

PROBLEM 1

The thermal coefficient K of a 1-D slab is defined on $\mathcal{D} = (0, 1)$ and is characterized by a shifted log-normal random process

$$K(x, \omega) = 2 + \exp(G(x, \omega)), \quad x \in \mathcal{D} \quad (1)$$

where $G(x, \omega)$ is a Gaussian random process also defined on \mathcal{D} . The mean and covariance functions of $G(x, \omega)$ are

$$\langle G(x, \cdot) \rangle = 1.0, \quad x \in \mathcal{D} \quad (2)$$

and

$$C_{GG}(x_1, x_2) = \sigma^2 \exp\left(\frac{-|x_1 - x_2|}{\ell}\right), \quad (x_1, x_2) \in \mathcal{D} \times \mathcal{D} \quad (3)$$

respectively. We would like to compute the statistics of the temperature field $u(x)$ by solving the governing steady-state stochastic heat equation

$$\begin{aligned} -\frac{\partial}{\partial x} \left(K(x, \omega) \frac{\partial u(x, \omega)}{\partial x} \right) &= 1.0, \quad x \in \mathcal{D}, \\ u(0, \omega) &= 0, \\ u(1, \omega) &= 0, \end{aligned} \quad (4)$$

using the stochastic Galerkin approach.

For this problem, we let $\sigma = 2$ and $\ell = 2.0$, and use the codes from Homework #1 to generate a $d = 2$ Karhunen-Löve expansion of $G(x, \omega)$ via the analytical solution. The approximate log-normal is then computed through its polynomial chaos expansion (PCE)

$$K_{p_k}(x, \mathbf{y}) = \sum_{i=0}^{p_k} K_i(x) \Psi_i(\mathbf{y}) \quad (5)$$

where K_0 and K_i ($i > 0$) are defined as

$$K_0 = 2 + \exp\left(1 + \frac{1}{2} \sum_{j=1}^d \lambda_j \phi_j^2(x)\right) \quad (6)$$

and

$$K_i(x) = \frac{K_0(x) - 2}{\sqrt{\sum_{j=1}^d (i_j!)}} \sum_{j=1}^d \left(\sqrt{\lambda_j} \phi_j(x) \right)^{i_j} \quad (7)$$

where $\{\lambda_i, \phi_i(x)\}_{i=1}^d$ are eigen-pairs of the covariance kernel C_{GG} , and i_j denotes the polynomial order of $\Psi_i(\mathbf{y})$ along the direction $j \in \{1, \dots, d\} = \{1, 2\}$. We set $p_k = 14$ as the total order of the PC expansion, and write a finite difference code to solve the stochastic PDE in (4). This employs the stochastic Galerkin discretization of total order p

$$u_p(x, \mathbf{y}) = \sum_{j=0}^p u_j(x) \Psi_j(\mathbf{y}) \quad (8)$$

SOLUTION

In the stochastic Galerkin discretization, we first discretize our differential operator in space, extracting the stochastic component from (4) through the PCE and arriving at

$$\sum_{k=0}^{P_K} \frac{-\partial}{\partial x} \left(K_k(x) \frac{\partial \cdot}{\partial x} \right) u(x, \mathbf{y}) \Psi_k(\mathbf{y}) = f \quad (9)$$

$$\sum_{k=0}^{P_K} \underline{\underline{K}}_k \Psi_k(\mathbf{y}) \underline{u}(x, \mathbf{y}) = \underline{f} \quad (10)$$

Combining the sum over k into one composite matrix yields the pseudo-spectral system

$$\underline{\underline{K}}(\mathbf{y}) \underline{u}(x, \mathbf{y}) = \underline{f} \quad (11)$$

We then use another PC expansion on the solution $\underline{u}(x, \mathbf{y})$, approximating it as

$$\hat{\underline{u}}(\mathbf{y}) = \sum_{j=0}^P \underline{u}_j \Phi_j(\mathbf{y}) \quad (12)$$

Then, noting that the optimal solution's residual must be orthogonal to the PCE basis functions, we arrive at the system of equations

$$\langle \underline{\underline{K}}(\mathbf{y}) \hat{\underline{u}}(\mathbf{y}) - \underline{f}, \Psi_i(\mathbf{y}) \rangle = 0 \quad (13)$$

$$\sum_{j=0}^P \langle \underline{\underline{K}}(\mathbf{y}) \Psi_i(\mathbf{y}) \Psi_j(\mathbf{y}) \rangle \underline{u}_j(\mathbf{y}) = \langle \underline{f}, \Psi_i(\mathbf{y}) \rangle \quad (14)$$

which can be written in matrix form as

$$\begin{bmatrix} \underline{\underline{K}}_{0,0} & \cdots & \underline{\underline{K}}_{0,P} \\ \vdots & \ddots & \vdots \\ \underline{\underline{K}}_{P,0} & \cdots & \underline{\underline{K}}_{P,P} \end{bmatrix} \begin{bmatrix} \underline{u}_0 \\ \vdots \\ \underline{u}_P \end{bmatrix} = \begin{bmatrix} \underline{f}_0 \\ \vdots \\ \underline{f}_P \end{bmatrix} \quad (15)$$

where

$$\underline{\underline{K}}_{i,j} \equiv \sum_{k=0}^{P_K} \underline{\underline{K}}_k \overbrace{\langle \Psi_i \Psi_j \Psi_k \rangle}^{c_{ijk}} \quad (16)$$

$$\underline{f}_i = \begin{cases} \underline{f} & i = 0 \\ 0 & \text{else} \end{cases} \quad (17)$$

What makes the polynomial chaos approach so attractive is that, though it is intrusive and requires knowledge of the PDE being solved, components of an existing numerical solver can be leveraged. In this example, the $\underline{\underline{K}}_k$ matrices and the \underline{f} vector are the LHS and RHS of the matrix system constructed by a standard central difference heat equation code.

Additionally, because $c_{ijk} = 0$ when $k > 2P$, it is the case that if $p_k \geq 2P$, the Galerkin projection of the PDE with the thermal coefficient expansion truncated at P_K terms (K_{p_k}) is equivalent to the same expansion including infinite terms ($K_{p_k=\infty}$). Thus, no benefit is gained from increasing p_k beyond $2P$. Because we take $p \in \{0, 1, \dots, 7\}$ in this homework with $p_k = 14$, no computational effort is wasted.

The convergence of our solution is shown in Figure 1, wherein for a fixed total order p_k of the thermal coefficient K_{p_k} PC expansion, the the total order p of the Galerkin projection solution goes from 0 to 7. Both mean and variance, which are computed from the PCE as

$$\begin{aligned}\langle u(x) \rangle &\approx u_0(x) \\ \text{Var}(x) &= \langle u(x)^2 \rangle - \langle u(x) \rangle^2, \quad \langle u(x)^2 \rangle \approx \sum_{i=0}^p u_i(x)^2\end{aligned}\tag{18}$$

are seen to approach a limiting function of x as p is increased.

To calculate a “true” mean and variance, we use the tensor-product stochastic collocation approach with $n_q = 40$ quadrature points in each dimension and a Gauss-Hermite quadrature rule. This procedure is relatively simple, in that we sample the value of $\Psi_j(\mathbf{y})$ at the value of \mathbf{y} indicated by the abscissas, and then compute the first and second moments of u in the manner of (8). The mean and variance are related to these moments as above, and are plotted in Figure 2. This enables us to calculate and plot the relative error in the mean and variance as a function of x , and for $x = 0.5$, as p is increased from 0 to 7. These results are shown in Figures 3 and 4.

We would expect a much smoother mean and variance, which is reason enough to believe that the “true” mean and variance are not the exact solutions. However, they are sufficient for our purposes of approximating a “true” solutions. The cool thing about the PCE approach is that we’ve only solved a single linear system, rather than solving $40^2 = 1600$ different realizations of the PDE. Further reason to suspect the truth of this solution, or the accuracy of our method at capturing variance, is seen in Figure 4, which shows that as p increases past 3, little improvement in the approximation of $\text{Var}(u(x = 0.5))$ is attained.

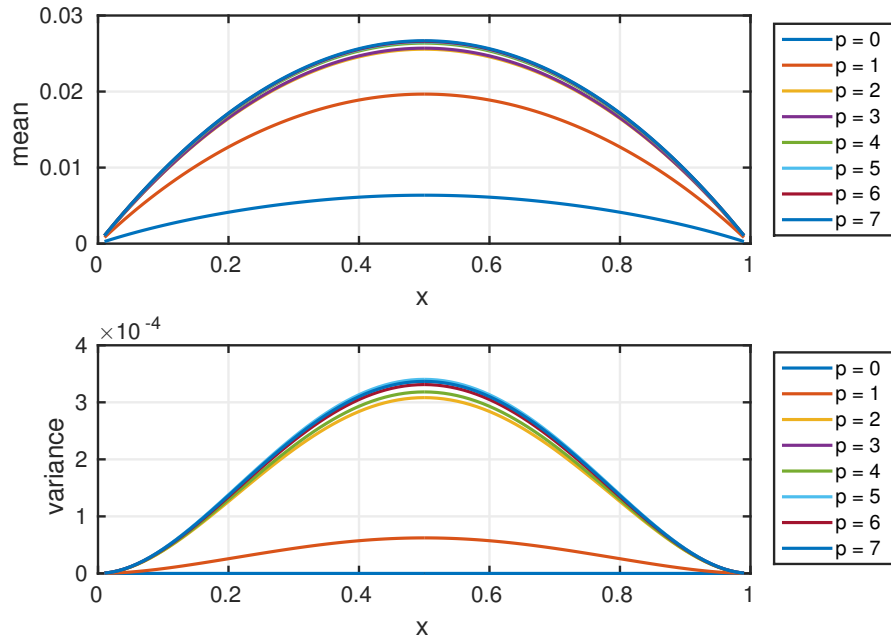


Figure 1: Convergence of temperature field mean and variance as the order p of the Galerkin projection is increased.

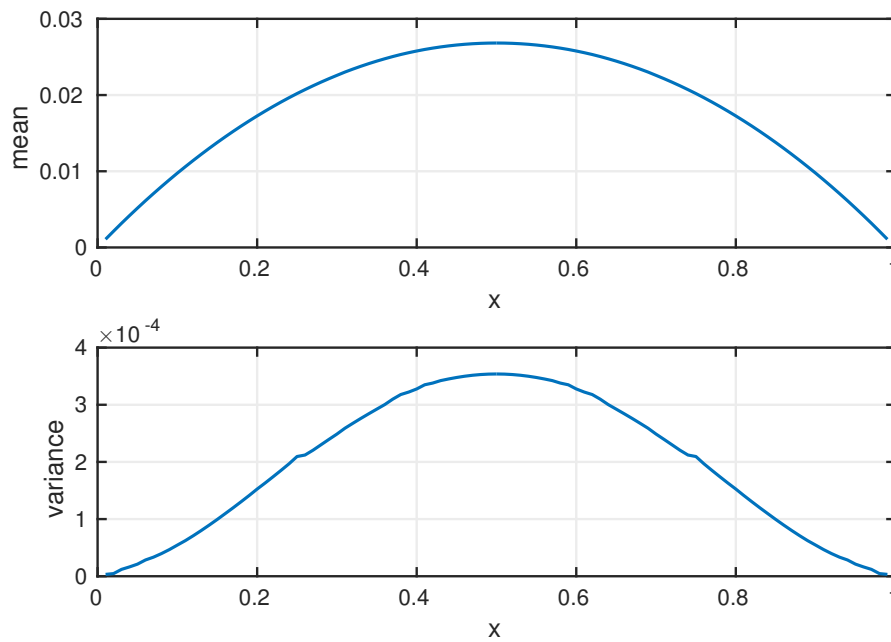


Figure 2: “True” mean and variance of the temperature field, as computed from Gauss-Hermite tensor-product stochastic collocation.

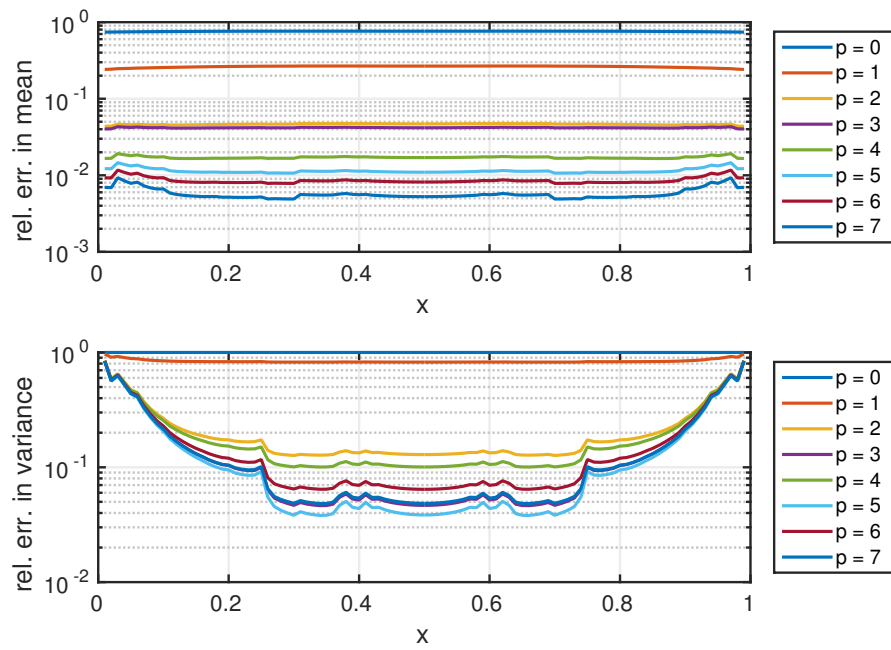


Figure 3: Relative error as a function of position computed from data in Figures 1 and 2.

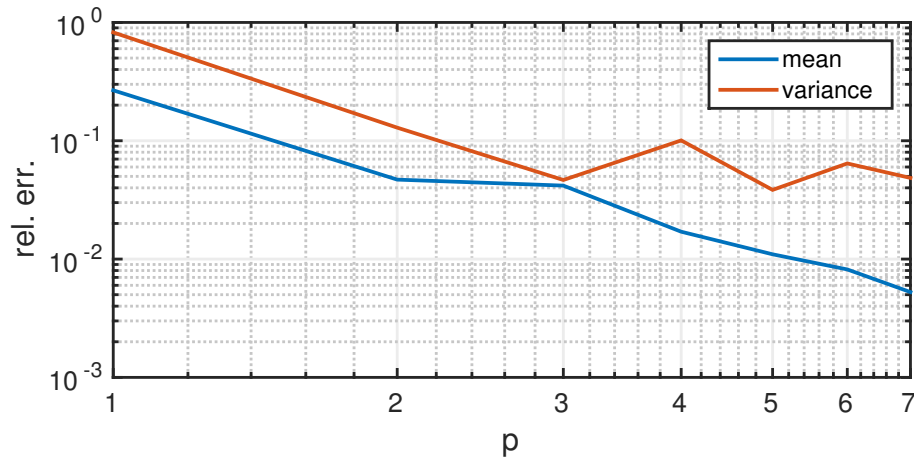


Figure 4: Relative error at $x = 0.5$ computed from data in Figures 1 and 2.