Ryan Skinner

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1 Introduction

We solve the following problems to better understand numerical techniques for solving ordinary differential equations. As will be described in the methods section, our tools primarily consist of second- and fourth-order Runge-Kutta methods, and Euler stability analysis.

1.1 *Problem 1*

Equations of motion for a rocket's vertical speed v can be written as

$$(m_c + m_p)\frac{dv}{dt} = -(m_c + m_p)g + m_p v_e - \tfrac{1}{2}\rho v\,|v|AC_D, \tag{1}$$

where z is the vertical coordinate, $v = dz/dt$ is the vertical speed, and

$$\begin{aligned}
m_c &= 51.02 \text{ kg} && \text{(rocket casing mass)}\\
g &= 9.8 \text{ m/s}^2 && \text{(gravitational acceleration)}\\
\rho &= 1.23 \text{ kg/m}^3 && \text{(air density)}\\
A &= 0.1 \text{ m}^2 && \text{(maximum cross-sectional area)}\\
v_e &= 360 \text{ m/s} && \text{(exhaust speed)}\\
C_D &= 0.15 && \text{(drag coefficient)}\\
m_{p0} &= 102.04 \text{ kg} && \text{(initial propellant mass).}
\end{aligned}$$

Furthermore, the instantaneous propellant mass at time t is given by

$$m_p(t) = m_{p0} - \int_0^t \dot{m}_p dt, \tag{2}$$

and the time-varying burn rate is

$$\dot{m}_p = \frac{m_{p0}}{4} \cdot \begin{cases} t & 0 \le t \le 1 \\ 1 & 1 \le t \le 4 \\ 5 - t & 4 \le t \le 5 \\ 0 & 5 \le t. \end{cases} \tag{3}$$

Use a second-order Runge-Kutta (RK2) integrator with $\Delta t = 0.1$ s to plot $z(t)$ and $v(t)$. Use these plots to find the maximum speed, and the height and time at which it is reached; the maximum height, and the time at which it is reached; and the time and velocity when the rocket hits the ground. Check these results with those obtained from MATLAB's `ODE45` solver.

This problem is fairly straight-forward. An RK2 scheme is relatively easy to implement, but we will need to be careful when accounting for the time-dependence of m_p. Furthermore, though it is not stated in the problem, when m_p reaches zero, the thrust term involving the exhaust speed v_e in (1) needs to "turn off."

1.2 Problem 2

Consider the stream function for a two-dimensional jet in self-similar form, which can be written as

$$f''(\eta) + f(\eta)f'(\eta) = 0, \qquad f(0) = 0, \qquad f'(0) = 1. \tag{4}$$

The velocities in the jet can be obtained via

$$\frac{U}{U_0} = f'(\eta), \qquad \frac{V}{U_0} = \eta f'(\eta) - \tfrac{1}{2} f(\eta). \tag{5}$$

Using numerical integration for $0 \leq \eta \leq 4$, with a step size of $\Delta\eta = 0.05$, plot the quantities U/U_0, V/V_0, and f as functions of η. Perform this analysis with both the second- and fourth-order Runge-Kutta methods, and compare their performance. Both solutions may be compared to those obtained from the best-fit expression

$$f(\eta) = \exp(-0.682\eta^2). \tag{6}$$

1.3 Problem 3

Show that the equation

$$y'' = -\frac{19}{4} y - 10y', \qquad y(0) = -9, \qquad y'(0) = 0 \tag{7}$$

is moderately stiff. Use Euler stability analysis to estimate the largest step size h_{max} for which the Runge-Kutta method will be stable. Then confirm this estimate by computing y using the fourth-order Runge-Kutta method with $h = \{\frac{1}{2} h_{\text{max}}, 2h_{\text{max}}\}$. Compare these solutions with the analytical solution.

2 Methodology

3 Results

4 Discussion

5 References

No external references were used other than the course notes for this assignment.

Appendix: MATLAB Code

The following code listings generate all figures presented in this homework assignment.