$$\overline{f+g} = \bar f + \bar g \qquad \overline{af} = a\bar f \quad (a \text{ const.})$$

$$\overline{fg} = \bar f\, \bar g . \qquad \bar{\bar a} = a \qquad \overline{\frac{\partial f}{\partial s}} = \frac{\partial \bar f}{\partial s}, \text{ when } s = x_i \text{ or } t.$$

$$\boxed{\cancel{\frac{\overline{u_i u_j}}{k} - \frac{2}{3}\delta_{ij} = \frac{-2 C_\rho k \bar S_{ij}}{\overline{u_k' u_\ell'}\, \bar S_{k\ell}}} \qquad \cancel{\mathrm{III}_a}}$$

(1.1) $u_i' \equiv U_i - \overline{U_i}$, $\Rightarrow u_i = \bar u_i + u_i'$

Starting with the Navier-Stokes equations... incompress.

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}, \qquad \frac{\partial u_i}{\partial x_i} = 0$$

NS

We take the ensemble average of both sides:

$$\overline{\frac{\partial u_i}{\partial t}} + \overline{u_j \frac{\partial u_i}{\partial x_j}} = \overline{\frac{-1}{\rho}\frac{\partial p}{\partial x_i}} + \overline{\nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}}, \qquad \overline{\frac{\partial u_i}{\partial x_i}} = 0$$

And simplify using the rules at the top of this page.

$$\frac{\partial \bar u_i}{\partial t} + \overline{u_j \frac{\partial u_i}{\partial x_j}} = \frac{-1}{\rho}\frac{\partial \bar p}{\partial x_i} + \nu \frac{\partial^2 \bar u_i}{\partial x_j \partial x_j}, \qquad \frac{\partial \bar u_i}{\partial x_i} = 0$$

$$\left(\frac{\partial \overline{u_i u_j}}{\partial x_j} = \overline{\frac{\partial (u_i u_j)}{\partial x_j}} = \overline{u_j \frac{\partial u_i}{\partial x_j}} + \overline{u_i \cancel{\frac{\partial u_j}{\partial x_j}}^{\text{cont.}}} = \overline{u_j \frac{\partial u_i}{\partial x_j}}, \right.$$

$$\overline{u_i u_j} = \overline{(\bar u_i + u_i')(\bar u_j + u_j')} = \overline{\bar u_i \bar u_j} + \overline{u_i' \bar u_j} + \overline{\bar u_i u_j'} + \overline{u_i' u_j'}$$

$$\left. = \bar u_i \bar u_j + \underbrace{\bar u_i' \bar u_j}_{0} + \underbrace{\bar u_i \bar u_j'}_{0} + \overline{u_i' u_j'} \right)$$

$\cancel{= \bar u_i \bar u_j + \overline{u_i' u_j'}}$ average of fluctuation velocities is zero.

$$\frac{\partial \bar u_i}{\partial t} + \frac{\partial \overline{u_i u_j}}{\partial x_j} = \frac{\partial \bar u_i}{\partial t} + \frac{\partial (\bar u_i \bar u_j + \overline{u_i' u_j'})}{\partial x_j}$$

$$= \frac{\partial \bar u_i}{\partial t} + \bar u_j \frac{\partial \bar u_i}{\partial x_j} + \bar u_i \cancel{\frac{\partial \bar u_j}{\partial x_j}}^{0 \text{ continuity.}} + \frac{\partial \overline{u_i' u_j'}}{\partial x_j}.$$

$$\Rightarrow \boxed{\frac{\partial \bar u_i}{\partial t} + \bar u_j \frac{\partial \bar u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar p}{\partial x_i} + \nu \frac{\partial^2 \bar u_i}{\partial x_j \partial x_j} - \frac{\partial \overline{u_i' u_j'}}{\partial x_j}.}$$

(1.2) Show for isotropic flows, $\overline{u_i'u_j'} = \frac{2}{3} k\,\delta_{ij}$.

Two-pt. corr. tensor for isotropic flow is

$$R_{ij}(r) = \overline{u'^2}\left[f(r)\,\delta_{ij} + \frac{r}{2}\frac{df}{dr}\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right)\right]. \quad \rightarrow f(0) = 1.$$

When $r=0$, $R_{ij}(0) = \overline{u'^2}\,\delta_{ij} = \frac{1}{3}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)\delta_{ij} = \frac{2}{3}k\,\delta_{ij}$.

$$= \frac{1}{3}\left(\overline{u_i'u_i'}\right)\delta_{ij} = \frac{2}{3}k\,\delta_{ij}.$$

In isotropic flow then, $\overline{u'^2} = \overline{v'^2} = \overline{w'^2} = \frac{2}{3}k$, $\overline{u'v'} = \overline{u'w'} = \overline{v'w'} = 0$.
where last part follows from orthogonal components being uncorrelated and thus $u_i'u_j'$ $(i \neq j)$ ~~being~~ having a mean of 0.

(1.3) Define the anisotropy tensor as $a_{ij} = \dfrac{\overline{u_i'u_j'}}{k} - \dfrac{2}{3}\delta_{ij}$,

$$\Rightarrow \overline{u_i'u_j'} = \frac{2}{3}k\,\delta_{ij} + k\,a_{ij}.$$

$$a_{ii} = \frac{\overbrace{\left(\overline{u_i'u_i'}\right)}^{\text{this is } 2k}}{k} - \frac{2}{3}\delta_{ii} = \frac{2k}{k} - \frac{2}{3}\delta_{ii} = 2 - 2 = 0.$$

For isotropic, ...

$$a_{ij} = \frac{\overline{u_i'u_j'}}{k} - \frac{2}{3}\delta_{ij} = \frac{2}{3}\frac{k\,\delta_{ij}}{k} - \frac{2}{3}\delta_{ij} = 0 \text{ when isotropic.}$$

$a_{ij} = \dfrac{\overline{u_i'u_j'}}{k} - \dfrac{2}{3}\delta_{ij}$. δ_{ij} is symmetric tensor $\delta_{ij} = \delta_{ji}$.
$\overline{u_i'u_j'}$ " " since multiplication is commutative: $\overline{u_i'u_j'} = \overline{u_j'u_i'}$.

~~then~~ $\Rightarrow a_{ij}$ symmetric b/c it is sum of sym. tensors.

~~[illegible] Diagonal terms must sum to zero always.~~

Since a_{ij} is symmetric, $a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$ 6 indep. terms.

(1.4) Start w/ transport eq'n for u_i'. NS for u_i − NS for $\bar{u}_i$.

$$\frac{\partial}{\partial t}u_i + u_j\frac{\partial u_i}{\partial x_j} - \frac{\partial}{\partial t}\bar{u}_i - \bar{u}_j\frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x_i} - \frac{\partial \bar{p}}{\partial x_i}\right) + \nu\left(\frac{\partial^2 u_i}{\partial x_j \partial x_j} - \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j}\right) + \frac{\partial(\overline{u_i' u_j'})}{\partial x_j}$$

$$\frac{\partial}{\partial t}(u_i - \bar{u}_i) + (\bar{u}_j + u_j')\frac{\partial u_i}{\partial x_j} - \bar{u}_j\frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial(p-\bar{p})}{\partial x_i} + \nu\frac{\partial^2(u_i - \bar{u}_i)}{\partial x_j \partial x_j} + \frac{\partial(\overline{u_i' u_j'})}{\partial x_j}$$

$$\frac{\partial u_i'}{\partial t} + \bar{u}_j\frac{\partial(u_i - \bar{u}_i)}{\partial x_j} + u_j'\frac{\partial u_i}{\partial x_j} = \text{" \qquad "}$$

$$\Rightarrow \left(\frac{\partial}{\partial t} + \bar{u}_j\frac{\partial}{\partial x_j}\right)u_i' = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} - \underbrace{u_j'\frac{\partial u_i}{\partial x_j}} + \nu\frac{\partial^2 u_i'}{\partial x_j \partial x_j} + \frac{\partial \overline{u_i' u_j'}}{\partial x_j}$$

$$= u_j'\frac{\partial(\bar{u}_i + u_i')}{\partial x_j} = u_j'\frac{\partial \bar{u}_i}{\partial x_j} + u_j'\frac{\partial u_i'}{\partial x_j}$$

$$S_{ij}' = \frac{1}{2}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)$$

$$= u_j'\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial u_i' u_j'}{\partial x_j} \quad \text{(continuity)}$$

$$= -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} - u_j'\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial u_i' u_j'}{\partial x_j} + \nu\frac{\partial u_i'}{\partial x_j \partial x_j} + \frac{\partial \overline{u_i' u_j'}}{\partial x_j}.$$

or
$$\boxed{\left(\frac{\partial}{\partial t} + \bar{u}_j\frac{\partial}{\partial x_j}\right)u_i' = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} - u_j'\frac{\partial \bar{u}_i}{\partial x_j} + \nu\frac{\partial u_i'}{\partial x_j \partial x_j} + \frac{\partial}{\partial x_j}\left(\overline{u_i' u_j'} - u_i' u_j'\right).} \quad ⊛$$

→ or not this and add in here a

(Transport eq.n for u_i')

① $\nu S_{ij} = \frac{\nu}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ ② $2\nu S_{ij}$

We want a transport eq. for the Reynolds stresses
$$\left(\frac{\partial}{\partial t} + \bar{u}_k\frac{\partial}{\partial x_k}\right)\overline{u_i' u_j'} = \overbrace{\overline{u_j'\left(\frac{\partial}{\partial t} + \bar{u}_k\frac{\partial}{\partial x_k}\right)u_i'}}^{①} + \overbrace{\overline{u_i'\left(\frac{\partial}{\partial t} + \bar{u}_k\frac{\partial}{\partial x_k}\right)u_j'}}^{②}$$

①: from premult.ply ⊛ then avg.ing.

$$① = -\frac{1}{\rho}\overline{u_j'\frac{\partial p'}{\partial x_i}} - \overline{u_j' u_k'}\frac{\partial \bar{u}_i}{\partial x_k} + \underbrace{\overline{u_j'\nu\frac{\partial u_i'}{\partial x_k \partial x_k}}} + \underbrace{\overline{u_j'\frac{\partial}{\partial x_k}\left(\overline{u_i' u_k'} - u_i' u_k'\right)}}$$

✓ ✓

or not this and add $2\nu S_{ij}'$ here

$$= -\overline{u_j'\frac{\partial(u_i' u_k')}{\partial x_k}} \quad \text{or} \rightarrow \quad = \overline{u_j'\frac{\partial}{\partial x_k}\left(2\nu S_{ij}' - u_i' u_k'\right)}$$

or ① $= \frac{-1}{\rho}\overline{u_j'\frac{\partial p'}{\partial x_i}} - \overline{u_j'u_k'}\frac{\partial \overline{u}_i}{\partial x_k} + \overline{u_j'\frac{\partial}{\partial x_k}\left(-u_i'u_k' + 2\nu s_{ij}'\right)}$ j, i flipped.

② $= \frac{-1}{\rho}\overline{u_i'\frac{\partial p'}{\partial x_j}} - \overline{u_i'u_k'}\frac{\partial \overline{u}_j}{\partial x_k} + \overline{u_i'\frac{\partial}{\partial x_k}\left(-u_j'u_k' + 2\nu s_{ji}'\right)}$

Our Reynolds stress transport equation then becomes...

$$\Rightarrow \left(\frac{\partial}{\partial t} + \overline{u}_k\frac{\partial}{\partial x_k}\right)\overline{u_i'u_j'} = \underbrace{\frac{-1}{\rho}\left(\overline{u_i'\frac{\partial p'}{\partial x_j}} + \overline{u_j'\frac{\partial p'}{\partial x_i}}\right)}_{\oplus}$$

$$\underbrace{-\left(\overline{u_i'u_k'}\frac{\partial \overline{u}_j}{\partial x_k} + \overline{u_j'u_k'}\frac{\partial \overline{u}_i}{\partial x_k}\right)}_{Ⓐ}$$

$$Ⓓ\left\{ + \left(\overline{u_i'\frac{\partial}{\partial x_k}\underbrace{(u_j'u_k')}_{(-)+2\nu s_{ij}'}} + \overline{u_j'\frac{\partial}{\partial x_k}\underbrace{(u_i'u_k')}_{(-)+2\nu s_{ij}'}}\right)\right.$$

$\overline{u_i'u_j'}$.

$$\Rightarrow Ⓓ = -\frac{\partial}{\partial x_k}\left(\overline{u_i'u_j'u_k'}\right) = -\frac{\partial}{\partial x_k}\left(\overline{u_i'u_j'u_k'}\right).$$

$$\oplus = \frac{-1}{\rho}\left[\frac{\partial}{\partial x_j}\left(\overline{p'u_i'}\right) + \frac{\partial}{\partial x_i}\left(\overline{p'u_j'}\right)\right]. \qquad \frac{\partial}{\partial x_j}(\overline{p'u_i'}) - \overline{p'\frac{\partial u_i'}{\partial x_j}}$$

$$\boxed{\begin{aligned}\text{Furthermore, } \oplus &= \frac{-1}{\rho}\left(\frac{\partial \overline{p'u_i'}}{\partial x_j} - \overline{p'\frac{\partial u_i'}{\partial x_j}} + \frac{\partial \overline{u_j'p'}}{\partial x_i} - \overline{p'\frac{\partial u_j'}{\partial x_i}}\right)\\ &= \frac{-1}{\rho}\left[\frac{\partial}{\partial x_k}\left(\overline{u_i'p'}\,\delta_{jk} + \overline{u_j'p'}\,\delta_{ik}\right) + \overline{p'(\;)}\right],\\ &\qquad \frac{1}{2}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right).\end{aligned}}$$

$\boxed{\text{OK}}$

$$\Rightarrow \left(\frac{\partial}{\partial t} + \overline{u}_k\frac{\partial}{\partial x_k}\right)\overline{u_i'u_j'} = \frac{-1}{\rho}\left[\frac{\partial}{\partial x_k}\left(\overline{u_i'p'}\,\delta_{jk} + \overline{u_j'p'}\,\delta_{ik}\right) + \overline{p'\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)}\right]$$

Transport for Reynolds Stresses (RSTE)

$$-\left(\overline{u_i'u_k'}\frac{\partial \overline{u}_j}{\partial x_k} + \overline{u_j'u_k'}\frac{\partial \overline{u}_i}{\partial x_k}\right) - \frac{\partial}{\partial x_k}\overline{u_i'u_j'u_k'}.$$

$$\cancel{= \frac{-1}{\rho}\left[\frac{\partial}{\partial x_k}\left(\overline{u_i'p'}\,\delta_{jk} + \overline{u_j'p'}\,\delta_{ik}\right) + \right.}$$

~~For transport of $K = \frac{1}{2}\overline{u_i'u_i'}$, take trace of RSTE.~~

$$\cancel{\frac{D}{Dt}K =}$$

$$\cancel{\frac{D}{Dt}(2K) = \frac{-1}{\rho}\left[\frac{\partial}{\partial x_k}\left(\overline{u_k'p'} + \overline{u_k'p'}\right) + \overline{p'\left(\frac{\partial u_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i}\right)}^{\to 0}\right]}$$

$$\cancel{- \left(\overline{u_i'u_k'}\frac{\partial u_i}{\partial x_k}\cdot 2\right) - \frac{\partial}{\partial x_k}\overline{u_i'u_i'u_k'}}$$

$$(\star) = \overline{u_i'\frac{\partial}{\partial x_k}\left(2\nu S_{jk}' - u_j'u_k'\right)} + \overline{u_j'\frac{\partial}{\partial x_k}\left(2\nu S_{ik}' - u_i'u_k'\right)}$$

$$= \overline{u_i'\frac{\partial}{\partial x_k}\left(2\nu S_{jk}'\right)} + \overline{u_j'\frac{\partial}{\partial x_k}\left(2\nu S_{ik}'\right)} - \frac{\partial}{\partial x_k}\left(\overline{u_i'u_j'u_k'}\right).$$

aside $\left[\; 2\nu\frac{\partial}{\partial x_k}\left(S_{jk}'u_i'\right) = 2\nu\left(u_i'\frac{\partial S_{jk}'}{\partial x_k} + S_{jk}'\frac{\partial u_i'}{\partial x_k}\right)\right]$

$$\left.\begin{aligned} &= 2\nu\frac{\partial}{\partial x_k}\left(\overline{S_{jk}'u_i'}\right) - 2\nu\overline{S_{jk}'\frac{\partial u_i'}{\partial x_k}} \\ &\quad + 2\nu\frac{\partial}{\partial x_k}\left(\overline{S_{ik}'u_j'}\right) - 2\nu\overline{S_{ik}'\frac{\partial u_j'}{\partial x_k}} - \frac{\partial}{\partial x_k}\overline{u_i'u_j'u_k'}.\end{aligned}\right\} (B)$$

Also,
$$(A) = \frac{-1}{\rho}\left(\frac{\partial\overline{p'u_i'}}{\partial x_j} - \overline{p'\frac{\partial u_i'}{\partial x_j}} + \frac{\partial\overline{u_j'p'}}{\partial x_i} - \overline{p'\frac{\partial u_j'}{\partial x_i}}\right)$$

$$= \frac{-1}{\rho}\left[\frac{\partial}{\partial x_k}\left(\overline{u_i'p'}\,\delta_{jk} + \overline{u_j'p'}\,\delta_{ik}\right) - 2\overline{p'S_{ij}'}\right]$$

$$\left.= \frac{-1}{\rho}\frac{\partial}{\partial x_k}\left(\overline{u_i'p'}\,\delta_{jk} + \overline{u_j'p'}\,\delta_{ik}\right) + \frac{2\overline{p'S_{ij}'}}{\rho}.\right\} (C)$$

$\longrightarrow$

$\boxed{\tau'_{ij} = 2\nu S'_{ij} \quad \text{in course notes.}}$

Thus...

$$\frac{D}{Dt}\overline{u_i'u_j'} = -\left(\overline{u_i'u_k'}\frac{\partial \bar{u}_j}{\partial x_k} + \overline{u_j'u_k'}\frac{\partial \bar{u}_i}{\partial x_k}\right) \qquad \text{from (A)}$$

$$-2\nu\left(\overline{S'_{ik}\frac{\partial u_j'}{\partial x_k}} + \overline{S'_{jk}\frac{\partial u_i'}{\partial x_k}}\right) \qquad \text{from (B)}$$

$$+\overline{2\frac{p'}{\rho}S'_{ij}} \qquad \text{from (C)}$$

$$+\frac{\partial}{\partial x_k}\left[-\overline{u_i'u_j'u_k'} + 2\nu\left(\overline{u_j'S'_{ik}} + \overline{u_i'S'_{jk}}\right) \qquad \text{from (B)}\right.$$

$$\left.-\frac{1}{\rho}\left(\overline{u_i'p'}\,\delta_{jk} + \overline{u_j'p'}\,\delta_{ik}\right)\right] \qquad \text{from (C)}$$

We want the transport equation for K.

Now, $K = \frac{1}{2}\overline{u_i'u_i'}$, so take trace of $\frac{D}{Dt}\overline{u_i'u_j'}$.

$$2\frac{D}{Dt}K = -2\left(\overline{u_i'u_k'}\frac{\partial \bar{u}_i}{\partial x_k}\right)$$

$$-4\nu\left(\overline{s'_{ik}\frac{\partial u_i'}{\partial x_k}}\right) \Big\} = -4\nu\overline{\left(\frac{\partial u_i'}{\partial x_k} + \frac{\partial u_k'}{\partial x_i}\right)\frac{\partial u_i'}{\partial x_k}} = -4\nu\left(\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}} + \overline{\frac{\partial u_k'}{\partial x_i}\frac{\partial u_i'}{\partial x_k}}\right)$$

$$+\overline{2\frac{p'}{\rho}s'_{ii}} \Big\} \; s'_{ii} = \left(\frac{\partial u_i'}{\partial x_i}\right) \to 0 \text{ by continuity.}$$

$$+\frac{\partial}{\partial x_k}\left[-\overline{u_i'u_i'u_k'} + 4\nu\left(\overline{u_i'S_{ik}}\right) - \frac{2}{\rho}\left(\overline{u_i'p'}\,\delta_{ik}\right)\right]$$

$$\frac{D}{Dt}K = -\overline{u_i'u_k'}\frac{\partial \bar{u}_i}{\partial x_k} - 2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}} - 2\nu\overline{\frac{\partial u_k'}{\partial x_i}\frac{\partial u_i'}{\partial x_k}} + 0$$

$$-2\frac{\partial}{\partial x_k}\overline{K u_k'} + 2\nu\left(\frac{\partial^2 K}{\partial x_i \partial x_i} + \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} + \overline{\frac{\partial u_k'}{\partial x_k}\frac{\partial u_k'}{\partial x_i}}\right) - \frac{1}{\rho}\frac{\partial}{\partial x_k}\overline{p'u_k'}.$$

$$2\frac{Dk}{Dt} = \overbrace{-2\,\overline{u_i'u_k'}\,\frac{\partial \bar{u}_i}{\partial x_k}}^{①} \overbrace{-4\nu\,\overline{s_{ik}'\frac{\partial u_i'}{\partial x_k}}}^{②} \overbrace{+2\,\overline{\frac{p'}{\rho}s_{ii}'}}^{③}$$

$$+\frac{\partial}{\partial x_k}\Big(\underbrace{-\overline{u_i'u_i'u_k'}}_{④} \underbrace{+4\nu\,\overline{u_i's_{ik}'}}_{⑤} \underbrace{-\frac{2}{\rho}\,\overline{u_i'p'}\,\delta_{ik}}_{⑥}\Big)$$

①: $-2\,\overline{u_i'u_k'}\,\frac{\partial \bar{u}_i}{\partial x_k} = -2\,\overline{u_i'u_j'}\,\frac{\partial \bar{u}_i}{\partial x_j}$ ok.

②: $-4\nu\,\overline{s_{ik}'\frac{\partial u_i'}{\partial x_k}} = -\frac{4\nu}{2}\overline{\left(\frac{\partial u_i'}{\partial x_k}+\frac{\partial u_k'}{\partial x_i}\right)\frac{\partial u_i'}{\partial x_k}} = -2\nu\left(\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}} + \underbrace{\overline{\frac{\partial u_k'}{\partial x_i}\frac{\partial u_i'}{\partial x_k}}}_{\text{cancel } \circledast}\right).$

③: $2\,\overline{\frac{p'}{\rho}s_{ii}'} = \frac{2}{\rho}\overline{p'\left(\frac{\partial u_i'}{\partial x_i}+\frac{\partial u_i'}{\partial x_i}\right)} = \frac{4}{\rho}\overline{p'\frac{\partial u_i'}{\partial x_i}} = 0$, continuity.

④: $\frac{\partial}{\partial x_k}\left(-\overline{u_i'u_i'u_k'}\right) = \frac{\partial}{\partial x_k}\left(-2\,\overline{k u_k'}\right)$. ok.

⑤: $\frac{\partial}{\partial x_k}\left(4\nu\,\overline{u_i's_{ik}'}\right) = \frac{4\nu}{2}\frac{\partial}{\partial x_k}\left(\overline{u_i'\left(\frac{\partial u_i'}{\partial x_k}+\frac{\partial u_k'}{\partial x_i}\right)}\right)$

$2\frac{\partial k}{\partial x_k} = \frac{\partial \overline{u_i'u_i'}}{\partial x_k} = 2\,\overline{u_i'\frac{\partial u_i'}{\partial x_k}}$

$$= 2\nu\frac{\partial}{\partial x_k}\left(\overline{u_i'\frac{\partial u_i'}{\partial x_k}} + \overline{u_i'\frac{\partial u_k'}{\partial x_i}}\right) = 2\nu\frac{\partial}{\partial x_k}\left(\frac{\partial k}{\partial x_k} + \overline{u_i'\frac{\partial u_k'}{\partial x_i}}\right)$$

$$= 2\nu\left[\frac{\partial}{\partial x_k}\left(\frac{\partial k}{\partial x_k}\right) + \underbrace{\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_k'}{\partial x_i}}}_{\text{cancel } \circledast} + \overline{u_i'\frac{\partial}{\partial x_i}\underbrace{\frac{\partial u_k'}{\partial x_k}}_{\to 0 \text{ cont.}}}\right]$$

⑥: $\frac{\partial}{\partial x_k}\left(-\frac{2}{\rho}\,\overline{u_i'p'}\,\delta_{ik}\right) = \frac{\partial}{\partial x_j}\left(\frac{-2}{\rho}\,\overline{u_j'p'}\right)$. ok.

$$\Rightarrow \frac{Dk}{Dt} = \underbrace{-\overline{u_i'u_j'}\frac{\partial \bar{u}_i}{\partial x_j}}_{①} \underbrace{-\nu\,\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}}_{②} + \frac{\partial}{\partial x_j}\Big(\underbrace{\nu\frac{\partial k}{\partial x_j}}_{⑤} - \underbrace{\overline{k'u_j'}}_{④} - \underbrace{\frac{1}{\rho}\,\overline{p'u_j'}}_{⑥}\Big). \ ✓$$

$$a_{ij} \equiv \frac{\overline{u_i'u_j'}}{k} - \frac{2}{3}\delta_{ij} .$$

Now governing eq'n for a_{ij}...

$$\frac{D(a_{ij})}{Dt} = \left(\frac{\partial}{\partial t} + \bar{u}_k \frac{\partial}{\partial x_k}\right)\left(\frac{\overline{u_i'u_j'}}{k} - \frac{2}{3}\delta_{ij}\right)$$

since δ_{ij} is const.

$$\frac{Da_{ij}}{Dt} = \frac{\partial}{\partial t}\left(\frac{\overline{u_i'u_j'}}{k}\right) + \bar{u}_k \frac{\partial}{\partial x_k}\left(\frac{\overline{u_i'u_j'}}{k}\right) - \cancelto{0}{\frac{2}{3}\frac{\partial}{\partial t}(\delta_{ij})} - \cancelto{0}{\bar{u}_k \frac{\partial}{\partial x_k}\left(\frac{2}{3}\delta_{ij}\right)}$$

$$= \frac{1}{k}\frac{\partial}{\partial t}\left(\overline{u_i'u_j'}\right) + \overline{u_i'u_j'}\frac{\partial}{\partial t}\left(\frac{1}{k}\right) + \frac{\bar{u}_k}{k}\frac{\partial}{\partial x_k}\left(\overline{u_i'u_j'}\right) + \overline{u_i'u_j'}\,\bar{u}_k\frac{\partial}{\partial x_k}\left(\frac{1}{k}\right)$$

$$= \frac{1}{k}\frac{D\overline{u_i'u_j'}}{Dt} + \overline{u_i'u_j'}\left(\frac{-1}{k^2}\right)\frac{\partial k}{\partial t} + \overline{u_i'u_j'}\,\bar{u}_k\left(\frac{-1}{k^2}\right)\frac{\partial k}{\partial x_k}$$

$$\frac{Da_{ij}}{Dt} = \underbrace{\frac{1}{k}\frac{D\overline{u_i'u_j'}}{Dt}}_{A} - \underbrace{\frac{\overline{u_i'u_j'}}{k^2}\frac{Dk}{Dt}}_{B} . \qquad \leftarrow (*)$$

Next, we just substitute in for our known $D\overline{u_i'u_j'}/Dt$ and Dk/Dt.

$$\begin{aligned}
\frac{Da_{ij}}{Dt} = \frac{1}{k}\Bigg[& -\overline{u_i'u_k'}\frac{\partial \bar{u}_j}{\partial x_k} - \overline{u_j'u_k'}\frac{\partial \bar{u}_i}{\partial x_k} - 2\nu\,\overline{S_{ik}'\frac{\partial u_j'}{\partial x_k}} - 2\nu\,\overline{S_{jk}'\frac{\partial u_i'}{\partial x_k}} \\
& + \frac{2}{\rho}\overline{p'S_{ij}'} - \frac{\partial}{\partial x_k}\left(\overline{u_i'u_j'u_k'}\right) + 2\nu\frac{\partial}{\partial x_k}\left(\overline{u_j'S_{ik}'}\right) \\
& + 2\nu\frac{\partial}{\partial x_k}\left(\overline{u_i'S_{jk}'}\right) - \frac{1}{\rho}\frac{\partial}{\partial x_k}\left(\overline{u_i'p'}\,\delta_{jk}\right) - \frac{1}{\rho}\frac{\partial}{\partial x_k}\left(\overline{u_j'p'}\,\delta_{ik}\right)\Bigg] \Bigg\} \; A \\
& - \frac{\overline{u_i'u_j'}}{k^2}\left[-\overline{u_i'u_j'}\frac{\partial \bar{u}_i}{\partial x_j} - \nu\,\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}} + \frac{\partial}{\partial x_j}\left(\nu\frac{\partial k}{\partial x_j} - \overline{k'u_j'} - \frac{1}{\rho}\overline{p'u_j'}\right)\right] \Big\} \; B
\end{aligned}$$

Let $\frac{2}{\rho}\overline{p's_{ij}'} \equiv \Pi_{ij}$, and let $\bar{S}_{ij} \equiv \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$, $\bar{W}_{ij} \equiv \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \bar{u}_j}{\partial x_i}\right)$

$$\Rightarrow \frac{\partial \bar{u}_i}{\partial x_j} = \bar{S}_{ij} + \bar{W}_{ij} .$$

Also $S_{ij}' = S_{ij} - \bar{S}_{ij}$

$$\frac{Da_{ij}}{Dt} = -\frac{\overline{u_i'u_k'}}{k}\left(\bar{S}_{jk}+\bar{W}_{jk}\right) - \frac{\overline{u_j'u_k'}}{k}\left(\bar{S}_{ik}+\bar{W}_{ik}\right)$$

$$-\frac{2\nu}{k}\overline{(s_{ik}-\bar{S}_{ik})\frac{\partial u_j'}{\partial x_k}} - \frac{2\nu}{k}\overline{(s_{jk}-\bar{S}_{jk})\frac{\partial u_i'}{\partial x_k}} + \frac{1}{k}\Pi_{ij}$$

$$-\frac{\partial}{\partial x_k}\left(\overline{u_i'u_j'u_k'}\right)\frac{1}{k} + \frac{2\nu}{k}\frac{\partial}{\partial x_k}\left(\overline{u_j's_{ik}'} + \overline{u_i's_{jk}'}\right)$$

$$-\frac{\partial}{\partial x_k}\left(\frac{1}{\rho}\overline{u_i'p'}\,\delta_{jk}\right)\frac{1}{k} - \frac{\partial}{\partial x_k}\left(\frac{1}{\rho}\overline{u_j'p'}\,\delta_{ik}\right)\frac{1}{k}$$

$$-\frac{\overline{u_i'u_j'}}{k^2}\left[-\overline{u_i'u_j'}\left(\bar{S}_{ij}+\bar{W}_{ij}\right) - \nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}} + \frac{\partial}{\partial x_j}\left(\nu\frac{\partial k}{\partial x_j} - \overline{k'u_j} - \frac{1}{\rho}\overline{p'u_j}\right)\right]$$

Note $\frac{\overline{u_i'u_j'}}{k} = a_{ij} + \frac{2}{3}\delta_{ij}\ldots$

$$A = \left(-a_{jk}-\frac{2}{3}\delta_{jk}\right)\left(\bar{S}_{jk}+\bar{W}_{jk}\right) + \left(-a_{ik}-\frac{2}{3}\delta_{ik}\right)\left(\bar{S}_{ik}+\bar{W}_{ik}\right)$$

$$= -a_{jk}\bar{S}_{jk} - \frac{2}{3}\bar{S}_{ii} - a_{jk}\bar{W}_{jk} - \frac{2}{3}\bar{W}_{ii}\xrightarrow{}0 - a_{ik}\bar{S}_{ik} - a_{ik}\bar{W}_{ik}$$

$$-\frac{2}{3}\bar{S}_{ii} - \frac{2}{3}\bar{W}_{ii}\xrightarrow{}0$$

$$\frac{\partial}{\partial x_k}\left[\overline{\frac{u_j'}{2}\left(\frac{\partial u_i'}{\partial x_k}+\frac{\partial u_k'}{\partial x_i}\right)}\right] = \frac{1}{2}\frac{\partial}{\partial x_k}\left[\overline{u_j'\frac{\partial u_i'}{\partial x_k}} + \overline{u_j'\frac{\partial u_k'}{\partial x_i}}\xrightarrow{}0 \text{ cont.}\right] = \frac{1}{2}\frac{\partial}{\partial x_k}\left[\frac{\partial\overline{u_i'u_j'}}{\partial x_k} - \overline{u_i'\frac{\partial u_j'}{\partial x_k}}\right]$$

AA

$$\frac{2\nu}{k}\frac{\partial}{\partial x_k}\left(\overline{u_j's_{ik}'}+\overline{u_i's_{jk}'}\right) = \frac{2\nu}{k}\frac{\partial}{\partial x_k}\left(\frac{1}{2}\overline{u_j'\left(\frac{\partial u_i'}{\partial x_k}+\frac{\partial u_k'}{\partial x_i}\right)} + \overline{\frac{u_i'}{2}\left(\frac{\partial u_j'}{\partial x_k}+\frac{\partial u_k'}{\partial x_j}\right)}\right) \quad \xrightarrow{}0 \text{ cont.}$$

$$= \frac{\nu}{k}\frac{\partial}{\partial x_k}\left(\frac{\partial\overline{u_i'u_j'}}{\partial x_k} - \overline{u_i'\frac{\partial u_j'}{\partial x_k}} + \frac{\partial\overline{u_i'u_j'}}{\partial x_k} - \overline{u_j'\frac{\partial u_i'}{\partial x_k}}\right)$$

$$= \frac{\nu}{k}\frac{\partial}{\partial x_k}\left(2\frac{\partial\overline{u_i'u_j'}}{\partial x_k} - \frac{\partial\overline{u_i'u_j'}}{\partial x_k}\right) = \frac{\nu}{k}\frac{\partial}{\partial x_k}\left(\frac{\partial\overline{u_i'u_j'}}{\partial x_k}\right) \checkmark$$

Note $\dfrac{\overline{u_i'u_j'}}{k} = a_{ij} + \dfrac{2}{3}\delta_{ij}$.

Let's start with the first term, in ⊛,

$$\frac{1}{k}\frac{D\overline{u_i'u_j'}}{Dt} = -\frac{\overline{u_i'u_k'}}{k}\frac{\partial \bar{u}_j}{\partial x_k} - \frac{\overline{u_j'u_k'}}{k}\frac{\partial \bar{u}_i}{\partial x_k} - \frac{2\nu}{k}\overline{s_{ik}'\frac{\partial u_j'}{\partial x_k}} - \frac{2\nu}{k}\overline{s_{jk}'\frac{\partial u_i'}{\partial x_k}}$$

$$+ \frac{2}{k\rho}\overline{p's_{ij}'} + \frac{\partial}{\partial x_k}\frac{1}{k}\Bigg(-\overline{u_i'u_j'u_k'} + 2\nu\,\overline{u_j's_{ik}'} + 2\nu\,\overline{u_i's_{jk}'} - \frac{1}{\rho}\left(\overline{u_i'p'}\,\delta_{jk} + \overline{u_j'p'}\,\delta_{ik}\right)\Bigg).$$

$$= -\left(a_{ik} + \tfrac{2}{3}\delta_{ik}\right)\left(\bar{S}_{jk} + \bar{W}_{jk}\right) - \left(a_{jk} + \tfrac{2}{3}\delta_{jk}\right)\left(\bar{S}_{ik} + \bar{W}_{ik}\right)$$

$$-\frac{2\nu}{k}\left(\overline{(s_{ik}')\frac{\partial u_j'}{\partial x_k}} + \overline{(s_{jk}')\frac{\partial u_i'}{\partial x_k}}\right) + \frac{1}{k}\Pi_{ij}$$

$$+\frac{1}{k}\frac{\partial}{\partial x_k}\Bigg(-\overline{u_i'u_j'u_k'} - \frac{\overline{p'u_j'}}{\rho}\delta_{ik} - \frac{\overline{p'u_i'}}{\rho}\delta_{jk} + \underbrace{2\nu\left(\overline{u_j's_{ik}'} + \overline{u_i's_{jk}'}\right)}_{\text{see AA}}\Bigg)$$

$$= -a_{ik}\bar{S}_{jk} - \tfrac{2}{3}\bar{S}_{ji} - a_{ik}\bar{W}_{jk} - \cancel{\tfrac{2}{3}\bar{W}_{ji}} \quad \text{since } \bar{W}_{ij} = -\bar{W}_{ji}.$$

$$-a_{jk}\bar{S}_{ik} - \tfrac{2}{3}\bar{S}_{ij} - a_{jk}\bar{W}_{ik} - \cancel{\tfrac{2}{3}\bar{W}_{ij}}$$

$$-\frac{2\nu}{k}\left[\overline{(s_{ik}')\frac{\partial u_j'}{\partial x_k}} + \overline{(s_{jk}')\frac{\partial u_i'}{\partial x_k}}\right] + \frac{1}{k}\Pi_{ij} + \frac{1}{k}D_{ij}.$$

where $D_{ij} = -\dfrac{\partial}{\partial x_k}\left(\overline{u_i'u_j'u_k'} + \dfrac{\overline{p'u_j'}}{\rho}\delta_{ik} + \dfrac{\overline{p'u_i'}}{\rho}\delta_{jk} - \nu\dfrac{\partial\overline{u_i'u_j'}}{\partial x_k}\right)$ AA

$$= -\left(a_{ik}\bar{S}_{kj} + \bar{S}_{ik}a_{kj}\right) - a_{ik}\bar{W}_{jk} - a_{jk}\bar{W}_{ik} - \frac{4}{3}\bar{S}_{ij} + \frac{1}{k}\Pi_{ij} + \frac{1}{k}D_{ij}$$

$$\cancel{-\frac{2\nu}{k}\left(\overline{s_{ik}\frac{\partial u_j}{\partial x_k}} - \overline{s_{ik}\frac{\partial u_j}{\partial x_k}} + \overline{s_{jk}\frac{\partial u_i}{\partial x_k}} - \overline{s_{jk}\frac{\partial u_i}{\partial x_k}}\right)} - \frac{2\nu}{k}\left(\overline{s_{ik}'\frac{\partial u_j'}{\partial x_k} + s_{jk}'\frac{\partial u_i'}{\partial x_k}}\right)$$

$$= -\left(a_{ik}\bar{S}_{kj} + \bar{S}_{ik}a_{kj}\right) - a_{ik}\bar{W}_{jk} - a_{jk}\bar{W}_{ik} - \frac{4}{3}\bar{S}_{ij} + \frac{1}{k}\Pi_{ij} + \frac{1}{k}D_{ij}$$

$$\cancel{-\frac{2\nu}{k}\left(\overline{s_{ik}\frac{\partial u_i}{\partial x_k} + s_{jk}\frac{\partial u_i}{\partial x_k}}\right)}$$

$$-\frac{2\nu}{k}\left(\overline{s_{ik}'\frac{\partial u_j'}{\partial x_k} + s_{jk}'\frac{\partial u_j'}{\partial x_k}}\right) \quad \text{note } = -\frac{1}{k}\left(2\nu\,\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}\right) \equiv -\frac{1}{k}\varepsilon_{ij}.$$

→

~~We next look at~~ $\cancel{\left(\bar{S}_{jk}\frac{\partial u_i'}{\partial x_k} + \bar{S}_{ik}\frac{\partial u_j'}{\partial x_k}\right)}$

This leaves us with the following expression for (A):

$$\frac{1}{k}\frac{D\overline{u_i'u_j'}}{Dt} = -\left(a_{ik}\bar{S}_{kj} + \bar{S}_{ik}a_{kj}\right) + \left(a_{ik}\bar{W}_{kj} - \bar{W}_{ik}a_{kj}\right)$$
$$+\frac{1}{k}\Pi_{ij} + \frac{1}{k}D_{ij} - \frac{1}{k}\varepsilon_{ij} - \frac{4}{3}\bar{S}_{ij}.$$

Now we consider (B). (B) $= -\dfrac{\overline{u_i'u_j'}}{k^2}\dfrac{Dk}{Dt}$ here let $i \to k$, $j \to l$.

$$(B) = \left(-\overline{u_i'u_j'}/k^2\right)\cdot\left(-\overline{u_i'u_j'}\frac{\partial \bar{u}_i}{\partial x_j} - \underbrace{\nu\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}}_{=\varepsilon} + \frac{\partial}{\partial x_j}\left(\nu\frac{\partial k}{\partial x_j} - \overline{k'u_j'} - \frac{1}{\rho}\overline{p'u_j'}\right)\right).$$

Let $\varepsilon = \varepsilon_{ii}/2 = \nu\dfrac{\partial u_i'}{\partial x_k}\dfrac{\partial u_i'}{\partial x_k}$. $D \equiv D_{ii}/2$

this is D!

$$(B) = \frac{\overline{u_i'u_j'}}{k^2}\overline{u_k'u_l'}\frac{\partial \bar{u}_k}{\partial x_l} + \frac{\overline{u_i'u_j'}}{k^2}\varepsilon \left\{\overbrace{-\frac{\overline{u_i'u_j'}}{k^2}\nu\frac{\partial^2 k}{\partial x_k \partial x_k}}^{=+\frac{\overline{u_i'u_j'}}{k^2}\left(\frac{-\nu}{2}\frac{\partial^2 \overline{u_i'u_i'}}{\partial x_k\partial x_k}\right)} + \frac{\overline{u_i'u_j'}}{k^2}\frac{\partial^2 \overline{k'u_j'}}{\partial x_k \partial x_k} + \frac{\partial}{\partial x_k}\overline{p'u_k'}\right\}.$$

$$= \frac{\overline{u_i'u_j'}}{k^2}\overline{u_k'u_l'}\frac{\partial \bar{u}_k}{\partial x_l} - \varepsilon\frac{\overline{u_i'u_j'}}{k^2} + D\frac{\overline{u_i'u_j'}}{k^2}$$

$\dfrac{\overline{u_i'u_j'}}{k} = a_{ij} + \dfrac{2}{3}\delta_{ij}$, so...

$$(B) = \left(a_{ij} + \frac{2}{3}\delta_{ij}\right)\frac{\overline{u_k'u_l'}}{k}\frac{\partial \bar{u}_k}{\partial x_l} - \frac{1}{k}\left(a_{ij} + \frac{2}{3}\delta_{ij}\right)\varepsilon + \frac{1}{k}\left(a_{ij} + \frac{2}{3}\delta_{ij}\right)D$$

prod. w/ symm. tensor

$$\text{B} = \left(a_{ij} + \tfrac{2}{3}\delta_{ij}\right)\left(a_{kl} + \tfrac{2}{3}\delta_{kl}\right)\left(\bar{S}_{kl} + \cancel{\bar{W}_{kl}}\right) + \frac{1}{k}\left(a_{ij} + \tfrac{2}{3}\delta_{ij}\right)(\varepsilon + D)$$

$$= \left(a_{ij} + \tfrac{2}{3}\delta_{ij}\right)\left[a_{kl}\bar{S}_{kl} + \tfrac{2}{3}\delta_{kl}\bar{S}_{kl}\right] - \frac{\varepsilon}{k}\left(a_{ij} + \tfrac{2}{3}\delta_{ij}\right) + \underbrace{\frac{D}{k}\left(a_{ij} + \tfrac{2}{3}\delta_{ij}\right)}$$

0: incompressible

$$= a_{ij}a_{kl}\bar{S}_{kl} + \tfrac{2}{3}\delta_{ij}a_{kl}\bar{S}_{kl} + \tfrac{2}{3}a_{ij}\cancel{\delta_{kl}\bar{S}_{kl}} + \tfrac{4}{9}\delta_{ij}\cancel{\delta_{kl}\bar{S}_{kl}}$$

$$- \frac{1}{k}a_{ij}\varepsilon - \frac{2}{3k}\delta_{ij}\varepsilon + \frac{1}{k}\left(a_{ij} + \tfrac{2}{3}\delta_{ij}\right)D.$$

$$= \left(\frac{+k\,a_{kl}\bar{S}_{kl}}{\varepsilon} - 1\right)\frac{\varepsilon}{k}a_{ij} - \frac{1}{k}\frac{2}{3}\varepsilon\,\delta_{ij} + \frac{1}{k}\left(a_{ij} + \tfrac{2}{3}\delta_{ij}\right)D + \frac{2}{3}\delta_{ij}a_{kl}\bar{S}_{kl}.$$

$$= \left(\frac{P}{\varepsilon} - 1\right)\frac{\varepsilon}{k}a_{ij} - \frac{\varepsilon}{k}\frac{2}{3}\delta_{ij} + \frac{1}{k}\left(a_{ij} + \tfrac{2}{3}\delta_{ij}\right)D + \frac{2}{3}\delta_{ij}\,a_{kl}\bar{S}_{kl}.$$

Combining,

$$\begin{aligned}\frac{Da_{ij}}{Dt} = \text{A} + \text{B} = &-\left(a_{ik}\bar{S}_{kj} + \bar{S}_{ik}a_{kj} - \tfrac{2}{3}\delta_{ij}a_{kl}\bar{S}_{kl}\right) \\ &+ \left(a_{ik}\bar{W}_{kj} - \bar{W}_{ik}a_{kj}\right) + \tfrac{1}{k}\Pi_{ij} + \tfrac{1}{k}D_{ij} - \tfrac{1}{k}\varepsilon_{ij} - \tfrac{4}{3}\bar{S}_{ij} \\ &- \left(\frac{P}{\varepsilon} - 1\right)\frac{\varepsilon}{k}a_{ij} + \frac{2}{3}\frac{\varepsilon}{k}\delta_{ij} - \frac{1}{k}\left(a_{ij} + \tfrac{2}{3}\delta_{ij}\right)D.\end{aligned}$$

$$\boxed{\begin{aligned}\frac{Da_{ij}}{Dt} = &-\left(\frac{P}{\varepsilon} - 1\right)\frac{\varepsilon}{k}a_{ij} - \frac{4}{3}\bar{S}_{ij} - \left(a_{ik}\bar{S}_{kj} + \bar{S}_{ik}a_{kj} - \tfrac{2}{3}a_{kl}\bar{S}_{kl}\delta_{ij}\right) \\ &+ \left(a_{ik}\bar{W}_{kj} - \bar{W}_{ik}a_{kj}\right) + \frac{1}{k}\Pi_{ij} - \frac{1}{k}\left(\varepsilon_{ij} - \frac{2}{3}\varepsilon\,\delta_{ij}\right) \\ &+ \frac{1}{k}\left[D_{ij} - \left(a_{ij} + \tfrac{2}{3}\delta_{ij}\right)D\right].\end{aligned}}$$

Assuming k and ε are known, the unclosed terms in $\circledast$ are...

- D_{ij}, D (contains a third order moment, a pressure-velocity correlation)
- P (~~multiple of~~ product of a_{kl} and $\bar{S}_{kl}$)
- $a_{ik}\bar{S}_{kj}$, etc (anything w/ a mult. by $\bar{S}$ or $\bar{W}$)
- $a_{ik}\bar{W}_{kj}$
- Π_{ij} (unclosed b/c product of p' and S'_{ij}).

(1.5) ε_{ij} can be treated as isotropic in many flows because dissipation primarily occurs in turbulence at scales $\boxed{\text{small enough to be treated as isotropic by K41 theory}}$.

Now we show that $\frac{-1}{k}\left(\varepsilon_{ij} - \frac{2}{3}\varepsilon\,\delta_{ij}\right) = 0$ under this assumption, where $\varepsilon = \varepsilon_{ii}/2$.

This is equivalent to showing that $\varepsilon_{ij} = \frac{1}{3}\varepsilon_{ii}\,\delta_{ij}$.

Since $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33}$, $\varepsilon_{ii} = 3\varepsilon_{11} \Rightarrow \boxed{\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = \frac{1}{3}\varepsilon_{ii}.}$

Furthermore, since $\varepsilon_{ij} = 2\nu\,\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}$, it is the case that

$\varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = \varepsilon_{21} = \varepsilon_{31} = \varepsilon_{32}$, and $\overline{\frac{\partial u_1'}{\partial x_k}\frac{\partial u_2'}{\partial x_k}} = \overline{\frac{\partial u_1'}{\partial u_k}\frac{\partial u_3'}{\partial x_k}} = \overline{\frac{\partial u_2'}{\partial x_k}\frac{\partial u_3'}{\partial x_k}}$.

However, since the fluctuation velocities are wholly uncorrelated in isotropic turbulence, these must be equal to zero, and thus there is $\boxed{\text{no anisotropic component}}$ of ε_{ij}. $\Rightarrow \boxed{\varepsilon_{ij} = \frac{1}{3}\varepsilon_{ii}\,\delta_{ij}}$.

1.6 $C_1 = 1.5, \quad C_2 = 0.8, \quad C_3 = 0.875, \quad C_4 = 0.655$

Let
$$\left.\begin{aligned}\Pi_{ij} = -C_1 \varepsilon a_{ij} + C_2 k \bar{S}_{ij} + C_3 k\left(a_{ik}\bar{S}_{kj} + \bar{S}_{ik}a_{kj} - \tfrac{2}{3} a_{kl}\bar{S}_{kl}\delta_{ij}\right) \\ + C_4 k\left(a_{ik}\bar{W}_{kj} - \bar{W}_{ik}a_{kj}\right),\end{aligned}\right\} \circledast$$

and let $\alpha_1 = \dfrac{P}{\varepsilon} - 1 + C_1, \quad \alpha_2 = C_2 - \dfrac{4}{3}, \quad \alpha_3 = C_3 - 1, \quad \alpha_4 = C_4 - 1.$

Starting with

$$\frac{Da_{ij}}{Dt} = -\left(\frac{P}{\varepsilon} - 1\right)\frac{\varepsilon}{k} a_{ij} - \frac{4}{3}\bar{S}_{ij} - \left(a_{ik}\bar{S}_{kj} + \bar{S}_{ik}a_{kj} - \frac{2}{3}a_{kl}\bar{S}_{kl}\delta_{ij}\right) + \left(a_{ik}\bar{W}_{kj} - \bar{W}_{ik}a_{kj}\right) + \frac{1}{k}\left[D_{ij} - \left(a_{ij} + \frac{2}{3}\delta_{ij}\right)D\right] + \frac{1}{k}\Pi_{ij},$$

We substitute in Π_{ij} from $\circledast$, yielding

$$\frac{Da_{ij}}{Dt} = -\left(\frac{P}{\varepsilon} - 1 + C_1\right)\frac{\varepsilon}{k}a_{ij} + \left(C_2 - \frac{4}{3}\right)\bar{S}_{ij} + (C_3 - 1)\left(a_{ik}\bar{S}_{kj} + \bar{S}_{ik}a_{kj} - \frac{2}{3}a_{kl}\bar{S}_{kl}\delta_{ij}\right) + (C_4 - 1)\left(a_{ik}\bar{W}_{kj} - \bar{W}_{ik}a_{kj}\right) + \frac{1}{k}\left[D_{ij} - \left(a_{ij} + \frac{2}{3}\delta_{ij}\right)D\right], \text{ or}$$

$$\boxed{\frac{Da_{ij}}{Dt} = -\alpha_1 \frac{\varepsilon}{k}a_{ij} + \alpha_2 \bar{S}_{ij} + \alpha_3\left(a_{ik}\bar{S}_{kj} + \bar{S}_{ik}a_{kj} - \frac{2}{3}a_{kl}\bar{S}_{kl}\delta_{ij}\right) - \alpha_4\left(a_{ik}\bar{W}_{kj} - \bar{W}_{ik}a_{kj}\right) + \frac{1}{k}\left[D_{ij} - \left(a_{ij} + \frac{2}{3}\delta_{ij}\right)D\right]}.$$

1.7 By neglecting the redistribution and transport terms, and writing simply $\dfrac{Da_{ij}}{Dt} = -\alpha_1 \dfrac{\varepsilon}{k}a_{ij} + \alpha_2 \bar{S}_{ij}$, we have neglected several physical effects:

- Near walls, the Reynolds number drops and diffusive effects become important, which we neglect by the viscous terms within D_{ij} and D.

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- Also near walls, real near-wall pressure effects are neglected, in that P/ε cannot fully capture those effects.

Basically, canonical HIT is simulated well with this model, but any low-Re or wall-bounded flows have regions that ~~they~~ it breaks down in.

These effects and flows are also relevant to 1.8.

1.8 We further assume the flow is in equillibrium, st.

$$\frac{D a_{ij}}{Dt} = 0 , \quad \text{meaning} \quad -\alpha_1 \frac{\varepsilon}{k} a_{ij} + \alpha_2 \bar{S}_{ij} = 0 .$$

$$\Rightarrow \boxed{a_{ij} = \frac{\alpha_2}{\alpha_1} \frac{k}{\varepsilon} \bar{S}_{ij}}$$

This is the closure under the equillibrium assumption.

This model obviously breaks down in physical situations where the <u>decay of anisotropy</u> (to a fully isotropic flow) takes place. It would be inaccurate when simulating flow in a stirred tank when stirring stopped, or simulating the decay of anisotropy behind a ~~grid~~ shear layer in a wind tunnel.

1.9 Further simplify w/ gradient transport hypothesis:

$$a_{ij} = -2 \frac{\nu_T}{k} \bar{S}_{ij} , \quad \nu_T = C_\mu \frac{k^2}{\varepsilon} .$$

$$\Rightarrow a_{ij} = -2 C_\mu \frac{k}{\varepsilon} \bar{S}_{ij} = \frac{\alpha_2}{\alpha_1} \frac{k}{\varepsilon} \bar{S}_{ij} \quad \Rightarrow \boxed{C_\mu = \frac{-1}{2} \frac{\alpha_2}{\alpha_1}}$$

$\longrightarrow$

If we assume C_μ is constant, this implies

$$\text{constant} = C_\mu = \frac{-1}{2}\frac{\alpha_2}{\alpha_1} = \frac{-1}{2}\frac{C_2 - 4/3}{P/\varepsilon - 1 + C_1} \Rightarrow \frac{P}{\varepsilon} - 1 + C_1 \sim C_2 - 4/3$$

$$\Rightarrow \frac{P}{\varepsilon} \sim \text{constant}.$$

First off, since C_μ sets P/ε as a constant if C_μ = constant, this neglects production near boundary layers (for instance, turbulent intensity introduced by small trees in an atmospheric boundary layer) which would require P/ε scale somehow near Dirichlet boundary conditions / walls. This simplifying assumption would perform well in flows that had no small-scale production (that is, it would do terribly for combustion) and probably <u>no walls</u>, and needed simulation speed increased. $P = -k a_{ij}\bar{S}_{ij}$

(since production rate increases with Reynolds stresses (anisotropy) and mean velocity gradients)

(1.10) If $a_{ij} = -2\frac{\nu_T}{k}\bar{S}_{ij}$, we can modify the RANS equation as...

$$\frac{D\bar{u}_i}{Dt} = \frac{-1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu\frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} - \frac{\partial \overline{u_i' u_j'}}{\partial x_j} \qquad \left(\text{Let } \hat{p} \equiv \bar{p} + \tfrac{2}{3}\rho k\right)$$

$$= \frac{-1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu\frac{\partial \bar{u}_i}{\partial x_j} + 2\nu_T\bar{S}_{ij} - \frac{2}{3}k\delta_{ij}\right), \quad \overline{u_i' u_j'} = \frac{2}{3}k\delta_{ij} + k a_{ij}$$

$$= \frac{-1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{-1}{\rho}\frac{\partial}{\partial x_i}\left(\tfrac{2}{3}\rho k\right) + \frac{\partial}{\partial x_j}\left(\nu\frac{\partial \bar{u}_i}{\partial x_j} + 2\nu_T\bar{S}_{ij}\right) \qquad = \frac{2}{3}k\delta_{ij} + k\left(-\frac{2}{k}\nu_T\bar{S}_{ij}\right)$$

$$= \frac{2}{3}k\delta_{ij} - 2\nu_T\bar{S}_{ij}.$$

$$= \frac{-1}{\rho}\frac{\partial \hat{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu\left(\frac{\partial \bar{u}_i}{\partial x_j} + \underbrace{\frac{\partial \bar{u}_j}{\partial x_i}}_{}\right) + 2\nu_T\bar{S}_{ij}\right)$$

=0 by cont., so can add it in w/in $\partial/\partial x_j$.

$$= \frac{-1}{\rho}\frac{\partial \hat{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2\nu\bar{S}_{ij} + 2\nu_T\bar{S}_{ij}\right)$$

$$\Rightarrow \boxed{\frac{D\bar{u}_i}{Dt} = \frac{-1}{\rho}\frac{\partial \hat{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[2\left[\nu + \nu_T\right]\bar{S}_{ij}\right]}.$$

This form of the RANS equation using our a_{ij} closure method may be popular for the following practical reasons:

- Since C_μ, K^2, ε are all > 0 (positive), ν_T is positive and lends additional stability to the numerical method by increasing the effectiveness of viscous diffusion mechanisms within the simulation

- The equation is linear in $\bar{u}_i$ and $\bar{p}$, so is easy to implement and quick to solve — this is particularly important for design and optimization applications.