Graduate Texts in Mathematics

Neal Koblitz

A Course in Number Theory and Cryptography

Second Edition

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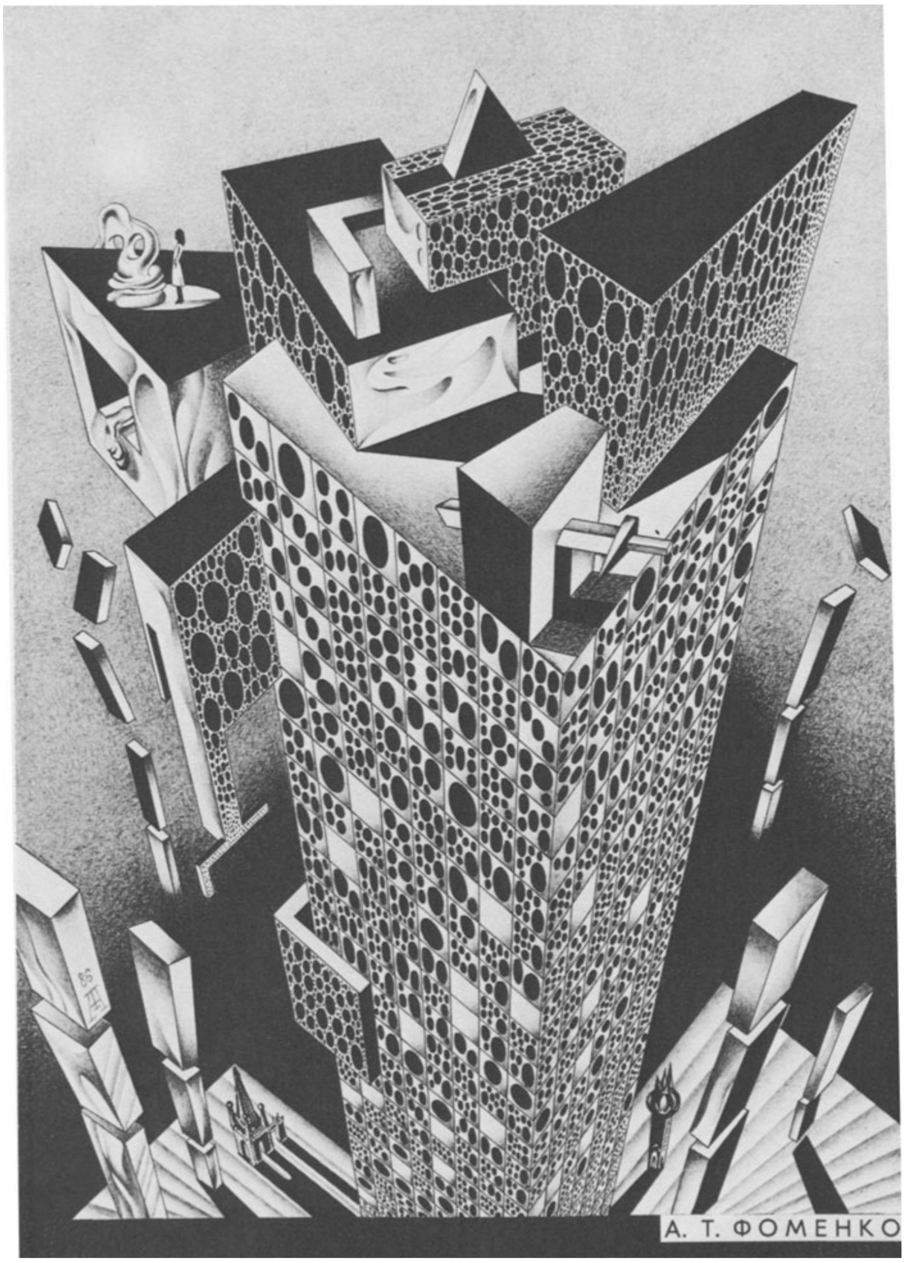
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Foreword

...both Gauss and lesser mathematicians may be justified in rejoicing that there is one science [number theory] at any rate, and that their own, whose very remoteness from ordinary human activities should keep it gentle and clean.

— G. H. Hardy, *A Mathematician's Apology*, 1940

G. H. Hardy would have been surprised and probably displeased with the increasing interest in number theory for application to "ordinary human activities" such as information transmission (error-correcting codes) and cryptography (secret codes). Less than a half-century after Hardy wrote the words quoted above, it is no longer inconceivable (though it hasn't happened yet) that the N.S.A. (the agency for U.S. government work on cryptography) will demand prior review and clearance before publication of theoretical research papers on certain types of number theory.

In part it is the dramatic increase in computer power and sophistication that has influenced some of the questions being studied by number theorists, giving rise to a new branch of the subject, called "computational number theory."

This book presumes almost no background in algebra or number theory. Its purpose is to introduce the reader to arithmetic topics, both ancient and very modern, which have been at the center of interest in applications, especially in cryptography. For this reason we take an algorithmic approach, emphasizing estimates of the efficiency of the techniques that arise from the theory. A special feature of our treatment is the inclusion (Chapter VI) of some very recent applications of the theory of elliptic curves. Elliptic curves have for a long time formed a central topic in several branches of theoretical mathematics; now the arithmetic of elliptic curves has turned out to have potential practical applications as well.

Extensive exercises have been included in all of the chapters in order to enable someone who is studying the material outside of a formal course structure to solidify her/his understanding.

The first two chapters provide a general background. A student who has had no previous exposure to algebra (field extensions, finite fields) or elementary number theory (congruences) will find the exposition rather condensed, and should consult more leisurely textbooks for details. On the other hand, someone with more mathematical background would probably want to skim through the first two chapters, perhaps trying some of the less familiar exercises.

Depending on the students' background, it should be possible to cover most of the first five chapters in a semester. Alternately, if the book is used in a sequel to a one-semester course in elementary number theory, then Chapters III–VI would fill out a second-semester course.

The dependence relation of the chapters is as follows (if one overlooks some inessential references to earlier chapters in Chapters V and VI):

$$\begin{tikzcd}
 & \text{Chapter I} \arrow[d, dash] & \\
 & \text{Chapter II} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\text{Chapter III} \arrow[d, dash] & \text{Chapter V} & \text{Chapter VI} \\
\text{Chapter IV} & &
\end{tikzcd}$$

This book is based upon courses taught at the University of Washington (Seattle) in 1985–86 and at the Institute of Mathematical Sciences (Madras, India) in 1987. I would like to thank Gary Nelson and Douglas Lind for using the manuscript and making helpful corrections.

The frontispiece was drawn by Professor A. T. Fomenko of Moscow State University to illustrate the theme of the book. Notice that the coded decimal digits along the walls of the building are not random.

This book is dedicated to the memory of the students of Vietnam, Nicaragua and El Salvador who lost their lives in the struggle against U.S. aggression. The author's royalties from sales of the book will be used to buy mathematics and science books for the universities and institutes of those three countries.

Seattle, May 1987

Preface to the Second Edition

As the field of cryptography expands to include new concepts and techniques, the cryptographic applications of number theory have also broadened. In addition to elementary and analytic number theory, increasing use has been made of algebraic number theory (primality testing with Gauss and Jacobi sums, cryptosystems based on quadratic fields, the number field sieve) and arithmetic algebraic geometry (elliptic curve factorization, cryptosystems based on elliptic and hyperelliptic curves, primality tests based on elliptic curves and abelian varieties). Some of the recent applications of number theory to cryptography — most notably, the number field sieve method for factoring large integers, which was developed since the appearance of the first edition — are beyond the scope of this book. However, by slightly increasing the size of the book, we were able to include some new topics that help convey more adequately the diversity of applications of number theory to this exciting multidisciplinary subject.

The following list summarizes the main changes in the second edition.

• Several corrections and clarifications have been made, and many references have been added.

• A new section on zero-knowledge proofs and oblivious transfer has been added to Chapter IV.

• A section on the quadratic sieve factoring method has been added to Chapter V.

• Chapter VI now includes a section on the use of elliptic curves for primality testing.

• Brief discussions of the following concepts have been added: k–threshold schemes, probabilistic encryption, hash functions, the Chor–Rivest knapsack cryptosystem, and the U.S. government's new Digital Signature Standard.

Seattle, May 1994

Contents

I
Some Topics in Elementary Number Theory

Most of the topics reviewed in this chapter are probably well known to most readers. The purpose of the chapter is to recall the notation and facts from elementary number theory which we will need to have at our fingertips in our later work. Most proofs are omitted, since they can be found in almost any introductory textbook on number theory. One topic that will play a central role later — estimating the number of bit operations needed to perform various number theoretic tasks by computer — is not yet a standard part of elementary number theory textbooks. So we will go into most detail about the subject of time estimates, especially in §1.

1 Time estimates for doing arithmetic

Numbers in different bases. A nonnegative integer n written to the *base* b is a notation for n of the form $(d_{k-1}d_{k-2}\cdots d_1d_0)_b$, where the d's are *digits*, i.e., symbols for the integers between 0 and $b-1$; this notation means that $n = d_{k-1}b^{k-1} + d_{k-2}b^{k-2} + \cdots + d_1 b + d_0$. If the first digit d_{k-1} is not zero, we call n a k-digit base-b number. Any number between b^{k-1} and b^k is a k-digit number to the base b. We shall omit the parentheses and subscript $(\cdots)_b$ in the case of the usual decimal system ($b = 10$) and occasionally in other cases as well, if the choice of base is clear from the context, especially when we're using the binary system ($b = 2$). Since it is sometimes useful to work in bases other than 10, one should get used to doing arithmetic in an arbitrary base and to converting from one base to another. We now review this by doing some examples.

Remarks. (1) Fractions can also be expanded in any base, i.e., they can be represented in the form $(d_{k-1}d_{k-2}\cdots d_1d_0.d_{-1}d_{-2}\cdots)_b$. (2) When $b > 10$ it is customary to use letters for the digits beyond 9. One could also use letters for *all* of the digits.

Example 1. (a) $(11001001)_2 = 201$.

(b) When $b = 26$ let us use the letters A—Z for the digits 0—25, respectively. Then $(\mathrm{BAD})_{26}$=679, whereas $(\mathrm{B.AD})_{26} = 1\frac{3}{676}$.

Example 2. Multiply 160 and 199 in the base 7. **Solution:**

$$\begin{array}{r}
316\\
\underline{403}\\
1254\\
\underline{16030}\\
161554
\end{array}$$

Example 3. Divide $(11001001)_2$ by $(100111)_2$, and divide $(\mathrm{HAPPY})_{26}$ by $(\mathrm{SAD})_{26}$.

Solution:

$$\begin{array}{rr}
 & 101\,\frac{110}{100111}\\
100111 & |\overline{11001001}\\
 & \underline{100111}\\
 & 101101\\
 & \underline{100111}\\
 & 110
\end{array}
\qquad\qquad
\begin{array}{rr}
 & \mathrm{KD}\,\frac{\mathrm{MLP}}{\mathrm{SAD}}\\
\mathrm{SAD} & |\overline{\mathrm{HAPPY}}\\
 & \underline{\mathrm{GYBE}}\phantom{\mathrm{Y}}\\
 & \mathrm{COLY}\\
 & \underline{\mathrm{CCAJ}}\\
 & \mathrm{MLP}
\end{array}$$

Example 4. Convert 10^6 to the bases 2, 7 and 26 (using the letters A—Z as digits in the latter case).

Solution. To convert a number n to the base b, one first gets the last digit (the ones' place) by dividing n by b and taking the remainder. Then replace n by the quotient and repeat the process to get the second-to-last digit d_1, and so on. Here we find that

$$10^6 = (11110100001001000000)_2 = (11333311)_7 = (\mathrm{CEXHO})_{26}.$$

Example 5. Convert $\pi = 3.1415926\cdots$ to the base 2 (carrying out the computation 15 places to the right of the point) and to the base 26 (carrying out 3 places to the right of the point).

Solution. After taking care of the integer part, the fractional part is converted to the base b by multiplying by b, taking the integer part of the result as d_{-1}, then starting over again with the fractional part of what you now have, successively finding d_{-2}, $d_{-3}, \ldots$. In this way one obtains:

$$3.1415926\cdots = (11.001001000011111\cdots)_2 = (\mathrm{D.DRS}\cdots)_{26}.$$

Number of digits. As mentioned before, an integer n satifying $b^{k-1} \leq n < b^k$ has k digits to the base b. By the definition of logarithms, this gives the following formula for the number of base-b digits (here "[ ]" denotes the greatest integer function):

$$\text{number of digits } = \left[log_b n\right] + 1 = \left[\frac{log\, n}{log\, b}\right] + 1,$$

where here (and from now on) "log" means the natural logarithm log_e.

Bit operations. Let us start with a very simple arithmetic problem, the addition of two binary integers, for example:

$$\begin{array}{r} {\scriptstyle 1111} \\ 1111000 \\ +\ \underline{0011110} \\ 10010110 \end{array}$$

Suppose that the numbers are both k bits long (the word "bit" is short for "binary digit"); if one of the two integers has fewer bits than the other, we fill in zeros to the left, as in this example, to make them have the same length. Although this example involves small integers (adding 120 to 30), we should think of k as perhaps being very large, like 500 or 1000.

Let us analyze in complete detail what this addition entails. Basically, we must repeat the following steps k times:

1. Look at the top and bottom bit, and also at whether there's a carry above the top bit.
2. If both bits are 0 and there is no carry, then put down 0 and move on.
3. If either (a) both bits are 0 and there is a carry, or (b) one of the bits is 0, the other is 1, and there is no carry, then put down 1 and move on.
4. If either (a) one of the bits is 0, the other is 1, and there is a carry, or else (b) both bits are 1 and there is no carry, then put down 0, put a carry in the next column, and move on.
5. If both bits are 1 and there is a carry, then put down 1, put a carry in the next column, and move on.

Doing this procedure once is called a *bit operation*. Adding two k-bit numbers requires k bit operations. We shall see that more complicated tasks can also be broken down into bit operations. The amount of time a computer takes to perform a task is essentially proportional to the number of bit operations. Of course, the constant of proportionality — the number of nanoseconds per bit operation — depends on the particular computer system. (This is an over-simplification, since the time can be affected by "administrative matters," such as accessing memory.) When we speak of estimating the "time" it takes to accomplish something, we mean finding an estimate for the number of bit operations required. In these estimates we shall neglect the time required for "bookkeeping" or logical steps other than the bit operations; in general, it is the latter which takes by far the most time.

Next, let's examine the process of *multiplying* a k-bit integer by an ℓ-bit integer in binary. For example,

$$\begin{array}{r}
11101 \\
\underline{1101} \\
11101 \\
111010 \\
\underline{11101} \\
101111001
\end{array}$$

Suppose we use this familiar procedure to multiply a k-bit integer n by an ℓ-bit integer m. We obtain at most ℓ rows (one row fewer for each 0-bit in m), where each row consists of a copy of n shifted to the left a certain distance, i.e., with zeros put on at the end. Suppose there are $\ell' \leq \ell$ rows. Because we want to break down all our computations into bit operations, we cannot simultaneously add together all of the rows. Rather, we move down from the 2nd row to the ℓ'-th row, adding each new row to the partial sum of all of the earlier rows. At each stage, we note how many places to the left the number n has been shifted to form the new row. We copy down the right-most bits of the partial sum, and then add to n the integer formed from the rest of the partial sum — as explained above, this takes k bit operations. In the above example 11101×1101, after adding the first two rows and obtaining 10010001, we copy down the last three bits 001 and add the rest (i.e., 10010) to $n = 11101$. We finally take this sum $10010 + 11101 = 101111$ and append 001 to obtain 101111001, the sum of the $\ell' = 3$ rows.

This description shows that the multiplication task can be broken down into $\ell' - 1$ additions, each taking k bit operations. Since $\ell' - 1 < \ell' \leq \ell$, this gives us the simple bound

$$\text{Time(multiply integer } k \text{ bits long by integer } \ell \text{ bits long)} < k\ell.$$

We should make several observations about this derivation of an estimate for the number of bit operations needed to perform a binary multiplication. In the first place, as mentioned before, we counted only the number of bit operations. We neglected to include the time it takes to shift the bits in n a few places to the left, or the time it takes to copy down the right-most digits of the partial sum corresponding to the places through which n has been shifted to the left in the new row. In practice, the shifting and copying operations are fast in comparison with the large number of bit operations, so we can safely ignore them. In other words, we shall *define* a "time estimate" for an arithmetic task to be an upper bound for the number of bit operations, without including any consideration of shift operations, changing registers ("copying"), memory access, etc. Note that this means that we would use the very same time estimate if we were multiplying a k-bit binary expansion of a fraction by an ℓ-bit binary expansion; the only additional feature is that we must note the location of the point separating integer from fractional part and insert it correctly in the answer.

In the second place, if we want to get a time estimate that is simple and convenient to work with, we should assume at various points that we're in the "worst possible case." For example, if the binary expansion of m has a lot of zeros, then ℓ' will be considerably less than ℓ. That is, we could use the estimate Time(multiply k-bit integer by ℓ-bit integer)$<$ $k \cdot$(number of 1-bits in m). However, it is usually not worth the improvement (i.e., lowering) in our time estimate to take this into account, because it is more useful to have a simple uniform estimate that depends only on the size of m and n and not on the particular bits that happen to occur.

As a special case, we have: Time(multiply k-bit by k-bit)$< k^2$.

Finally, our estimate $k\ell$ can be written in terms of n and m if we remember the above formula for the number of digits, from which it follows that $k = [log_2 n] + 1 \leq \frac{log\, n}{log\, 2} + 1$ and $\ell = [log_2 m] + 1 \leq \frac{log\, m}{log\, 2} + 1$.

Example 6. Find an upper bound for the number of bit operations required to compute $n!$.

Solution. We use the following procedure. First multiply 2 by 3, then the result by 4, then the result of that by 5,..., until you get to n. At the $(j-1)$-th step $(j = 2, 3, \ldots, n-1)$, you are multiplying $j!$ by $j+1$. Hence you have $n-2$ steps, where each step involves multiplying a partial product (i.e., $j!$) by the next integer. The partial products will start to be very large. As a worst case estimate for the number of bits a partial product has, let's take the number of binary digits in the very last product, namely, in $n!$.

To find the number of bits in a product, we use the fact that the number of digits in the product of two numbers is either the sum of the number of digits in each factor or else 1 fewer than that sum (see the above discussion of multiplication). From this it follows that the product of n k-bit integers will have at most nk bits. Thus, if n is a k-bit integer — which implies that every integer less than n has at most k bits — then $n!$ has at most nk bits.

Hence, in each of the $n-2$ multiplications needed to compute $n!$, we are multiplying an integer with at most k bits (namely $j+1$) by an integer with at most nk bits (namely $j!$). This requires at most nk^2 bit operations. We must do this $n-2$ times. So the total number of bit operations is bounded by $(n-2)nk^2 = n(n-2)([log_2 n] + 1)^2$. Roughly speaking, the bound is approximately $n^2(log_2 n)^2$.

Example 7. Find an upper bound for the number of bit operations required to multiply a polynomial $\sum a_i x^i$ of degree $\leq n_1$ and a polynomial $\sum b_j x^j$ of degree $\leq n_2$ whose coefficients are positive integers $\leq m$. Suppose $n_2 \leq n_1$.

Solution. To compute $\sum_{i+j=\nu} a_i b_j$, which is the coefficient of x^ν in the product polynomial (here $0 \leq \nu \leq n_1 + n_2$) requires at most $n_2 + 1$ multiplications and n_2 additions. The numbers being multiplied are bounded by m, and the numbers being added are each at most m^2; but since we have to add the partial sum of up to n_2 such numbers we should take n_2m^2 as our bound on the size of the numbers being added. Thus, in computing the coefficient of x^ν the number of bit operations required is at most

$$(n_2+1)(log_2 m+1)^2+n_2(log_2(n_2m^2)+1).$$

Since there are n_1+n_2+1 values of ν, our time estimate for the polynomial multiplication is

$$(n_1+n_2+1)\big((n_2+1)(log_2 m+1)^2+n_2(log_2(n_2m^2)+1)\big).$$

A slightly less rigorous bound is obtained by dropping the 1's, thereby obtaining an expression having a more compact appearance:

$$\frac{n_2(n_1+n_2)}{log\,2}\left(\frac{(log\,m)^2}{log\,2}+(log\,n_2+2\log\,m)\right).$$

Remark. If we set $n=n_1\geq n_2$ and make the assumption that $m\geq 16$ and $m\geq\sqrt{n_2}$ (which usually holds in practice), then the latter expression can be replaced by the much simpler $4n^2(log_2 m)^2$. This example shows that there is generally no single "right answer" to the question of finding a bound on the time to execute a given task. One wants a function of the bounds on the imput data (in this problem, n_1, n_2 and m) which is fairly simple and at the same time gives an upper bound which for most input data is more-or-less the same order of magnitude as the number of bit operations that turns out to be required in practice. Thus, for example, in Example 7 we would not want to replace our bound by, say, $4n^2m$, because for large m this would give a time estimate many orders of magnitude too large.

So far we have worked only with addition and multiplication of a k-bit and an ℓ-bit integer. The other two arithmetic operations — subtraction and division — have the same time estimates as addition and multiplication, respectively: Time(subtract k-bit from ℓ-bit)$\leq$ $\max(k,\ell)$; Time(divide k-bit by ℓ-bit)$\leq kl$. More precisely, to treat subtraction we must extend our definition of a bit operation to include the operation of subtracting a 0- or 1-bit from another 0- or 1-bit (with possibly a "borrow" of 1 from the previous column). See Exercise 8.

To analyze division in binary, let us orient ourselves by looking at an illustration, such as the one in Example 3. Suppose $k\geq\ell$ (if $k<\ell$, then the division is trivial, i.e., the quotient is zero and the entire dividend is the remainder). Finding the quotient and remainder requires at most $k-\ell+1$ subtractions. Each subtraction requires ℓ or $\ell+1$ bit operations; but in the latter case we know that the left-most column of the difference will always be a 0-bit, so we can omit that bit operation (thinking of it as "bookkeeping" rather than calculating). We similarly ignore other administrative details, such as the time required to compare binary integers (i.e., take just enough bits of the dividend so that the resulting integer is greater than the divisor), carry down digits, etc. So our estimate is simply $(k-\ell+1)\ell$, which is $\le k\ell$.

Example 8. Find an upper bound for the number of bit operations it takes to compute the binomial coefficient $\binom{n}{m}$.

Solution. Since $\binom{n}{m} = \binom{n}{n-m}$, without loss of generality we may assume that $m \le n/2$. Let us use the following procedure to compute $\binom{n}{m} =$ $= n(n-1)(n-2)\cdots(n-m+1)/(2\cdot3\cdots m)$. We have $m-1$ multiplications followed by $m-1$ divisions. In each case the maximum possible size of the first number in the multiplication or division is $n(n-1)(n-2)\cdots(n-m+1) < n^m$, and a bound for the second number is n. Thus, by the same argument used in the solution to Example 6, we see that a bound for the total number of bit operations is $2(m-1)m([log_2 n]+1)^2$, which for large m and n is essentially $2m^2(log_2 n)^2$.

We now discuss a very convenient notation for summarizing the situation with time estimates.

The big-O notation. Suppose that $f(n)$ and $g(n)$ are functions of the positive integers n which take *positive* (but not necessarily integer) values for all n. We say that $f(n) = O(g(n))$ (or simply that $f = O(g)$) if there exists a constant C such that $f(n)$ is always less than $C\cdot g(n)$. For example, $2n^2 + 3n - 3 = O(n^2)$ (namely, it is not hard to prove that the left side is always less than $3n^2$).

Because we want to use the big-O notation in more general situations, we shall give a more all-encompassing definition. Namely, we shall allow f and g to be functions of several variables, and we shall not be concerned about the relation between f and g for small values of n. Just as in the study of limits as $n \longrightarrow \infty$ in calculus, here also we shall only be concerned with large values of n.

Definition. Let $f(n_1, n_2, \ldots, n_r)$ and $g(n_1, n_2, \ldots, n_r)$ be two functions whose domains are subsets of the set of all r-tuples of positive integers. Suppose that there exist constants B and C such that whenever all of the n_j are greater than B the two functions are defined and positive, and $f(n_1, n_2, \ldots, n_r) < C\,g(n_1, n_2, \ldots, n_r)$. In that case we say that f is *bounded* by g and we write $f = O(g)$.

Note that the "=" in the notation $f = O(g)$ should be thought of as more like a "<" and the big-O should be thought of as meaning "some constant multiple."

Example 9. (a) Let $f(n)$ be *any* polynomial of degree d whose leading coefficient is positive. Then it is easy to prove that $f(n) = O(n^d)$. More generally, one can prove that $f = O(g)$ in any situation when $f(n)/g(n)$ has a finite limit as $n \longrightarrow \infty$.

(b) If ϵ is any positive number, no matter how small, then one can prove that $log\, n = O(n^\epsilon)$ (i.e., for large n, the log function is smaller than any power function, no matter how small the power). In fact, this follows because $lim_{n\to\infty} \frac{log\, n}{n^\epsilon} = 0$, as one can prove using l'Hôpital's rule.

(c) If $f(n)$ denotes the number k of binary digits in n, then it follows from the above formulas for k that $f(n) = O(log\, n)$. Also notice that the same relation holds if $f(n)$ denotes the number of base-b digits, where b is any fixed base. On the other hand, suppose that the base b is not kept fixed but is allowed to increase, and we let $f(n, b)$ denote the number of base-b digits. Then we would want to use the relation $f(n, b) = O(\frac{log\, n}{log\, b})$.

(d) We have: Time$(n \cdot m) = O(log\, n \cdot log\, m)$, where the left hand side means the number of bit operations required to multiply n by m.

(e) In Exercise 6, we can write: Time$(n!) = O\big((n\, log\, n)^2\big)$.

(f) In Exercise 7, we have:

$$\text{Time}\Big(\sum a_i x^i \cdot \sum b_j x^j\Big) = O\Big(n_1 n_2\big((log\, m)^2 + log(\min(n_1, n_2))\big)\Big).$$

In our use, the functions $f(n)$ or $f(n_1, n_2, \ldots, n_r)$ will often stand for the amount of time it takes to perform an arithmetic task with the integer n or with the set of integers $n_1, n_2, \ldots, n_r$ as input. We will want to obtain fairly simple-looking functions $g(n)$ as our bounds. When we do this, however, we do not want to obtain functions $g(n)$ which are much larger than necessary, since that would give an exaggerated impression of how long the task will take (although, from a strictly mathematical point of view, it is not incorrect to replace $g(n)$ by any larger function in the relation $f = O(g)$).

Roughly speaking, the relation $f(n) = O(n^d)$ tells us that the function f increases approximately like the d-th power of the variable. For example, if $d = 3$, then it tells us that doubling n has the effect of increasing f by about a factor of 8. The relation $f(n) = O(log^d n)$ (we write $log^d n$ to mean $(log\, n)^d$) tells us that the function increases approximately like the d-th power of the number of binary digits in n. That is because, up to a constant multiple, the number of bits is approximately $log\, n$ (namely, it is within 1 of being $log\, n/log\, 2 = 1.4427\, log\, n$). Thus, for example, if $f(n) = O(log^3 n)$, then doubling the number of bits in n (which is, of course, a much more drastic increase in the size of n than merely doubling n) has the effect of increasing f by about a factor of 8.

Note that to write $f(n) = O(1)$ means that the function f is bounded by some constant.

Remark. We have seen that, if we want to multiply two numbers of about the same size, we can use the estimate Time(k-bit$\cdot k$-bit)$=O(k^2)$. It should be noted that much work has been done on increasing the speed of multiplying two k-bit integers when k is large. Using clever techniques of multiplication that are much more complicated than the grade-school method we have been using, mathematicians have been able to find a procedure for multiplying two k-bit integers that requires only $O(k\, log\, k\; log\, log\, k)$ bit operations. This is better than $O(k^2)$, and even better than $O(k^{1+\epsilon})$ for any $\epsilon > 0$, no matter how small. However, in what follows we shall always be content to use the rougher estimates above for the time needed for a multiplication.

In general, when estimating the number of bit operations required to do something, the first step is to decide upon and write down an outline of a detailed procedure for performing the task. An explicit step-by-step procedure for doing calculations is called an *algorithm.* Of course, there may be many different algorithms for doing the same thing. One may choose to use the one that is easiest to write down, or one may choose to use the fastest one known, or else one may choose to compromise and make a trade-off between simplicity and speed. The algorithm used above for multiplying n by m is far from the fastest one known. But it is certainly a lot faster than repeated addition (adding n to itself m times).

Example 10. Estimate the time required to convert a k-bit integer to its representation in the base 10.

Solution. Let n be a k-bit integer written in binary. The conversion algorithm is as follows. Divide $10 = (1010)_2$ into n. The remainder — which will be one of the integers 0, 1, 10, 11, 100, 101, 110, 111, 1000, or 1001 — will be the ones digit d_0. Now replace n by the quotient and repeat the process, dividing that quotient by $(1010)_2$, using the remainder as d_1 and the quotient as the next number into which to divide $(1010)_2$. This process must be repeated a number of times equal to the number of decimal digits in n, which is $\left[\frac{log\, n}{log\, 10}\right]+1 = O(k)$. Then we're done. (We might want to take our list of decimal digits, i.e., of remainders from all the divisions, and convert them to the more familiar notation by replacing $0, 1, 10, 11, \ldots, 1001$ by $0, 1, 2, 3, \ldots, 9$, respectively.) How many bit operations does this all take? Well, we have $O(k)$ divisions, each requiring $O(4k)$ operations (dividing a number with at most k bits by the 4-bit number $(1010)_2$). But $O(4k)$ is the same as $O(k)$ (constant factors don't matter in the big-O notation), so we conclude that the total number of bit operations is $O(k) \cdot O(k) = O(k^2)$. If we want to express this in terms of n rather than k, then since $k = O(log\, n)$, we can write

$$\text{Time(convert } n \text{ to decimal)} = O(log^2 n).$$

Example 11. Estimate the time required to convert a k-bit integer n to its representation in the base b, where b might be very large.

Solution. Using the same algorithm as in Example 10, except dividing now by the ℓ-bit integer b, we find that each division now takes longer (if ℓ is large), namely, $O(k\ell)$ bit operations. How many times do we have to divide? Here notice that the number of base-b digits in n is $O(k/\ell)$ (see Example 9(c)). Thus, the total number of bit operations required to do all of the necessary divisions is $O(k/\ell) \cdot O(k\ell) = O(k^2)$. This turns out to be the same answer as in Example 10. That is, our estimate for the conversion time does not depend upon the base to which we're converting (no matter how large it may be). This is because the greater time required to find each digit is offset by the fact that there are fewer digits to be found.

Example 12. Express in terms of the O-notation the time required to compute (a) $n!$, (b) $\binom{n}{m}$ (see Examples 6 and 8).

Solution. (a) $O(n^2 log^2 n)$, (b) $O(m^2 log^2 n)$.

In concluding this section, we make a definition that is fundamental in computer science and the theory of algorithms.

Definition. An algorithm to perform a computation involving integers $n_1, n_2, \ldots, n_r$ of $k_1, k_2, \ldots, k_r$ bits, respectively, is said to be a *polynomial time* algorithm if there exist integers $d_1, d_2, \ldots, d_r$ such that the number of bit operations required to perform the algorithm is $O\big(k_1^{d_1} k_2^{d_2} \cdots k_r^{d_r}\big)$.

Thus, the usual arithmetic operations $+$, $-$, $\times$, $\div$ are examples of polynomial time algorithms; so is conversion from one base to another. On the other hand, computation of $n!$ is not. (However, if one is satisfied with knowing $n!$ to only a certain number of significant figures, e.g., its first 1000 binary digits, then one can obtain that by a polynomial time algorithm using Stirling's approximation formula for $n!$.)

Exercises

1. Multiply $(212)_3$ by $(122)_3$.
2. Divide $(40122)_7$ by $(126)_7$.
3. Multiply the binary numbers 101101 and 11001, and divide 10011001 by 1011.
4. In the base 26, with digits A—Z representing 0—25, (a) multiply YES by NO, and (b) divide JQVXHJ by WE.
5. Write $e = 2.7182818\cdots$ (a) in binary 15 places out to the right of the point, and (b) to the base 26 out 3 places beyond the point.
6. By a "pure repeating" fraction of "period" f in the base b, we mean a number between 0 and 1 whose base-b digits to the right of the point repeat in blocks of f. For example, $1/3$ is pure repeating of period 1 and $1/7$ is pure repeating of period 6 in the decimal system. Prove that a fraction c/d (in lowest terms) between 0 and 1 is pure repeating of period f in the base b if and only if $b^f - 1$ is a multiple of d.
7. (a) The "hexadecimal" system means $b = 16$ with the letters A–F representing the tenth through fifteenth digits, respectively. Divide $(131B6C3)_{16}$ by $(1A2F)_{16}$.

   (b) Explain how to convert back and forth between binary and hexadecimal representations of an integer, and why the time required is far less than the general estimate given in Example 11 for converting from binary to base-b.
8. Describe a subtraction-type bit operation in the same way as was done for an addition-type bit operation in the text (the list of five alternatives).

9. (a) Using the big-O notation, estimate in terms of a simple function of n the number of bit operations required to compute 3^n in binary.
(b) Do the same for n^n.
10. Estimate in terms of a simple function of n and N the number of bit operations required to compute N^n.
11. The following formula holds for the sum of the first n perfect squares:

$$\sum_{j=1}^{n} j^2 = n(n+1)(2n+1)/6.$$

(a) Using the big-O notation, estimate (in terms of n) the number of bit operations required to perform the computations in the left side of this equality.
(b) Estimate the number of bit operations required to perform the computations on the right in this equality.
12. Using the big-0 notation, estimate the number of bit operations required to multiply an $r \times n$-matrix by an $n \times s$-matrix, where all matrix entries are $\leq m$.
13. The object of this exercise is to estimate as a function of n the number of bit operations required to compute the product of all prime numbers less than n. Here we suppose that we have already compiled an extremely long list containing all primes up to n.
(a) According to the Prime Number Theorem, the number of primes less than or equal to n (this is denoted $\pi(n)$) is asymptotic to $n/log\, n$. This means that the following limit approaches 1 as $n \longrightarrow \infty$: $lim\ \frac{\pi(n)}{n/log\, n}$. Using the Prime Number Theorem, estimate the number of binary digits in the product of all primes less than n.
(b) Find a bound for the number of bit operations in one of the multiplications that's required in the computation of this product.
(c) Estimate the number of bit operations required to compute the product of all prime numbers less than n.
14. (a) Suppose you want to test if a large odd number n is a prime by trial division by all odd numbers $\leq \sqrt{n}$. Estimate the number of bit operations this will take.
(b) In part (a), suppose you have a list of prime numbers up to $\sqrt{n}$, and you test primality by trial division by those primes (i.e., no longer running through all odd numbers). Give a time estimate in this case. Use the Prime Number Theorem.
15. Estimate the time required to test if n is divisible by a prime $\leq m$. Suppose that you have a list of all primes $\leq m$, and again use the Prime Number Theorem.
16. Let n be a very large integer written in binary. Find a simple algorithm that computes $[\sqrt{n}]$ in $O(log^3 n)$ bit operations (here $[\ ]$ denotes the greatest integer function).

2 Divisibility and the Euclidean algorithm

Divisors and divisibility. Given integers a and b, we say that a *divides* b (or "b is *divisible* by a") and we write $a|b$ if there exists an integer d such that $b = ad$. In that case we call a a *divisor* of b. Every integer $b > 1$ has at least two positive divisors: 1 and b. By a *proper divisor* of b we mean a positive divisor not equal to b itself, and by a *nontrivial divisor* of b we mean a positive divisor not equal to 1 or b. A *prime* number, by definition, is an integer greater than one which has no positive divisors other than 1 and itself; a number is called *composite* if it has at least one nontrivial divisor. The following properties of divisibility are easy to verify directly from the definition:

1. If $a|b$ and c is any integer, then $a|bc$.
2. If $a|b$ and $b|c$, then $a|c$.
3. If $a|b$ and $a|c$, then $a|b \pm c$.

If p is a prime number and α is a nonnegative integer, then we use the notation $p^\alpha||b$ to mean that p^α is the highest power of p dividing b, i.e., that $p^\alpha|b$ and $p^{\alpha+1}\not|b$. In that case we say that p^α *exactly divides* b.

The *Fundamental Theorem of Arithmetic* states that any natural number n can be written uniquely (except for the order of factors) as a product of prime numbers. It is customary to write this factorization as a product of distinct primes to the appropriate powers, listing the primes in increasing order. For example, $4200 = 2^3 \cdot 3 \cdot 5^2 \cdot 7$.

Two consequences of the Fundamental Theorem (actually, equivalent assertions) are the following properties of divisibility:

4. If a prime number p divides ab, then either $p|a$ or $p|b$.
5. If $m|a$ and $n|a$, and if m and n have no divisors greater than 1 in common, then $mn|a$.

Another consequence of unique factorization is that it gives a systematic method for finding all divisors of n once n is written as a product of prime powers. Namely, any divisor d of n must be a product of the same primes raised to powers not exceeding the power that exactly divides n. That is, if $p^\alpha||n$, then $p^\beta||d$ for some β satisfying $0 \leq \beta \leq \alpha$. To find the divisors of 4200, for example, one takes 2 to the 0-, 1-, 2- or 3-power, multiplied by 3 to the 0- or 1-power, times 5 to the 0-, 1- or 2-power, times 7 to the 0- or 1- power. The number of possible divisors is thus the product of the number of possibilities for each prime power, which, in turn, is $\alpha+1$. That is, a number $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ has $(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_r+1)$ different divisors. For example, there are 48 divisors of 4200.

Given two integers a and b, not both zero, the *greatest common divisor* of a and b, denoted $g.c.d.(a,b)$ (or sometimes simply (a,b)) is the largest integer d dividing both a and b. It is not hard to show that another equivalent definition of $g.c.d.(a,b)$ is the following: it is the only positive integer d which divides a and b and is divisible by any other number which divides both a and b.

If you happen to have the prime factorization of a and b in front of you, then it's very easy to write down $g.c.d.(a, b)$. Simply take all primes which occur in both factorizations raised to the minimum of the two exponents. For example, comparing the factorization $10780 = 2^2 \cdot 5 \cdot 7^2 \cdot 11$ with the above factorization of 4200, we see that $g.c.d.(4200, 10780) = 2^2 \cdot 5 \cdot 7 = 140$.

One also occasionally uses the *least common multiple* of a and b, denoted $l.c.m.(a, b)$. It is the smallest positive integer that both a and b divide. If you have the factorization of a and b, then you can get $l.c.m.(a, b)$ by taking all of the primes which occur in *either* factorization raised to the *maximum* of the exponents. It is easy to prove that $l.c.m.(a, b) = |ab|/g.c.d.(a, b)$.

The Euclidean algorithm. If you're working with very large numbers, it's likely that you won't know their prime factorizations. In fact, an important area of research in number theory is the search for quicker methods of factoring large integers. Fortunately, there's a relatively quick way to find $g.c.d.(a, b)$ even when you have no idea of the prime factors of a or b. It's called the *Euclidean algorithm.*

The Euclidean algorithm works as follows. To find $g.c.d.(a, b)$, where $a > b$, we first divide b into a and write down the quotient q_1 and the remainder r_1: $a = q_1 b + r_1$. Next, we perform a second division with b playing the role of a and r_1 playing the role of b: $b = q_2 r_1 + r_2$. Next, we divide r_2 into r_1: $r_1 = q_3 r_2 + r_3$. We continue in this way, each time dividing the last remainder into the second-to-last remainder, obtaining a new quotient and remainder. When we finally obtain a remainder that divides the previous remainder, we are done: that final nonzero remainder is the greatest common divisor of a and b.

Example 1. Find $g.c.d.(1547, 560)$.

Solution:

$$\begin{aligned}1547 &= 2 \cdot 560 + 427\\ 560 &= 1 \cdot 427 + 133\\ 427 &= 3 \cdot 133 + 28\\ 133 &= 4 \cdot 28 + 21\\ 28 &= 1 \cdot 21 + 7.\end{aligned}$$

Since $7|21$, we are done: $g.c.d.(1547, 560) = 7$.

Proposition I.2.1. *The Euclidean algorithm always gives the greatest common divisor in a finite number of steps. In addition, for $a > b$*

$$\text{Time(finding } g.c.d.(a, b) \text{ by the Euclidean algorithm)} = O(log^3(a)).$$

Proof. The proof of the first assertion is given in detail in many elementary number theory textbooks, so we merely summarize the argument. First, it is easy to see that the remainders are strictly decreasing from one step to the next, and so must eventually reach zero. To see that the last remainder is the g.c.d., use the second definition of the g.c.d. That is, if any number divides both a and b, it must divide r_1, and then, since it divides

b and r_1, it must divide r_2, and so on, until you finally conclude that it must divide the last nonzero remainder. On the other hand, working from the last row up, one quickly sees that the last remainder must divide all of the previous remainders and also a and b. Thus, it is the g.c.d., because the g.c.d. is the only number which divides both a and b and at the same time is divisible by any other number which divides a and b.

We next prove the time estimate. The main question that must be resolved is how many divisions we're performing. We claim that the remainders are not only decreasing, but they're decreasing rather rapidly. More precisely:

Claim. $r_{j+2} < \frac{1}{2}r_j$.

Proof of claim. First, if $r_{j+1} \le \frac{1}{2}r_j$, then immediately we have $r_{j+2} < r_{j+1} \le \frac{1}{2}r_j$. So suppose that $r_{j+1} > \frac{1}{2}r_j$. In that case the next division gives: $r_j = 1 \cdot r_{j+1} + r_{j+2}$, and so $r_{j+2} = r_j - r_{j+1} < \frac{1}{2}r_j$, as claimed.

We now return to the proof of the time estimate. Since every two steps must result in cutting the size of the remainder at least in half, and since the remainder never gets below 1, it follows that there are at most $2 \cdot \left[log_2 a\right]$ divisions. This is $O(log\, a)$. Each division involves numbers no larger than a, and so takes $O(log^2 a)$ bit operations. Thus, the total time required is $O(log\, a) \cdot O(log^2 a) = O(log^3 a)$. This concludes the proof of the proposition.

Remark. If one makes a more careful analysis of the number of bit operations, taking into account the decreasing size of the numbers in the successive divisions, one can improve the time estimate for the Euclidean algorithm to $O(log^2 a)$.

Proposition I.2.2. *Let $d = g.c.d.(a, b)$, where $a > b$. Then there exist integers u and v such that $d = ua + bv$. In other words, the g.c.d. of two numbers can be expressed as a linear combination of the numbers with integer coefficients. In addition, finding the integers u and v can be done in $O(log^3 a)$ bit operations.*

Outline of proof. The procedure is to use the sequence of equalities in the Euclidean algorithm from the bottom up, at each stage writing d in terms of earlier and earlier remainders, until finally you get to a and b. At each stage you need a multiplication and an addition or subtraction. So it is easy to see that the number of bit operations is once again $O(log^3 a)$.

Example 1 (continued). To express 7 as a linear combination of 1547 and 560, we successively compute:

$$\begin{aligned}7 &= 28 - 1 \cdot 21 = 28 - 1(133 - 4 \cdot 28)\\ &= 5 \cdot 28 - 1 \cdot 133 = 5(427 - 3 \cdot 133) - 1 \cdot 133\\ &= 5 \cdot 427 - 16 \cdot 133 = 5 \cdot 427 - 16(560 - 1 \cdot 427)\\ &= 21 \cdot 427 - 16 \cdot 560 = 21(1547 - 2 \cdot 560) - 16 \cdot 560\\ &= 21 \cdot 1547 - 58 \cdot 560.\end{aligned}$$

Definition. We say that two integers a and b are *relatively prime* (or that "a is prime to b") if $g.c.d.(a, b) = 1$, i.e., if they have no common divisor greater than 1.

Corollary. *If $a > b$ are relatively prime integers, then 1 can be written as an integer linear combination of a and b in polynomial time, more precisely, in $O(log^3 a)$ bit operations.*

Definition. Let n be a positive integer. The *Euler phi-function* $\varphi(n)$ is defined to be the number of nonnegative integers b less than n which are prime to n:

$$\varphi(n) \underset{\text{def}}{=} \Big|\{0 \le b < n \mid g.c.d.(b,n) = 1\}\Big|.$$

It is easy to see that $\varphi(1) = 1$ and that $\varphi(p) = p - 1$ for any prime p. We can also see that for any prime power

$$\varphi(p^\alpha) = p^\alpha - p^{\alpha-1} = p^\alpha\Big(1 - \frac{1}{p}\Big).$$

To see this, it suffices to note that the numbers from 0 to $p^\alpha - 1$ which are *not* prime to p^α are precisely those that are divisible by p, and there are $p^{\alpha-1}$ of those.

In the next section we shall show that the Euler φ-function has a "multiplicative property" that enables us to evaluate $\varphi(n)$ quickly, provided that we have the prime factorization of n. Namely, if n is written as a product of powers of distinct primes p^α, then it turns out that $\varphi(n)$ is equal to the product of the $\varphi(p^\alpha)$.

Exercises

1. (a) Prove the following properties of the relation $p^\alpha||b$: (i) if $p^\alpha||a$ and $p^\beta||b$, then $p^{\alpha+\beta}||ab$; (ii) if $p^\alpha||a$, $p^\beta||b$ and $\alpha < \beta$, then $p^\alpha||a \pm b$.
   (b) Find a counterexample to the assertion that, if $p^\alpha||a$ and $p^\alpha||b$, then $p^\alpha||a + b$.
2. How many divisors does 945 have? List them all.
3. Let n be a positive odd integer.
   (a) Prove that there is a 1-to-1 correspondence between the divisors of n which are $< \sqrt{n}$ and those that are $> \sqrt{n}$. (This part does not require n to be odd.)
   (b) Prove that there is a 1-to-1 correspondence between all of the divisors of n which are $\ge \sqrt{n}$ and all the ways of writing n as a difference $s^2 - t^2$ of two squares of nonnegative integers. (For example, 15 has two divisors 6, 15 that are $\ge \sqrt{15}$, and $15 = 4^2 - 1^2 = 8^2 - 7^2$.)
   (c) List all of the ways of writing 945 as a difference of two squares of nonnegative integers.
4. (a) Show that the power of a prime p which exactly divides $n!$ is equal to $[n/p] + [n/p^2] + [n/p^3] + \cdots$. (Notice that this is a finite sum.)
   (b) Find the power of each prime 2, 3, 5, 7 that exactly divides 100!, and then write out the entire prime factorization of 100!.