Part 3

(3.1) RANS: $\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = \frac{-1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} - \frac{\partial \overline{u_i' u_j'}}{\partial x_j}$.

In homogeneous flows, fluctuation statistics, such as $\overline{u_i' u_j'}$ are spatially invariant, meaning their derivatives w.r.t. position vanish. Thus

$$\frac{\partial \overline{u_i' u_j'}}{\partial x_j} \to 0, \quad \text{and} \quad \underset{\text{(homo.)}}{\text{RANS}} \to \frac{\bar{D}\bar{u}_i}{\bar{D}t} = \frac{-1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j}$$

This means that the evolution of $\bar{u}_i$ is <u>uncoupled</u> from that of $\overline{u_i' u_j'}$ in homogeneous turbulent flows.

This far simplifies RANS model testing, because, first, there is one less coupling in the governing equations that needs to be addressed and tracked during numerical solution.

(3.2) To obtain closure for a_{ij}, we commonly write model equations for k and ε as

$$\text{(A)} \quad \frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = P - \varepsilon + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_T}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] \quad \text{and}$$

$$\text{(B)} \quad \frac{\partial \varepsilon}{\partial t} + \bar{u}_j \frac{\partial \varepsilon}{\partial x_j} = C_{\varepsilon 1} P \frac{\varepsilon}{k} - C_{\varepsilon 2}\frac{\varepsilon^2}{k} + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right],$$

where $P = -k\, a_{ij} \bar{S}_{ij}$, $C_{\varepsilon 1} = 1.44$, $C_{\varepsilon 2} = 1.92$, $\sigma_k = 1.0$, $\sigma_\varepsilon = 1.3$.

In homogeneous flows, $\partial/\partial x_i$ of fluctuation statistics are zero, so

since $k = \frac{1}{2}\overline{u_i' u_i'}$

$$\text{(A)} \to \frac{\bar{D}}{\bar{D}t}k = P - \varepsilon, \quad \text{or even} \quad \frac{\partial k}{\partial t} = P - \varepsilon$$

$\longrightarrow$

~~Even here, though,~~ $\cancel{\varepsilon \equiv \frac{2\nu}{2}\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}} = 0}$, ~~so~~

$\cancel{Ⓐ \rightarrow \frac{\partial k}{\partial t} = P = -k a_{ij}\bar{S}_{ij} = -k\left(\frac{\overline{u_i'u_j'}}{k} - \frac{2}{3}\delta_{ij}\right) = 0}$ ~~... eh.~~

And since $\varepsilon \equiv \nu \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}}$ is a fluctuation statistic, $\frac{\partial \varepsilon}{\partial x_j} = 0$

$$\Rightarrow Ⓑ \rightarrow \frac{\partial \varepsilon}{\partial t} = C_{\varepsilon 1} P \frac{\varepsilon}{k} - C_{\varepsilon 2}\frac{\varepsilon^2}{k}.$$

3.3 Initial values of k and ε are k_0 and ε_0, and let

$$\breve{k} \equiv \frac{k}{k_0}, \quad \breve{\varepsilon} \equiv \frac{\varepsilon}{\varepsilon_0}, \quad \tau = \frac{t}{(k_0/\varepsilon_0)}.$$

Start by non-dimensionalizing Ⓐ with $k = k_0\breve{k}$, $\varepsilon = \varepsilon_0\breve{\varepsilon}$, $t = \frac{\tau k_0}{\varepsilon_0}$.

$$\frac{\partial k}{\partial t} = P - \varepsilon, \quad \frac{\partial k}{\partial t} = -k a_{ij}\bar{S}_{ij} - \varepsilon, \quad \frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial t}(k_0\breve{k}) = -k_0\breve{k} a_{ij}\bar{S}_{ij} - \varepsilon_0\breve{\varepsilon},$$

$$\text{or } \frac{\partial \breve{k}}{\partial \tau}\cdot\frac{\varepsilon_0 \cancel{k_0}}{\cancel{k_0}} = -k_0\breve{k} a_{ij}\bar{S}_{ij} - \varepsilon_0\breve{\varepsilon}$$

$$\Rightarrow \boxed{\frac{\partial \breve{k}}{\partial \tau} = -\breve{k} a_{ij}\left(\frac{\bar{S}_{ij} k_0}{\varepsilon_0}\right) - \breve{\varepsilon}}.$$

For Ⓑ,

$$\frac{\partial \varepsilon}{\partial t} = C_{\varepsilon 1} P \frac{\varepsilon}{k} - C_{\varepsilon 2}\frac{\varepsilon^2}{k} \rightarrow \frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial t}\varepsilon_0\breve{\varepsilon} = C_{\varepsilon 1}(-\cancel{k_0\breve{k}} a_{ij}\bar{S}_{ij})\frac{\varepsilon_0\breve{\varepsilon}}{\cancel{k_0\breve{k}}} - C_{\varepsilon 2}\frac{\varepsilon_0^2\breve{\varepsilon}^2}{k_0\breve{k}}$$

$$\Rightarrow \frac{\partial \breve{\varepsilon}}{\partial t} = \frac{k_0}{\varepsilon_0^2}\left(-C_{\varepsilon 1} a_{ij}\bar{S}_{ij}\varepsilon_0\breve{\varepsilon} - C_{\varepsilon 2}\frac{\varepsilon_0^2\breve{\varepsilon}^2}{k_0\breve{k}}\right)$$

$$\Rightarrow \boxed{\frac{\partial \breve{\varepsilon}}{\partial \tau} = -C_{\varepsilon 1}\breve{\varepsilon} a_{ij}\left(\frac{\bar{S}_{ij} k_0}{\varepsilon_0}\right) - C_{\varepsilon 2}\frac{\breve{\varepsilon}^2}{\breve{k}}}.$$

For the closure for a_{ij}, from simplified differential form,

$$\frac{\partial a_{ij}}{\partial t} + \cancel{\overline{u}_k \frac{\partial a_{ij}}{\partial x_k}}^{\to 0,\text{ homo.}} = -\alpha_1 \frac{\varepsilon}{k} a_{ij} + \alpha_2 \overline{S}_{ij}$$

$$\frac{\partial}{\partial \tau}\frac{\partial t}{\partial t} a_{ij} = -\alpha_1 \frac{\varepsilon_0 \hat{\varepsilon}}{k_0 \hat{k}} a_{ij} + \alpha_2 \overline{S}_{ij}$$

$$\frac{\partial a_{ij}}{\partial \tau} = -\alpha_1 \frac{\hat{\varepsilon}}{\hat{k}} a_{ij} + \frac{\alpha_2 k_0}{\varepsilon_0} \overline{S}_{ij}$$

$$\Rightarrow \boxed{\frac{\partial a_{ij}}{\partial \tau} = -\alpha_1 \frac{\hat{\varepsilon}}{\hat{k}} a_{ij} + \alpha_2 \left(\frac{\overline{S}_{ij} k_0}{\varepsilon_0}\right)}.$$

Differential K-ε (DKE) model

where $\alpha_1 = P/\varepsilon - 1 + C_1$,

$$= -\frac{k_0 \hat{k} a_{ij} \overline{S}_{ij}}{\varepsilon_0 \hat{\varepsilon}} - 1 + C_1 \Rightarrow \boxed{\alpha_1 = -\frac{\hat{k}}{\hat{\varepsilon}} a_{ij} \left(\frac{\overline{S}_{ij} k_0}{\varepsilon_0}\right) - 1 + C_1}$$

and $\alpha_2 = C_2 - 4/3$.

Standard k-ε (SKE) model

For the equillibrium closure for a_{ij},

$$a_{ij} = -2 C_\mu \frac{k^2}{\varepsilon} \frac{1}{k} \overline{S}_{ij} = \boxed{a_{ij} = -2 C_\mu \frac{\hat{k}}{\hat{\varepsilon}} \left(\frac{\overline{S}_{ij} k_0}{\varepsilon_0}\right)}.$$

3.4 Consider a homogeneous turbulent flow that undergoes periodic shearing such that $\overline{S}_{12} = \overline{S}_{21}$ is the only nonzero component of $\overline{S}_{ij}$, and

$$\overline{S}_{12} = \begin{cases} 0 & \text{for } t < 0 \\ (S/2)\sin(\omega t) & \text{for } t \geq 0 \end{cases}, \quad \begin{array}{l} S \equiv \text{shearing magnitude} \\ \omega \equiv \text{" " frequency} \end{array}$$

So for $t \geq 0$, $$\frac{\partial \hat{k}}{\partial t} = -\hat{k}\, 2 a_{12} \left(\frac{k_0}{\varepsilon_0} \frac{S}{2} \sin(\omega t)\right) - \hat{\varepsilon}$$

$$= -\hat{k} a_{12} \left(\frac{S k_0}{\varepsilon_0}\right) \sin\left[\left(\frac{\omega}{S}\right)\left(\frac{S k_0}{\varepsilon_0}\right)\left(\frac{\varepsilon_0 t}{k_0}\right)\right] - \hat{\varepsilon}$$

$$\Rightarrow \boxed{\frac{\partial \hat{k}}{\partial \tau} = -\hat{k} a_{12} S^* \sin\left[\frac{\omega}{S} S^* \tau\right] - \hat{\varepsilon}}.$$

$$\frac{\partial \check{\varepsilon}}{\partial \tau} = -C_{\varepsilon 1} \cancel{2} \check{\varepsilon} a_{12} \left(\frac{S}{\cancel{2}} \frac{k_o}{\varepsilon_o} \sin(\omega t) \right) - C_{\varepsilon 2} \frac{\check{\varepsilon}^2}{\check{k}}$$

$$= -C_{\varepsilon 1} \check{\varepsilon} a_{12} \left(\frac{S k_o}{\varepsilon_o} \right) \sin\left[\frac{\omega}{S} \left(\frac{S k_o}{\varepsilon_o} \right) \frac{\varepsilon_o t}{k_o} \right] - C_{\varepsilon 2} \frac{\check{\varepsilon}^2}{\check{k}}$$

$$\Rightarrow \boxed{\frac{\partial \check{\varepsilon}}{\partial t} = -C_{\varepsilon 1} \check{\varepsilon} a_{12} S^* \sin\left[\frac{\omega}{S} S^* \tau \right] - C_{\varepsilon 2} \frac{\check{\varepsilon}^2}{\check{k}}} .$$

For the SKE model,

$$a_{12} = -2 C_\mu \frac{\check{k}}{\check{\varepsilon}} \left(\frac{S}{2} \frac{k_o}{\varepsilon_o} \sin(\omega t) \right) = \frac{-2 C_\mu}{2} \frac{\check{k}}{\check{\varepsilon}} \left(\frac{S k_o}{\varepsilon_o} \right) \sin \underbrace{\left(\frac{\omega}{S} S^* \tau \right)}^{\text{as before}}$$

$$\Rightarrow \boxed{a_{12} = -C_\mu \frac{\check{k}}{\check{\varepsilon}} S^* \sin\left[\frac{\omega}{S} S^* \tau \right]} .$$

For the DKE model,

$$\frac{\partial a_{12}}{\partial \tau} = -\alpha_1 \frac{\check{\varepsilon}}{\check{k}} a_{12} + \alpha_2 \frac{1}{2} \left(\frac{S k_o}{\varepsilon_o} \right) \sin(\omega t)$$

$$\Rightarrow \boxed{\frac{\partial a_{12}}{\partial \tau} = -\alpha_1 \frac{\check{\varepsilon}}{\check{k}} a_{12} + \frac{\alpha_2}{2} S^* \sin\left[\frac{\omega}{S} S^* \tau \right]} .$$

and $\alpha_1 = -\frac{\check{k}}{\check{\varepsilon}} a_{12} \frac{2}{2} \left(\frac{S k_o}{\varepsilon_o} \right) \sin(\omega t) - 1 + C_1$

$$\Rightarrow \boxed{\alpha_1 = -\frac{\check{k}}{\check{\varepsilon}} a_{12} S^* \sin\left[\frac{\omega}{S} S^* \tau \right] - 1 + C_1} .$$