Project #2 Assignment (MCEN7221/ASEN6037)
Final project must be uploaded to D2L by midnight on Friday, May 8, 2015.

The accurate prediction of turbulent flows using numerical simulations is a fundamental challenge in a wide range of problems. This includes the design of more efficient propulsion systems, optimization of energy output and turbine lifespan in large-scale wind farms, and reliable forecasting of the climate and weather, all of which require high-fidelity predictions of turbulent flows. Numerical simulations can allow low cost and rapid design of these applications, but accurate descriptions of turbulence are required in the presence of complex processes such as unsteady separation, fluid-structure interactions, combustion, and shock-boundary layer interactions.

The difficulty associated with these computations stems, in large part, from the wide range of spatial and temporal scales present in most practical problems. The size of a supersonic combustion ramjet (scramjet) engine, for example, is orders of magnitude larger than the smallest turbulent and flame scales in the inlet and combustor. Although it is theoretically possible to resolve this full range of scales using direct numerical simulations (DNS), the associated demand for computational resources is prohibitive in nearly all practical applications. Consequently, approaches which model some, or all, of the relevant scales – such as large eddy simulations (LES) and Reynolds averaged Navier Stokes (RANS) approaches – continue to be widely used. Even as computing power increases over the coming decades, LES and RANS will remain popular; rather than resolving more scales as in DNS, the extra resources will be used for increasingly realistic and repetitive simulations involving more complicated, larger-scale geometries and additional physical processes. As simulations become more ambitious, however, the need for more accurate turbulence models in RANS, LES, and hybrid RANS/LES approaches will grow considerably.

In this project you will examine RANS turbulence modeling from a number of different perspectives. You will first derive a hierarchy of commonly-used RANS models. These models will then be calibrated and validated using DNS data before final testing in a computational fluid dynamics code. There are four parts to this project. The first part addresses the derivation of RANS models, the second part addresses the *a priori* testing of RANS models using DNS data for a turbulent channel flow, the third part tests several of the models using DNS of periodically sheared homogeneous turbulence, and the last part involves computational fluid dynamics of a simple inhomogeneous flow problem using a software package of your choice.

Your entire project, including figures and all code used to perform the analysis, should be uploaded as a single Word or PDF file to D2L by midnight on Friday, May 8, 2015.

Part 1: Derivation of RANS Models

1.1 By taking an ensemble average of the Navier-Stokes equation, show that for incompressible turbulent flows the governing equation for the ensemble mean velocity $\overline{u}_i$ is given by

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{u}_i}{\partial x_j \partial x_j} - \frac{\partial \overline{u_i' u_j'}}{\partial x_j}\,, \tag{1}$$

where $u_i' \equiv u_i - \overline{u}_i$ is the fluctuation velocity. This is the equation that must be solved in essentially all RANS simulation approaches. Closure of the governing equation requires a model for $\overline{u_i' u_j'}$.

1.2 Show that in isotropic turbulent flows $\overline{u_i' u_j'}$ can be written as simply

$$\overline{u_i' u_j'} = \frac{2}{3} k \delta_{ij} \quad \text{(Isotropic Turbulence)}\,, \tag{2}$$

where k is the turbulence kinetic energy defined as

$$k \equiv \frac{1}{2}\overline{u_i' u_i'} \,. \tag{3}$$

1.3 Using the result from Eq. (2), it is possible to write $\overline{u_i' u_j'}$ as the sum of isotropic and anisotropic parts as

$$\overline{u_i' u_j'} = \underbrace{\frac{2}{3}k\delta_{ij}}_{isotropic} + \underbrace{k a_{ij}}_{anisotropic} \quad , \tag{4}$$

where a_{ij} is the Reynolds stress anisotropy tensor defined as

$$a_{ij} \equiv \frac{\overline{u_i' u_j'}}{k} - \frac{2}{3}\delta_{ij} \,. \tag{5}$$

Show that $a_{ii} = 0$, that $a_{ij} = 0$ when the flow is isotropic, that a_{ij} is a symmetric tensor, and thus that there are only six independent components of a_{ij}. Since the governing equation for k is easily modeled, closure of Eq. (1) thus amounts to developing a closure model for a_{ij}.

1.4 By first deriving the governing equations for $\overline{u_i' u_j'}$ and k, show that the *exact* governing equation for a_{ij} in incompressible flows is given by

$$\begin{aligned}\frac{\partial a_{ij}}{\partial t} + \overline{u}_k \frac{\partial a_{ij}}{\partial x_k} = &-\left(\frac{P}{\varepsilon} - 1\right)\frac{\varepsilon}{k}a_{ij} - \frac{4}{3}\overline{S}_{ij} - \left(a_{ik}\overline{S}_{kj} + \overline{S}_{ik}a_{kj} - \frac{2}{3}a_{kl}\overline{S}_{kl}\delta_{ij}\right) \\ &+ \left(a_{ik}\overline{W}_{kj} - \overline{W}_{ik}a_{kj}\right) + \frac{1}{k}\Pi_{ij} - \frac{1}{k}\left(\varepsilon_{ij} - \frac{2}{3}\varepsilon\delta_{ij}\right) + \frac{1}{k}\left[D_{ij} - \left(a_{ij} + \frac{2}{3}\delta_{ij}\right)D\right] ,\end{aligned} \tag{6}$$

where P, ε_{ij}, Π_{ij}, and D_{ij} are defined as

$$P \equiv -k a_{kl}\overline{S}_{kl} \,, \quad \varepsilon_{ij} \equiv 2\nu \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} \,, \quad \Pi_{ij} \equiv \frac{2}{\rho}\overline{p' S_{ij}'} \,,$$

$$D_{ij} \equiv -\frac{\partial}{\partial x_k}\left(\overline{u_i' u_j' u_k'} + \frac{\overline{p' u_j'}}{\rho}\delta_{ik} + \frac{\overline{p' u_i'}}{\rho}\delta_{jk} - \nu \frac{\partial \overline{u_i' u_j'}}{\partial x_k}\right) ,$$

and $D \equiv D_{ii}/2$, $\varepsilon \equiv \varepsilon_{ii}/2$, and $\Pi_{ii} = 0$. The mean strain, $\overline{S}_{ij}$, and rotation, $\overline{W}_{ij}$, rate tensors are defined as

$$\overline{S}_{ij} \equiv \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) , \quad \overline{W}_{ij} \equiv \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} - \frac{\partial \overline{u}_j}{\partial x_i}\right) , \tag{7}$$

and $S_{ij}' \equiv S_{ij} - \overline{S}_{ij}$. Assuming that k and ε are known, what are the unclosed terms in Eq. (6)?

1.5 Provide an argument as to why ε_{ij} can be treated as isotropic in many turbulent flows and use the isotropic form for this tensor to rewrite Eq. (6) as

$$\begin{aligned}\frac{\partial a_{ij}}{\partial t} + \overline{u}_k \frac{\partial a_{ij}}{\partial x_k} = &-\left(\frac{P}{\varepsilon} - 1\right)\frac{\varepsilon}{k}a_{ij} - \frac{4}{3}\overline{S}_{ij} - \left(a_{ik}\overline{S}_{kj} + \overline{S}_{ik}a_{kj} - \frac{2}{3}a_{kl}\overline{S}_{kl}\delta_{ij}\right) \\ &+ \left(a_{ik}\overline{W}_{kj} - \overline{W}_{ik}a_{kj}\right) + \frac{1}{k}\Pi_{ij} + \frac{1}{k}\left[D_{ij} - \left(a_{ij} + \frac{2}{3}\delta_{ij}\right)D\right] ,\end{aligned} \tag{8}$$

1.6 The pressure-strain rate correlation is typically modeled in terms of a_{ij}, S_{ij}, and W_{ij} as

$$\Pi_{ij} = -C_1 \varepsilon a_{ij} + C_2 k \overline{S}_{ij} - C_3 k \left(a_{ik}\overline{S}_{kj} + \overline{S}_{ik} a_{kj} - \frac{2}{3} a_{kl}\overline{S}_{kl}\delta_{ij} \right) + C_4 k \left(a_{ik}\overline{W}_{kj} - \overline{W}_{ik} a_{kj} \right) , \quad (9)$$

where the C_i are coefficients that are typically determined from experiments, DNS, or theoretical arguments; in the following we will use the values

$$C_1 = 1.5 , \quad C_2 = 0.8 , \quad C_3 = 0.875 , \quad C_4 = 0.655 .$$

Use the form for Π_{ij} from Eq. (9) to show that Eq. (8) can be written as

$$\begin{aligned} \frac{\partial a_{ij}}{\partial t} + \overline{u}_k \frac{\partial a_{ij}}{\partial x_k} = &-\alpha_1 \frac{\varepsilon}{k} a_{ij} + \alpha_2 \overline{S}_{ij} + \alpha_3 \left(a_{ik}\overline{S}_{kj} + \overline{S}_{ik} a_{kj} - \frac{2}{3} a_{kl}\overline{S}_{kl}\delta_{ij} \right) \\ &- \alpha_4 \left(a_{ik}\overline{W}_{kj} - \overline{W}_{ik} a_{kj} \right) + \frac{1}{k}\Pi_{ij} + \frac{1}{k}\left[D_{ij} - \left(a_{ij} + \frac{2}{3}\delta_{ij} \right) D \right] , \end{aligned} \quad (10)$$

where

$$\alpha_1 = \frac{P}{\varepsilon} - 1 + C_1 , \quad \alpha_2 = C_2 - \frac{4}{3} , \quad \alpha_3 = C_3 - 1 , \quad \alpha_4 = C_4 - 1 .$$

The terms involving combinations of a_{ij}, $\overline{S}_{ij}$, and $\overline{W}_{ij}$ in Eq. (10) are termed *redistribution terms* since they effectively redistribute the anisotropy between the various components of a_{ij}. Apart from the transport term, this equation is closed provided that we know k and ε.

1.7 In many turbulent flows, both the redistribution and transport (i.e., D_{ij} and D) terms can be either approximately or exactly neglected. In such instances, the governing equation for a_{ij} can be simply written as

$$\frac{\partial a_{ij}}{\partial t} + \overline{u}_k \frac{\partial a_{ij}}{\partial x_k} = -\alpha_1 \frac{\varepsilon}{k} a_{ij} + \alpha_2 \overline{S}_{ij} , \quad (11)$$

What *physical* effects have been neglected in obtaining Eq. (11)? In what types of flows might the neglect of these effects pose problems, and in what types of flows is the neglect of these effects fully correct?

1.8 It is exceptionally common to further assume that the flow is in *equilibrium* such that

$$\frac{\partial a_{ij}}{\partial t} + \overline{u}_k \frac{\partial a_{ij}}{\partial x_k} = 0 . \quad (12)$$

Show that the resulting closure under such an assumption is given by

$$a_{ij} = \frac{\alpha_2}{\alpha_1}\frac{k}{\varepsilon}\overline{S}_{ij} . \quad (13)$$

Comment on all of the physical effects that have been lost or neglected in deriving Eq. (13) from the exact governing equation for a_{ij} in Eq. (6). In what types of flows might the closure assumption in Eq. (13) be inaccurate?

1.9 In what is commonly known as the Boussinesq, or gradient transport, hypothesis, Eq. (13) can be written in terms of an eddy viscosity, ν_T, as

$$a_{ij} = -2\frac{\nu_T}{k}\overline{S}_{ij} , \quad (14)$$

where

$$\nu_T = C_\mu \frac{k^2}{\varepsilon} \,. \tag{15}$$

What is C_μ in terms of the coefficients α_i? If we assume that C_μ is constant, what additional effects have been lost by writing a_{ij} in this fashion?

1.10 Show that the form for a_{ij} from Eq. (14) allows Eq. (1) to be written as

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[2\left(\nu + \nu_T\right)\overline{S}_{ij}\right] \,, \tag{16}$$

where

$$\tilde{p} \equiv \overline{p} + \frac{2}{3}\rho k \,. \tag{17}$$

Based on the form of this equation for $\overline{u}_i$, comment on why the closure from Eq. (14) may be popular from a practical perspective.

Part 2: *A priori* Testing of RANS Models in Turbulent Channel Flow

2.1 Consider fully-developed turbulent channel flow. In such a flow, which of the components of $\overline{u}_i$, $\overline{S}_{ij}$, and $\overline{W}_{ij}$ are non-zero? What are $\partial/\partial t$ and $\overline{u}_j \partial/\partial x_j$ equal to? Based on these results, comment on the validity of the equilibrium assumption used to obtain the closure in Eq. (13).

2.2 Use the turbulent channel flow DNS data available here:

`http://turbulence.ices.utexas.edu/MKM_1999.html`

to plot the nonzero components of $\overline{u}_i$, $\overline{u_i' u_j'}$, k, ε, a_{ij}, and $\overline{S}_{ij}$ as functions of both y^+ and y/h for $Re_\tau = 590$, where h is the half-height of the channel.

2.3 Using the DNS data for a_{12} and $\overline{S}_{12}$, calculate the *constant* value of C_μ that gives the closest agreement between the closure from Eqs. (14) and (15) and the computational data *away from the channel walls*.

2.4 Considering a_{12} only, in what way does the closure from Eq. (14) fail near the channel walls? Why does this failure occur in terms of the physical effects that have been neglected in obtaining Eq. (14), and what should have been retained in the definition of C_μ to improve the closure?

2.5 A popular proposal for improving the closure from Eq. (14) near channel walls is to limit C_μ such that

$$C_\mu = \begin{cases} 0.09 & \text{for} \quad (Sk/\varepsilon) \le 3.4 \\ 0.31(Sk/\varepsilon)^{-1} & \text{for} \quad (Sk/\varepsilon) > 3.4 \end{cases} \,. \tag{18}$$

What effect does this have on the accuracy of the model for a_{12} near the channel walls? Would you consider this a "good" approach for improving the near-wall accuracy? Why or why not?

2.6 From a physical perspective, why do turbulence models have difficulty for $y^+ < 30$? Can you propose a mathematical function that more accurately predicts a_{ij} very close to the channel walls?

2.7 What does the closure from Eq. (14) predict for the normal stresses $\overline{u_1'^2}$, $\overline{u_2'^2}$, and $\overline{u_3'^2}$ in the channel? How do these predictions compare with the DNS results? How can you explain the observed discrepancies and similarities between the modeled and DNS results based on the assumptions used to obtain Eq. (14)?

2.8 Based on the comparisons between the DNS and model results for the closure in Eq. (14), comment on the overall accuracy and behavior of this closure in turbulent flows. For what problems could you use this model and expect reasonably accurate results? For what problems and flow conditions would the model break down? Given its shortcomings, why is such a model still useful for studies of engineering turbulent flows, and why is it still widely used?

Part 3: Testing of RANS Models in an Unsteady Homogeneous Flow

3.1 Consider the first-order moment equation in Eq. (1). How can this equation be simplified for homogeneous turbulent flows? What does this mean about the coupling between the evolutions of $\overline{u}_i$ and $\overline{u_i'u_j'}$ in homogeneous turbulent flows? How does this simplify RANS model testing?

3.2 In order to obtain a closure for a_{ij}, model equations are needed for both k and ε. It is common to write these equations as

$$\frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j} = P - \varepsilon + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_T}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right], \tag{19}$$

and

$$\frac{\partial \varepsilon}{\partial t} + \overline{u}_j \frac{\partial \varepsilon}{\partial x_j} = C_{\varepsilon 1} P \frac{\varepsilon}{k} - C_{\varepsilon 2}\frac{\varepsilon^2}{k} + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right], \tag{20}$$

where $P = -k a_{ij}\overline{S}_{ij}$ and the constants are typically given the values

$$C_{\varepsilon 1} = 1.44\,, \quad C_{\varepsilon 2} = 1.92\,, \quad \sigma_k = 1.0\,, \quad \sigma_\varepsilon = 1.3\,.$$

Show how these equations can be simplified for homogeneous turbulent flows.

3.3 It is common in numerical simulations to normalize all flow variables so that they are dimensionless. If we normalize using the initial values of k and ε, denoted k_0 and ε_0, show that the homogeneous equations for k and ε from Problem 3.2 can be written in non-dimensional form as

$$\frac{\partial \tilde{k}}{\partial \tau} = -\tilde{k} a_{ij}\left(\frac{\overline{S}_{ij}k_0}{\varepsilon_0}\right) - \tilde{\varepsilon}\,, \tag{21}$$

$$\frac{\partial \tilde{\varepsilon}}{\partial \tau} = -C_{\varepsilon 1}\tilde{\varepsilon} a_{ij}\left(\frac{\overline{S}_{ij}k_0}{\varepsilon_0}\right) - C_{\varepsilon 2}\frac{\tilde{\varepsilon}^2}{\tilde{k}}\,, \tag{22}$$

where $\tilde{k} \equiv k/k_0$, $\tilde{\varepsilon} = \varepsilon/\varepsilon_0$ and $\tau = t/(k_0/\varepsilon_0)$. Show also that the simplified differential closure for a_{ij} from Eq. (11) can be written as

$$\frac{\partial a_{ij}}{\partial \tau} = -\alpha_1 \frac{\tilde{\varepsilon}}{\tilde{k}} a_{ij} + \alpha_2\left(\frac{\overline{S}_{ij}k_0}{\varepsilon_0}\right), \quad \alpha_1 = -\frac{\tilde{k}}{\tilde{\varepsilon}} a_{ij}\left(\frac{\overline{S}_{ij}k_0}{\varepsilon_0}\right) - 1 + C_1\,, \tag{23}$$

and the equilibrium closure for a_{ij} from Eq. (14) can be written as

$$a_{ij} = -2C_\mu \frac{\tilde{k}}{\tilde{\varepsilon}}\left(\frac{\overline{S}_{ij}k_0}{\varepsilon_0}\right). \tag{24}$$

Along with the equations for $\tilde{k}$ and $\tilde{\varepsilon}$, either Eq. (23) or Eq. (24) give us a closed set of equations for the evolution of a_{ij} provided that the evolution of $\overline{S}_{ij}k_0/\varepsilon_0$ is known. In the following, the closure based on Eq. (23) will be denoted the *differential* $k-\varepsilon$ (DKE) model and the closure based on Eq. (24) will be denoted the *standard* $k-\varepsilon$ (SKE) model.

3.4 Consider a homogeneous turbulent flow that undergoes periodic shearing such that $\overline{S}_{12} = \overline{S}_{21}$ is the only nonzero component of $\overline{S}_{ij}$ and the time evolution of $\overline{S}_{12}$ is given by

$$\overline{S}_{12} = \begin{cases} 0 & \text{for} \quad t < 0 \\ (S/2)\sin(\omega t) & \text{for} \quad t \geq 0 \end{cases}, \tag{25}$$

where S is the shearing magnitude and ω is the shearing frequency. Show that for $t \geq 0$, the equations for $\tilde{k}$ and $\tilde{\varepsilon}$ are given by

$$\frac{\partial \tilde{k}}{\partial \tau} = -\tilde{k} a_{12} S^* \sin\left[\left(\frac{\omega}{S}\right) S^* \tau\right] - \tilde{\varepsilon}\,, \quad \frac{\partial \tilde{\varepsilon}}{\partial \tau} = -C_{\varepsilon 1} \tilde{\varepsilon} a_{12} S^* \sin\left[\left(\frac{\omega}{S}\right) S^* \tau\right] - C_{\varepsilon 2} \frac{\tilde{\varepsilon}^2}{\tilde{k}}\,, \tag{26}$$

where $S^* \equiv S k_0 / \varepsilon_0$ and the SKE and DKE models are given, respectively, by

$$a_{12} = -C_\mu \frac{\tilde{k}}{\tilde{\varepsilon}} S^* \sin\left[\left(\frac{\omega}{S}\right) S^* \tau\right]\,, \tag{27}$$

and

$$\frac{\partial a_{12}}{\partial \tau} = -\alpha_1 \frac{\tilde{\varepsilon}}{\tilde{k}} a_{12} + \frac{\alpha_2}{2} S^* \sin\left[\left(\frac{\omega}{S}\right) S^* \tau\right]\,, \quad \alpha_1 = -\frac{\tilde{k}}{\tilde{\varepsilon}} a_{12} S^* \sin\left[\left(\frac{\omega}{S}\right) S^* \tau\right] - 1 + C_1\,. \tag{28}$$

3.5 Assuming that $a_{12} = 0$ at $t = 0$, numerically integrate the set of ordinary differential equations given by Eqs. (26)–(28) for the SKE and DKE models. Set $S^* \equiv S k_0/\varepsilon_0 = 3.3$ and examine the flow evolution for $(\omega/S) = [0.01, 0.1, 0.5, 1, 10]$. Plot the evolution of a_{12} as a function of $S \cdot t = S^* \tau$ for each of these values of ω/S and for both the SKE and DKE models. Assume that $C_\mu = 0.09$ in the SKE model.

3.6 Using the program DataThief (`http://www.datathief.org/`), or some other means, extract the evolution of $a_{12} = 2b_{12}$ from Figures 14, 15, and 25 of the paper "Direct numerical simulations of homogeneous turbulence subject to periodic shear," by D. Yu and S.S. Girimaji in *Journal of Fluid Mechanics*, Vol. 566, pp. 117-151 (2006). Compare these DNS results with the SKE and DKE model results from Problem 3.5. Which model matches the DNS data more closely and why? Consider how the model accuracies change with ω/S and discuss the effects on the amplitudes and phases of the model predictions.

3.7 What does the study of unsteady periodically sheared turbulence tell us about the applicability of the equilibrium approximation in unsteady turbulent flows? For what problems is this approximation acceptable and unacceptable? How generally accurate is the assumption that $a_{ij} \sim \overline{S}_{ij}$ in turbulent flows?

Part 4: RANS Model Testing in a Computational Fluid Dynamics Code

4.1 Choose a computational fluid dynamics (CFD) software package with which you are familiar or in which you are interested (e.g., Fluent, StarCCM, OpenFOAM, Overflow, etc.) and describe in detail the full range of turbulence models included in the code. What is the formulation of each of these models, what physics are included and neglected in each model, are the models differential or equilibrium formulations, are LES or hybrid RANS/LES models available? Be as comprehensive and precise as possible in your description of the models.

4.2 [EXTRA CREDIT] Choose a simple turbulence test case and simulate it using at least two different models in your chosen code. The test case can be as simple as a turbulent channel flow or boundary layer and may also be a pre-existing test case available with the code (several such cases are provided, for example, with OpenFOAM). The emphasis here is on understanding the differences produced by different models. How large are the differences between the models, and how do these differences relate to the physics included and neglected in the models? It is enough to simply compare the mean velocities and turbulence kinetic energy produced by the different models.