Volumes and Surfaces of Solids of Revolution

1.
V. and S. of Solid of Revolution

Solid of revolution: The body generated by revolving a plane area about a fixed line (axis of revolution) lying in its own plane, is called a solid of revolution.

Volume of solid of revolution:

1.) The volume of the solid of revolution about x-axis of the area bounded by the curve $y = f(x)$, the lines $x = a$, $x = b$ and the x-axis.

$$= \int_a^b \pi y^2 \, dx = \int_a^b \pi \, [f(x)]^2 \, dx$$

If axis of revolution is $y = k$, then

$$\text{Volume} = \int_a^b \pi \, (y-k)^2 \, dx$$

2.) If axis of revolution is y-axis and solid is generated by revolving the area bounded by the curve $x = g(y)$, the lines $y = c$ and $y = d$ and the y-axis, then

Volume is given by $= \int_c^d \pi x^2 \, dy = \int_c^d \pi [g(y)]^2 dy$

If the axis of revolution being parallel to y-axis i.e. $x = k$, then

$$\text{Volume} = \int_c^d \pi \, (x-k)^2 \, dy$$

3.) If the equation of the generating curve is given in polar form i.e. $r = f(\theta)$ and the curve revolves about the initial line (x-axis), then

$$\text{Volume} = \int_a^b \pi y^2 \, dx$$

or $\quad V = \int_{\alpha}^{\beta} \pi \, (r \sin\theta)^2 \, \frac{d}{d\theta}(r\cos\theta)\, d\theta$

where α, β are the values of θ corresponding to the points $x = a$ and $x = b$.

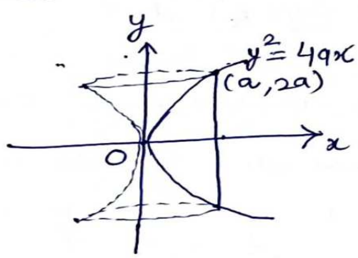
4.) If the parametric form of the curve is given i.e.
$x = \phi(t)$ and $y = \psi(t)$
then, Volume $= \pi \int_{t_1}^{t_2} y^2 \frac{dx}{dt}\, dt$

(where axis of revolution is x-axis)

Problems.

Q.1 The part of the parabola $y^2 = 4ax$ cut off by the latus rectum revolves about the tangent at the vertex. Find the volume of the reel thus generated.

Solution:



The required Volume $= 2\int_0^{2a} \pi x^2 dy$

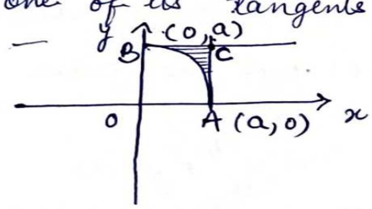
$= 2\pi \int_0^{2a} \frac{y^4}{16a^2} dy$

$= \frac{2\pi}{16a^2} \left(\frac{y^5}{5}\right)_0^{2a}$

Volume $= \frac{4}{5}\pi a^3$

Q.2. A figure is bounded by the quadrant of a circle and the tangents at its extremeties. Find the volume generated when this figure is revolved about one of its tangents.

Solⁿ:



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Let the circle be $x^2+y^2=a^2$
and the area bounded by arc AB and tangents AC and BC is revolved about the tangent AC i.e. $x=a$

Now Required Volume $= \int_0^a \pi (x-a)^2\, dy$

$$= \pi \int_0^a (x^2+a^2-2ax)\, dy$$

$$= \pi \int_0^a \{a^2-y^2+a^2-2a\sqrt{(a^2-y^2)}\}\, dy$$

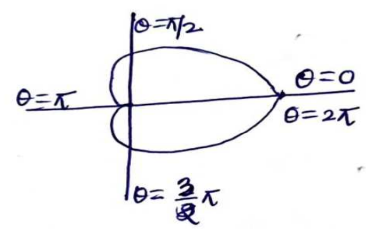
$$= \pi \left[2a^2y - \frac{y^3}{3} - 2a\left\{\frac{y}{2}\sqrt{a^2-y^2} + \frac{a^2}{2}\sin^{-1}\frac{y}{a}\right\}\right]_0^a$$

$$= \pi\left[2a^3 - \frac{a^3}{3} - 2a\left(a^2\frac{\pi}{4}\right)\right] = \pi\left[\frac{5}{3}a^3 - \frac{\pi}{2}a^3\right]$$

or $V = \frac{\pi a^3}{6}(10-3\pi)$

Q.3 The Cardiod $r = a(1+\cos\theta)$ revolves about the initial line. Find the volume of the solid generated.

Solution:



Required volume $= \int_{\theta=0}^{\pi} \pi y^2 \frac{dx}{d\theta}\, d\theta$

$$V = \pi \int_0^{\pi} (r\sin\theta)^2 \frac{d}{d\theta}(r\cos\theta)\, d\theta$$

$$= \pi \int_0^{\pi} a^2(1+\cos\theta)^2 \sin^2\theta \frac{d}{d\theta}\{a(1+\cos\theta)\cos\theta\}\, d\theta$$

$$= \pi a^3 \int_0^{\pi} (1+\cos\theta)^2(1-\cos^2\theta)\{-\sin\theta - 2\cos\theta\sin\theta\}\, d\theta$$

$$V = \pi a^3 \int_0^{\pi} (1+\cos\theta)^2(1-\cos^2\theta)(1+2\cos\theta)\sin\theta\, d\theta$$

Let $\cos\theta = t$, then $-\sin\theta\, d\theta = dt$

Now $V = \pi a^3 \int_1^{-1} (1+t)^2(1-t^2)(1+2t)\, dt$

or $V = -2\pi a^3 \int_0^1 (-5t^4 + 4t^2 + 1)\,dt$

$V = \frac{8}{3}\pi a^3$

Q.4 Find the volume of the solid generated by the revolution of the curve $x(y^2+a^2) = ay^2$ about its asymptote.

Solution: (i) Symmetry : about x- axis

(ii) Curve passes through (0,0) and $x=0$ is the tangent at (0,0)

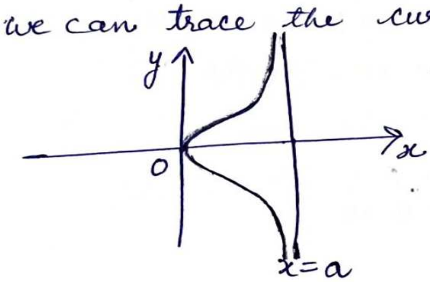
(iii) Curve meets the axes only at (0,0)

(iv) Asymptote to the curve parallel to y-axis is $x=a$

(v) $x = \frac{ay^2}{(y^2+a^2)}$ or $y = \frac{\pm\sqrt{x}\,a}{\sqrt{(a-x)}}$

Curve exists when $0 < x < a$

Now we can trace the curve



$$\text{Volume} = 2\int_0^\infty \pi (x-a)^2\,dy$$

$$= 2\pi \int_0^\infty \left[\frac{ay^2}{(y^2+a^2)} - a\right]^2 dy$$

$$V = 2\pi \int_0^\infty \frac{a^6}{(y^2+a^2)^2}\,dy$$

Let $y = a\tan\theta$, then $dy = a\sec^2\theta\,d\theta$

$$V = 2\pi a^3 \int_0^{\pi/2} \cos^2\theta\,d\theta$$

$$= 2\pi a^3 \frac{\Gamma(\frac{1}{2})\,\Gamma(3/2)}{2\,\Gamma(4/2)}$$

$\left[\because \int_0^{\pi/2} \sin^n\theta \cos^m\theta\,d\theta = \frac{\Gamma(\frac{n+1}{2})\,\Gamma(\frac{m+1}{2})}{2\,\Gamma(\frac{m+n+2}{2})}\right]$

$$V = 2\pi a^3 \frac{\sqrt{\pi}\,\frac{1}{2}\sqrt{\pi}}{2.1} = \frac{\pi^2}{2}a^3$$

$\left[\because \Gamma(\frac{1}{2}) = \sqrt{\pi}\ \&\ \Gamma(n+1) = n\Gamma(n)\right]$
$\Gamma(1) = 1$

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V.R.S. of Solid of revolution

Surface of revolution: The surface generated by the perimeter of the plane curve which is revolved about a fixed line lying in its own plane is called its surface of revolution.

1.) If solid is generated by the revolution about the x-axis of the area bounded by $y=f(x)$, $x=a$, $x=b$ and the x-axis, then surface is given by

$$= 2\pi \int_a^b y \frac{ds}{dx} dx$$

$$\text{where } \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

when the axis of revolution is the y-axis, then

$$\text{surface} = 2\pi \int_a^b x \frac{ds}{dy} dy$$

$$\text{where } \frac{ds}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}$$

2.) In polar form, revolution about initial line, then

$$\text{surface} = 2\pi \int_\alpha^\beta y \frac{ds}{d\theta} d\theta$$

$$\text{where } \frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$$

3.) If parametric form is given, then

$$\text{surface} = 2\pi \int_{t_1}^{t_2} y \frac{ds}{dt} dt \quad \text{(revolution about x-axis)}$$

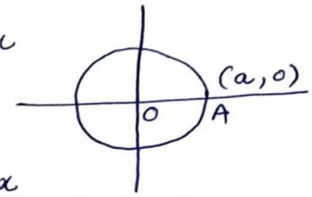
$$\text{where } \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

Problems.

Q.5 Find the surface of a sphere of radius a

Solution: Let the circle be $x^2+y^2=a^2$ and a sphere is generated by the revolution of the circle about the x-axis.

Required surface $= 2\int_{x=0}^{a} 2\pi y \frac{ds}{dx}\,dx$



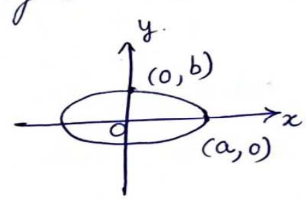
$$= 4\pi \int_0^a y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 4\pi \int_0^a y\,\frac{a}{y}\,dx$$

$$= 4\pi a^2.$$

Q.6. Find the surface of the solid formed by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about major axis.

Solution. $S = 2\int_0^a 2\pi y\,\frac{ds}{dx}\,dx$



$$S = 4\pi \int_0^a y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$\because$ $y = \frac{b}{a}\sqrt{a^2-x^2}$

$$\frac{dy}{dx} = -\frac{b}{a}\frac{x}{\sqrt{a^2-x^2}}$$

or $1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{b^2x^2}{a^2(a^2-x^2)}$

Now $S = 4\pi\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\;\frac{\sqrt{a^4-a^2x^2+b^2x^2}}{a\sqrt{a^2-x^2}}\,dx$

$$= \frac{4\pi b}{a^2}\int_0^a \sqrt{a^4-(a^2-b^2)\cdot x^2}\,dx$$

$$= \frac{4\pi ba}{a^2}\int_0^a \sqrt{a^2-e^2x^2}\,dx \qquad \left[\because (a^2-b^2) = a^2e^2\right]$$

$$S = \frac{4\pi b}{a}\,\frac{1}{e}\left[\frac{ex}{2}\sqrt{a^2-e^2x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{ex}{a}\right)\right]_{x=0}^{a}$$

$$S = \frac{4\pi b}{2ae}\left[ea^2\sqrt{1-e^2}+a^2\sin^{-1}e\right]$$

or $S = 2\pi ab\left[\sqrt{1-e^2}+\frac{1}{e}\sin^{-1}e\right]$

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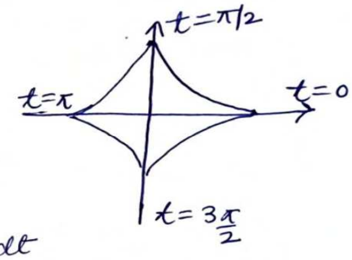
7
V & S. of solid of revolution

Q.7. Find the surface of the solid generated by the revolution of the astroid $x = a\cos^3 t$, $y = a\sin^3 t$ about the axis.

Solution



$$S = 2\int_{t=0}^{\pi/2} 2\pi y \frac{ds}{dt}\,dt$$

$$= 4\pi \int_{t=0}^{\pi/2} a\sin^3 t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

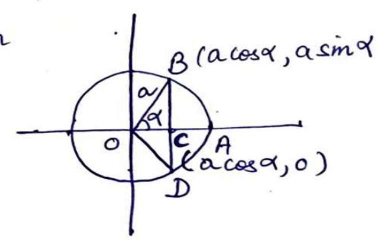
$$= 4\pi a \int_0^{\pi/2} \sin^3 t \;\; 3a \sin t \cos t \, dt$$

$$= 12\pi a^2 \int_0^{\pi/2} \sin^4 t \cos t \, dt$$

$$S = 12\pi a^2 \left[\frac{\sin^5 t}{5}\right]_0^{\pi/2} = \frac{12}{5}\pi a^2$$

Q.8 A circular arc revolves about its chord. Prove that the area of the surface generated is $4\pi a^2(\sin\alpha - \alpha\cos\alpha)$ where a is the radius and 2α the angle subtended by the arc at the centre.

Solution. Let the arc BD subtend an angle 2α at the centre.



Equation of circle is given by

$x = a\cos\theta$, $y = a\sin\theta$

Now Required surface

$$S = 2\int_{\theta=0}^{\alpha} 2\pi (x - a\cos\alpha)\frac{ds}{d\theta}\,d\theta$$

$$= 4\pi \int_{\theta=0}^{\alpha} (a\cos\theta - a\cos\alpha)\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$= 4\pi \int_{\theta=0}^{\alpha} (a\cos\theta - a\cos\alpha)\, a\, d\theta$$

$$= 4\pi a^2 \cdot (\sin\theta - \theta\cos\alpha)_0^{\alpha}$$

$$S = 4\pi a^2(\sin\alpha - \alpha\cos\alpha) \quad \text{Proved}.$$

V&S. of solid of revolution.

Q.9. Prove that the surface and volume of the solid generated by revolving the loop of the curve $x = t^2$, $y = t - \frac{t^3}{3}$ about the x-axis are respectively 3π and $\frac{3}{4}\pi$.

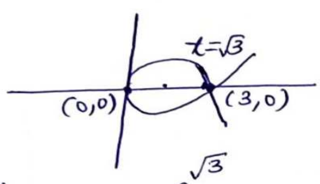
Solⁿ. Given curve $y = t\left(1 - \frac{t^2}{3}\right)$

or $y^2 = t^2\left(1 - \frac{t^2}{3}\right)^2$

or $y^2 = x\left(1 - \frac{x}{3}\right)^2$

(i) Symmetry : about x-axis
(ii) Curve passes through (0,0) and $x=0$ is the tangent at (0,0).
(iii) Curve passes through (0,0) and (3,0) also
(iv) There is no asymptote.
(v) Curve does not exist when $x < 0$

Now we can trace the curve,



$\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = 1 - t^2$

$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$$\text{Surface} = \int_{t=0}^{\sqrt{3}} 2\pi y \frac{ds}{dt}\, dt$$

$$= 2\pi \int_0^{\sqrt{3}} \left(t - \frac{t^3}{3}\right)\sqrt{(1+t^2)^2}\, dt$$

$$= 2\pi \int_0^{\sqrt{3}} \left(t - \frac{t^3}{3}\right)(1+t^2)\, dt$$

$$= 2\pi \int_0^{\sqrt{3}} \left(t + t^3 - \frac{t^3}{3} - \frac{t^5}{3}\right) dt$$

$$= 3\pi$$

$$\text{Volume} = \pi \int_{t=0}^{\sqrt{3}} y^2 \frac{dx}{dt}\, dt$$

$$= \pi \int_0^{\sqrt{3}} \left(t - \frac{t^3}{3}\right)^2 2t\, dt$$

$$= \pi \int_0^{\sqrt{3}} \left(t^2 + \frac{t^6}{9} - 2\frac{t^4}{3}\right) 2t\, dt$$

$$= \frac{3}{4}\pi$$ Proved.