1 Introduction

1.1 Problem 1

The partial differential equation (PDE) that governs one-dimensional heat transfer, with constants and boundary conditions given for a 2 cm-thick steel pipe, is

$$\frac{\partial u}{\partial t} = \frac{k}{c\rho}\frac{\partial^2 u}{\partial x^2}\,, \qquad \begin{array}{l} u(0,t)=0\,, \\ u(2,t)=0\,, \\ u(x,0)=100sin(\pi x/2)\,, \end{array} \qquad \begin{array}{rll} k=&0.13 & \text{cal / sec cm °C}\,, \\ c=&0.11 & \text{cal / g °C}\,, \\ \rho=&7.8 & \text{g / cm}^3\,, \end{array} \tag{1}$$

where $u(x,t)$ is the temperature in °C. Numerically integrate this PDE using the explicit forward-time, central-space (FTCS, or Euler) method with $\Delta x = 0.1$ cm. Choose Δt such that the diffusion criterion is (a) $d = 0.5$ and (b) $d = 1.0$. Compare your solution for (a) at $t = \{1,2,4,8\}$ sec with the analytical solution

$$u = 100\exp(-0.3738t)\sin(\pi x/2)\,. \tag{2}$$

Note that for (b), the diffusion criterion is higher than the one allowed by the von Neumann method. For this case, plot your solutions at approximately $t = \{0.1, 0.5, 1, 2\}$ sec, and comment on stability.

1.2 Problem 2

Solve the system from Problem 1 using the implicit Crank-Nicolson method with forward time differencing. The Thomas algorithm can be utilized for this purpose. Try values for the diffusion criterion of $d = \{0.5, 1, 10\}$, and comment on the solution's stability.

1.3 Problem 3

Solve the system from Problem 1, but with an adiabatic boundary condition at the upper boundary,

$$u(0,t)=0\,, \qquad \frac{\partial u}{\partial x}(2,t)=0\,. \tag{3}$$

2 Methodology

2.1 Problem 1

Discretizing (1) using the forward-time, central-space (FTCS) method with explicit time advancement yields

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} = \alpha\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}\,, \tag{4}$$

where the subscript denotes the spatial grid point, and the superscript indexes the time step. Further note it is convenient to define the coefficient of thermal diffusivity as $\alpha \equiv k/c\rho = 0.1515$ m/s^2. Solving for the unknown quantity, we obtain

$$u_i^{n+1} = du_{i-1}^n + (1-2d)u_i^n + du_{i+1}^n\,, \tag{5}$$

where the diffusion number is defined as

$$d = \frac{\alpha \Delta t}{\Delta x^2} \quad \rightarrow \quad \Delta t = \frac{d \Delta x^2}{\alpha} \, . \tag{6}$$

Using (5), it is trivial to step forward in time with Dirichlet boundary conditions on u. We use (6) to obtain the requested diffusion numbers by setting (a) $\Delta t = 0.033$ and (b) $\Delta t = 0.066$.

2.2 Problem 2

2.3 Problem 3

3 Results

3.1 Problem 1

Results for the FTCS method are presented in Figure 1.

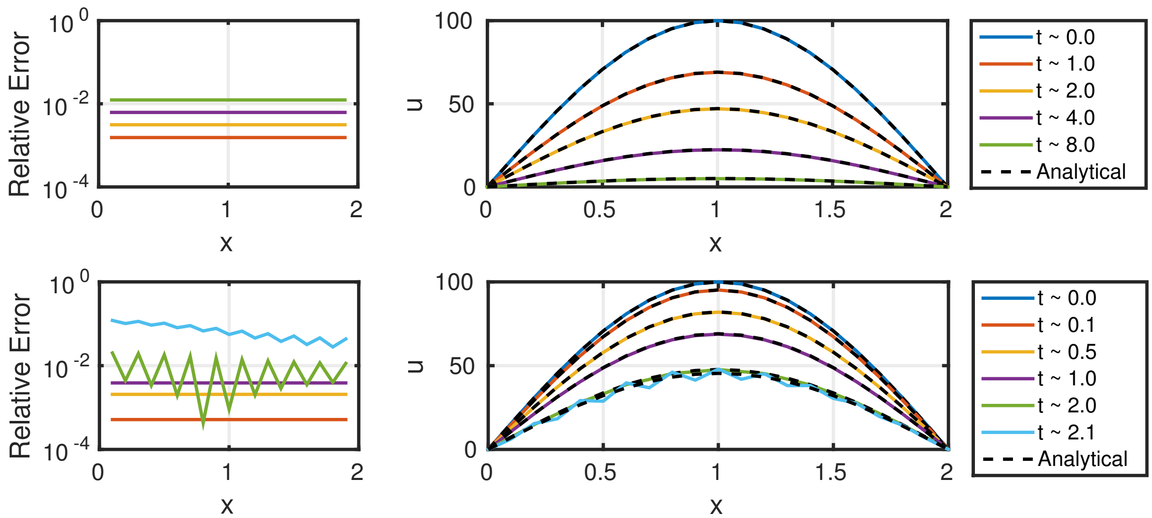

Figure 1: FTCS solutions to (1) compared to analytical solutions (2), and the point-wise relative error over time. Upper plots are for $d = 0.5$ ($\Delta t = 0.033$), and lower plots correspond to $d = 0.1$ ($\Delta t = 0.066$).

3.2 Problem 2

3.3 Problem 3

4 Discussion

4.1 Problem 1

As can be seen in Figure 1, FTCS results for a diffusion number of $d = 0.5$ have fairly low relative error at all time-steps considered, and no numerical instability is present. For a condition number of $d = 1.0$, which should be unstable according to the von Neumann method, we see instabilities manifest at $t \sim 2.0$. Relative error grows rapidly, as can be seen at $t \sim 2.1$. Later times are not shown, because the solution blows up.

4.2 Problem 2

4.3 Problem 3

5 References

No external references were used other than the course notes for this assignment.

Appendix: MATLAB Code

The following code listings generate all figures presented in this homework assignment.