Fourier Series :

Power series convert any function into algebraic infinite series. while Fourier series convert the algebraic function into infinite no of sine and cosine series.

Fourier Series

- In many engineering problems it is necessary to express a function in a series of sines and cosines functions
- This types of series called Fourier series was first developed by the French Mathematician Joseph Fourier in 1822.

Euler's Formula

The fourier series for the function $f(x)$ interval $\alpha < x < \alpha + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \cdot \sin nx \longrightarrow (1)$$

Where $a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx$ & $b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx$

Theorem : (Convergence of Fourier Series)

- Any function $f(x)$ can be developed as a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where a_0, a_n & b_n are constants provided

(i) $f(x)$ is periodic, single valued and finite

(ii) $f(x)$ has finite no of discontinuities in any one period.

(iii) $f(x)$ has atmost a finite number of maxima and minima.

i.e. If (i), (ii), (iii) are satisfied, then the fourier series of $f(x)$ Converges to $f(x)$ at all points where $f(x)$ is continuous.

Fourier Series

Periodic function → A function $f(x)$ which satisfies the relation $f(x+T) = f(x)$ for all real x is called a periodic function. The smallest positive number T, for which this relation holds is called the period of $f(x)$.

If T be the period of $f(x)$, then $f(x) = f(x+T) = f(x+2T) = \cdots = f(x+nT)$

Also $f(x) = f(x-T) = f(x-2T) = \cdots = f(x-nT)$

$\therefore$ $f(x) = f(x \pm nT)$, where 'n' is a positive integer.

Thus, $f(x)$ repeats itself after a period of T.

e.g. $\sin x$, $\cos x$, $\sec x$ and $\operatorname{cosec} x$ are the periodic function with period 2π, while $\tan x$ and $\cot x$ are the periodic function with period $\pi \left(\frac{2\pi}{n}\right)$

Fourier Series : If $f(x)$ be a periodic function with period 2π in the interval $\alpha < x < \alpha + 2\pi$, then the infinite trigonometric series given by.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \cdot \sin nx \longrightarrow (1)$$

where a_0, a_n & b_n are the fourier coefficient of $f(x)$ and are determined by the following formulae

'**Euler Formulaes**'

$$a_0 = \frac{1}{2\pi} \int_{\alpha}^{\alpha+2\pi} f(x)\,dx \longrightarrow (2)$$

$$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx\,dx \longrightarrow (3)$$

$$b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx\,dx \longrightarrow (4)$$

Scanned with CamScanner

If $\alpha = 0$ then interval is $0 < x < 2\pi$

and a_0, a_n & b_n are determined by the following result

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

& $$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

If $\alpha = -\pi$, then the interval is $-\pi < x < \pi$

and a_0, a_n & b_n are determined by the following result

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cdot\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{\pi}^{+\pi} f(x)\sin nx\,dx$$

→ If $f(x)$ is an even function then $\int_{-a}^{+a} f(x)\,dx = 2\int_0^a f(x)\,dx$

→ If $f(x)$ is an odd function then $\int_{-a}^{+a} f(x)\,dx = 0$

Q)) Find the fourier series for the function $f(x) = e^{-ax}$, $-\pi < x < \pi$, 'a' is a constant and hence deduce a series for $\frac{\pi}{\sinh\pi}$

Sol^n Let the fourier series for the function $f(x) = e^{-ax}$ in $-\pi < x < \pi$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\cdot\sin nx \longrightarrow ①$$

$$e^{-ax} = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx \longrightarrow ②$$

Scanned with CamScanner

where a_0, a_n, b_n are Fourier constants, are to be determined by the Euler's Formulae

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\,dx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-ax}\,dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{+\pi} = \frac{1}{2\pi a}\left[e^{a\pi} - e^{-a\pi}\right]$$

$$\boxed{a_0 = \frac{1}{\pi a}\sinh \pi a}$$

Further $a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{+\pi} e^{-ax}\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}(-a\cos nx + n\sin nx)\right]_{-\pi}^{+\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-a\pi}(-a\cos n\pi + n\sin n\pi) - e^{a\pi}(-a\cos n\pi - n\sin n\pi)}{(a^2+n^2)}\right]$$

$$= \frac{1}{\pi(a^2+n^2)}\left[e^{-a\pi}(-a)\cos n\pi + a e^{a\pi}\cos n\pi\right]$$

$\boxed{\sin n\pi = 0}$

$$= \frac{1}{\pi(a^2+n^2)}\cos n\pi\,(a e^{a\pi} - a e^{-a\pi})$$

$\boxed{e^{ax} - e^{-ax} = 2\sinh ax}$

$$= \frac{2a\cos n\pi\,\sinh a\pi}{\pi(a^2+n^2)} = \frac{2a(-1)^n\sinh(a\pi)}{\pi(a^2+n^2)}$$

$\cos n\pi = (-1)^n$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{+\pi} e^{-ax}\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{(a^2+n^2)}(-a\sin nx - n\cos nx)\right]_{-\pi}^{+\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-a\pi}}{a^2+n^2}(-a\sin n\pi - n\cos n\pi) - e^{a\pi}(-a\sin n\pi - n\cos n\pi)\right]$$

$$= \frac{1}{\pi(a^2+n^2)}\left[e^{-a\pi}(-n\cos n\pi) + e^{a\pi} n\cos n\pi\right]$$

$$= \frac{1}{\pi(a^2+n^2)} n\cos n\pi\,(e^{a\pi} - e^{-a\pi})$$

$$= \frac{2n\cos n\pi\,\sinh a\pi}{\pi(a^2+n^2)}$$

$$b_n = \frac{2n(-1)^n\cdot\sinh a\pi}{\pi(a^2+n^2)}$$

Hence from ① the Fourier Series is given by

$$f(x) = a_0 + \Sigma a_n\cos nx + \Sigma b_n\sin nx$$

$$e^{-ax} = \frac{\sinh(\pi a)}{\pi a} + \sum_{n=1}^{\infty}\frac{2a(-1)^n\sinh(a\pi)\cos nx}{\pi(a^2+n^2)}$$

$$+ \sum_{n=1}^{\infty}\frac{2n(-1)^n\sinh(a\pi)\sin nx}{\pi(a^2+n^2)}$$

$$\Rightarrow e^{-ax} = \frac{\sinh(a\pi)}{a\pi} + \frac{2a\sinh(a\pi)}{\pi}\left\{-\frac{\cos x}{1^2+a^2} + \frac{\cos 2x}{2^2+a^2} - \frac{\cos 3x}{3^2+a^2} + \cdots\right\}$$

$$+ \frac{2\sinh(a\pi)}{\pi}\left\{-\frac{\sin x}{1^2+a^2} + \frac{2\sin 2x}{2^2+a^2} - \frac{3\sin 3x}{3^2+a^2} + \cdots\right\}$$

put $x=0$

$$1 = \frac{\sinh(a\pi)}{a\pi} + \frac{2a\sinh(a\pi)}{\pi}\left\{-\frac{1}{1^2+a^2} + \frac{1}{2^2+a^2} - \frac{1}{3^2+a^2} + \cdots\right\} \longrightarrow (3)$$

$\because$ $x=0$ is a point of continuity of $f(x) = e^{-ax}$, $-\pi < x < \pi$. So by Dirichlet's conditions the series (3) will converges on $f(x) = 0 = e^0 = 1$ at $x=0$

$$\Rightarrow \frac{\pi}{\sinh(a\pi)} = \frac{1}{a} + 2a\left\{-\frac{1}{1^2+a^2} + \frac{1}{2^2+a^2} + \cdots\right\}$$

put $\boxed{a=1}$

$$\frac{\pi}{\sinh\pi} = 1 + 2\left\{-\frac{1}{2} + \frac{1}{5} - \frac{1}{10} + \frac{1}{17} \cdots\right\}$$

$$\Rightarrow \frac{\pi}{\sinh\pi} = 1 - 2\left\{\frac{1}{2} - \frac{1}{5} + \frac{1}{10} - \frac{1}{17} + \cdots\right\}$$

Q1) Find Fourier series for the function $f(x) = \begin{cases} \pi + x; & -\pi < x < 0 \\ \pi - x; & 0 < x < \pi \end{cases}$

Sol^n Let the Fourier series for the function $f(x)$ in the interval $-\pi < x < \pi$ be

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (1)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} (\pi + x)\,dx + \int_{0}^{\pi} (\pi - x)\,dx\right]$$

$$= \frac{1}{2\pi}\left[\pi x + \frac{x^2}{2}\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_{0}^{\pi}$$

$$= \frac{1}{2\pi}\left[\pi^2 - \frac{\pi^2}{2} + \pi^2 - \frac{\pi^2}{2}\right] = \pi/2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (\pi + x)\cos nx\,dx + \int_{0}^{\pi} (\pi - x)\cos nx\,dx\right]$$

Integrating by parts we have

$$= \frac{1}{\pi}\left[(\pi + x)\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[(\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_{0}^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{\cos n\pi}{n^2} - \frac{\cos n\pi}{n^2} + \frac{1}{n^2}\right] = \frac{2}{\pi n^2}\left[1 - (-1)^n\right]$$

$\{\cos n\pi = (-1)^n\}$

$$a_n = \begin{cases} 0, & n \text{ is even} \\ \frac{4}{\pi n^2}, & n \text{ is odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (\pi + x)\sin nx\,dx + \int_{0}^{\pi} (\pi - x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{(\pi + x)\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-\frac{(\pi - x)\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\cos 0 + \frac{2\pi\cos n\pi}{n}\right] + \frac{1}{\pi}\left[-0 + \frac{\pi}{n}\cos 0\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n} + \frac{\pi}{n}\right] = 0$$

put the value of a_0, a_n & b_n in (1) we have

$$f(x) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

$$= \frac{\pi}{2} + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \cdots\right]$$

Scanned with CamScanner