Problem 1

Given the joint CDF of random variables X_1 and X_2,

$$F_{X_1,X_2}(x_1,x_2) = 1 - \exp(-x_1) - \exp(-x_2) + \exp(-x_1 - x_2 - x_1 x_2), \quad x_1, x_2 \geq 0, \tag{1}$$

we are tasked with finding the marginal CDF $F_{X_1}(x_1)$ and the conditional CDF $F_{X_2|X_1}(x_2|x_1)$, and subsequently generating realizations of X_1, X_2 using the inversion method.

The marginal CDF of X_1 is trivially calculated in the limit $x_2 \to \infty$ as

$$\boxed{F_{X_1}(x_1)} = F_{X_1,X_2}(x_1,\infty) = 1 - \exp(-x_1). \tag{2}$$

Applying the relation

$$F_{X_2|X_1}(x_2|x_1) = \left(\int_0^{x_2} f_{X_1,X_2}(x_1,t_2)\,dt_2 \right) \Big/ f_{X_1}(x_1), \tag{3}$$

where the marginal and joint pdfs are

$$f_{X_1}(x_1) = \partial_{x_1} F_{X_1}(x_1)$$
$$f_{X_1,X_2}(x_1,x_2) = \partial_{x_1}\partial_{x_2} F_{X_1,X_2}(x_1,x_2),$$

it can be shown that

$$\boxed{F_{X_2|X_1}(x_2|x_1)} = 1 - (1 + x_2)\exp(-[1 + x_1]x_2), \tag{4}$$

which is impossible to invert analytically, though computational root-finding methods show success.

Realizations of X_1, X_2 are generated in the standard manner: for each $i = 1, ..., N$, a random variable $U_1^i \sim U[0,1]$ is generated, and set equal to $F_{X_1}(x_1)$, which can be solved for realization x_1^i. Another random variable $U_2^i \sim U[0,1]$ is generated and set equal to $F_{X_2|X_1}(x_2|x_1)$, which is then solved numerically for x_2^i, given x_1^i. We choose Matlab's `fzero` function as our root finder.

In Figure 1, we generate $N = 10{,}000$ realizations and compare the cumulative expectation of the first n samples to the analytical expectations

$$\begin{aligned} \langle x_1 \rangle &= 1.0 \\ \langle x_1 \rangle &= 1.0 \\ \langle x_1 x_2 \rangle &= 0.596347 \end{aligned} \tag{5}$$

All three quantities approach their analytical values, and the relative error in each quantity is seen to decrease as n increases.

This method of verification is by no means rigorous. The mean square error of the empirical CDF or pdf would be a better way of checking the validity of our answers, but this suffices for the purposes of this exercise.

Problem 2

This problem concerns a derivation from scratch of the Bayesian MAP estimate of a random variable V, assuming a Gaussian prior $V \sim N(V_0, \sigma_0^2)$. Further details are worked by hand on the attached sheets.

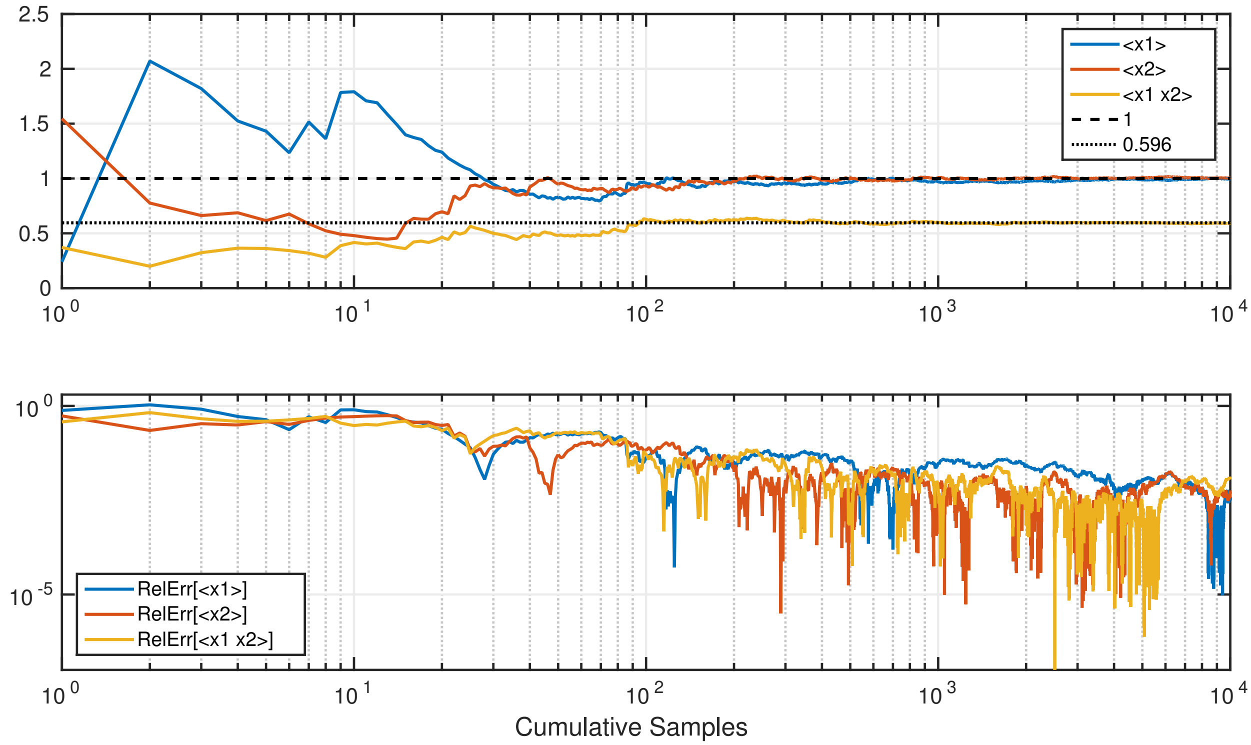

Figure 1: Expectation values of various functions of x_1 and x_2, using the first n cumulative samples. Analytical expectation values plotted to show convergence, as well as relative error.

PROBLEM 3

Bloop!