Homework 5

ASEN 5417: Numerical Methods

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1 Introduction

1.1 Problem 1

The partial differential equation (PDE) that governs one-dimensional heat transfer, with constants and boundary conditions given for a 2 cm-thick steel pipe, is

$$\frac{\partial u}{\partial t} = \frac{k}{c\rho}\frac{\partial^2 u}{\partial x^2}\,, \qquad \begin{aligned} u(0,t) &= 0\,, \\ u(2,t) &= 0\,, \\ u(x,0) &= 100 sin(\pi x/2)\,, \end{aligned} \qquad \begin{aligned} k &= 0.13 && \text{cal / sec cm °C}\,, \\ c &= 0.11 && \text{cal / g °C}\,, \\ \rho &= 7.8 && \text{g / cm}^3\,, \end{aligned} \tag{1}$$

where $u(x,t)$ is the temperature in °C. Numerically integrate this PDE using the explicit forward-time, central-space (FTCS, or Euler) method with $\Delta x = 0.1$ cm. Choose Δt such that the diffusion criterion is (a) $d = \frac{1}{2}$ and (b) $d = 1$. Compare your solution for (a) at $t = \{1,2,4,8\}$ sec with the analytical solution

$$u = 100\,\mathrm{exp}(-0.3738t)\,\mathrm{sin}(\pi x/2)\,. \tag{2}$$

Note that for (b), the diffusion criterion is higher than the one allowed by the von Neumann method. For this case, plot your solutions at approximately $t = \{\frac{1}{10}, \frac{1}{2}, 1, 2\}$ sec, and comment on stability.

1.2 Problem 2

Solve the system from Problem 1 using the implicit Crank-Nicolson method with forward time differencing. The Thomas algorithm can be utilized for this purpose. Try values for the diffusion criterion of $d = \{\frac{1}{2}, 1, 10\}$, and comment on the solution's stability.

1.3 Problem 3

Solve the system from Problem 1, but with an adiabatic boundary condition at the upper boundary,

$$u(0,t) = 0\,, \qquad \frac{\partial u}{\partial x}(2,t) = 0\,. \tag{3}$$

2 Methodology

2.1 Problem 1

Discretizing (1) using the forward-time, central-space (FTCS) method with explicit time advancement yields

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}\,, \tag{4}$$

where the subscript denotes the spatial grid point, and the superscript indexes the time step. Further note it is convenient to define the coefficient of thermal diffusivity as $\alpha \equiv k/c\rho = 0.1515$ m/s^2. Solving for the unknown quantity, we obtain

$$u_i^{n+1} = du_{i-1}^n + (1-2d)u_i^n + du_{i+1}^n\,, \tag{5}$$

where the diffusion number is defined as

$$d = \frac{\alpha \Delta t}{\Delta x^2} \quad \rightarrow \quad \Delta t = \frac{d \Delta x^2}{\alpha} \,. \tag{6}$$

Using (5), it is trivial to step forward in time with Dirichlet boundary conditions on u. We use (6) to obtain the requested diffusion numbers by setting (a) $\Delta t = 0.033$ and (b) $\Delta t = 0.066$.

2.2 Problem 2

The implicit Crank-Nicolson method with forward time-differencing is most-commonly used to solve the diffusion equation, which can be written as

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\alpha}{2\Delta x^2}\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} + u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) \,. \tag{7}$$

It is a simple matter to re-write this as a linear equation relating the value of u_i at the next time step to its spatial neighbors at the next time step, and its spatial neighbors at the current time step:

$$u_i^{n+1} = \beta\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} + u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) + u_i^n \tag{8}$$

$$\underbrace{(-\beta)}_{b_i} u_{i-1}^{n+1} + \underbrace{(1+2\beta)}_{a_i} u_i^{n+1} + \underbrace{(-\beta)}_{c_i} u_{i+1}^{n+1} = \underbrace{\beta\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) + u_i^n}_{d_i} \,, \tag{9}$$

where

$$\beta = \frac{\alpha \Delta t}{2\Delta x^2} \,, \tag{10}$$

is known, along with all point-wise values of u^n. We solve the system of equations using Dirichlet boundary conditions with the same method detailed in Homework 3.

2.3 Problem 3

3 Results

3.1 Problem 1

Results for the FTCS method are presented in Figure 1.

3.2 Problem 2

Results for the Crank-Nicolson method with Dirichlet boundary conditions are presented in Figure 2.

3.3 Problem 3

4 Discussion

4.1 Problem 1

As can be seen in Figure 1, FTCS results for a diffusion number of $d = \frac{1}{2}$ have fairly low relative error at all time-steps considered, and no numerical instability is present. For a condition number of $d = 1$, which should be unstable according to the von Neumann method, we see high-frequency instabilities manifest at $t \sim 2.0$. Relative error grows rapidly, as can be seen at $t \sim 2.1$. Later times are not shown, because the solution quickly becomes non-physical.

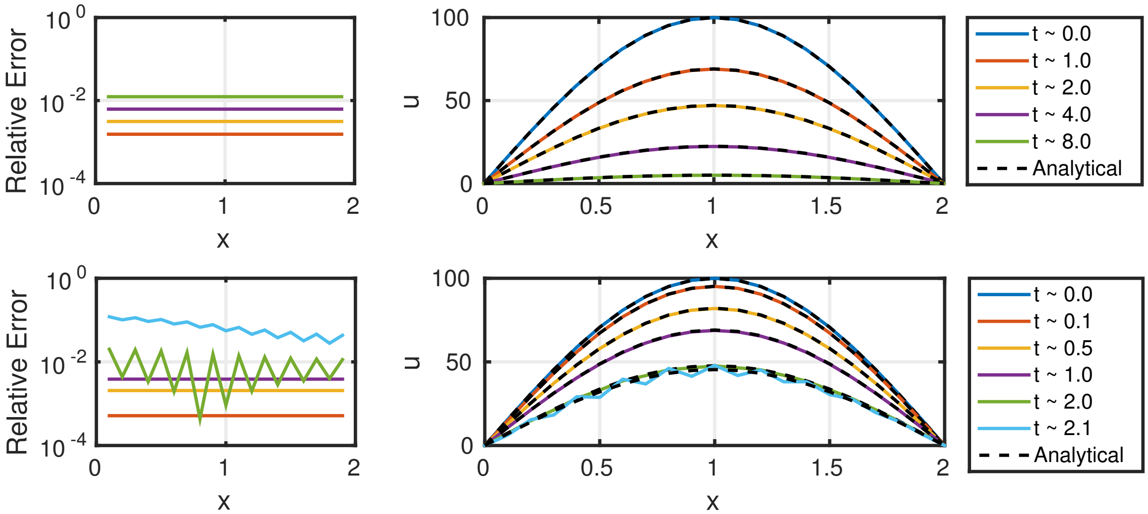

Figure 1: FTCS solutions to (1) compared to analytical solutions (2), and the point-wise relative error over time. Upper plots are for $d = \frac{1}{2}$ ($\Delta t = 0.033$), and lower plots correspond to $d = 1$ ($\Delta t = 0.066$).

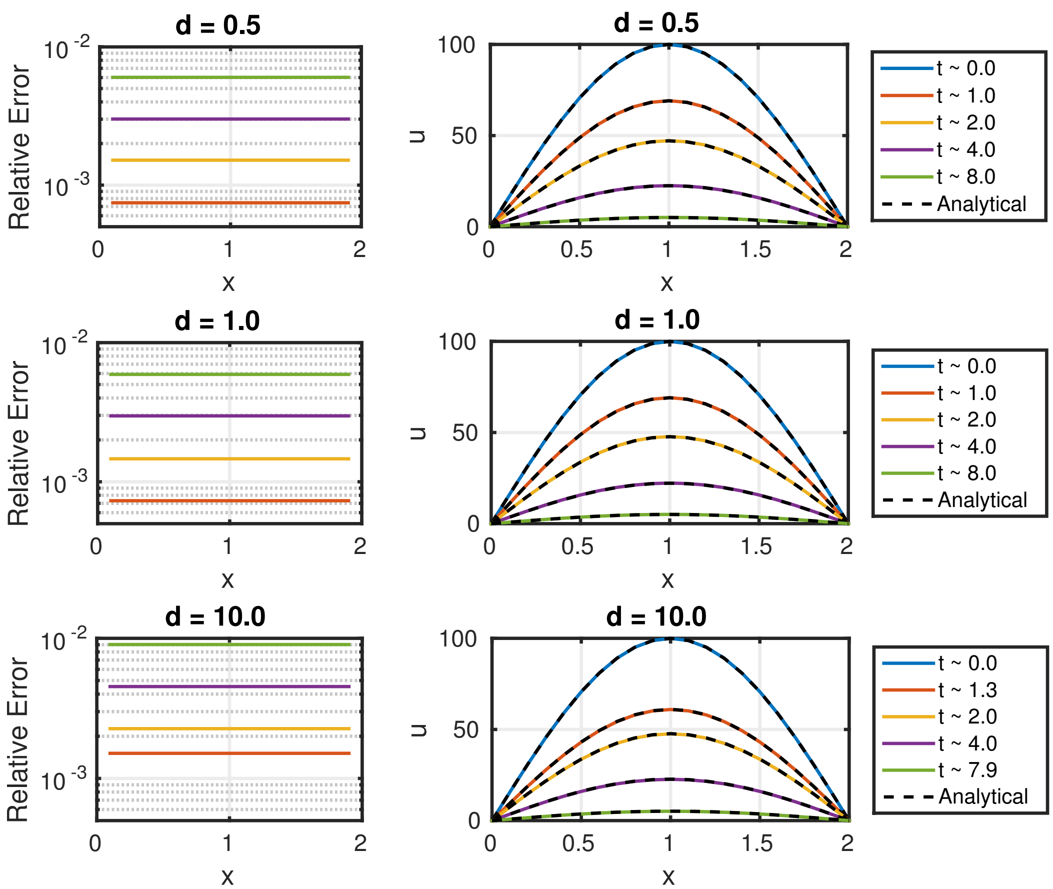

Figure 2: Crank-Nicolson solutions to (1) compared to analytical solutions (2), and the point-wise relative error over time. Diffusion numbers d are annotated. For $d = 10$, $\Delta t = 0.6601$.

4.2 Problem 2

As shown in Figure 2, the Crank-Nicolson method produces results with point-wise errors slightly better than the FTCS method, at diffusion numbers at least 10× greater than the maximum stable diffusion number of the FTCS method. We conclude that the Crank-Nicolson method is very stable. Though we do need to solve a matrix system at each time step, Δt can be much higher than the FTCS method, and thus the Crank-Nicolson method has the potential for much higher efficiency.

4.3 Problem 3

5 References

No external references were used other than the course notes for this assignment.

Appendix: MATLAB Code

The following code listings generate all figures presented in this homework assignment.