Exercise 2

a) What is the connection between SVD of A and SVD of A^T

Proof: $A = U\Sigma V^T \Rightarrow A^T = (U\Sigma V^T)^T$

$$= V\Sigma U^T$$

$\Rightarrow$ A and A^T have the same Sing. values

b) SVD A connec. to eigvalue dec. of A^TA resp AA^T

Proof: $A^TA = V\Sigma U^T U\Sigma V^T$

$$= V\Sigma^2 V^T$$

$$\Leftrightarrow A^TAV = V\Sigma^2$$

$\Rightarrow \sigma_j = \sqrt{\lambda_j(A^TA)}$ (right sing. values)

$AA^T = AA^T = U\Sigma V^T V\Sigma U^T$

$$= U\Sigma^2 U^T$$

$$\Leftrightarrow AAU = U\Sigma^2$$

$\Rightarrow \sigma_j = \sqrt{\lambda_j(AA^T)}$ (left sing values)

c) Show that $A = U \Sigma V^T \iff A = \sum_i^r \sigma_i u_i v_i^T$.
(Sum of rank-1 matrices

Proof: $A = U \Sigma V^T$

$$= [u_1 \; u_2 \; \dots \; u_m] \begin{bmatrix} \tilde{\Sigma} & 0_{r \times (n-r)} \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix} \cdot \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix}$$

with $\tilde{\Sigma}$ Diag Matrix with sing values.

Now if we take a look at

$$[u_1 \; u_2 \; \dots \; u_m] \; \tilde{\Sigma} = \sum_{i=1}^r \sigma_i u_i$$

Therefore

$$A = U \Sigma V^T = \sum_{i=1}^r \sigma_i u_i v_i^T$$