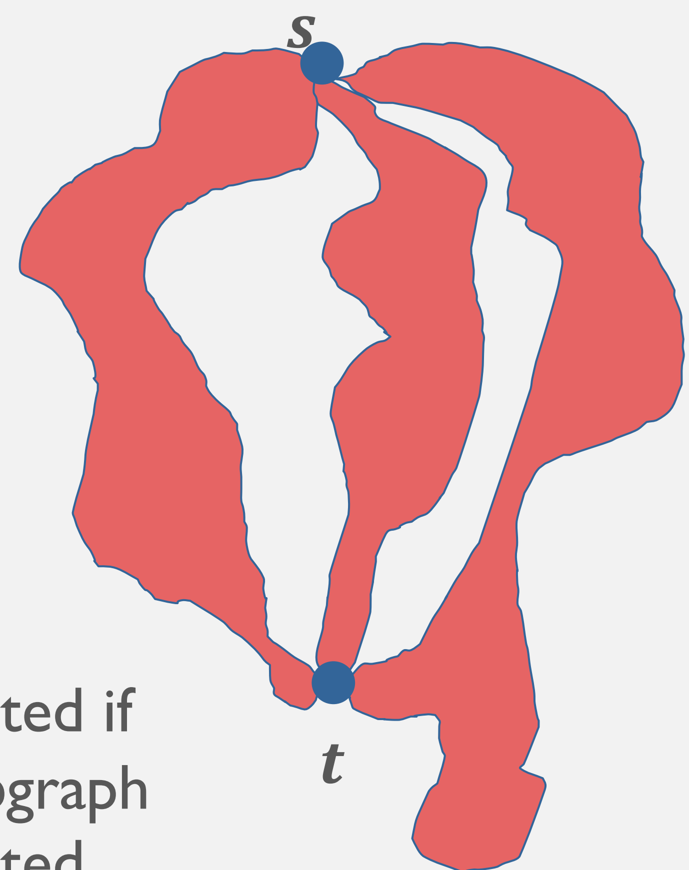

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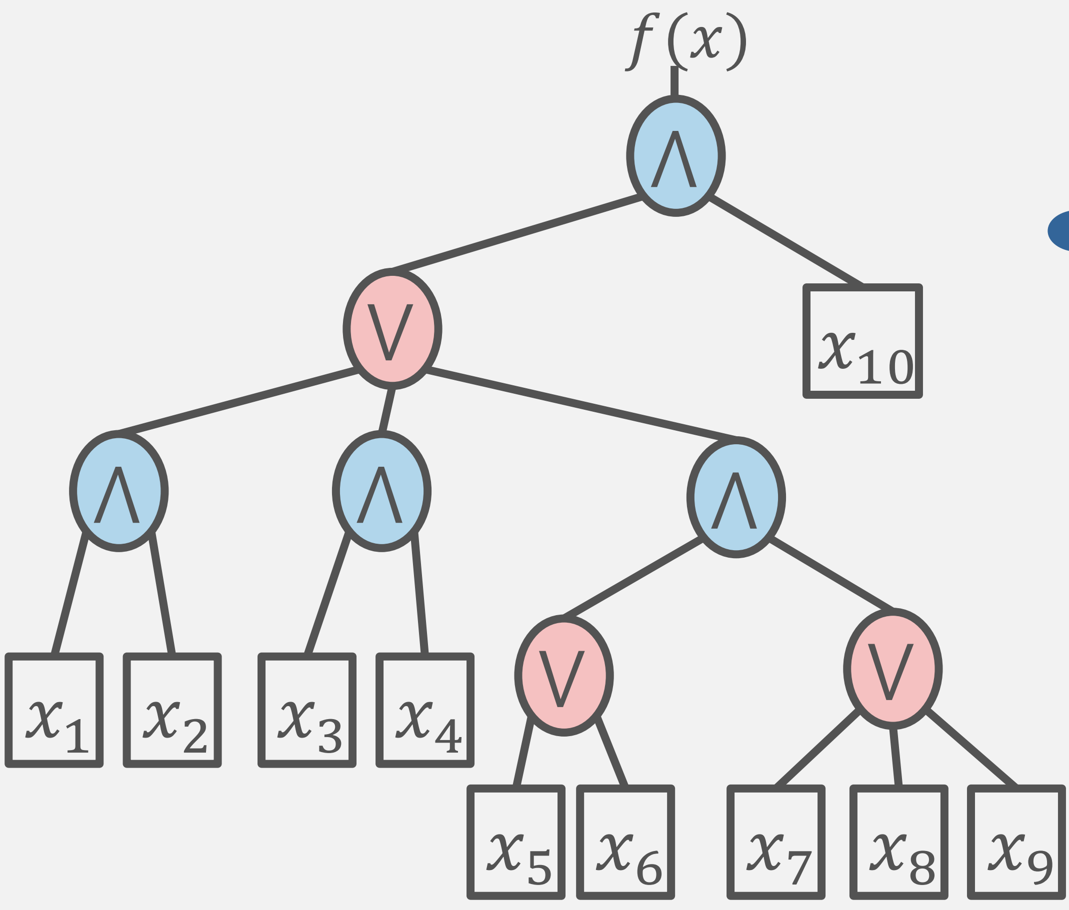
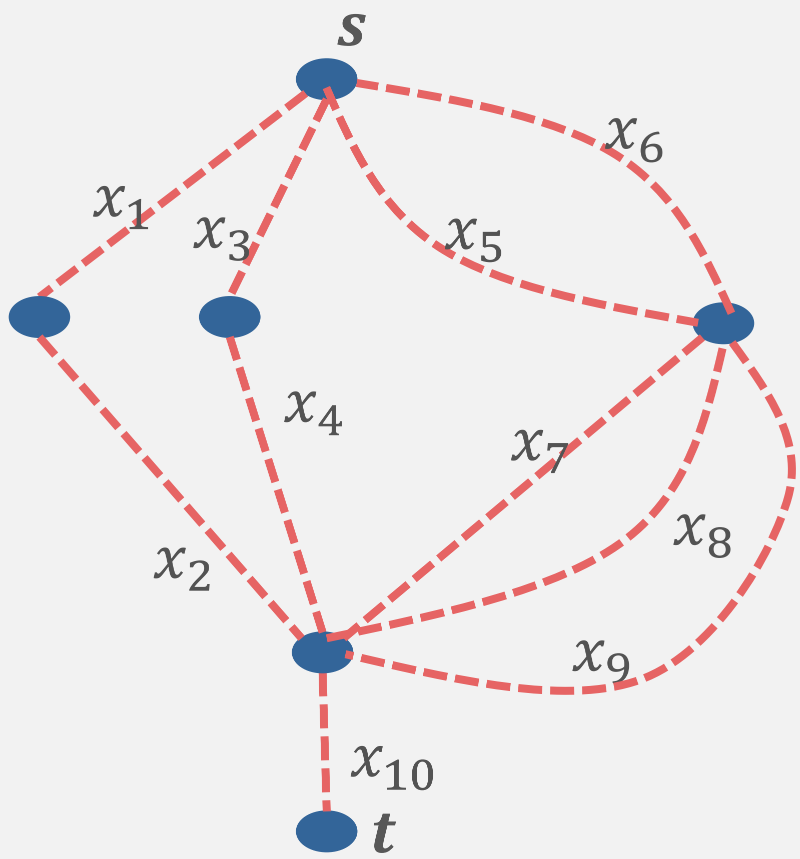
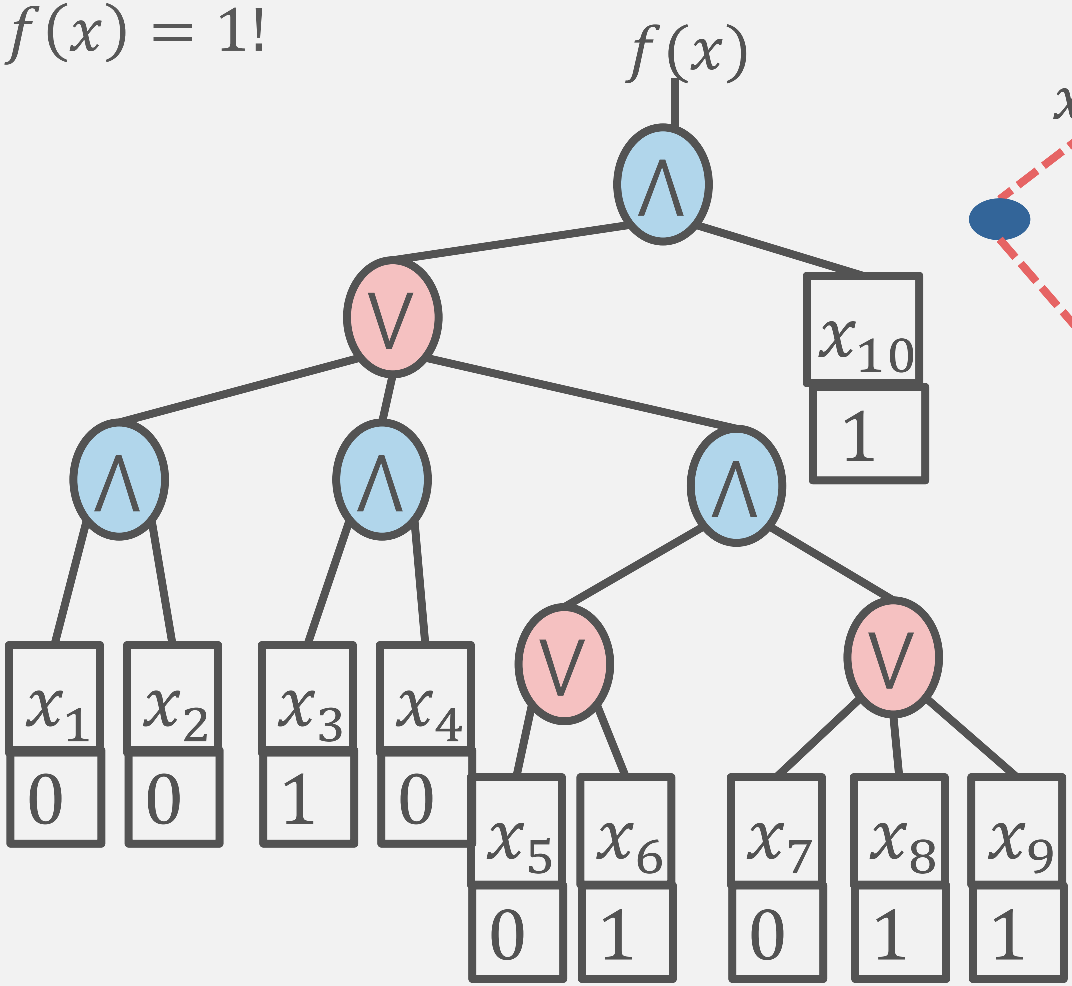
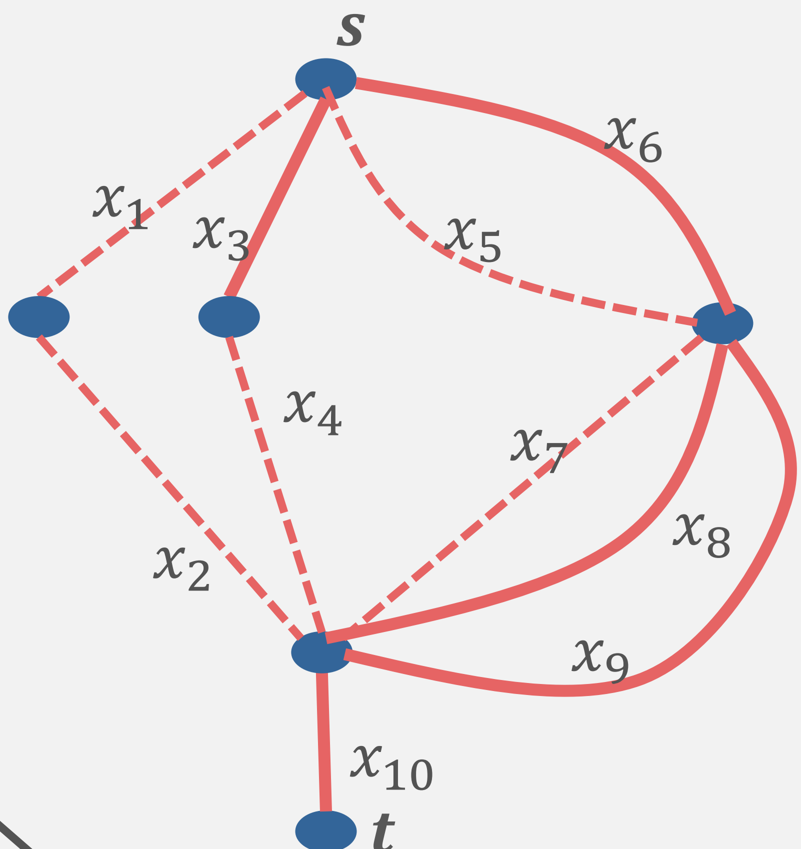




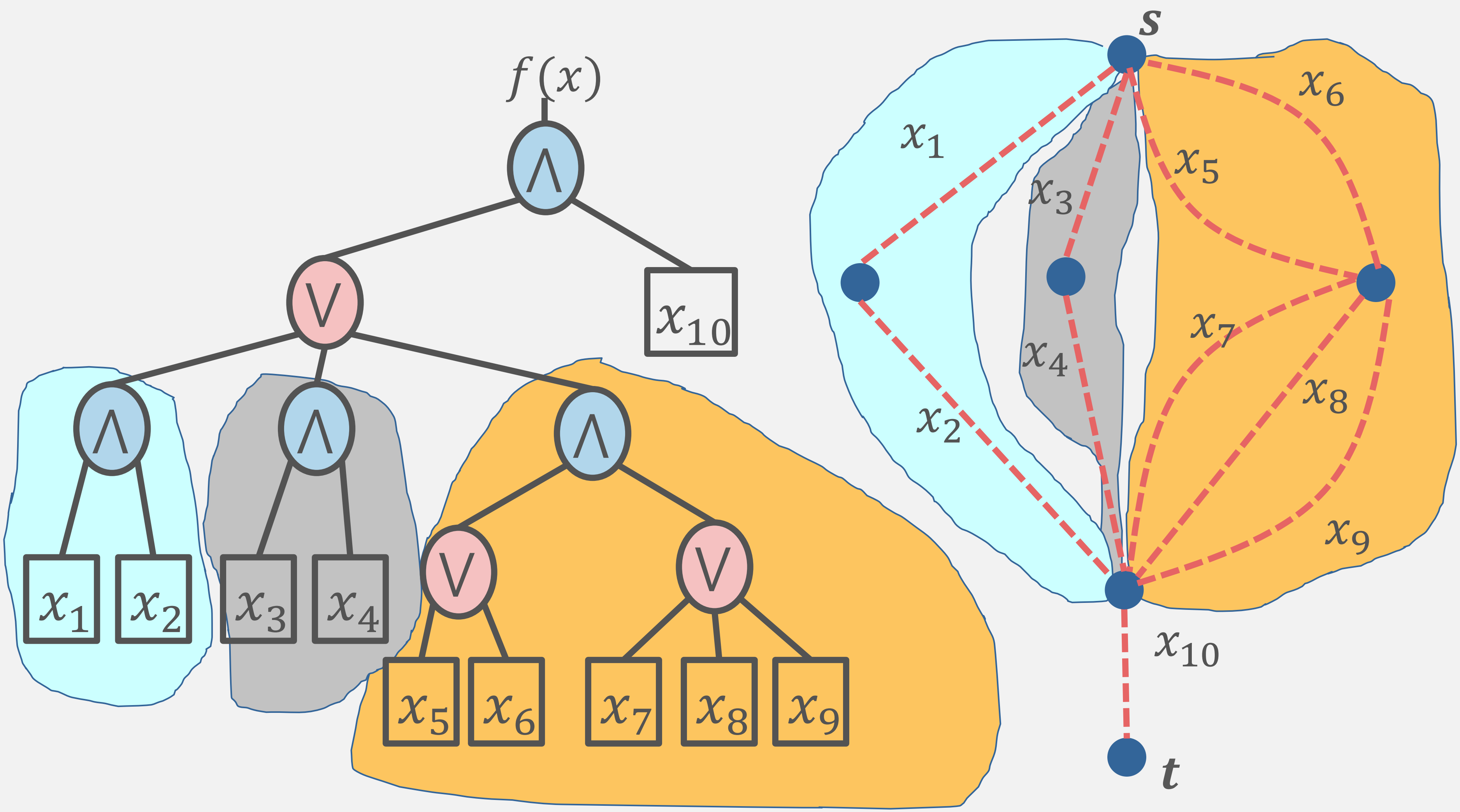
Boolean Formulas

s and t are connected iff $f(x) = 1$!





Boolean Formulas

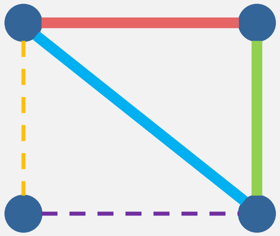


Boolean Formula Applications

- Logic
- Designing electrical circuits
- Game theory (deciding who will win a game)
- Combinatorics and graph problems
- Linear programming
- Testing potential solution to an NP-complete problem

Cycle Detection

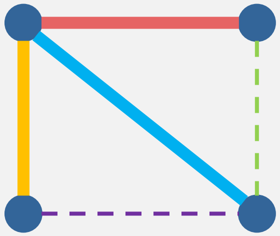
Is there a cycle?



Yes

Cycle Detection

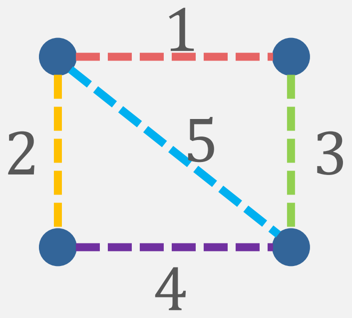
Is there a cycle?



No

Cycle Detection

Is there a cycle through edge 1?



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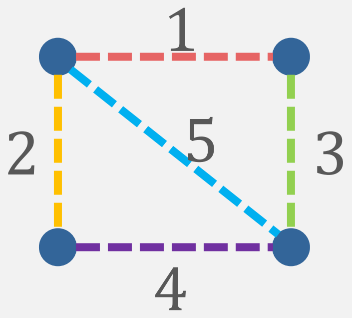

There is a cycle through Edge 1 iff

- Edge 1 is present
- Path between the endpoints of Edge 1 not using Edge 1

Cycle Detection

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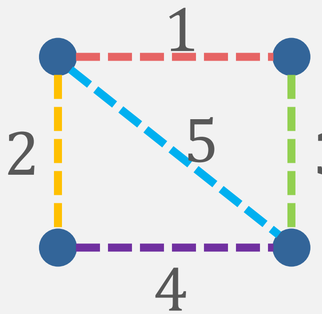

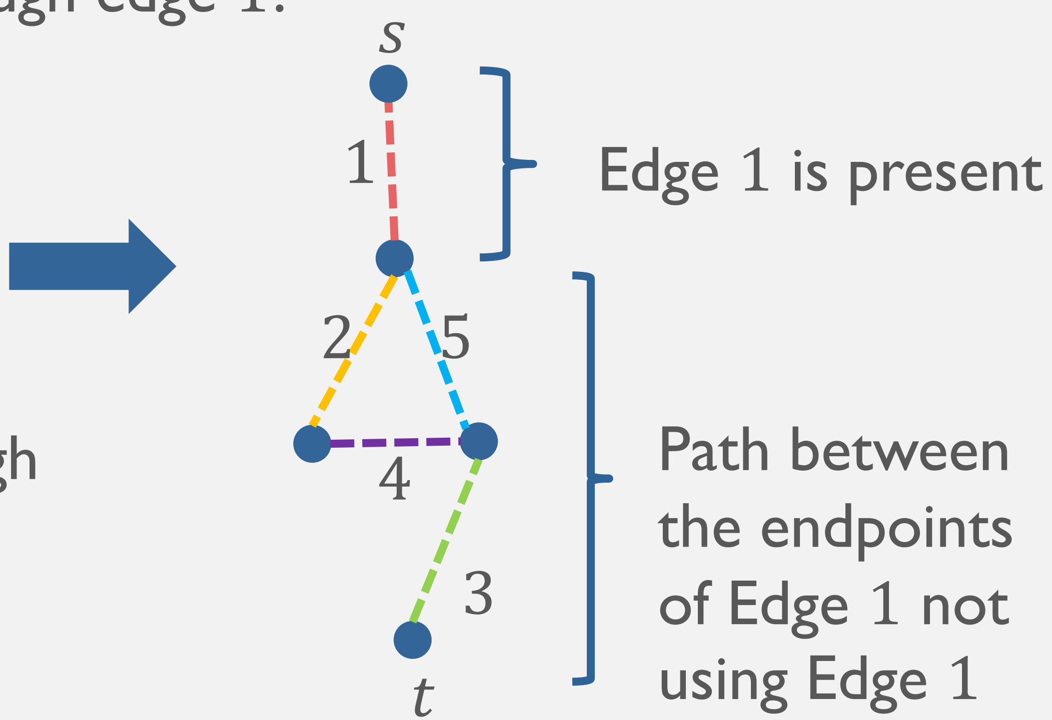

There is a cycle through Edge 1 iff

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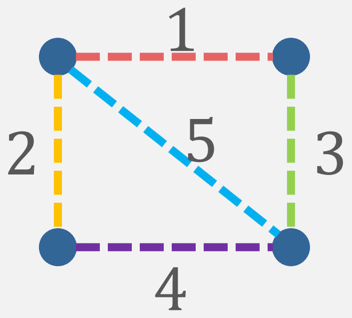

There is a cycle if there is a cycle through some edge

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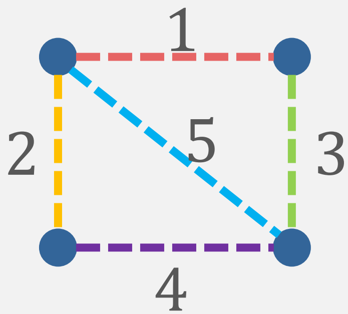

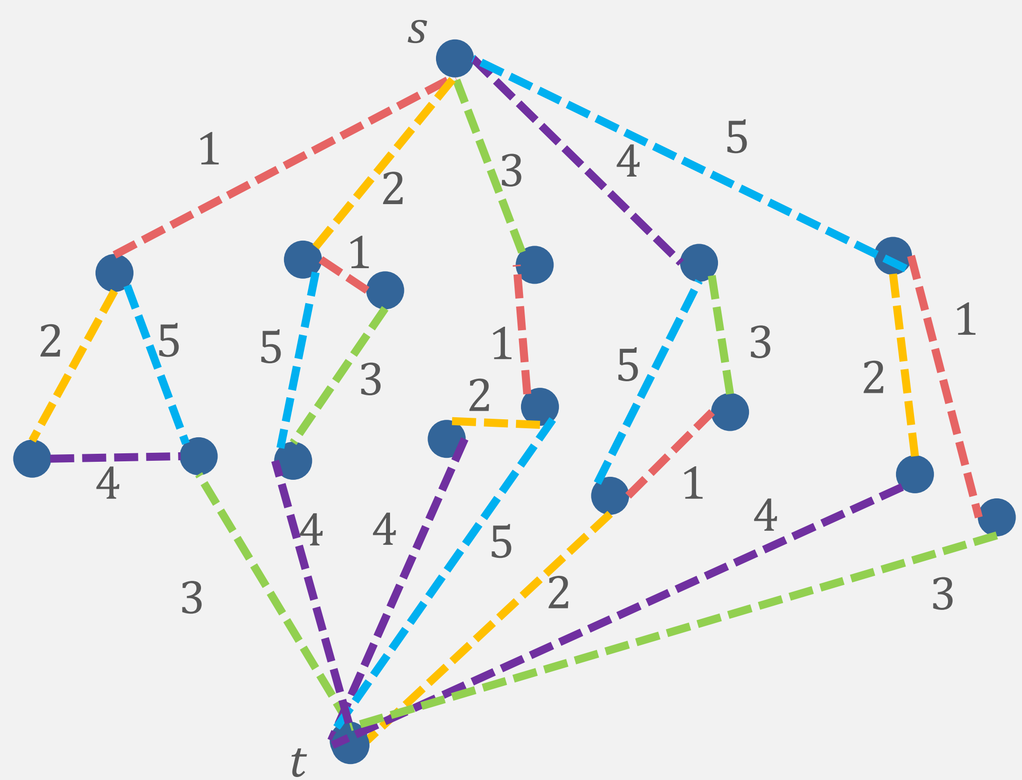

There is a cycle if there is a cycle through some edge

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C. Extra example

Algorithm Complexity:

Space Complexity: $O(\log(\#\ edges\ in\ skeleton\ graph))$

Algorithm Complexity:

Query Complexity:

$$O\left(\sqrt{\max_{connected\ G} R_{s,t}(G)}\ \sqrt{\max_{not\ connected\ G} C_{s,t}(G)}\right)$$

Effective resistance   Effective capacitance

Algorithm Complexity:

Query Complexity:

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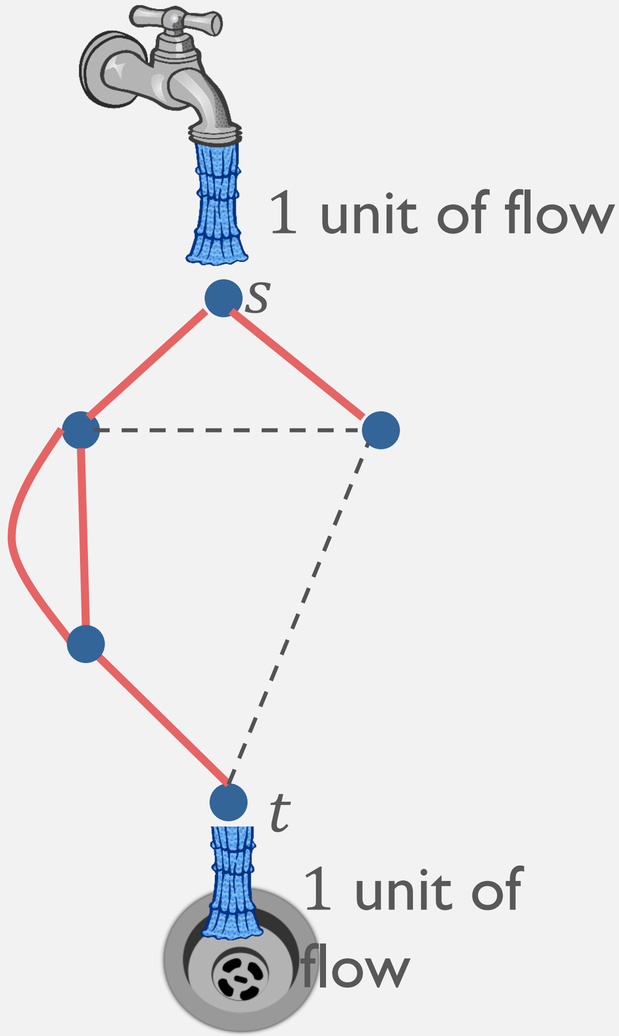
Effective resistance

[Belovs, Reichardt, '12]

Effective capacitance

[Jarret, Jeffery, Kimmel, Piedrafita, '18]

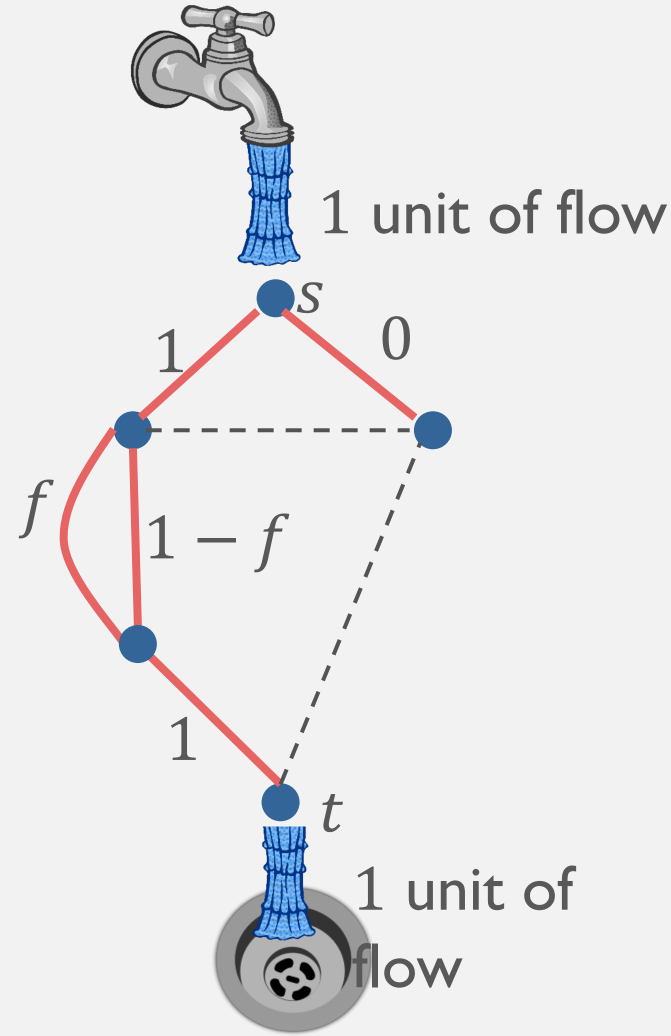
Effective Resistance



Effective Resistance

Valid flow:

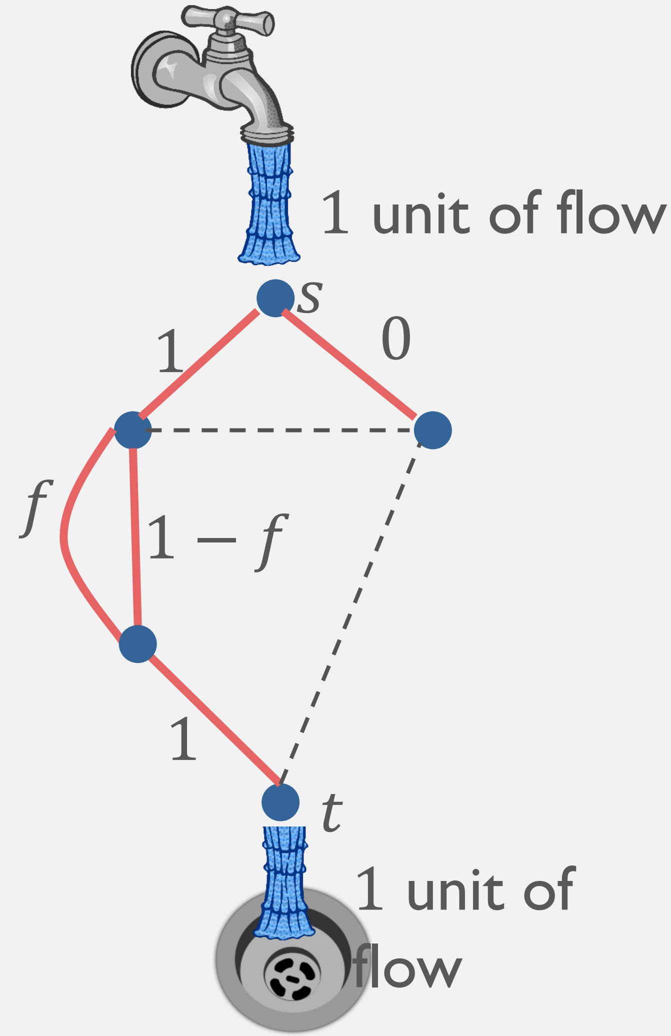
- 1 unit in at s
- 1 unit out at t
- At all other nodes, zero net flow



Effective Resistance

Flow energy:

$$\sum_{edges} (flow\ on\ edge)^2$$



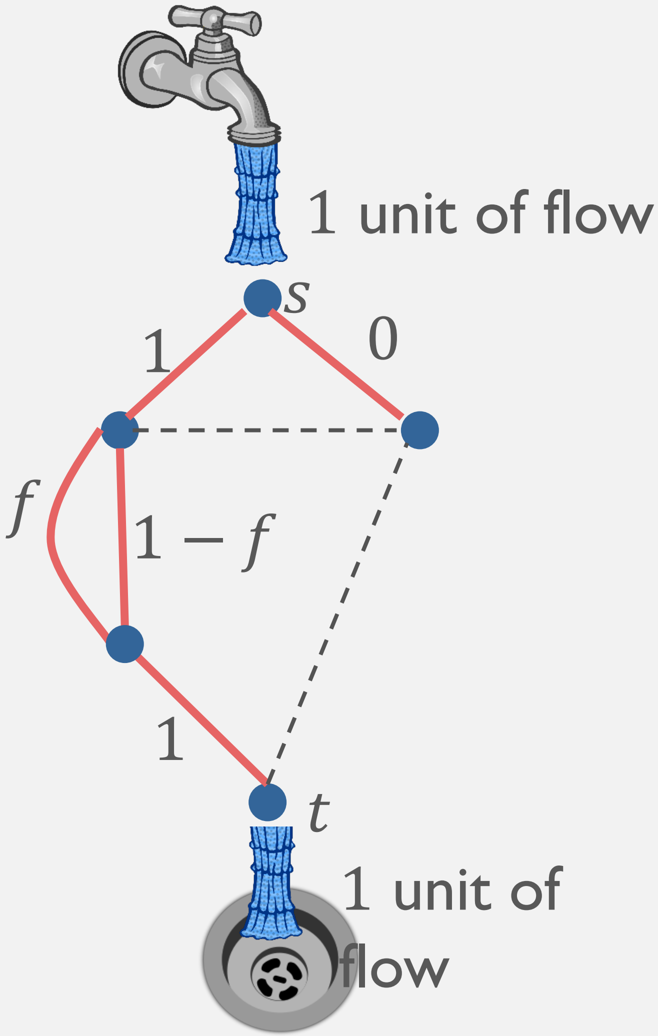
Effective Resistance

Flow energy:

$$\sum_{edges} (flow\ on\ edge)^2$$

Effective Resistance: $R_{s,t}(G)$

- Smallest energy of any valid flow from s to t on G.



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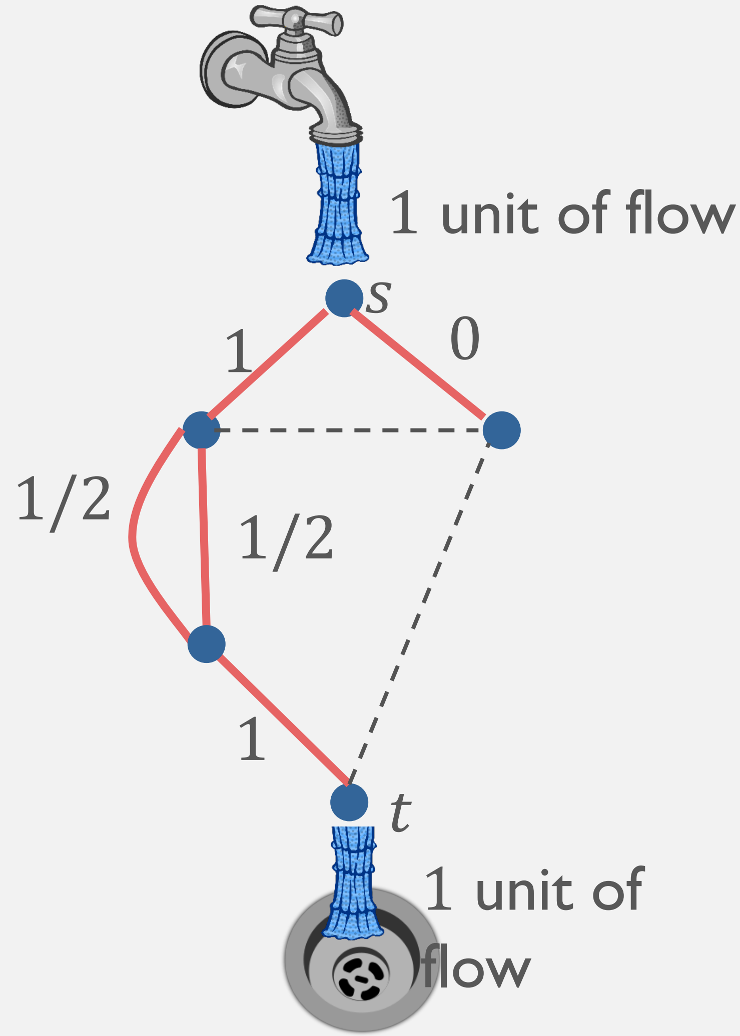


Algorithm Complexity:

Query Complexity:

$$O\left(\sqrt{\max_{connected\ G} R_{s,t}(G)}\ \sqrt{\max_{not\ connected\ G} C_{s,t}(G)}\right)$$

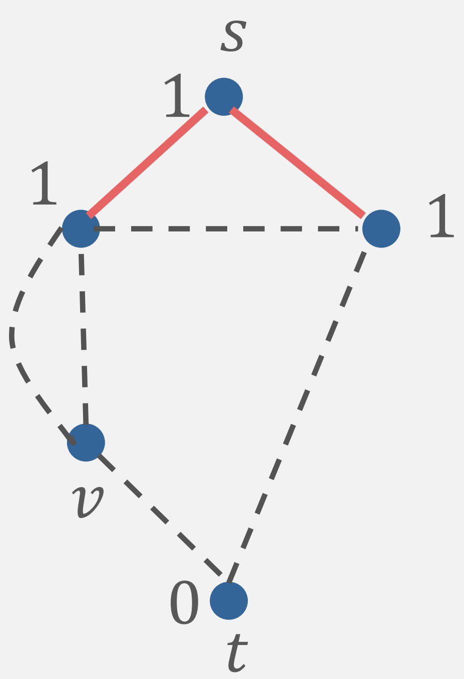
Effective resistance

Effective capacitance

Effective Capacitance

Generalized cut:

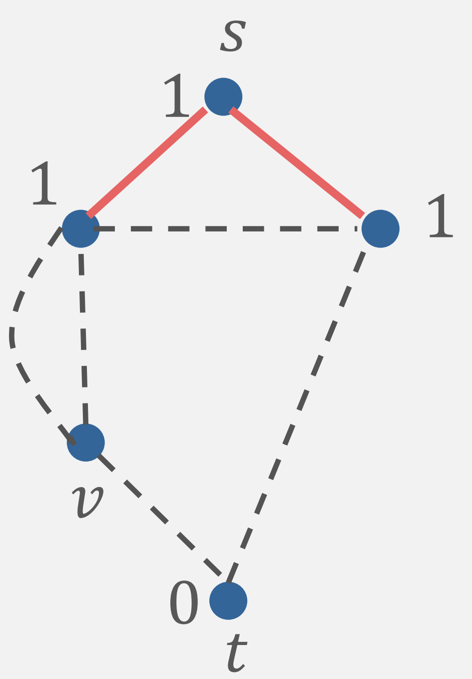
- 1 at s
- 0 at t
- Difference is 0 across edge



Effective Capacitance

Potential energy:

$$\sum_{\substack{edges\ in \\ skeleton\ graph}} (cut\ difference)^2$$



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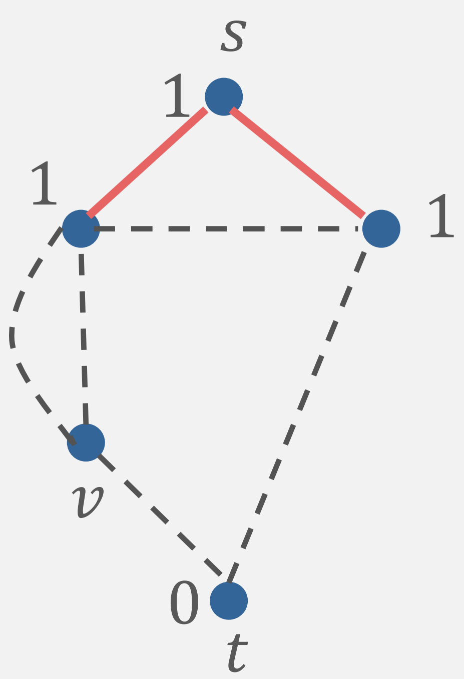

Effective Capacitance

Potential energy:

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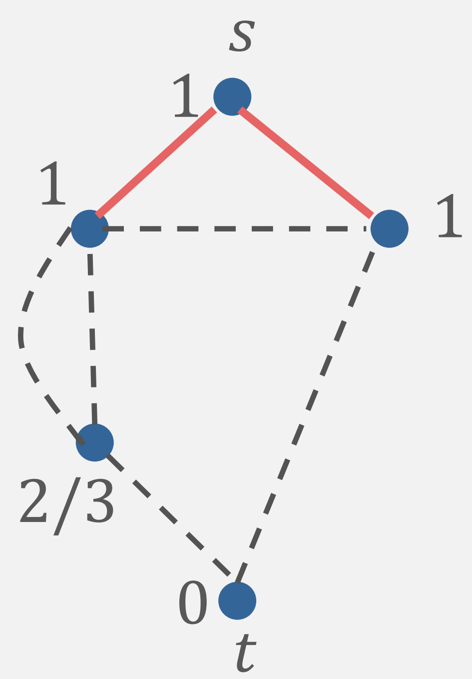
Effective Capacitance: $C_{s,t}(G)$

- Smallest potential energy of any valid generalized cut between s and t on G.



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Query Complexity:

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Effective resistance

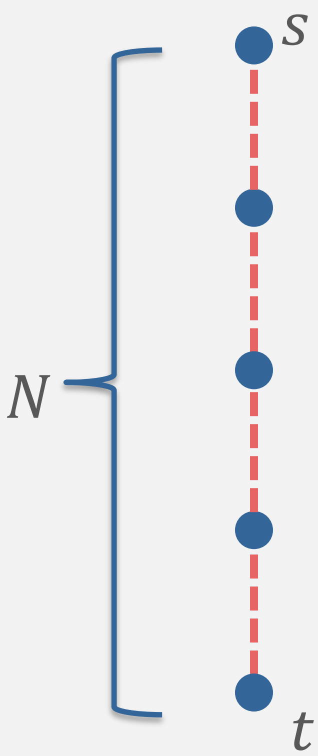
Effective capacitance

Example

Decide $AND(x_1, x_2, \ldots, x_N)$, if

- All $x_i = 1$, or
- At least k input bits are 0.

Example



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$\Updownarrow$

Decide if

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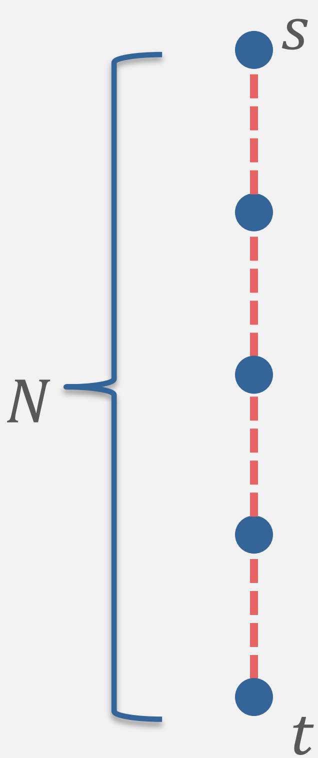

Decide if

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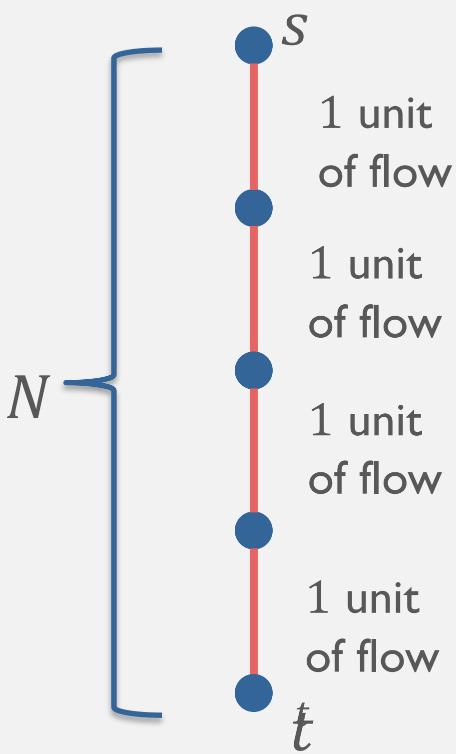

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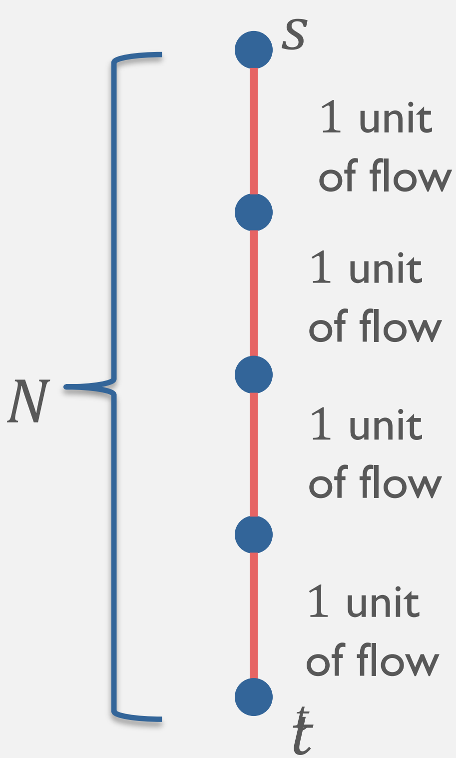

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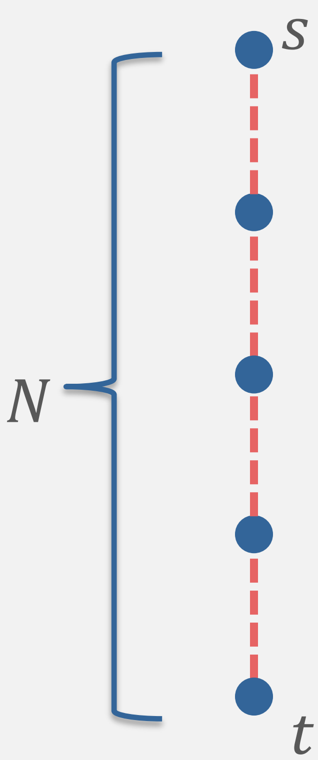
$$O\left(\sqrt{\max_{connected\ G} R_{s,t}(G)}\ \sqrt{\max_{not\ connected\ G} C_{s,t}(G)}\right)$$

$$\max_{connected\ G} R_{s,t}(G) = N$$

Example



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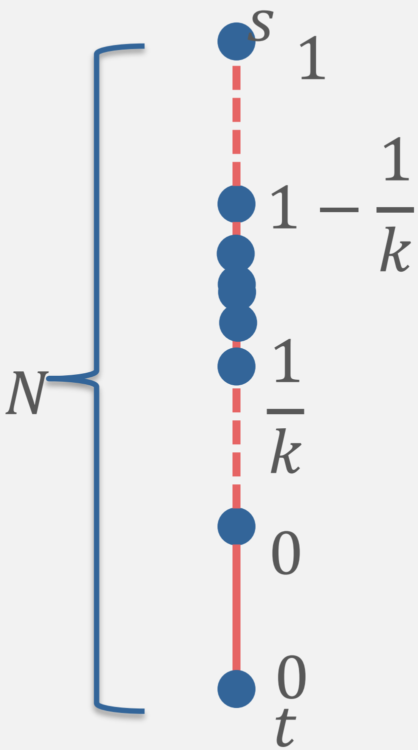

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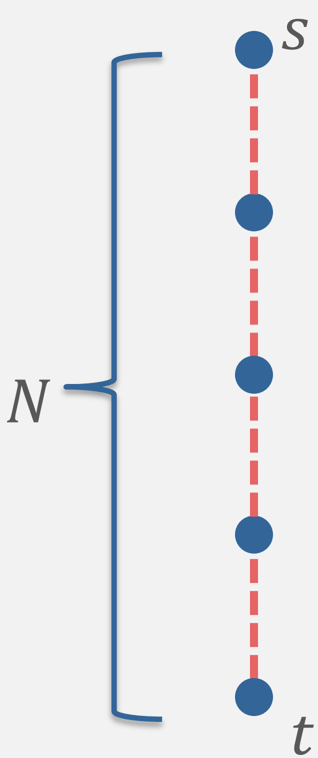
$$O\left(\sqrt{\max_{connected\ G} R_{s,t}(G)}\ \sqrt{\max_{not\ connected\ G} C_{s,t}(G)}\right)$$

$$\max_{not\ connected\ G} C_{s,t}(G) = k \times \left(\frac{1}{k}\right)^2 = \frac{1}{k}$$

Example



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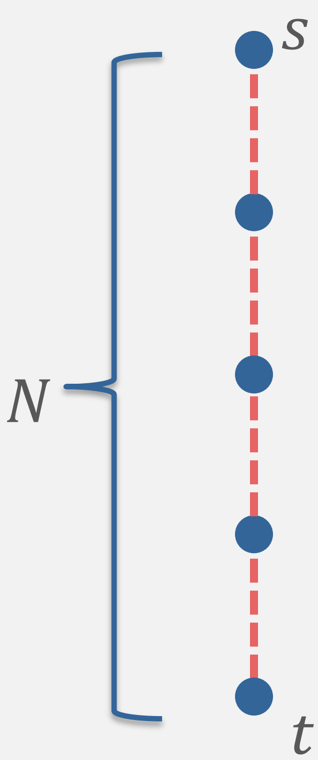
$$O\left(\sqrt{\underbrace{\max_{connected\ G} R_{s,t}(G)}_{N}}\sqrt{\underbrace{\max_{not\ connected\ G} C_{s,t}(G)}_{1/k}}\right)$$

Example



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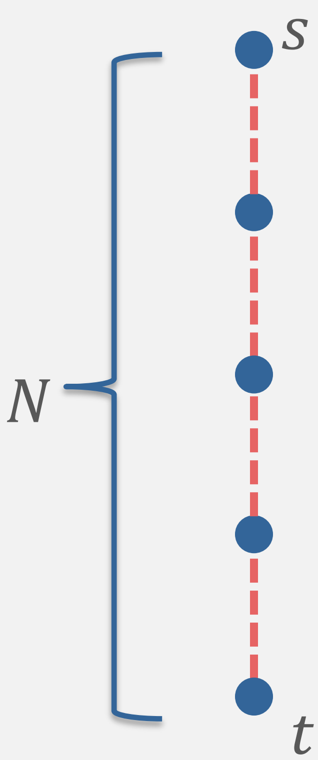
N $\qquad$ $1/k$

Quantum complexity is $O\left(\sqrt{N/k}\right)$ (optimal)

Example



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$$N \qquad\qquad\qquad\qquad 1/k$$

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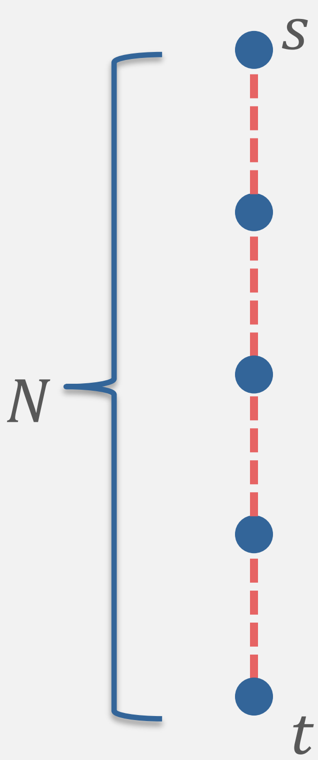

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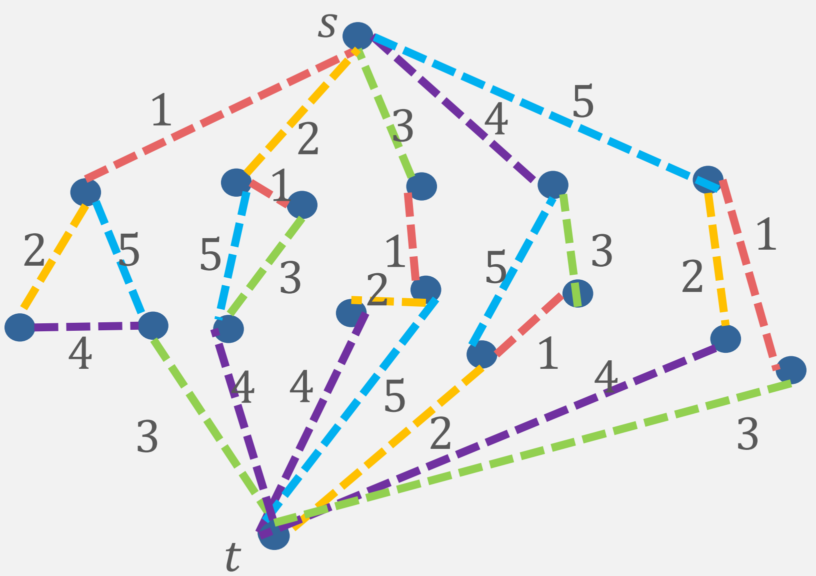
$$O\left(\underbrace{\sqrt{\max_{connected\ G} R_{s,t}(G)}}_{N}\ \underbrace{\sqrt{\max_{not\ connected\ G} C_{s,t}(G)}}_{1/k}\right)$$

Quantum complexity is $O\left(\sqrt{N/k}\right)$ (optimal)

Randomized classical complexity is $\Omega(N/k)$

Example

Cycle Detection $O\left(\sqrt{\max_{\text{connected } G} R_{s,t}(G)}\ \sqrt{\max_{\text{not connected } G} C_{s,t}(G)}\right)$



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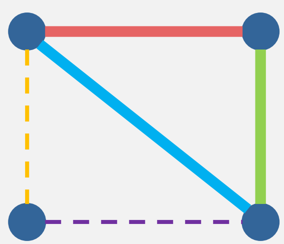

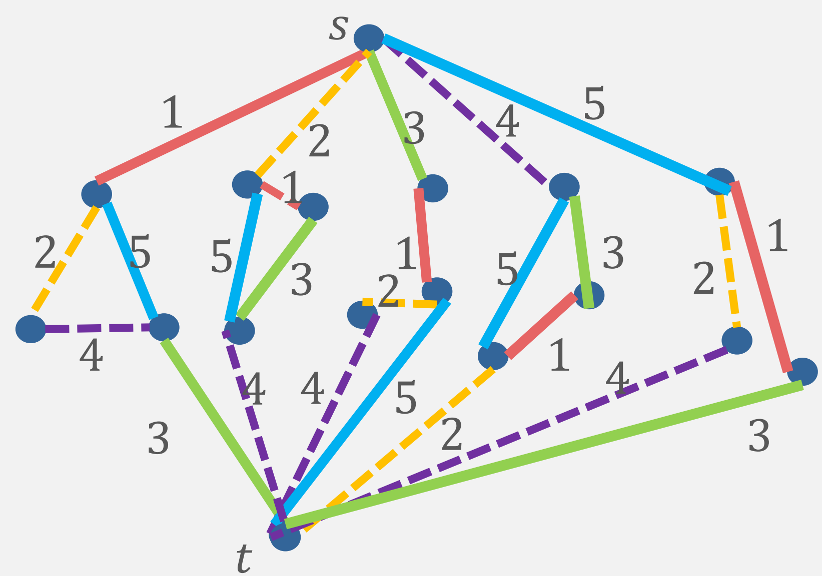

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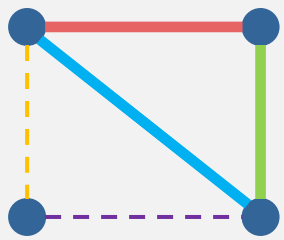
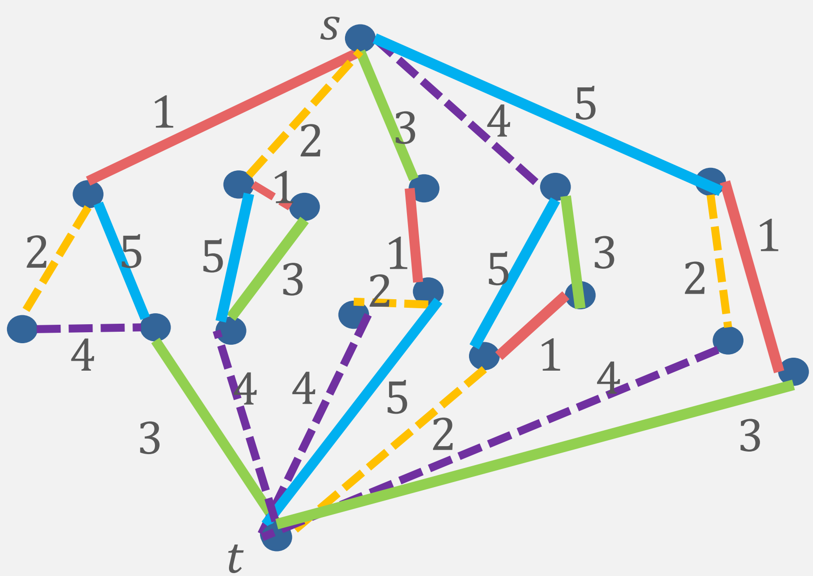
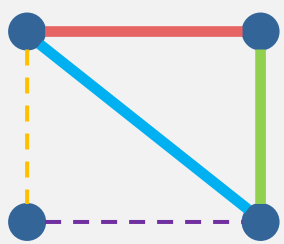


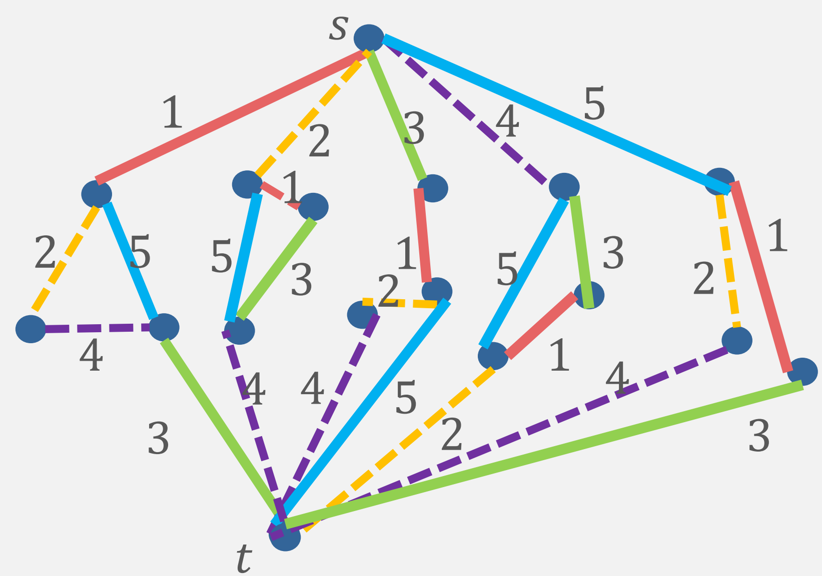


$R_{s,t}(G) = 1$

Example

Cycle Detection $O\left(\sqrt{\max_{connected\ G} R_{s,t}(G)}\ \sqrt{\max_{not\ connected\ G} C_{s,t}(G)}\right)$





More generally: $R_{s,t}(G) = (circuit\ rank)^{-1} \leq 1$

Circuit rank = min # of edges that must be cut to create a cycle free graph

Example

Cycle Detection $O\left(\sqrt{\max_{connected\ G} R_{s,t}(G)}\ \sqrt{\max_{not\ connected\ G} C_{s,t}(G)}\right)$

$R_{s,t}(G) = (circuit\ rank)^{-1}$

Circuit rank = min # of edges that must be cut to create a cycle free graph

- Quantum algorithm picks out critical topological parameter
- If promised large circuit rank (if cycle exists), then cycle detection algorithm runs faster
- Proved by 2nd year university students

Example

Cycle Detection $O\left(\sqrt{\max_{connected\ G} R_{s,t}(G)}\ \sqrt{\max_{not\ connected\ G} C_{s,t}(G)}\right)$

$R_{s,t}(G) = (circuit\ rank)^{-1}$

$C_{s,t}(G) = O(n^3)$

Query complexity: $O\left(n^{3/2}\right)$

(optimal – logarithmic improvement over previous algorithm)

Bonus Algorithm:

Quantum query algorithm to estimate effective resistance or effective capacitance of G. (Jeffery, Ito '15)

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Quantum query algorithm to estimate effective resistance or effective capacitance of G. (Jeffery, Ito '15)

Because effective resistance depends directly on circuit rank, we now have a quantum algorithm to estimate circuit rank.

Recap

st-connectivity makes a good algorithmic primitive

1. Widely applicable
2. Easy to analyze (without knowing quantum mechanics)

Open Questions and Current Directions

- Time complexity (current research at QuSoft)
- How to choose edge weights?
- When is st-connectivity reduction optimal?
- What is the classical time/query complexity of st-connectivity in the black box model? Under the promise of small capacitance/resistance?

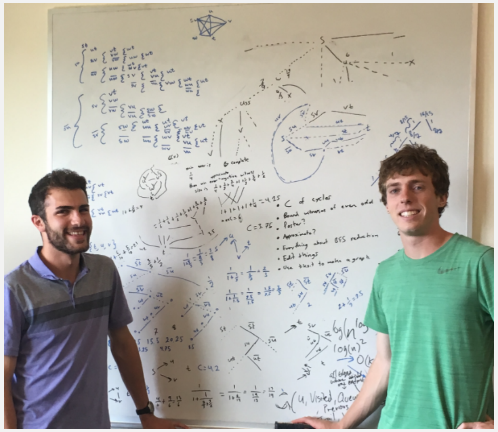
Thank you!

Stacey Jeffery

Michael Jarret

Lizeth Lucero

Alvaro Piedrafita



Teal Witter

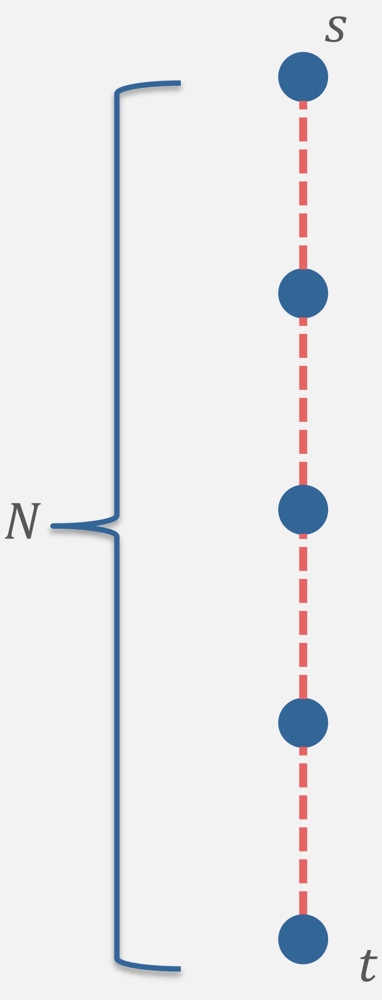
Kai De Lorenzo

Example

What is quantum complexity of deciding $AND(x_1, x_2, \ldots, x_N)$, promised

- All $x_i = 1$, or
- At least $\sqrt{N}$ input variables are 0.

Example



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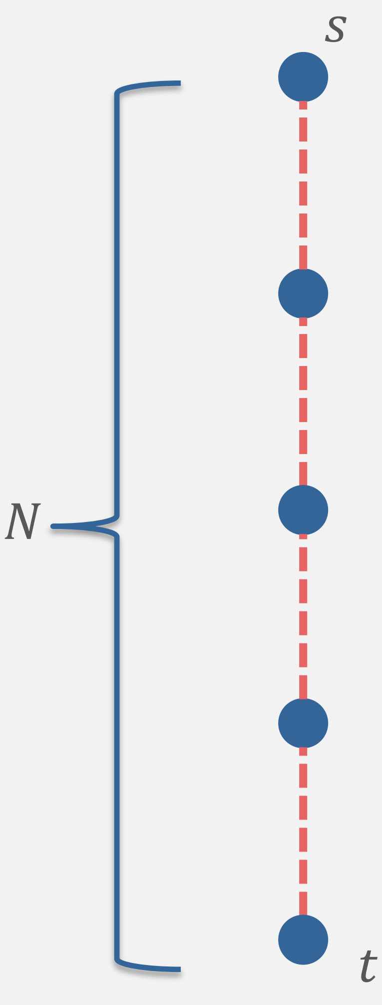

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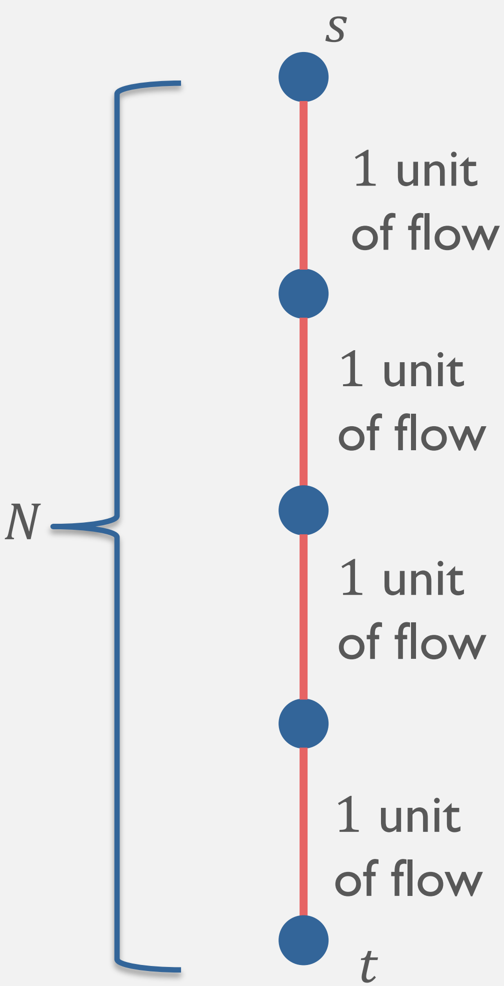
Example



What is quantum complexity of deciding if

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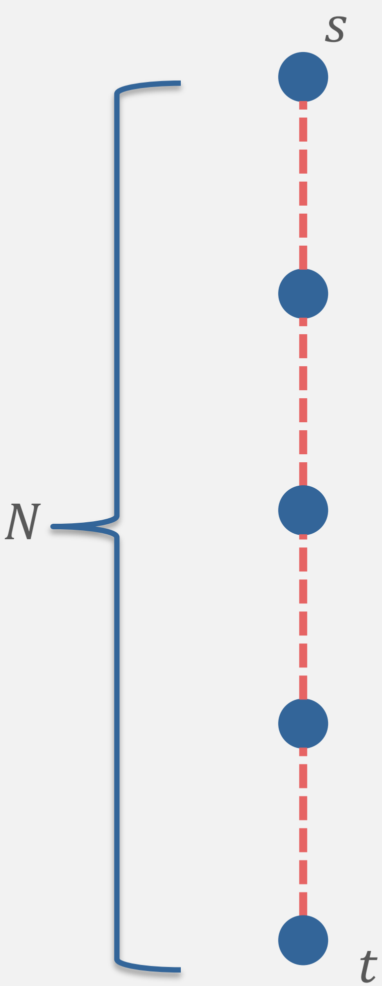
$$\sqrt{\max_{G\in\mathcal{H}:connected} R_{s,t}(G)}\ \sqrt{\max_{G'\in\mathcal{H}:not\ connected} C_{s,t}(G')}$$

$$\max_{G\in\mathcal{H}:connected} R_{s,t}(G) = N$$

Example



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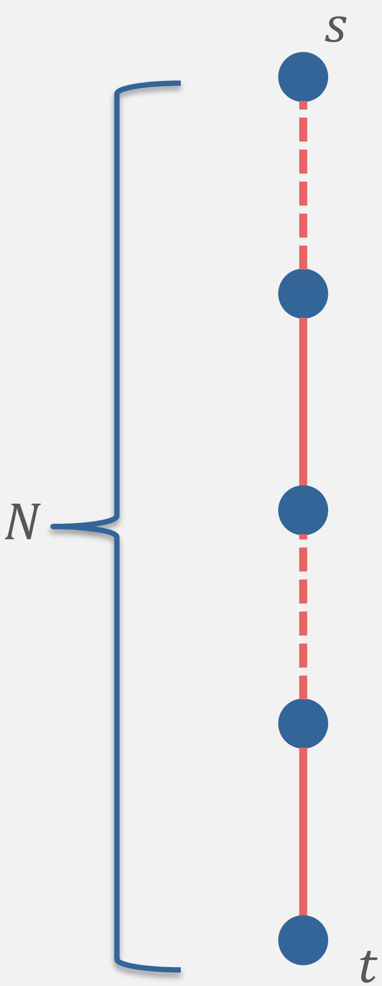

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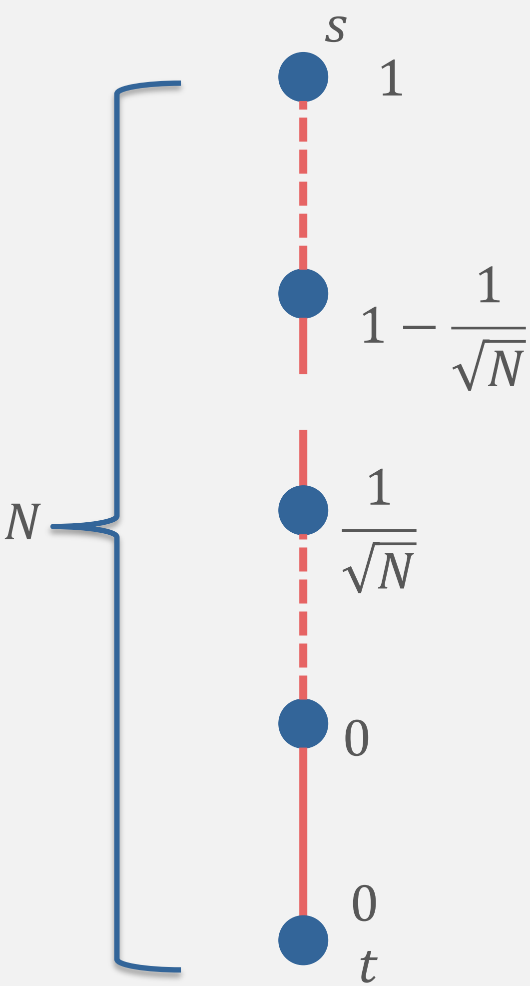

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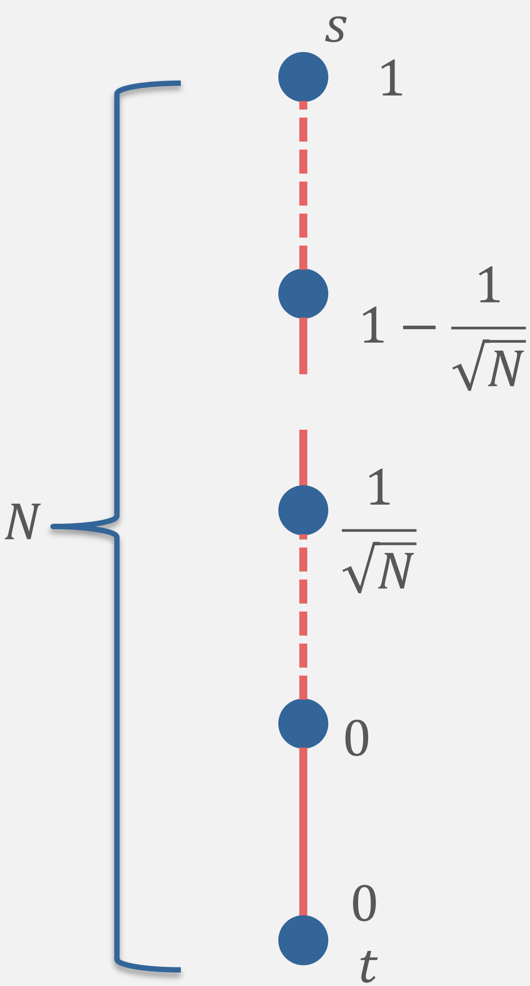

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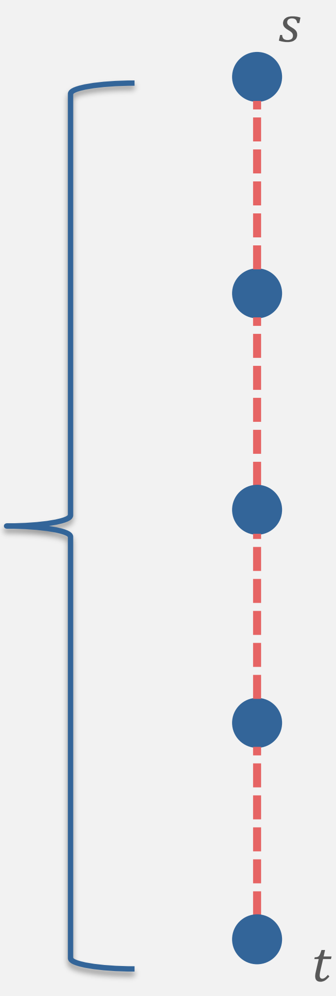
$$\max_{G'\in\mathcal{H}:not\ connected} C_{s,t}(G') = \sqrt{N}\times\left(\frac{1}{\sqrt{N}}\right)^2 = \frac{1}{\sqrt{N}}$$

Example



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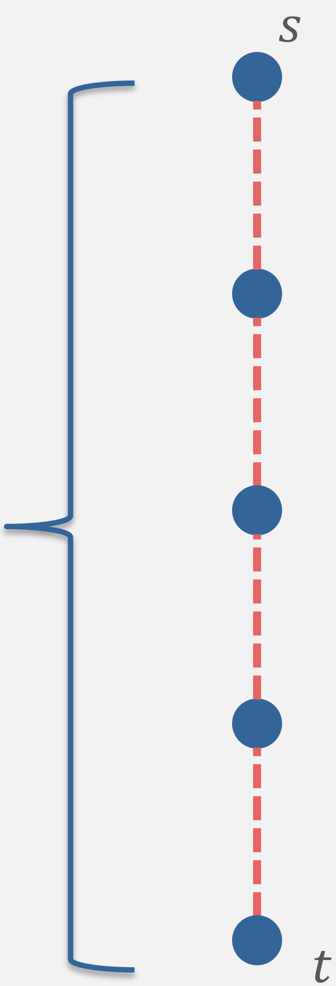
$$\sqrt{\max_{G\in\mathcal{H}:connected} R_{s,t}(G)} \quad \sqrt{\max_{G\in\mathcal{H}:not\ connected} R_{s',t'}(G')}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$N \qquad\qquad 1/\sqrt{N}$$

Quantum complexity is $O\left(N^{1/4}\right)$

Example



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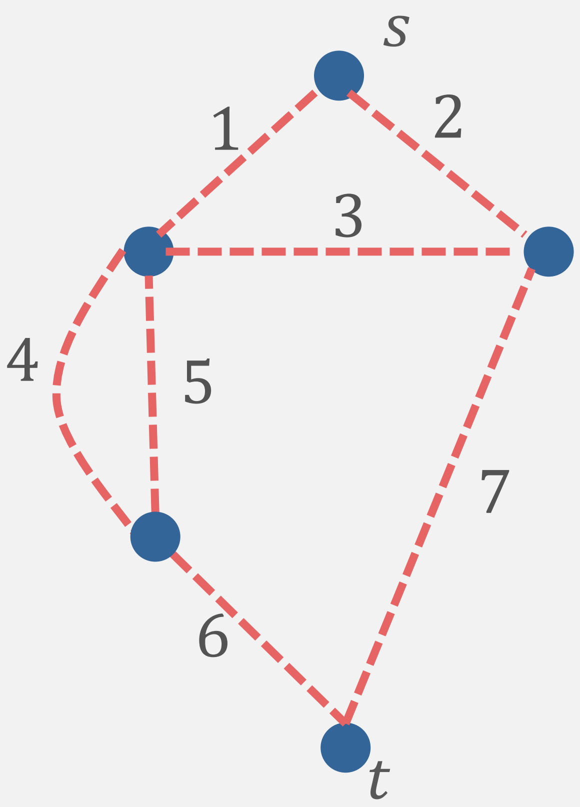
New Example

Connectivity – is every vertex connected to every other vertex?



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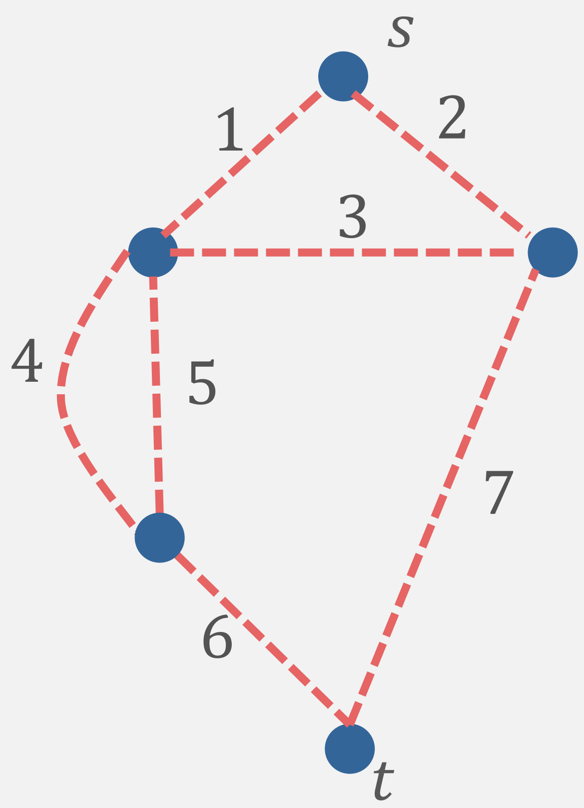
Connectivity – is every vertex connected to every other vertex?

Connectivity=
$(st - conn) \wedge (su - conn) \wedge (uv - conn) \ldots$



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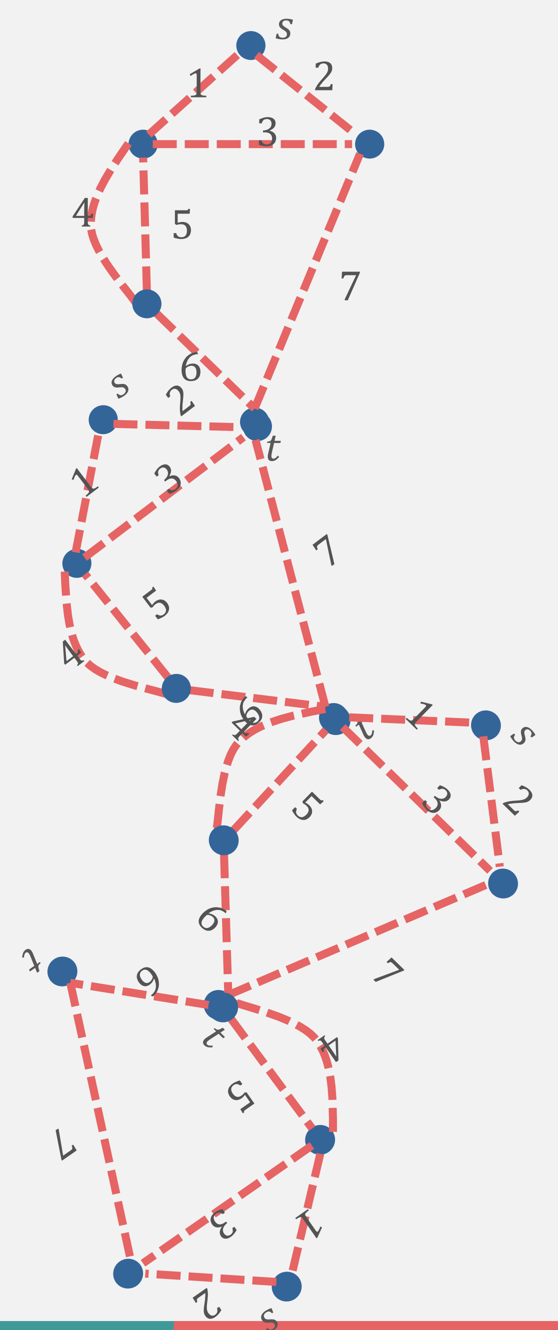
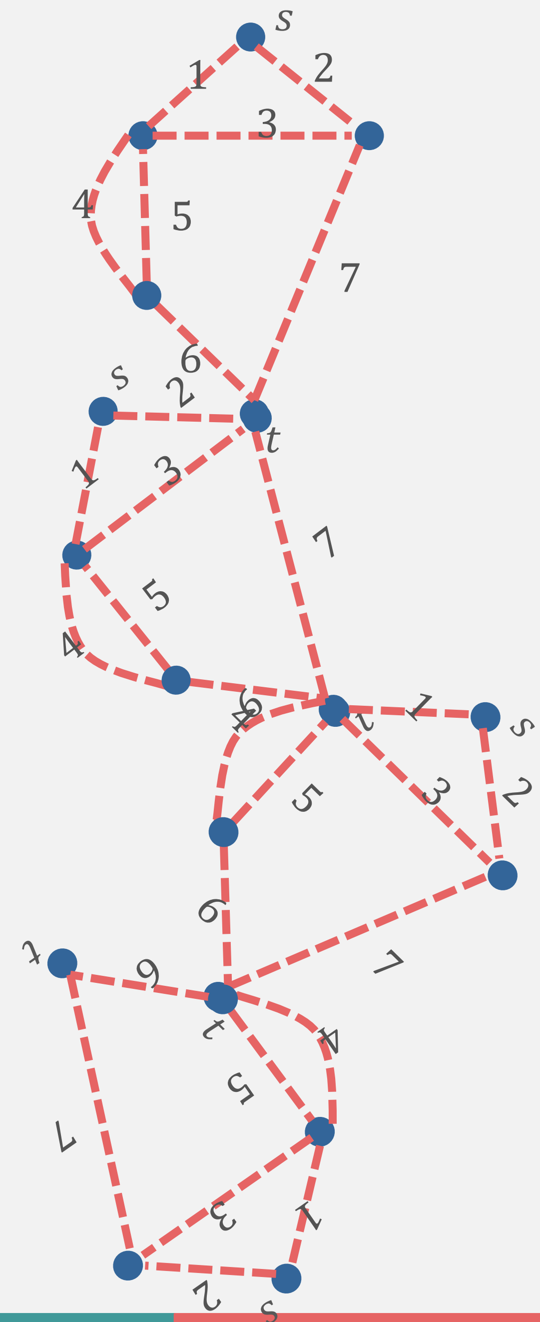


New Example

Connectivity – is every vertex connected to every other vertex?

Results:

- Worst case: $O(n^{3/2})$ (n = # vertices)
- Promised
  - YES – diameter is D
  - NO – every connected component has at most n^* vertices
  - $O\left(\sqrt{nn^*D}\right)$



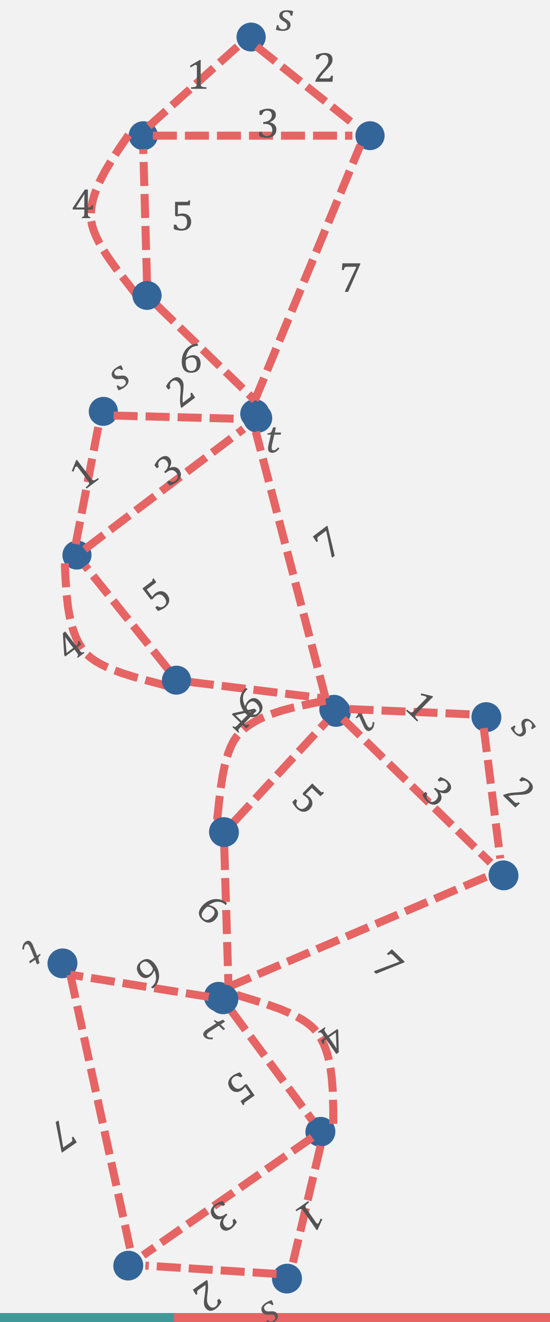
New Example

Connectivity – is every vertex connected to every other vertex?

Results:

- Worst case: $O(n^{3/2})$ (n = # vertices)
- Promised
  - YES – diameter is D
  - NO – every connected component has at most K vertices
  - $O\left(\sqrt{nKD}\right)$

(Diameter result previously discovered by Arins using slightly different approach)



The Algorithm

Span Program

- Span vectors
- Target vector

The input to the problem determines which subset of span vectors are allowed.

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Thus span program encodes a function.

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If target vector is in span of the allowed span vectors, then function evaluates to 1 on that input. Otherwise, evaluates to 0.

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Infinite number of span programs can encode the same function

Given a span program, can create a quantum algorithm to evaluate the corresponding function (create a quantum walk whose dispersion operators are based on the vectors)

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The efficiency of the span program is a (relatively) simple function of the vectors.

There is always a span program algorithm that is optimal (and many that are not optimal.)