PROBLEM 1

The thermal coefficient K of a 1-D slab is defined on $\mathcal{D} = (0,1)$ and is characterized by a shifted log-normal random process

$$K(x,\omega) = 2 + \exp(G(x,\omega)), \quad x \in \mathcal{D} \tag{1}$$

where $G(x,\omega)$ is a Gaussian random process also defined on $\mathcal{D}$. The mean and covariance functions of $G(x,\omega)$ are

$$\langle G(x,\cdot)\rangle = 1.0, \quad x \in \mathcal{D} \tag{2}$$

and

$$C_{GG}(x_1,x_2) = \sigma^2 \exp\left(\frac{-|x_1 - x_2|}{\ell}\right), \quad (x1, x2) \in \mathcal{D}\times\mathcal{D} \tag{3}$$

respectively. We would like to compute the statistics of the temperature field $u(x)$ by solving the governing steady-state stochastic heat equation

$$\begin{aligned} -\frac{\partial}{\partial x}\left(K(x,\omega)\frac{\partial u(x,\omega)}{\partial x}\right) &= 1.0, \qquad x \in \mathcal{D}, \\ u(0,\omega) &= 0, \\ u(1,\omega) &= 0, \end{aligned} \tag{4}$$

using the **least-squares regression** approach.

For this problem, we let $\sigma = 2$ and $\ell = 2.0$, and use the codes from Homework #1 to generate a $d = 2$ Karhunen-Loève expansion of $G(x,\omega)$ via the analytical solution. The approximate log-normal is then computed through its polynomial chaos expansion (PCE)

$$K_{p_k}(x,\mathbf{y}) = \sum_{i=0}^{P_K} K_i(x)\Psi_i(\mathbf{y}) \tag{5}$$

where K_0 and K_i $(i > 0)$ are defined as

$$K_0 = 2 + \exp\left(1 + \frac{1}{2}\sum_{j=1}^{d}\lambda_j\phi_j^2(x)\right) \tag{6}$$

and

$$K_i(x) = \frac{K_0(x) - 2}{\sqrt{\sum_{j=1}^{d}(i_j!)}}\sum_{j=1}^{d}\left(\sqrt{\lambda_j}\phi_j(x)\right)^{i_j} \tag{7}$$

where $\{\lambda_i, \phi_i(x)\}_{i=1}^{d}$ are eigen-pairs of the covariance kernel C_{GG}, and i_j denotes the polynomial order of $\Psi_i(\mathbf{y})$ along the direction $j \in \{1,\ldots,d\} = \{1,2\}$. We set $p_k = 14$ as the total order of the PC expansion, and write a finite difference code to solve the PDE in (4) for a given K_{p_k}, which we are able to compute for arbitrary $\mathbf{y}$. Note that $\mathbf{y}$ is a Gaussian random vector. We are only interested in the values of our solution at $x = 0.5$, so we employ the stochastic Galerkin discretization of total order p (total number of terms P):

$$u_p(\mathbf{y}) = \sum_{j=0}^{P}\underbrace{u_j(x=0.5)}_{c_j}\Psi_j(\mathbf{y}) \tag{8}$$

SOLUTION

In the least-squares linear regression approach, we sample random vectors $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_N$. For each $\mathbf{y}_i$, we compute the solution $u(x = 0.5, \mathbf{y}_i)$ and the basis functions $\Psi_j(\mathbf{y}_i)$. These are used to form our LHS measurement matrix and RHS solution sample vector in a linear system that we seek to solve for the coefficient vector $\underline{c}$:

$$\underbrace{\begin{bmatrix} \psi_1(\mathbf{y}_1) & \cdots & \psi_P(\mathbf{y}_1) \\ \vdots & \ddots & \vdots \\ \psi_1(\mathbf{y}_N) & \cdots & \psi_P(\mathbf{y}_N) \end{bmatrix}}_{\Psi} \underbrace{\begin{bmatrix} \hat{c}_1 \\ \vdots \\ \hat{c}_P \end{bmatrix}}_{\hat{\mathbf{c}}} = \underbrace{\begin{bmatrix} u(\mathbf{y}_1) \\ \vdots \\ u(\mathbf{y}_N) \end{bmatrix}}_{\mathbf{u}} \tag{9}$$

The least squares solution is obtained by solving the matrix system

$$(\Psi^T \Psi)\hat{\mathbf{c}} = \Psi^T \mathbf{u} \tag{10}$$

which is over-determined if $N > P$ and under-determined and susceptible to instability if $N < P$.

To construct the matrix, we loop over N samples of our Gaussian random variable $\mathbf{y}$. For each sample $\mathbf{y}_i$, we compute the values of $\Psi_1, \ldots, \Psi_P$ and place them in the Ψ-matrix. The heat equation solution is computed with the same realization of $\mathbf{y}_i$. However, note that no solution PCE basis functions are used in the computation of $u(\mathbf{y}_i)$, only the basis functions used to represent K_{p_k} per (5).

The results for total solution polynomial order of $p = 3$ ($P = 10$) and N ranging from $0.5P$ to more than $4P$ are shown in Figure 1. We see the mean and variance approach the values we obtained using the stochastic Galerkin approach in Homework #4. It appears that a converged variance is reached with approximately $N = 4P = 40$ samples, which is in keeping with the recommended value of (4–6)P that was discussed in class.

Looking at the relative error in the variance from 2, we see that as the polynomial order of the stochastic Galerkin PC expansion increases from $p = 1$ to $p = 4$, the solution improves in accuracy if for each employ $N = 4P$ samples. However, this is just one realization of the plot; even when averaging 1000 N-sample computations for variance, our relative error was not always monotonically decreasing.

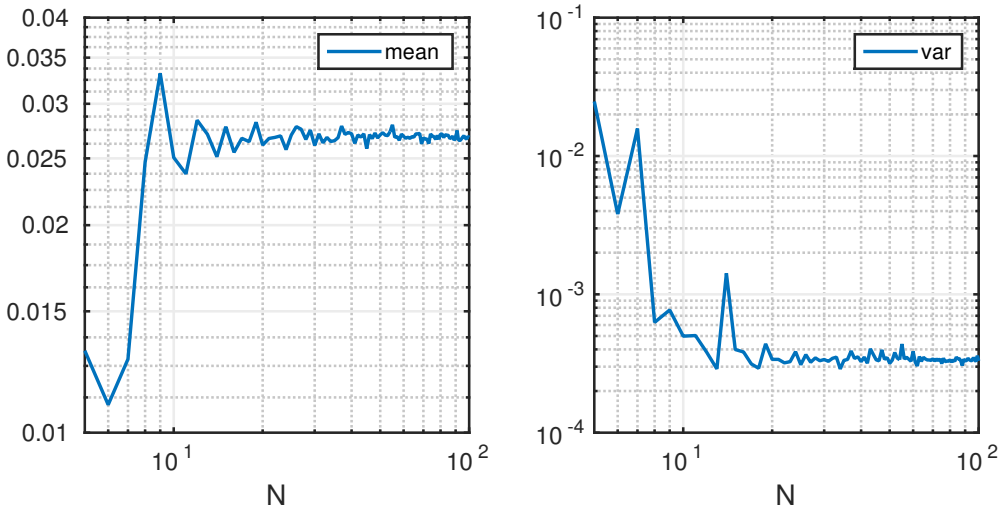

Figure 1: For total solution polynomial order $p = 3$, convergence in the mean and variance of $u(x = 0.5, \mathbf{y})$ as a function of samples N.

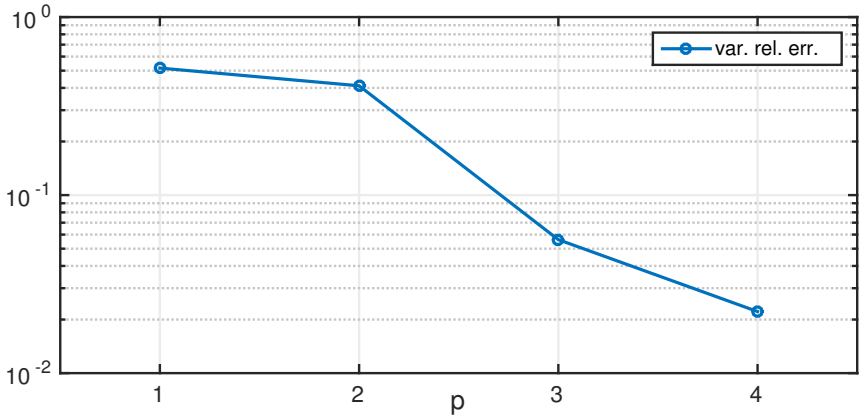

Figure 2: Relative error of $\mathrm{Var}(u(x = 0.5, \cdot))$ as the solution's total polynomial order p is increased. $N = 4P$ realizations are computed for each p to achieve an over-sampled system for linear regression.