1 INTRODUCTION

Consider the linear convection-diffusion equation

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}\,, \qquad v = \sin(\pi y)\,, \qquad \alpha = \frac{1}{\mathrm{Pr}\,\mathrm{Re}}\,, \tag{1}$$

subject to the initial conditions

$$\begin{aligned} &\text{(a)} \quad T(y, t=0) = \cos(2\pi y)\sin(\pi y) \\ &\text{(b)} \quad T(y, t=0) = \cos(2\pi y)\,, \end{aligned} \tag{2}$$

and parameters

$$\begin{aligned} \mathrm{Re} &= 1 && \text{(Reynolds number, molten glass)} \\ \mathrm{Pr} &= 25 && \text{(Prandtl number, molten glass)} \\ \Delta t &= 0.001 && \text{(Time step)} \\ L_y &= 2 && \text{(Domain } y\text{-length)} \\ N &= 2^n + 1 && \text{(Number of } y\text{-points, where } n = 5, 6)\,. \end{aligned} \tag{3}$$

1.1 PROBLEM 1

Use the Fourier pseudo-spectral method to numerically integrate (1) with the given parameters. Use the Euler explicit method for time advancement. Higher resolution with $n = 6$ will improve the accuracy of the method for the initial condition (a). Plot T as a function of time t at $t = \{0.2, 2, 5, 10\}$.

1.2 PROBLEM 2

Use the FTCS Euler explicit method with second-order finite differences for the same computation, and compare results to the Fourier pseudo-spectral method using the same mesh resolution.

2 METHODOLOGY

2.1 PROBLEM 1

We re-write (1) as

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial y^2} - v\frac{\partial T}{\partial y}\,. \tag{4}$$

The first and second spatial derivatives of T are calculated by taking the Fourier transform of T, multiplying the Fourier coefficients by ik_n and $-k_n^2$, respectively, and then taking the inverse Fourier transform. With values of $\partial T/\partial t$ known at all grid points now, values of T at the next time step are calculated using the explicit Euler method,

$$T^{n+1} = T^n + \Delta t\frac{\partial T}{\partial t}\,. \tag{5}$$

2.2 Problem 2

To implement the FTCS explicit method, we approximate (1) using forward-differences in time and central-differences in space as

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} + \nu \frac{T_{i+1}^n - T_{i-1}^n}{2\Delta y} = \alpha \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta y^2}\,, \tag{6}$$

which can be solved for T_i^{n+1} as

$$T_i^{n+1} = T_i^n + \Delta t \left(\alpha \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta y^2} - \nu \frac{T_{i+1}^n - T_{i-1}^n}{2\Delta y} \right). \tag{7}$$

This constitutes an explicit equation for T^{n+1}, and thus we can step forward in time without the need to solve any matrix equations. We enforce periodic boundary conditions by wrapping T_{i-1} to T_N when $i = 1$ and vice versa.

3 Results

4 Discussion

The FTCS method for initial condition (b) has difficulty capturing the diffusive behavior accurately; the limiting Fourier value is $T \sim 0.765$ for both $n = \{5, 6\}$, whereas the FTCS method's limiting values are $T \sim \{0.806, 0.788\}$ respectively.

5 References

No external references were used other than the course notes for this assignment.

Appendix: MATLAB Code

The following code listings generate all figures presented in this homework assignment.

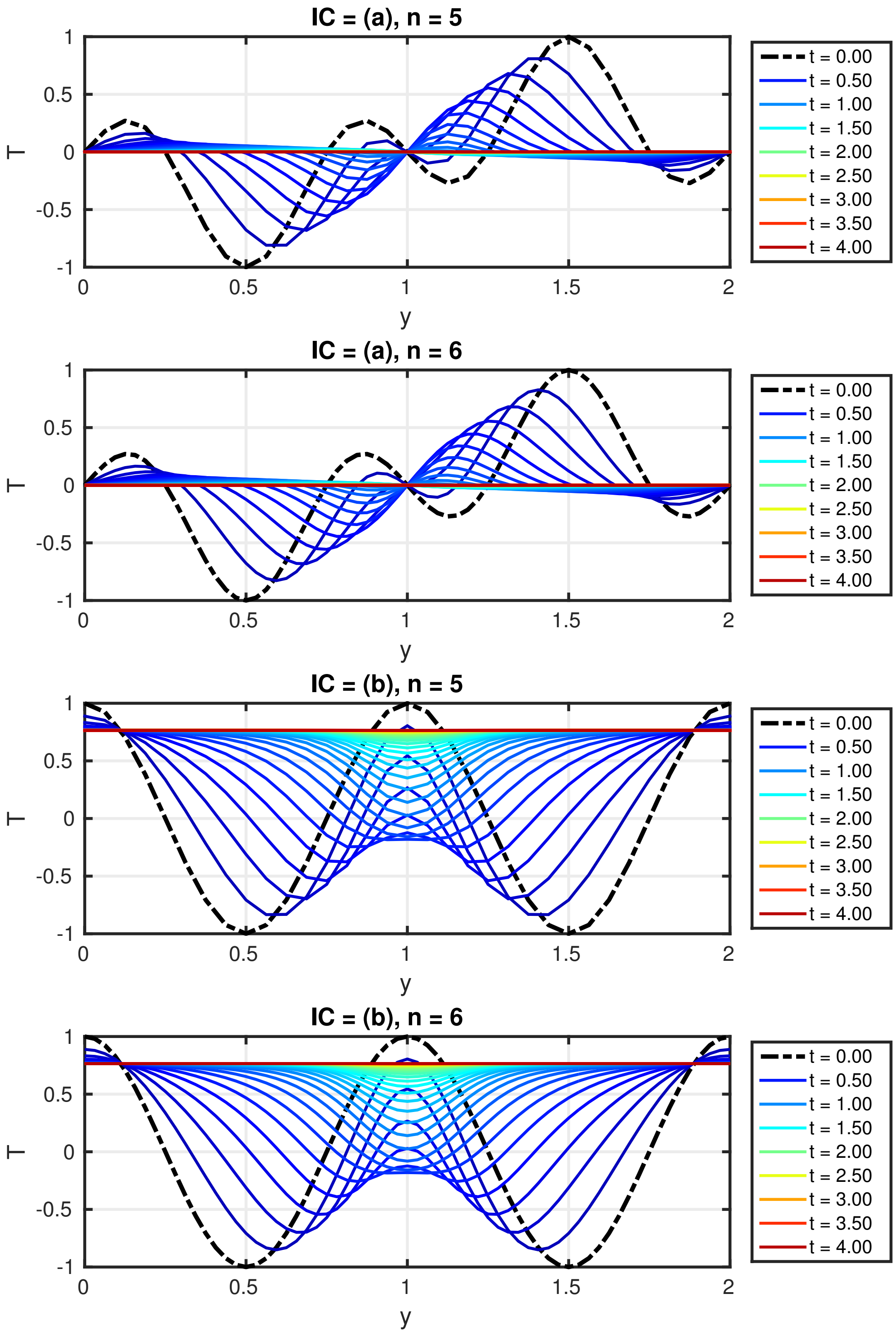

Figure 1: Solutions to the linear convection-diffusion equation using the Fourier pseudo-spectral method for different initial conditions and mesh resolution parameters. Most interesting behavior occurs when $t < 4$. Trends can be extrapolated to future times $t = 5, 10$, which are not shown. Initial condition displayed as black dot-dashed line (·−·−).

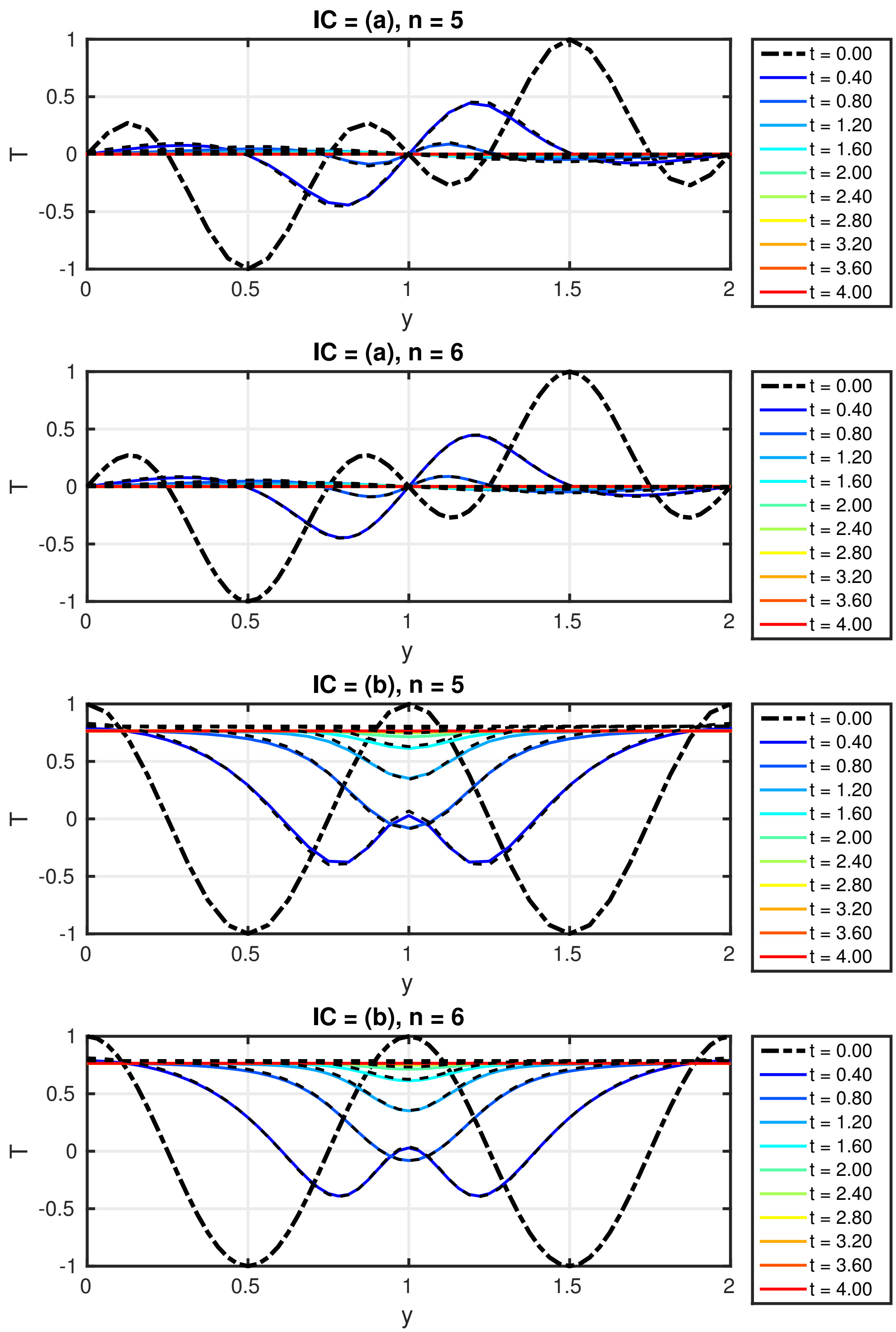

Figure 2: Solution comparison between the Fourier pseudo-spectral method (colored lines) and the FTCS method (−−−) for $t < 4$. Initial condition displayed as black dot-dashed line (·−·−). Agreement between the Fourier and FTCS methods is decent. The limiting Fourier value is $T \sim 0.765$ for both $n = \{5, 6\}$, whereas the FTCS method's limiting values are $T \sim \{0.806, 0.788\}$ respectively. Further analysis is deferred to Figure 3.

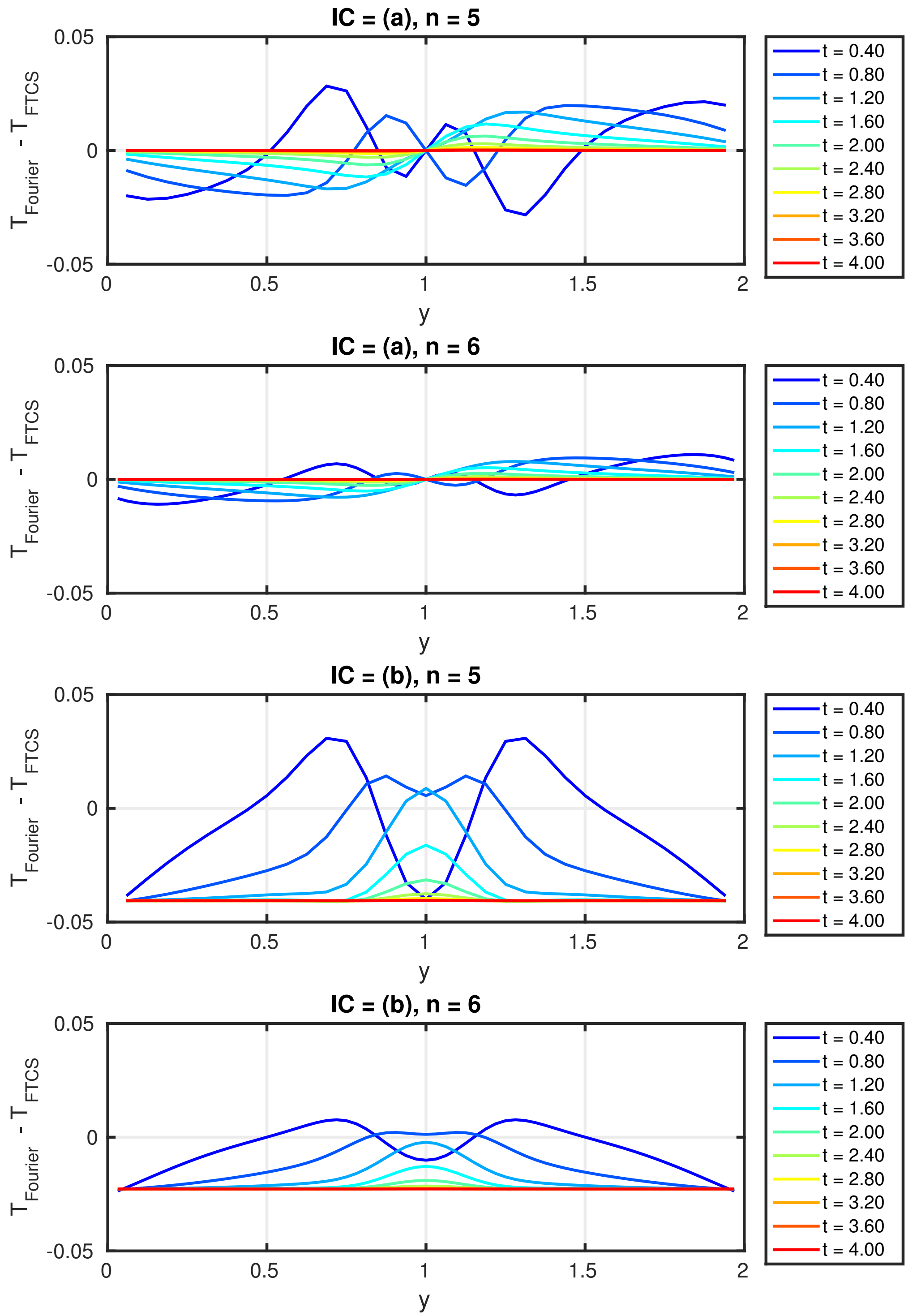

Figure 3: Difference between the Fourier pseudo-spectral and FTCS solutions at evenly-spaced discrete times $t < 4$, omitting the initial condition. Different initial conditions and spatial resolutions produce different behavior.