$$\overline{f+g} = \overline{f} + \overline{g} \qquad \overline{af} = a\overline{f} \quad (a \text{ const.})$$

$$\overline{a} = a \qquad \overline{\frac{\partial f}{\partial s}} = \frac{\partial \overline{f}}{\partial s}, \text{ when } s = x_i \text{ or } t.$$

$$\overline{\overline{f}\,g} = \overline{f}\,\overline{g}.$$

$$\cancel{\frac{\overline{u_i u_j}}{k} - \frac{2}{3}\delta_{ij} = \frac{-2 C_\mu k S_{ij}}{\overline{u_k u_l}\, S_{kl}}}$$

~~III$_a$~~

(1.1) $u_i' \equiv U_i - \overline{U_i}, \Rightarrow u_i = \overline{u_i} + u_i'$

Starting with the Navier-Stokes equations...

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}, \quad \underbrace{\frac{\partial u_i}{\partial x_i} = 0}_{\text{incompress.}}$$

NS

We take the ensemble average of both sides:

$$\overline{\frac{\partial u_i}{\partial t}} + \overline{u_j \frac{\partial u_i}{\partial x_j}} = \overline{-\frac{1}{\rho}\frac{\partial p}{\partial x_i}} + \overline{\nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}}, \quad \overline{\frac{\partial u_i}{\partial x_i}} = 0$$

And simplify using the rules at the top of this page.

$$\frac{\partial \overline{u_i}}{\partial t} + \overline{u_j \frac{\partial u_i}{\partial x_j}} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{u_i}}{\partial x_j \partial x_j}, \quad \frac{\partial \overline{u_i}}{\partial x_i} = 0$$

$$\left(\frac{\partial \overline{u_i u_j}}{\partial x_j} = \overline{\frac{\partial (u_i u_j)}{\partial x_j}} = \overline{u_j \frac{\partial u_i}{\partial x_j}} + \overline{u_i \cancel{\frac{\partial u_j}{\partial x_j}}^{\text{const.}}} = \overline{u_j \frac{\partial u_i}{\partial x_j}}, \right.$$

$$\overline{u_i u_j} = \overline{(\overline{u_i} + u_i')(\overline{u_j} + u_j')} = \overline{\overline{u_i}\,\overline{u_j}} + \overline{u_i'\overline{u_j}} + \overline{\overline{u_i}\,u_j'} + \overline{u_i' u_j'}$$

$$\left. = \overline{u_i}\,\overline{u_j} + \underbrace{\overline{u_i'}\,\overline{u_j}}_{0} + \underbrace{\overline{u_i}\,\overline{u_j'}}_{0} + \overline{u_i' u_j'} \right)$$

average of fluctuation velocities is zero.

$$\frac{\partial \overline{u_i}}{\partial t} + \frac{\partial \overline{u_i u_j}}{\partial x_j} = \frac{\partial \overline{u_i}}{\partial t} + \frac{\partial (\overline{u_i}\,\overline{u_j} + \overline{u_i' u_j'})}{\partial x_j}$$

$$= \frac{\partial \overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} + \overline{u_i}\cancel{\frac{\partial \overline{u_j}}{\partial x_j}}^{0 \text{ continuity.}} + \frac{\partial \overline{u_i' u_j'}}{\partial x_j}.$$

$$\Rightarrow \boxed{\frac{\partial \overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{u_i}}{\partial x_j \partial x_j} - \frac{\partial \overline{u_i' u_j'}}{\partial x_j}.}$$

(1.2) Show for isotropic flows, $\overline{u_i' u_j'} = \frac{2}{3} K \delta_{ij}$.

Two-pt. corr. tensor for isotropic flow is

$$R_{ij}(r) = \overline{u'^2}\left[f(r)\delta_{ij} + \frac{r}{2}\frac{df}{dr}\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right)\right]. \quad \to f(0) = 1.$$

When $r=0$, $R_{ij}(0) = \overline{u'^2}\delta_{ij} = \frac{1}{3}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)\delta_{ij} = \frac{2}{3}K\delta_{ij}$.

$$= \frac{1}{3}\left(\overline{u_i' u_i'}\right)\delta_{ij} = \frac{2}{3}k\delta_{ij}.$$

In isotropic flow then, $\overline{u'^2} = \overline{v'^2} = \overline{w'^2} = \frac{2}{3}K$, $\overline{u'v'} = \overline{u'w'} = \overline{v'w'} = 0$.

where last part follows from orthogonal components being uncorrelated and thus $u_i' u_j'$ $(i \neq j)$ ~~[illegible]~~ having a mean of 0.

(1.3) Define the anisotropy tensor as $a_{ij} = \frac{\overline{u_i' u_j'}}{K} - \frac{2}{3}\delta_{ij}$,

$$\Rightarrow \overline{u_i' u_j'} = \frac{2}{3}k\delta_{ij} + K a_{ij}.$$

$$a_{ii} = \frac{\overbrace{\left(\overline{u_i' u_i'}\right)}^{\text{this is } 2K}}{K} - \frac{2}{3}\delta_{ii} = \frac{2K}{K} - \frac{2}{3}\delta_{ii} = 2-2 = 0.$$

For isotropic, ...

$$a_{ij} = \frac{\overline{u_i' u_j'}}{K} - \frac{2}{3}\delta_{ij} = \frac{2}{3}\frac{K\delta_{ij}}{k} - \frac{2}{3}\delta_{ij} = 0 \text{ when isotropic.}$$

$a_{ij} = \frac{\overline{u_i' u_j'}}{k} - \frac{2}{3}\delta_{ij}$. δ_{ij} is symmetric tensor $\delta_{ij} = \delta_{ji}$.

$\overline{u_i' u_j'}$ " " since multiplication is commutative: $\overline{u_i' u_j'} = \overline{u_j' u_i'}$.

~~[illegible]~~ $\Rightarrow a_{ij}$ symmetric b/c it is sum of sym. tensors.

~~[illegible] Diagonal terms must sum to zero always.~~

Since a_{ij} is symmetric, $a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$ 6 indep. terms.

1.4 start w/ transport eq'n for u_i'. NS for u_i − NS for $\bar{u}_i$.

$$\frac{\partial}{\partial t}u_i + u_j\frac{\partial u_i}{\partial x_j} - \frac{\partial}{\partial t}\bar{u}_i - \bar{u}_j\frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x_i} - \frac{\partial \bar{p}}{\partial x_i}\right) + \nu\left(\frac{\partial^2 u_i}{\partial x_j \partial x_j} - \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j}\right) + \frac{\partial \overline{u_i' u_j'}}{\partial x_j}$$

$$\frac{\partial}{\partial t}(u_i - \bar{u}_i) + (\bar{u}_j + u_j')\frac{\partial u_i}{\partial x_j} - \bar{u}_j\frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial (p-\bar{p})}{\partial x_i} + \nu\frac{\partial^2 (u_i - \bar{u}_i)}{\partial x_j \partial x_j} + \frac{\partial \overline{u_i' u_j'}}{\partial x_j}$$

$$\frac{\partial u_i'}{\partial t} + \bar{u}_j\frac{\partial (u_i - \bar{u}_i)}{\partial x_j} + u_j'\frac{\partial u_i}{\partial x_j} = \quad '' \qquad\qquad ''$$

$$\Rightarrow \left(\frac{\partial}{\partial t} + \bar{u}_j\frac{\partial}{\partial x_j}\right)u_i' = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} - \underbrace{u_j'\frac{\partial u_i}{\partial x_j}} + \nu\frac{\partial^2 u_i'}{\partial x_j \partial x_j} + \frac{\partial \overline{u_i' u_j'}}{\partial x_j}$$

$$= u_j'\frac{\partial(\bar{u}_i + u_i')}{\partial x_j} = u_j'\frac{\partial \bar{u}_i}{\partial x_j} + u_j'\frac{\partial u_i'}{\partial x_j}$$

$$S_{ij}' = \frac{1}{2}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)$$

$$= u_j'\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial u_i' u_j'}{\partial x_j} \quad \text{(continuity)}$$

$$= -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} - u_j'\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial u_i' u_j'}{\partial x_j} + \nu\frac{\partial u_i'}{\partial x_j \partial x_j} + \frac{\partial \overline{u_i' u_j'}}{\partial x_j}.$$

or

$$\boxed{\left(\frac{\partial}{\partial t} + \bar{u}_j\frac{\partial}{\partial x_j}\right)u_i' = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} - u_j'\frac{\partial \bar{u}_i}{\partial x_j} + \underbrace{\nu\frac{\partial u_i'}{\partial x_j \partial x_j}} + \frac{\partial}{\partial x_j}\left(\overline{u_i' u_j'} - u_i' u_j'\right).} \quad ⊛$$

→ or not this and add in here $2\nu S_{ij}'$

(Transport eq.n for u_i')

① $\nu S_{ij}' = \frac{\nu}{2}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)$ ②

We want a transport eq. for the Reynolds stresses

$$\left(\frac{\partial}{\partial t} + \bar{u}_k\frac{\partial}{\partial x_k}\right)\overline{u_i' u_j'} = \overbrace{\overline{u_j'\left(\frac{\partial}{\partial t} + \bar{u}_k\frac{\partial}{\partial x_k}\right)u_i'}}^{①} + \overbrace{\overline{u_i'\left(\frac{\partial}{\partial t} + \bar{u}_k\frac{\partial}{\partial x_k}\right)u_j'}}^{②}$$

①: from premult. by ⊛ then avg.ing.

$$① = -\frac{1}{\rho}\overline{u_j'\frac{\partial p'}{\partial x_i}} - \overline{u_j' u_k'}\frac{\partial \bar{u}_i}{\partial x_k} + \underbrace{\overline{u_j'\nu\frac{\partial u_i'}{\partial x_k \partial x_k}}} + \underbrace{\overline{u_j'\frac{\partial}{\partial x_k}\left(\overline{u_i' u_k'} - u_i' u_k'\right)}}$$

✓ ✓

or not this and add $2\nu S_{ij}'$ here

$$= -\overline{u_j'\frac{\partial (u_i' u_k')}{\partial x_k}} \quad \text{or} \rightarrow \quad = \overline{u_j'\frac{\partial}{\partial x_k}\left(2\nu S_{ij}' - u_i' u_j'\right)}$$

or ① $\displaystyle = \frac{-1}{\rho}\,\overline{u_j'\frac{\partial p'}{\partial x_i}} - \overline{u_j' u_k'}\,\frac{\partial \bar{u}_i}{\partial x_k} + \overline{u_j'\frac{\partial}{\partial x_k}\left(-u_i' u_k' + 2\nu s_{ij}'\right)}$

② $\displaystyle = \frac{-1}{\rho}\,\overline{u_i'\frac{\partial p'}{\partial x_j}} - \overline{u_i' u_k'}\,\frac{\partial \bar{u}_j}{\partial x_k} + \overline{u_i'\frac{\partial}{\partial x_k}\left(-u_j' u_k' + 2\nu s_{ji}'\right)}$

j, i flipped.

Our Reynolds stress transport equation then becomes...

$$\Rightarrow \left(\frac{\partial}{\partial t} + \bar{u}_k\frac{\partial}{\partial x_k}\right)\overline{u_i'u_j'} = \underbrace{\frac{-1}{\rho}\left(\overline{u_i'\frac{\partial p'}{\partial x_j}} + \overline{u_j'\frac{\partial p'}{\partial x_i}}\right)}_{\oplus}$$

$$\underbrace{-\left(\overline{u_i'u_k'}\frac{\partial \bar{u}_j}{\partial x_k} + \overline{u_j'u_k'}\frac{\partial \bar{u}_i}{\partial x_k}\right)}_{A}$$

$$+\left(\overline{u_i'\frac{\partial}{\partial x_k}\underbrace{(u_j'u_k')}_{(-)+2\nu s_{ij}'}} + \overline{u_j'\frac{\partial}{\partial x_k}\underbrace{(u_i'u_k')}_{(-)+2\nu s_{ij}'}}\right).$$

~~Transport of $u_i'u_j'$.~~

$$\cancel{\Rightarrow \oplus = -\frac{\partial}{\partial x_k}\left(\overline{u_i'u_j'u_k'}\right) = -\frac{\partial}{\partial x_k}\left(\overline{u_i'u_j'u_k'}\right).}$$

$$\cancel{\frac{\partial}{\partial x_j}(\overline{p'u_i'}) - \overline{p'\frac{\partial u_i'}{\partial x_j}}}$$

$$\cancel{\oplus = \frac{-1}{\rho}\left[\frac{\partial}{\partial x_j}\left(\overline{p'u_i'}\right) + \frac{\partial}{\partial x_i}\left(\overline{p'u_j'}\right)\right].}$$

Furthermore, $\displaystyle \oplus = \frac{-1}{\rho}\left(\frac{\partial \overline{p'u_i'}}{\partial x_j} - \overline{p'\frac{\partial u_i'}{\partial x_j}} + \frac{\partial \overline{u_j'p'}}{\partial x_i} - \overline{p'\frac{\partial u_j'}{\partial x_i}}\right)$

$$= \frac{-1}{\rho}\left[\frac{\partial}{\partial x_k}\left(\overline{u_i'p'}\,\delta_{jk} + \overline{u_j'p'}\,\delta_{ik}\right) + \overline{p'(\;)}\right],$$

$$\frac{1}{2}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right).$$

OK

$$\cancel{\Rightarrow \left(\frac{\partial}{\partial t} + \bar{u}_k\frac{\partial}{\partial x_k}\right)\overline{u_i'u_j'} = \frac{-1}{\rho}\left[\frac{\partial}{\partial x_k}\left(\overline{u_i'p'}\,\delta_{jk} + \overline{u_j'p'}\,\delta_{ik}\right) + \overline{p'\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)}\right]}$$

~~Transport for Reynolds Stresses (RSTE)~~

$$\cancel{-\left(\overline{u_i'u_k'}\frac{\partial \bar{u}_j}{\partial x_k} + \overline{u_j'u_k'}\frac{\partial \bar{u}_i}{\partial x_k}\right) - \frac{\partial}{\partial x_k}\overline{u_i'u_j'u_k'}}$$

$$\cancel{= \frac{-1}{\rho}\left[\frac{\partial}{\partial x_k}\left(\overline{u_i'p'}\,\delta_{jk} + \overline{u_j'p'}\,\delta_{ik}\right) + \right.}$$

~~For transport of $K = \frac{1}{2}\overline{u_i'u_i'}$, take trace of RSTE.~~

$$\cancel{\frac{D}{Dt}K}$$

$$\cancel{\frac{D}{Dt}(2k) = \frac{-1}{\rho}\left[\frac{\partial}{\partial x_k}\left(\overline{u_k'p'} + \overline{u_k'p'}\right) + \overline{p'\left(\frac{\partial u_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i}\right)}^{\,0}\right]}$$

$$\cancel{-\left(\overline{u_i'u_k'}\frac{\partial U_i}{\partial x_k}\cdot 2\right) - \frac{\partial}{\partial x_k}\overline{u_i'u_i'u_k'}}$$

$$\circledast = \overline{u_i'\frac{\partial}{\partial x_k}\left(2\nu S_{jk}' - u_j'u_k'\right)} + \overline{u_j'\frac{\partial}{\partial x_k}\left(2\nu S_{ik}' - u_i'u_k'\right)}$$

$$= \overline{u_i'\frac{\partial}{\partial x_k}\left(2\nu S_{jk}'\right)} + \overline{u_j'\frac{\partial}{\partial x_k}\left(2\nu S_{ik}'\right)} - \frac{\partial}{\partial x_k}\left(\overline{u_i'u_j'u_k'}\right).$$

aside $\left[\; 2\nu\frac{\partial}{\partial x_k}\left(S_{jk}'u_i'\right) = 2\nu\left(u_i'\frac{\partial S_{jk}'}{\partial x_k} + S_{jk}'\frac{\partial u_i'}{\partial x_k}\right)\right]$

$$\left.\begin{aligned} &= 2\nu\frac{\partial}{\partial x_k}\left(\overline{S_{jk}'u_i'}\right) - 2\nu\,\overline{S_{jk}'\frac{\partial u_i'}{\partial x_k}} \\ &\quad + 2\nu\frac{\partial}{\partial x_k}\left(\overline{S_{ik}'u_j'}\right) - 2\nu\,\overline{S_{ik}'\frac{\partial u_j'}{\partial x_k}} - \frac{\partial}{\partial x_k}\overline{u_i'u_j'u_k'}. \end{aligned}\right\} \text{B}$$

Also, $\oplus = \dfrac{-1}{\rho}\left(\dfrac{\partial\overline{p'u_i'}}{\partial x_j} - \overline{p'\dfrac{\partial u_i'}{\partial x_j}} + \dfrac{\partial\overline{u_j'p'}}{\partial x_i} - \overline{p'\dfrac{\partial u_j'}{\partial x_i}}\right)$

$$= \frac{-1}{\rho}\left[\frac{\partial}{\partial x_k}\left(\overline{u_i'p'}\,\delta_{jk} + \overline{u_j'p'}\,\delta_{ik}\right) - 2\overline{p'S_{ij}'}\right]$$

$$\left.= \frac{-1}{\rho}\frac{\partial}{\partial x_k}\left(\overline{u_i'p'}\,\delta_{jk} + \overline{u_j'p'}\,\delta_{ik}\right) + \frac{2}{\rho}\overline{p'S_{ij}'}\;.\right\} \text{C}$$

$\longrightarrow$

$\boxed{\tau'_{ij} = 2\nu S'_{ij} \quad \text{in course notes.}}$

Thus... $$\frac{D}{Dt}\overline{u_i' u_j'} = -\left(\overline{u_i' u_k'}\frac{\partial \overline{u_j}}{\partial x_k} + \overline{u_j' u_k'}\frac{\partial \overline{u_i}}{\partial x_k}\right) \qquad \text{from (A)}$$

$$-2\nu\left(\overline{S_{ik}'\frac{\partial u_j'}{\partial x_k} + S_{jk}'\frac{\partial u_i'}{\partial x_k}}\right) \qquad \text{from (B)}$$

$$+\overline{2\frac{p'}{\rho}S_{ij}'} \qquad \text{from (C)}$$

$$+\frac{\partial}{\partial x_k}\left[-\overline{u_i' u_j' u_k'} + 2\nu\left(\overline{u_j' S_{ik}'} + \overline{u_i' S_{jk}'}\right)\right. \qquad \text{from (B)}$$

$$\left.-\frac{1}{\rho}\left(\overline{u_i' p'}\,\delta_{jk} + \overline{u_j' p'}\,\delta_{ik}\right)\right] \qquad \text{from (C)}$$

We want the transport equation for K.

Now, $K = \frac{1}{2}\overline{u_i' u_i'}$, so take trace of $\frac{D}{Dt}\overline{u_i' u_j'}$.

$$2\frac{D}{Dt}K = -2\left(\overline{u_i' u_k'}\frac{\partial \overline{u_i}}{\partial x_k}\right)$$

$$-4\nu\left(\overline{S_{ik}'\frac{\partial u_i'}{\partial x_k}}\right) \Big\} = -4\nu\left(\frac{\partial u_i'}{\partial x_k} + \frac{\partial u_k'}{\partial x_i}\right)\frac{\partial u_i'}{\partial x_k} = -4\nu\left(\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k} + \frac{\partial u_k'}{\partial x_i}\frac{\partial u_i'}{\partial x_k}\right)$$

$$+\overline{2\frac{p'}{\rho}S_{ii}'} \Big\} \; S_{ii}' \left(\frac{\partial u_i'}{\partial x_i}\right) \to 0 \text{ by continuity.}$$

$$+\frac{\partial}{\partial x_k}\left[-\overline{u_i' u_i' u_k'} + 4\nu\left(\overline{u_i' S_{ik}'}\right) - \frac{2}{\rho}\left(\overline{u_i' p'}\,\delta_{ik}\right)\right]$$

$$\frac{D}{Dt}K = -\overline{u_i' u_k'}\frac{\partial \overline{u_i}}{\partial x_k} - 2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}} - 2\nu\overline{\frac{\partial u_k'}{\partial x_i}\frac{\partial u_i'}{\partial x_k}} + 0$$

$$-2\frac{\partial}{\partial x_k}\overline{K' u_k'} + 2\nu\left(\frac{\partial^2 \overline{K}}{\partial x_i \partial x_i} + \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} + \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_k'}{\partial x_i}}\right) - \frac{1}{\rho}\frac{\partial}{\partial x_k}\overline{p' u_k'}.$$

$$2\frac{Dk}{Dt} = \overbrace{-2\,\overline{u_i'u_k'}\,\frac{\partial \bar{u}_i}{\partial x_k}}^{(1)} \; \overbrace{-\,4\nu\,\overline{s_{ik}'\frac{\partial u_i'}{\partial x_k}}}^{(2)} \; \overbrace{+\,2\,\overline{\frac{p'}{\rho}\,s_{ii}'}}^{(3)}$$

$$+\frac{\partial}{\partial x_k}\Big(\underbrace{-\overline{u_i'u_i'u_k'}}_{(4)} \; \underbrace{+\,4\nu\,\overline{u_i's_{ik}'}}_{(5)} \; \underbrace{-\,\frac{2}{\rho}\,\overline{u_i'p'}\,\delta_{ik}}_{(6)}\Big)$$

(1): $-2\,\overline{u_i'u_k'}\,\frac{\partial \bar{u}_i}{\partial x_k} = -2\,\overline{u_i'u_j'}\,\frac{\partial \bar{u}_i}{\partial x_j}$ ok.

cancel $\circledast$

(2): $-4\nu\,\overline{s_{ik}'\frac{\partial u_i'}{\partial x_k}} = -\frac{4\nu}{2}\,\overline{\left(\frac{\partial u_i'}{\partial x_k}+\frac{\partial u_k'}{\partial x_i}\right)\frac{\partial u_i'}{\partial x_k}} = -2\nu\left(\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}} + \overline{\frac{\partial u_k'}{\partial x_i}\frac{\partial u_i'}{\partial x_k}}\right).$

(3): $2\,\overline{\frac{p'}{\rho}\,s_{ii}'} = \frac{2}{\rho}\,\overline{p'\left(\frac{\partial u_i'}{\partial x_i}+\frac{\partial u_i'}{\partial x_i}\right)} = \frac{4}{\rho}\,\overline{p'\frac{\partial u_i'}{\partial x_i}} = 0$, continuity.

(4): $\frac{\partial}{\partial x_k}\left(-\overline{u_i'u_i'u_k'}\right) = \frac{\partial}{\partial x_k}\left(-2\,\overline{k u_k'}\right)$. ok.

$2\frac{\partial k}{\partial x_k} = \frac{\partial \overline{u_i'u_i'}}{\partial x_k} = 2\,\overline{u_i'\frac{\partial u_i'}{\partial x_k}}$

(5): $\frac{\partial}{\partial x_k}\left(4\nu\,\overline{u_i's_{ik}'}\right) = \frac{4\nu}{2}\frac{\partial}{\partial x_k}\overline{\left(u_i'\left(\frac{\partial u_i'}{\partial x_k}+\frac{\partial u_k'}{\partial x_i}\right)\right)}$

$$= 2\nu\frac{\partial}{\partial x_k}\left(\overline{u_i'\frac{\partial u_i'}{\partial x_k}} + \overline{u_i'\frac{\partial u_k'}{\partial x_i}}\right) = 2\nu\frac{\partial}{\partial x_k}\left(\frac{\partial k}{\partial x_k} + \overline{u_i'\frac{\partial u_k'}{\partial x_i}}\right)$$

$$= 2\nu\left[\frac{\partial}{\partial x_k}\left(\frac{\partial k}{\partial x_k}\right) + \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_k'}{\partial x_i}} + \overline{u_i'\frac{\partial}{\partial x_i}\underbrace{\frac{\partial u_k'}{\partial x_k}}_{\to 0 \text{ cont.}}}\right]$$

cancel $\circledast$

(6): $\frac{\partial}{\partial x_k}\left(-\frac{2}{\rho}\,\overline{u_i'p'}\,\delta_{ik}\right) = \frac{\partial}{\partial x_j}\left(\frac{-2}{\rho}\,\overline{u_j'p'}\right)$. ok.

$$\Rightarrow \frac{Dk}{Dt} = \underbrace{-\overline{u_i'u_j'}\,\frac{\partial \bar{u}_i}{\partial x_j}}_{(1)} \; \underbrace{-\,\nu\,\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}}_{(2)} + \frac{\partial}{\partial x_j}\left(\underbrace{\nu\frac{\partial k}{\partial x_j}}_{(5)} - \underbrace{\overline{k'u_j'}}_{(4)} \; \underbrace{-\,\frac{1}{\rho}\,\overline{p'u_j'}}_{(6)}\right). \; ✓$$

$$a_{ij} \equiv \frac{\overline{u_i'u_j'}}{K} - \frac{2}{3}\delta_{ij} .$$

Now governing eq'n for a_{ij}...

$$\frac{D(a_{ij})}{Dt} = \left(\frac{\partial}{\partial t} + \bar{u}_K \frac{\partial}{\partial x_K}\right)\left(\frac{\overline{u_i'u_j'}}{K} - \frac{2}{3}\delta_{ij}\right)$$

since δ_{ij} is const.

$$\frac{Da_{ij}}{Dt} = \frac{\partial}{\partial t}\left(\frac{\overline{u_i'u_j'}}{K}\right) + \bar{u}_K \frac{\partial}{\partial x_K}\left(\frac{\overline{u_i'u_j'}}{K}\right) - \cancelto{0}{\frac{2}{3}\frac{\partial}{\partial t}(\delta_{ij})} - \cancelto{0}{\bar{u}_K \frac{\partial}{\partial x_K}\left(\frac{2}{3}\delta_{ij}\right)}$$

$$= \frac{1}{K}\frac{\partial}{\partial t}\left(\overline{u_i'u_j'}\right) + \overline{u_i'u_j'}\frac{\partial}{\partial t}\left(\frac{1}{K}\right) + \frac{\bar{u}_K}{K}\frac{\partial}{\partial x_K}\left(\overline{u_i'u_j'}\right) + \overline{u_i'u_j'}\,\bar{u}_K \frac{\partial}{\partial x_K}\left(\frac{1}{K}\right)$$

$$= \frac{1}{K}\frac{D\overline{u_i'u_j'}}{Dt} + \overline{u_i'u_j'}\left(\frac{-1}{K^2}\right)\frac{\partial K}{\partial t} + \overline{u_i'u_j'}\,\bar{u}_K\left(\frac{-1}{K^2}\right)\frac{\partial K}{\partial x_K}$$

$$\frac{Da_{ij}}{Dt} = \underbrace{\frac{1}{K}\frac{D\overline{u_i'u_j'}}{Dt}}_{(A)} \underbrace{- \frac{\overline{u_i'u_j'}}{K^2}\frac{DK}{Dt}}_{(B)} . \qquad \leftarrow (*)$$

Next, we just substitute in for our known $D\overline{u_i'u_j'}/Dt$ and DK/Dt.

$$\frac{Da_{ij}}{Dt} = \left.\frac{1}{K}\left[-\overline{u_i'u_K'}\frac{\partial \bar{u}_j}{\partial x_K} - \overline{u_j'u_K'}\frac{\partial \bar{u}_i}{\partial x_K} - 2\nu\,\overline{s_{iK}'\frac{\partial u_j'}{\partial x_K}} - 2\nu\,\overline{s_{jK}'\frac{\partial u_i'}{\partial x_K}} + \frac{2}{\rho}\overline{p's_{ij}'} - \frac{\partial}{\partial x_K}\left(\overline{u_i'u_j'u_K'}\right) + 2\nu\frac{\partial}{\partial x_K}\left(\overline{u_j's_{iK}'}\right) + 2\nu\frac{\partial}{\partial x_K}\left(\overline{u_i's_{jK}'}\right) - \frac{1}{\rho}\frac{\partial}{\partial x_K}\left(\overline{u_i'p'}\delta_{jK}\right) - \frac{1}{\rho}\frac{\partial}{\partial x_K}\left(\overline{u_j'p'}\delta_{iK}\right)\right]\right\}(A)$$

$$\left. - \frac{\overline{u_i'u_j'}}{K^2}\left[-\overline{u_i'u_j'}\frac{\partial \bar{u}_i}{\partial x_j} - \nu\,\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}} + \frac{\partial}{\partial x_j}\left(\nu\frac{\partial K}{\partial x_j} - \overline{K'u_j'} - \frac{1}{\rho}\overline{p'u_j'}\right)\right]\right\}(B)$$

Let $\frac{2}{\rho}\overline{p's_{ij}'} \equiv \Pi_{ij}$, and let $\bar{S}_{ij} \equiv \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$, $\bar{W}_{ij} \equiv \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \bar{u}_j}{\partial x_i}\right)$

$$\Rightarrow \frac{\partial \bar{u}_i}{\partial x_j} = \bar{S}_{ij} + \bar{W}_{ij} .$$

Also $S_{ij}' = S_{ij} - \bar{S}_{ij}$

$$\frac{Da_{ij}}{Dt} = -\frac{\overline{u_i' u_k'}}{k}\left(\bar{S}_{jk} + \bar{W}_{jk}\right) - \frac{\overline{u_j' u_k'}}{k}\left(\bar{S}_{ik} + \bar{W}_{ik}\right)$$

$$-\frac{2\nu}{k}\overline{(S_{ik}-\bar{S}_{ik})\frac{\partial u_j'}{\partial x_k}} - \frac{2\nu}{k}\overline{(S_{jk}-\bar{S}_{jk})\frac{\partial u_i'}{\partial x_k}} + \frac{1}{k}\Pi_{ij}$$

$$-\frac{\partial}{\partial x_k}\left(\overline{u_i' u_j' u_k'}\right)\frac{1}{k} + \frac{2\nu}{k}\frac{\partial}{\partial x_k}\left(\overline{u_j' s_{ik}'} + \overline{u_i' s_{jk}'}\right)$$

$$-\frac{\partial}{\partial x_k}\left(\frac{1}{\rho}\overline{u_i' p'}\,\delta_{jk}\right)\frac{1}{k} - \frac{\partial}{\partial x_k}\left(\frac{1}{\rho}\overline{u_j' p'}\,\delta_{ik}\right)\frac{1}{k}$$

$$-\frac{\overline{u_i' u_j'}}{k^2}\left[-\overline{u_i' u_j'}\left(\bar{S}_{ij} + \bar{W}_{ij}\right) - \nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}} + \frac{\partial}{\partial x_j}\left(\nu\frac{\partial k}{\partial x_j} - \overline{k' u_j'} - \frac{1}{\rho}\overline{p' u_j'}\right)\right]$$

Note $\frac{\overline{u_i' u_j'}}{k} = a_{ij} + \frac{2}{3}\delta_{ij} \ldots$

$$A = \left(-a_{jk} - \frac{2}{3}\delta_{jk}\right)\left(\bar{S}_{jk} + \bar{W}_{jk}\right) + \left(-a_{ik} - \frac{2}{3}\delta_{ik}\right)\left(\bar{S}_{ik} + \bar{W}_{ik}\right)$$

$$= -a_{jk}\bar{S}_{jk} - \frac{2}{3}\bar{S}_{ii} - a_{jk}\bar{W}_{jk} - \frac{2}{3}\overline{W}_{ii} \xrightarrow{} 0 - a_{ik}\bar{S}_{ik} - a_{ik}\bar{W}_{ik}$$

$$-\frac{2}{3}\bar{S}_{ii} - \frac{2}{3}\overline{W}_{ii} \xrightarrow{} 0$$

$$\frac{\partial}{\partial x_k}\left[\overline{\frac{u_j'}{2}\left(\frac{\partial u_i'}{\partial x_k} + \frac{\partial u_k'}{\partial x_i}\right)}\right] = \frac{1}{2}\frac{\partial}{\partial x_k}\left[\overline{u_j'\frac{\partial u_i'}{\partial x_k}} + \overline{u_j'\frac{\partial u_k'}{\partial x_i}}\xrightarrow{} 0 \text{ cont.}\right] = \frac{1}{2}\frac{\partial}{\partial x_k}\left[\frac{\partial \overline{u_i' u_j'}}{\partial x_k} - \overline{u_i'\frac{\partial u_j'}{\partial x_k}}\right]$$

$$\frac{2\nu}{k}\frac{\partial}{\partial x_k}\left(\overline{u_j' s_{ik}'} + \overline{u_i' s_{jk}'}\right) = \frac{2\nu}{k}\frac{\partial}{\partial x_k}\left(\overline{\frac{1}{2}u_j'\left(\frac{\partial u_i'}{\partial x_k} + \frac{\partial u_k'}{\partial x_i}\right)} + \overline{\frac{u_i'}{2}\left(\frac{\partial u_j'}{\partial x_k} + \frac{\partial u_k'}{\partial x_j}\right)}\right) \quad \to 0 \text{ cont.} \quad \to 0 \text{ cont.}$$

$$= \frac{\nu}{k}\frac{\partial}{\partial x_k}\left(\frac{\partial \overline{u_i' u_j'}}{\partial x_k} - \overline{u_i'\frac{\partial u_j'}{\partial x_k}} + \frac{\partial \overline{u_i' u_j'}}{\partial x_k} - \overline{u_j'\frac{\partial u_i'}{\partial x_k}}\right)$$

$$= \frac{\nu}{k}\frac{\partial}{\partial x_k}\left(2\frac{\partial \overline{u_i' u_j'}}{\partial x_k} - \frac{\partial \overline{u_i' u_j'}}{\partial x_k}\right) = \frac{\nu}{k}\frac{\partial}{\partial x_k}\left(\frac{\partial \overline{u_i' u_j'}}{\partial x_k}\right) \checkmark$$

AA

Note $\dfrac{\overline{u_i'u_j'}}{k} = a_{ij} + \frac{2}{3}\delta_{ij}$.

Let's start with the first term, in ⊛,

$$\frac{1}{k}\frac{D\overline{u_i'u_j'}}{Dt} = -\frac{\overline{u_i'u_k'}}{k}\frac{\partial \bar{u}_j}{\partial x_k} - \frac{\overline{u_j'u_k'}}{k}\frac{\partial \bar{u}_i}{\partial x_k} - \frac{2\nu}{k}\overline{s_{ik}'\frac{\partial u_j'}{\partial x_k}} - \frac{2\nu}{k}\overline{s_{jk}'\frac{\partial u_i'}{\partial x_k}}$$
$$+ \frac{2}{k\rho}\overline{p's_{ij}'} + \frac{\partial}{\partial x_k}\frac{1}{k}\left(-\overline{u_i'u_j'u_k'} + 2\nu\,\overline{u_j's_{ik}'} + 2\nu\,\overline{u_i's_{jk}'} - \frac{1}{\rho}\left(\overline{u_i'p'}\,\delta_{jk} + \overline{u_j'p'}\,\delta_{ik}\right)\right).$$

$$= -\left(a_{ik} + \tfrac{2}{3}\delta_{ik}\right)\left(\bar{S}_{jk} + \bar{W}_{jk}\right) - \left(a_{jk} + \tfrac{2}{3}\delta_{jk}\right)\left(\bar{S}_{ik} + \bar{W}_{ik}\right)$$
$$-\frac{2\nu}{k}\left(\overline{\left(s_{ik}'\right)\frac{\partial u_j'}{\partial x_k}} + \overline{\left(s_{jk}'\right)\frac{\partial u_i'}{\partial x_k}}\right) + \frac{1}{k}\Pi_{ij}$$
$$+\frac{1}{k}\frac{\partial}{\partial x_k}\left(-\overline{u_i'u_j'u_k'} - \frac{\overline{p'u_j'}}{\rho}\delta_{ik} - \frac{\overline{p'u_i'}}{\rho}\delta_{jk} + 2\nu\underbrace{\left(\overline{u_j's_{ik}'} + \overline{u_i's_{jk}'}\right)}_{\text{see AA}}\right)$$

$$= -a_{ik}\bar{S}_{jk} - \tfrac{2}{3}\bar{S}_{ij} - a_{ik}\bar{W}_{jk} - \cancel{\tfrac{2}{3}\bar{W}_{ji}}$$ since $\bar{W}_{ij} = -\bar{W}_{ji}$.
$$-a_{jk}\bar{S}_{ik} - \tfrac{2}{3}\bar{S}_{ij} - a_{jk}\bar{W}_{ik} - \cancel{\tfrac{2}{3}\bar{W}_{ij}}$$
$$-\frac{2\nu}{k}\left[\overline{\left(s_{ik}'\right)\frac{\partial u_j'}{\partial x_k}} + \overline{\left(s_{jk}'\right)\frac{\partial u_i'}{\partial x_k}}\right] + \frac{1}{k}\Pi_{ij} + \frac{1}{k}D_{ij},$$

where $D_{ij} = -\dfrac{\partial}{\partial x_k}\left(\overline{u_i'u_j'u_k'} + \dfrac{\overline{p'u_j'}}{\rho}\delta_{ik} + \dfrac{\overline{p'u_i'}}{\rho}\delta_{jk} - \nu\dfrac{\partial \overline{u_i'u_j'}}{\partial x_k}\right)$

$$= -\left(a_{ik}\bar{S}_{kj} + \bar{S}_{ik}a_{kj}\right) - a_{ik}\bar{W}_{jk} - a_{jk}\bar{W}_{ik} - \tfrac{4}{3}\bar{S}_{ij} + \tfrac{1}{k}\Pi_{ij} + \tfrac{1}{k}D_{ij}$$
$$\cancel{-\frac{2\nu}{k}\left(\overline{s_{ik}\frac{\partial u_j}{\partial x_k}} - s_{ik}\frac{\partial u_j}{\partial x_k} + s_{jk}\frac{\partial u_i}{\partial x_k} - s_{jk}\frac{\partial u_i}{\partial x_k}\right)} - \frac{2\nu}{k}\left(\overline{s_{ik}'\frac{\partial u_j'}{\partial x_k}} + \overline{s_{jk}'\frac{\partial u_i'}{\partial x_k}}\right)$$

$$= -\left(a_{ik}\bar{S}_{kj} + \bar{S}_{ik}a_{kj}\right) - a_{ik}\bar{W}_{jk} - a_{jk}\bar{W}_{ik} - \tfrac{4}{3}\bar{S}_{ij} + \tfrac{1}{k}\Pi_{ij} + \tfrac{1}{k}D_{ij}$$
$$\cancel{-\frac{2\nu}{k}\left(\overline{s_{ik}\frac{\partial u_i}{\partial x_k} + s_{jk}\frac{\partial u_i}{\partial x_k}}\right)}$$
$$-\frac{2\nu}{k}\left(\overline{s_{ik}'\frac{\partial u_j'}{\partial x_k}} + \overline{s_{jk}'\frac{\partial u_j'}{\partial x_k}}\right)$$

note $= -\dfrac{1}{k}\left(2\nu\,\overline{\dfrac{\partial u_i'}{\partial x_k}\dfrac{\partial u_j'}{\partial x_k}}\right) \equiv -\dfrac{1}{k}\varepsilon_{ij}$.

~~We next look at~~ $\cancel{\left(\bar{S}_{jk}\frac{\partial u_i'}{\partial x_k} + \bar{S}_{ik}\frac{\partial u_j'}{\partial x_k}\right)}$

This leaves us with the following expression for (A):

$$\frac{1}{k}\frac{D\overline{u_i'u_j'}}{Dt} = -\left(a_{ik}\bar{S}_{kj} + \bar{S}_{ik}a_{kj}\right) + \left(a_{ik}\bar{W}_{kj} - \bar{W}_{ik}a_{kj}\right)$$
$$+\frac{1}{k}\Pi_{ij} + \frac{1}{k}D_{ij} - \frac{1}{k}\varepsilon_{ij} - \frac{4}{3}\bar{S}_{ij}.$$

Now we consider (B), $\quad (B) = -\frac{\overline{u_i'u_j'}}{k^2}\frac{Dk}{Dt}$ $\qquad$ here let $i \to k$, $j \to l$.

$$(B) = \left(-\overline{u_i'u_j'}/k^2\right)\cdot\left(-\overline{u_i'u_j'}\frac{\partial \bar{u}_i}{\partial x_j} - \underbrace{\nu\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}}_{=\varepsilon} + \frac{\partial}{\partial x_j}\left(\nu\frac{\partial k}{\partial x_j} - \overline{k'u_j'} - \frac{1}{\rho}\overline{p'u_j'}\right)\right).$$

Let $\varepsilon = \varepsilon_{ii}/2 = \nu\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}$. $\qquad D \equiv D_{ii}/2$

$$(B) = \frac{\overline{u_i'u_j'}}{k^2}\overline{u_k'u_l'}\frac{\partial \bar{u}_k}{\partial x_l} + \frac{\overline{u_i'u_j'}}{k^2}\varepsilon \left\{\overbrace{-\frac{\overline{u_i'u_j'}}{k^2}\nu\frac{\partial^2 k}{\partial x_k\partial x_k}}^{=+\frac{\overline{u_i'u_j'}}{k^2}\left(\frac{-\nu}{2}\frac{\partial^2\overline{u_i'u_i'}}{\partial x_k\partial x_k}\right)} + \frac{\overline{u_i'u_j'}}{k^2}\frac{\partial^2\overline{k'u_j'}}{\partial x_k\partial x_k} + \frac{\partial}{\partial x_k}\overline{p'u_k'}\right\}.$$

this is D!

$$= \frac{\overline{u_i'u_j'}}{k^2}\overline{u_k'u_l'}\frac{\partial \bar{u}_k}{\partial x_l} - \varepsilon\frac{\overline{u_i'u_j'}}{k^2} + D\frac{\overline{u_i'u_j'}}{k^2}$$

$\frac{\overline{u_i'u_j'}}{k} = a_{ij} + \frac{2}{3}\delta_{ij}$, so...

$$(B) = \left(a_{ij} + \tfrac{2}{3}\delta_{ij}\right)\frac{\overline{u_k'u_l'}}{k}\frac{\partial \bar{u}_k}{\partial x_l} - \frac{1}{k}\left(a_{ij} + \tfrac{2}{3}\delta_{ij}\right)\varepsilon + \frac{1}{k}\left(a_{ij} + \tfrac{2}{3}\delta_{ij}\right)D$$

prod. w/ symm. tensor

$$Ⓑ = \left(a_{ij} + \tfrac{2}{3}\delta_{ij}\right)\left(a_{kl} + \tfrac{2}{3}\delta_{kl}\right)\left(\overline{S}_{kl} + \cancel{\overline{W}_{kl}}\right) + \frac{1}{k}\left(a_{ij} + \tfrac{2}{3}\delta_{ij}\right)(\varepsilon + D)$$

$$= \left(a_{ij} + \tfrac{2}{3}\delta_{ij}\right)\left[a_{kl}\overline{S}_{kl} + \tfrac{2}{3}\delta_{kl}\overline{S}_{kl}\right] - \frac{\varepsilon}{k}\left(a_{ij} + \tfrac{2}{3}\delta_{ij}\right) + \underbrace{\frac{D}{k}\left(a_{ij} + \tfrac{2}{3}\delta_{ij}\right)}$$

0 ← incompressible → 0

$$= a_{ij}a_{kl}\overline{S}_{kl} + \tfrac{2}{3}\delta_{ij}a_{kl}\overline{S}_{kl} + \tfrac{2}{3}a_{ij}\cancel{\delta_{kl}\overline{S}_{kl}} + \tfrac{4}{9}\delta_{ij}\cancel{\delta_{kl}\overline{S}_{kl}}$$

$$- \frac{1}{k}a_{ij}\varepsilon - \frac{2}{3k}\delta_{ij}\varepsilon + \frac{1}{k}\left(a_{ij} + \tfrac{2}{3}\delta_{ij}\right)D.$$

$$= \left(\frac{+k\,a_{kl}\overline{S}_{kl}}{\varepsilon} - 1\right)\frac{\varepsilon}{k}a_{ij} - \frac{1}{k}\frac{2}{3}\varepsilon\delta_{ij} + \frac{1}{k}\left(a_{ij} + \tfrac{2}{3}\delta_{ij}\right)D + \frac{2}{3}\delta_{ij}a_{kl}\overline{S}_{kl}.$$

$$= \left(\frac{P}{\varepsilon} - 1\right)\frac{\varepsilon}{k}a_{ij} - \frac{\varepsilon}{k}\frac{2}{3}\delta_{ij} + \frac{1}{k}\left(a_{ij} + \tfrac{2}{3}\delta_{ij}\right)D + \frac{2}{3}\delta_{ij}a_{kl}\overline{S}_{kl}.$$

Combining,

$$\frac{Da_{ij}}{Dt} = Ⓐ + Ⓑ = -\left(a_{ik}\overline{S}_{kj} + \overline{S}_{ik}a_{kj} - \tfrac{2}{3}\delta_{ij}a_{kl}\overline{S}_{kl}\right) + \left(a_{ik}\overline{W}_{kj} - \overline{W}_{ik}a_{kj}\right) + \frac{1}{k}\Pi_{ij} + \frac{1}{k}D_{ij} - \frac{1}{k}\varepsilon_{ij} - \frac{4}{3}\overline{S}_{ij} - \left(\frac{P}{\varepsilon} - 1\right)\frac{\varepsilon}{k}a_{ij} + \frac{2}{3}\frac{\varepsilon}{k}\delta_{ij} - \frac{1}{k}\left(a_{ij} + \tfrac{2}{3}\delta_{ij}\right)D.$$

$$\boxed{\begin{aligned}\frac{Da_{ij}}{Dt} = &-\left(\frac{P}{\varepsilon} - 1\right)\frac{\varepsilon}{k}a_{ij} - \frac{4}{3}\overline{S}_{ij} - \left(a_{ik}\overline{S}_{kj} + \overline{S}_{ik}a_{kj} - \frac{2}{3}a_{kl}\overline{S}_{kl}\delta_{ij}\right)\\ &+ \left(a_{ik}\overline{W}_{kj} - \overline{W}_{ik}a_{kj}\right) + \frac{1}{k}\Pi_{ij} - \frac{1}{k}\left(\varepsilon_{ij} - \frac{2}{3}\varepsilon\delta_{ij}\right)\\ &+ \frac{1}{k}\left[D_{ij} - \left(a_{ij} + \frac{2}{3}\delta_{ij}\right)D\right].\end{aligned}}$$

Assuming k and ε are known, the unclosed terms in ✪ are...

- D_{ij}, D (contains a third order moment, a pressure-velocity correlation)
- P (~~multiple of~~ product of a_{kl} and $\bar{S}_{kl}$)
- $a_{ik}\bar{S}_{kj}$, etc (anything w/ a mult. by $\bar{S}$ or $\bar{W}$)
- $a_{ik}\bar{W}_{kj}$
- Π_{ij} (unclosed b/c product of p' and S'_{ij}).

1.5 ε_{ij} can be treated as isotropic in many flows because dissipation primarily occurs in turbulence at scales $\boxed{\text{small enough to be treated as isotropic by K41 theory}}$.

Now we show that $\frac{-1}{k}\left(\varepsilon_{ij} - \frac{2}{3}\varepsilon\,\delta_{ij}\right) = 0$ under this assumption, where $\varepsilon = \varepsilon_{ii}/2$.

This is equivalent to showing that $\varepsilon_{ij} = \frac{1}{3}\varepsilon_{ii}\delta_{ij}$.

Since $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33}$, $\varepsilon_{ii} = 3\varepsilon_{11} \Rightarrow \boxed{\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = \frac{1}{3}\varepsilon_{ii}}$.

Furthermore, since $\varepsilon_{ij} = 2\nu \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}$, it is the case that

$\varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = \varepsilon_{21} = \varepsilon_{31} = \varepsilon_{32}$, and $\overline{\frac{\partial u_1'}{\partial x_k}\frac{\partial u_2'}{\partial x_k}} = \overline{\frac{\partial u_1'}{\partial u_k}\frac{\partial u_3'}{\partial x_k}} = \overline{\frac{\partial u_2'}{\partial x_k}\frac{\partial u_3'}{\partial x_k}}$.

However, since the fluctuation velocities are wholly uncorrelated in isotropic turbulence, these must be equal to zero, and thus there is $\boxed{\text{no anisotropic component}}$ of ε_{ij}. $\Rightarrow \boxed{\varepsilon_{ij} = \frac{1}{3}\varepsilon_{ii}\delta_{ij}}$.

(1.6) $C_1 = 1.5, \quad C_2 = 0.8, \quad C_3 = 0.875, \quad C_4 = 0.655$

Let
$$\left.\begin{aligned}\Pi_{ij} = -C_1 \varepsilon a_{ij} + C_2 k \bar{S}_{ij} + C_3 k \left(a_{ik}\bar{S}_{kj} + \bar{S}_{ik}a_{kj} - \tfrac{2}{3} a_{kl}\bar{S}_{kl}\delta_{ij}\right) \\ + C_4 k\left(a_{ik}\bar{W}_{kj} - \bar{W}_{ik}a_{kj}\right),\end{aligned}\right\} \circledast$$

and let $\alpha_1 = \frac{P}{\varepsilon} - 1 + C_1$, $\alpha_2 = C_2 - \frac{4}{3}$, $\alpha_3 = C_3 - 1$, $\alpha_4 = C_4 - 1$.

Starting with
$$\frac{Da_{ij}}{Dt} = -\left(\frac{P}{\varepsilon} - 1\right)\frac{\varepsilon}{k} a_{ij} - \frac{4}{3}\bar{S}_{ij} - \left(a_{ik}\bar{S}_{kj} + \bar{S}_{ik}a_{kj} - \frac{2}{3} a_{kl}\bar{S}_{kl}\delta_{ij}\right)$$
$$+ \left(a_{ik}\bar{W}_{kj} - \bar{W}_{ik}a_{kj}\right) + \frac{1}{k}\left[D_{ij} - \left(a_{ij} + \frac{2}{3}\delta_{ij}\right)D\right] + \frac{1}{k}\Pi_{ij},$$

We substitute in Π_{ij} from $\circledast$, yielding
$$\frac{Da_{ij}}{Dt} = -\left(\frac{P}{\varepsilon} - 1 + C_1\right)\frac{\varepsilon}{k} a_{ij} + \left(C_2 - \frac{4}{3}\right)\bar{S}_{ij} + (C_3 - 1)\left(a_{ik}\bar{S}_{kj} + \bar{S}_{ik}a_{kj} - \frac{2}{3}a_{kl}\bar{S}_{kl}\delta_{ij}\right)$$
$$+ (C_4 - 1)\left(a_{ik}\bar{W}_{kj} - \bar{W}_{ik}a_{kj}\right)$$
$$+ \frac{1}{k}\left[D_{ij} - \left(a_{ij} + \frac{2}{3}\delta_{ij}\right)D\right], \quad \text{or}$$

$$\boxed{\begin{aligned}\frac{Da_{ij}}{Dt} = -\alpha_1 \frac{\varepsilon}{k} a_{ij} + \alpha_2 \bar{S}_{ij} + \alpha_3\left(a_{ik}\bar{S}_{kj} + \bar{S}_{ik}a_{kj} - \tfrac{2}{3}a_{kl}\bar{S}_{kl}\delta_{ij}\right) \\ -\alpha_4\left(a_{ik}\bar{W}_{kj} - \bar{W}_{ik}a_{kj}\right) + \tfrac{1}{k}\left[D_{ij} - \left(a_{ij} + \tfrac{2}{3}\delta_{ij}\right)D\right]\end{aligned}}.$$

✓

(1.7) By neglecting the redistribution and transport terms, and writing simply $\frac{Da_{ij}}{Dt} = -\alpha_1 \frac{\varepsilon}{k} a_{ij} + \alpha_2 \bar{S}_{ij}$, we have neglected several physical effects:

- Near walls, the Reynolds number drops and diffusive effects become important, which we neglect by the viscous terms within D_{ij} and D.

→

- Also near walls, real near-wall pressure effects are neglected, in that P/ε cannot fully capture those effects.

Basically, canonical HIT is simulated well with this model, but any low-Re or wall-bounded flows have regions that it breaks down in.

These effects and flows are also relevant to 1.8.

(1.8) We further assume the flow is in equillibrium, s.t.

$$\frac{D a_{ij}}{Dt} = 0 , \quad \text{meaning} \quad -\alpha_1 \frac{\varepsilon}{k} a_{ij} + \alpha_2 \bar{S}_{ij} = 0 .$$

$$\Rightarrow \boxed{a_{ij} = \frac{\alpha_2}{\alpha_1} \frac{k}{\varepsilon} \bar{S}_{ij}}$$

This is the closure under the equillibrium assumption.

This model obviously breaks down in physical situations where the <u>decay of anisotropy</u> (to a fully isotropic flow) takes place. It would be inaccurate when simulating flow in a stirred tank when stirring stopped, or simulating the decay of anisotropy behind a ~~grid~~ shear layer in a wind tunnel.

(1.9) Further simplify w/ gradient transport hypothesis:

$$a_{ij} = -2 \frac{\nu_T}{k} \bar{S}_{ij} \quad , \quad \nu_T = C_\mu \frac{k^2}{\varepsilon} .$$

$$\Rightarrow a_{ij} = -2 C_\mu \frac{k}{\varepsilon} \bar{S}_{ij} = \frac{\alpha_2}{\alpha_1} \frac{k}{\varepsilon} \bar{S}_{ij} \quad \Rightarrow \boxed{C_\mu = \frac{-1}{2} \frac{\alpha_2}{\alpha_1}}$$

$\longrightarrow$

If we assume C_μ is constant, this implies

$$\text{constant} = C_\mu = \frac{-1}{2}\frac{\alpha_2}{\alpha_1} = \frac{-1}{2}\frac{C_2 - 4/3}{P/\varepsilon - 1 + C_1} \Rightarrow \frac{P}{\varepsilon} - 1 + C_1 \sim C_2 - 4/3$$

$$\Rightarrow \frac{P}{\varepsilon} \sim \text{constant}.$$

First off, since C_μ sets P/ε as a constant if C_μ = constant, this neglects production near boundary layers (for instance, turbulent intensity introduced by small trees in an atmospheric boundary layer) which would require P/ε scale somehow near Dirichlet boundary conditions / walls. This simplifying assumption would perform well in flows that had no small-scale production (that is, it would do terribly for combustion) and probably <u>no walls</u>, and needed simulation speed increased. $P = -k a_{ij} \bar{S}_{ij}$

(since production rate increases with Reynolds stresses (anisotropy) and mean velocity gradients)

(1.10) If $a_{ij} = -2\frac{\nu_T}{k}\bar{S}_{ij}$, we can modify the RANS equation as...

$$\frac{D\bar{u}_i}{Dt} = \frac{-1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} - \frac{\partial \overline{u_i' u_j'}}{\partial x_j} \qquad \left(\text{Let } \hat{p} \equiv \bar{p} + \tfrac{2}{3}\rho k\right)$$

$$= \frac{-1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu\frac{\partial \bar{u}_i}{\partial x_j} + 2\nu_T \bar{S}_{ij} - \frac{2}{3}k\delta_{ij}\right), \quad \overline{u_i' u_j'} = \frac{2}{3}k\delta_{ij} + k a_{ij}$$

$$= \frac{2}{3}k\delta_{ij} + k\left(\frac{-2}{k}\nu_T \bar{S}_{ij}\right)$$

$$= \frac{2}{3}k\delta_{ij} - 2\nu_T \bar{S}_{ij}.$$

$$= \frac{-1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{-1}{\rho}\frac{\partial\left(\frac{2}{3}\rho k\right)}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu\frac{\partial \bar{u}_i}{\partial x_j} + 2\nu_T \bar{S}_{ij}\right)$$

$$= \frac{-1}{\rho}\frac{\partial \hat{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu\left(\frac{\partial \bar{u}_i}{\partial x_j} + \underbrace{\frac{\partial \bar{u}_j}{\partial x_i}}_{=0 \text{ by cont., so can add it in w/in } \partial/\partial x_j.}\right) + 2\nu_T \bar{S}_{ij}\right)$$

$$= \frac{-1}{\rho}\frac{\partial \hat{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2\nu\bar{S}_{ij} + 2\nu_T \bar{S}_{ij}\right)$$

$$\Rightarrow \boxed{\frac{D\bar{u}_i}{Dt} = \frac{-1}{\rho}\frac{\partial \hat{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[2\left[\nu + \nu_T\right]\bar{S}_{ij}\right]}.$$

This form of the RANS equation using our a_{ij} closure method may be popular for the following practical reasons:

- Since C_μ, K^2, ε are all > 0 (positive), ν_T is positive and lends additional stability to the numerical method by increasing the effectiveness of viscous diffusion mechanisms within the simulation

- The equation is linear in $\overline{u}_i$ and $\overline{p}$, so is easy to implement and quick to solve — this is particularly important for design and optimization applications.