Homework 6

ASEN 5417: Numerical Methods

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1 Introduction

The equations that govern open channel flow at an inclination of θ degrees with the horizontal can be transformed into a Poisson equation by scaling the streamwise velocity u as

$$U(Y,Z) = \frac{u(Y,Z)}{L^2 \rho g \sin(\theta/\mu)} , \tag{1}$$

where L is the length of the square channel, ρ is the fluid density, g is gravitational acceleration, and μ is the fluid's dynamic viscosity. The cross-sectional dimensions in y and z are also normalized by $Y = y/L$ and $Z = Z/L$. Through these scaling procedures, the governing equations map onto a unit square as

$$U_{,YY} + U_{,ZZ} = -1 , \tag{2}$$

$$\begin{aligned} U(0,Z) &= 0 , \\ U_{,Y}(1,Z) &= 0 , \\ U(Y,0) = U(Y,1) &= 0 . \end{aligned} \tag{3}$$

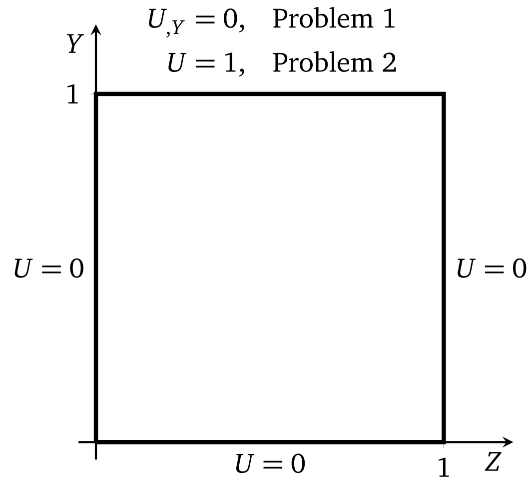

Figure 1: Boundary value problem for Homework 6. No-slip conditions are imposed at three side walls. For Problem 2, the upper boundary is a moving wall.

1.1 Problem 1

Numerically integrate (2) with the stated boundary conditions (3) using the ADI method with $N = 101$ grid points in each direction. Implement LU decomposition to solve the tridiagonal systems, and determine convergence by a reduction of the original error by three orders of magnitude.

Plot the contours of U in the Y-Z plane at convergence.

1.2 Problem 2

Change the upper boundary condition to represent a solid lid moving at a constant velocity with

$$U_{,Y}(1,Z) = 0 \quad \longrightarrow \quad U(1,Z) = 1 . \tag{4}$$

Solve this problem using the SOR method. As in Problem 1, define convergence as a reduction by three orders of magnitude of the initial error in the maximum norm.

Obtain the best estimate for the acceleration parameter ω by numerical experimentation. That is, plot the number of iterations required for convergence as a function of ω, and determine the value of ω that minimizes this function. How does this value of ω compare to the theoretical value?

Plot the contours of U in the Y-Z plane at convergence, and compare the results to Problem 1.

2 Methodology

2.1 Problem 1

The alternating-direction implicit (ADI) method assumes a pseudo-time derivative ($\partial/\partial t$), such that (2) becomes

$$T_{,t} = U_{,YY} + U_{,ZZ} = -1 \,, \tag{5}$$

and our solution for (2) can be interpreted as the steady-state solution to (5) as $t \to \infty$. This equation is parabolic in space and elliptic in time. Of course, the transient is not physical, so it acceptable to advance the solution in time using the fully implicit Euler method.

The ADI method breaks the problem into two directions and solves each over two half-time steps. Discretizing the problem using second-order central differences, defining the inhomogeneous source term as $\xi(x,y) = 1$, and letting the grid spacing in both directions be equal ($h \equiv \Delta y = \Delta z$), we obtain

$$U^{n+1/2}_{(i+1,j)} - (2+\rho)U^{n+1/2}_{(i,j)} + U^{n+1/2}_{(i-1,j)} = -U^{n}_{(i,j+1)} + (2-\rho)U^{n}_{(i,j)} - U^{n}_{(i,j-1)} - h^2\xi_{(i,j)} \,, \tag{6}$$

$$U^{n+1}_{(i,j+1)} - (2+\rho)U^{n+1}_{(i,j)} + U^{n+1}_{(i,j-1)} = -U^{n+1/2}_{(i+1,j)} + (2-\rho)U^{n+1/2}_{(i,j)} - U^{n+1/2}_{(i-1,j)} - h^2\xi_{(i,j)} \,, \tag{7}$$

where $\rho = h^2/\Delta t$. The first pass loops over all j values, and for each j solves for U at all i-locations. The second equation does the opposite. Initial guesses of the solution along the domain must be provided, but the RHS is always known and solution proceeds in the standard manner for implicit central differences. Boundary conditions are incorporated in the standard fashion as well, by modifying the RHS and diagonal terms as needed. To determine convergence, we define our error norm for the n^{th} iteration as

$$\epsilon_n = \frac{1}{\epsilon_1} \sum_{i=1}^{N} \sum_{j=1}^{N} \left| U^{n}_{(i,j)} - U^{n-1}_{(i,j)} \right| \,, \quad n = 2, 3, \dots \,, \tag{8}$$

and cease solution when $\epsilon_n \le 10^{-3}$. Here, ϵ_1 is defined as the RHS sums evaluated for $n = 1$, and the initial guess is defined as $U^0_{(i,j)} = 0$.

Peaceman and Rachford (1955) show that the ADI method converges for any value of the iteration parameter ρ. The optimal value is a function of the iteration number:

$$\rho = 4 \sin^2 \frac{k\pi}{2N} \,, \qquad k = 1, 2, ..., n \quad \text{(until convergence)} \,. \tag{9}$$

If we were to change ρ on each iteration, the **L** and **U** matrices involved in LU-decomposition would need to change each iteration as well, and we would sacrifice the efficiency gains of LU-decomposition. Instead, we choose a constant value of $\rho = 0.002$ that gives reasonably fast convergence (9 iterations) based on experimentation. The details of LU-decomposition will not be discussed here, as they were presented in full in Homework 4.

2.2 Problem 2

We modify the equations from Problem 1 ever so slightly for the SOR method, defining our convergence criterion as

$$\epsilon_n = \frac{1}{\epsilon_1} \max_{(i,j)} \left| U^{n}_{(i,j)} - U^{n-1}_{(i,j)} \right| \,. \tag{10}$$

Similar to Problem 1, ϵ_1 is defined as the RHS max operation evaluated for $n = 1$, and the initial guess is defined as $U^0_{(i,j)} = 0$ except for when $j = N$, in which case the $U = 1$ boundary condition is imposed.

3 Results

3.1 Problem 1

A contour plot of the solution for U obtained from the ADI method is presented in Figure 2.

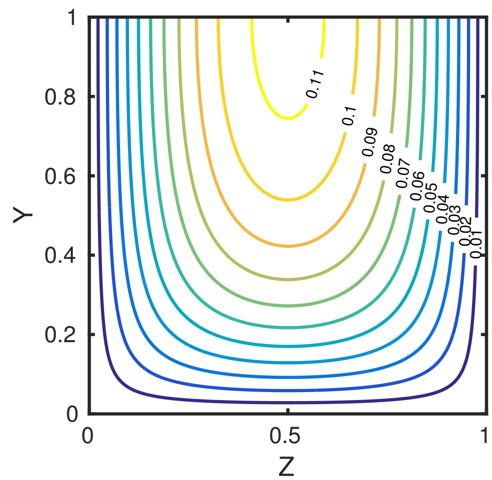

Figure 2: Contours of U for Problem 1.

3.2 Problem 2

Iterations to convergence as a function of ω, as well as a contour plot of the solution for U obtained from the SOR method is presented in Figure 3.

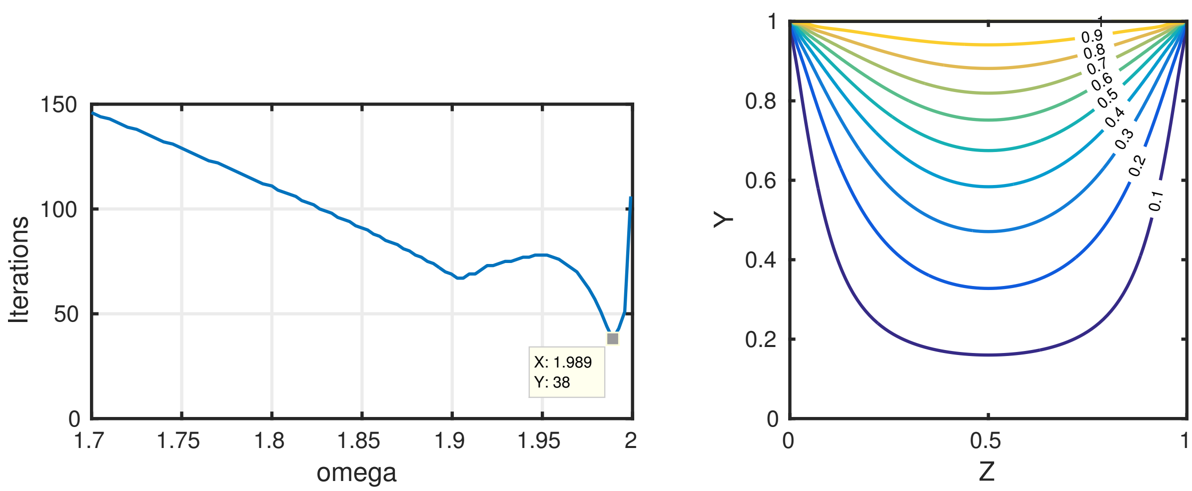

Figure 3: Convergence behavior and contours of U for Problem 2.

4 Discussion

4.1 Problem 1

4.2 Problem 2

5 References

No external references were used other than the course notes for this assignment.

Appendix: MATLAB Code

The following code listings generate all figures presented in this homework assignment.