Первое задание.

№1 $\xi \sim R(0, \theta)$

$\tilde\theta_1 = 2\bar{x}$ $\quad$ $\tilde\theta_3 = x_{max}$ $\quad$ $\tilde\theta_5 = \left(x_1 + \dfrac{\sum_{k=2}^{n} x_k}{(n-1)}\right)$

$\tilde\theta_2 = x_{min}$ $\quad$ $\tilde\theta_4 = x_{min} + x_{max}$

$M\xi = \theta/2$; $M\xi^2 = \int_0^\theta x^2 \cdot \frac{1}{\theta} dx = \frac{\theta^2}{3}$

тогда $D\xi = \frac{\theta^2}{3} - \frac{\theta^2}{4} = \frac{\theta^2}{12}$

а) 1) $M[\tilde\theta_1] = \theta$

$M[\frac{2}{n}\sum^n x_i] = \frac{2}{n}\sum^n Mx_i = \frac{2}{n}\cdot n \cdot M\xi = \theta \to$ несмещ.

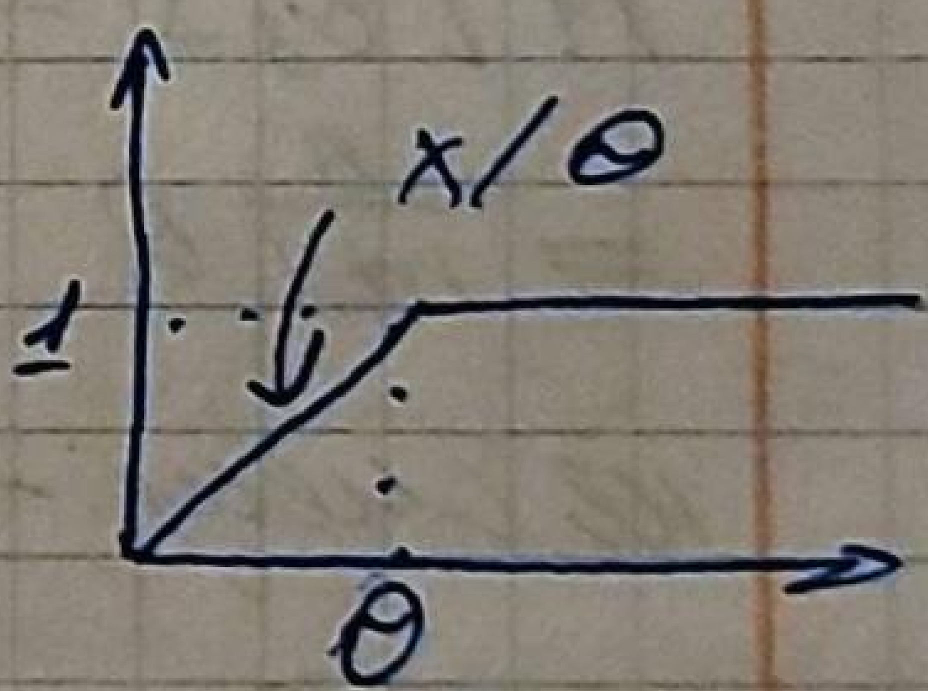
$D[\tilde\theta_1] = D[\frac{2}{n}\sum^n x_i] = \frac{4}{n^2}\sum D x_i = \frac{4}{n^2} D\xi = \frac{\theta^2}{3n} \to 0$ при $n \to \infty$

по дост. условию оценка состоятельна

2) $\xi \sim F(x)$ независ.

$\min(\xi_1 \dots \xi_n) \sim 1 - (1 - F(y))^n = \Phi(y)$

$f(y) = \Phi'(y) = n\,(1 - (F(y)))^{n-1} \cdot p(y)$



$\to p(y) = \frac{1}{\theta}\,(0, \theta)$

$M[\tilde\theta_2] = \int_0^\theta y\,p(y)\,dy = \int_0^\theta y\,n\left(1 - \frac{y}{\theta}\right)^{n-1} \cdot \frac{1}{\theta}\,dy = \left\{t = \frac{y}{\theta}\right\} =$

$= \int_0^1 t\,n\,(1-t)^{n-1}\,\theta\,dt = n\theta\,\frac{\Gamma(2)\Gamma(n)}{\Gamma(n+2)} = \frac{n\theta}{n(n+1)} = \frac{\theta}{n+1}$ –

смещ. $\Rightarrow$ надо исправить $\tilde\theta_2' = (n+1)\cdot x_{min}$

$M[\tilde\theta_2^2] = \int_0^\theta y^2 n\left(1 - \frac{y}{\theta}\right)^{n-1} \cdot \frac{1}{\theta}\,dy = \int_0^1 t^2 (1-t)^{n-1} \cdot \theta^2 n\,dt =$

$= \theta^2 \cdot \frac{n}{(n+2)(n+1)^2}$

тогда $D[\tilde{\theta}_2] = \frac{2\theta^2}{(n+2)(n+1)} - \frac{\theta^2}{(n+1)^2} = \theta^2 \cdot \frac{n}{(n+2)(n+1)^2}$

$D[(n+1)\tilde{\theta}_2] = D[\tilde{\theta}_2'] = \theta^2 \cdot \frac{n}{n+2} \underset{n\to\infty}{\not\to} 0$

Пусть $\theta_2' \xrightarrow{p} \theta \Rightarrow P(|\theta_2' - \theta| > \varepsilon) \underset{n\to\infty}{\to} 0$

$P(|\theta_2' - \theta| \geq \varepsilon) \geq P(\theta_2' \geq \theta + \varepsilon) =$

$= P((n+1)\theta_2 > \theta + \varepsilon) = P(x_1 \geq \frac{\theta+\varepsilon}{n+1} \ldots x_n \geq \frac{\theta+\varepsilon}{n+1}) =$

$= (P(\xi \geq \frac{\theta+\varepsilon}{n+1}))^n = (1 - F(\frac{\theta+\varepsilon}{n+1}))^n = (1 - \frac{\theta+\varepsilon}{\theta(n+1)})^n$

$\longrightarrow e^{-\frac{\theta+\varepsilon}{\theta}} > 0$ при $n \to \infty$

$\Rightarrow$ оценка не явл. состоятельной

3) $\xi \sim F(x) \Rightarrow \max(\xi_1 \ldots \xi_n) \sim (F(z))^n = \Psi(z)$

$\Psi(z) = nF^{n-1}(z) \cdot p(z) = n\left(\frac{z}{\theta}\right)^{n-1} \cdot \frac{1}{\theta} \, (0, \theta)$

$M[\tilde{\theta}_3] = \int_0^\theta z^2 n \left(\frac{z}{\theta}\right)^{n-1} \cdot \frac{1}{\theta} dz = \int_0^1 n\theta^2 t^{n+1} \cdot dt =$

$= \frac{n}{n+1}\theta$

не смещ. $\to \tilde{\theta}_3' = \frac{n+1}{n} X_{max}$

$M[\tilde{\theta}_3^2] = \int_0^\theta z^2 n \left(\frac{z}{\theta}\right)^{n-1} \cdot \frac{1}{\theta} dz = \frac{n}{n+2}\theta^2$

$D[\tilde{\theta}_3] = \frac{n\theta^2}{n+2} - \frac{n^2}{(n+1)^2}\theta^2 = \frac{n\theta^2}{(n+2)(n+1)^2} \not\to 0$

но если $D[\tilde{\theta}_3'] = D[\frac{n+1}{n}\tilde{\theta}_3] = \frac{\theta^2}{n(n+2)} \underset{n\to\infty}{\to} 0$

4) $\tilde{\theta}_4 = x_{min} + x_{max}$

$M[\tilde{\theta}_4] = M[\tilde{\theta}_2] + M[\tilde{\theta}_3] = \frac{\theta}{n+1} + \frac{\theta n}{n+1} = \theta$

$\mathcal{D}[\tilde{\theta}_4] = \mathcal{D}[\tilde{\theta}_2] + \mathcal{D}[\tilde{\theta}_3] + 2\operatorname{cov}(\tilde{\theta}_2, \tilde{\theta}_3)$

$\operatorname{cov}(\tilde{\theta}_2, \tilde{\theta}_3) = M(\tilde{\theta}_2 \cdot \tilde{\theta}_3) - M(\tilde{\theta}_2) \cdot M(\tilde{\theta}_3)$

$$K(y,z) = \begin{cases} F^n(z) - (F(z) - F(y))^n, & z \geq y \\ F^n(z), & z < y \end{cases}$$

$$k(y,z) = \partial^2 K / \partial y \partial z = \frac{\partial}{\partial z}\left(n(F(z)-F(y))^{n-1} \cdot F'(y)\right) =$$

$$= n(n-1)(F(z)-F(y))^{n-2} \cdot \underset{\overset{\shortparallel}{1/\theta}}{F'(y)} \cdot \underset{\overset{\shortparallel}{1/\theta}}{F'(z)} \qquad (z \geq y)$$

$$M[\tilde{\theta}_2 \cdot \tilde{\theta}_3] = \iint\limits_{-\infty}^{+\infty} yz\, k(y,z)\, dy\, dz = \int_0^\theta dz \int_0^z yz\, n(n-1) \cdot$$

$$\cdot \left(\frac{z}{\theta} - \frac{y}{\theta}\right)^{n-2} \cdot \frac{1}{\theta}\, dy = \{t = y/z\} = \int_0^1 n(n-1) t z^2 \left(\frac{z}{\theta} - \frac{y}{\theta}\right) \frac{1}{\theta^2}\, dt$$

$$= n(n-1) \frac{z^{n+1}}{\theta^n} \int_0^1 t(1-t)^{n-2}\, dt = z^{n+1}/\theta^n$$

$$\Rightarrow M(\tilde{\theta}_2 \cdot \tilde{\theta}_3) = \int_0^\theta \frac{z^{n+1}}{\theta^n}\, dz = \frac{1}{n+2}\theta^2$$

тогда $\operatorname{cov}(\tilde{\theta}_2, \tilde{\theta}_3) = \theta^2 \cdot \frac{1}{(n+2)(n+1)^2}$

$$\mathcal{D}[\tilde{\theta}_4] = \theta^2 \left(\frac{n}{(n+2)(n+1)^2} + \frac{n}{(n+2)(n+1)^2} + 2\frac{1}{(n+2)(n+1)^2}\right) =$$

$= 2\theta^2 \cdot \frac{1}{(n+2)(n+1)} \to 0$ при $n \to \infty \Rightarrow$ состоятельная

5) $\tilde{\theta}_5 = x_1 + \frac{1}{n-1}\sum\limits_{k=2}^{n} x_k$

$$M[\tilde{\theta}_5] = Mx_1 + \frac{1}{n-1} \cdot (n-1) \cdot Mx_k = \frac{\theta}{2} + \frac{\theta}{2} = \theta$$

$$\mathcal{D}[\tilde{\theta}_5] = \mathcal{D}x_1 + \frac{1}{(n-1)^2} \cdot (n-1)\mathcal{D}\xi = \frac{\theta^2}{12}\left(1 + \frac{1}{n-1}\right) \not\to 0$$

Если по опр: $\tilde{\theta}_5 = x_1 + \frac{1}{n-1}\sum\limits_{k=2} x_k \to x_1 + \frac{\theta}{2} \Rightarrow$ не сост.

Сравним эффективность: $D[\tilde{\theta}_1] = \frac{\theta^2}{3n}$

$$D[\tilde{\theta}_3'] = \frac{\theta^2}{n(n+2)}$$

$$D[\tilde{\theta}_4] = \frac{2\theta^2}{(n+2)(n+1)}$$

$\frac{1}{n} < \frac{2}{n+1} \rightarrow n+1 < 2n \ \forall n$

$\Rightarrow$ выкидываем $\tilde{\theta}_4$

остается $\frac{1}{3}$ и $\frac{1}{n+2}$; $\frac{1}{n+2} < \frac{1}{3} \Rightarrow$

самая эффективная $\tilde{\theta}_3'$

№2.

$$p(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

c) найти ядерную оценку плотности

$$p'(z) = \frac{1}{h}\int_{-\infty}^{+\infty} p(x) \cdot q\left(\frac{z-x}{h}\right) = \cancel{\ldots} \frac{3}{4}$$

$$q(y) = \frac{3}{4}(1-y^2) \quad (-1, 1)$$

д) Воспользуемся ЦПТ $\frac{1}{n}\cdot\frac{1/n\sum x_i - M\xi}{\sqrt{D\xi}}\cdot\sqrt{n} \sim N(0,1)$

$$M\xi = \int_0^\infty x e^{-x}\,dx = -xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x}\,dx = 1$$

$$M\xi^2 = \int_0^\infty x^2 e^{-x}\,dx = 2$$

$$D\xi = 1 \to \sqrt{D\xi} = 1 \to \cancel{\frac{1}{25}\cdot\sqrt{25}\cdot\frac{\bar{x}-1}{1} \sim N(0,1)}$$

тогда $\sqrt{25}\cdot\frac{\bar{x}-1}{1} \to N(0,1) \Rightarrow \bar{x} \sim N\left(M\xi, \frac{1}{25}\right)$

$$= N(1, 1/25)$$

f) $$p_{\xi_i \text{ и } \xi_j} = n(n-1)\cdot C^{j-i-1}_{n-i-1}\, p(x)\cdot p(y)\cdot F^{i-1}(x)\cdot$$

$$\cdot(F(y)-F(x))^{j-i-1}\cdot(1-F(y))^{n-j} = 600\, C^{j-i-1}_{24-i}\, e^{-x-y}\cdot$$

$$\cdot(1-e^{-x})^{i-1}\cdot(e^{-y\cdot 24})\cdot(e^{-x}-e^{-y})^{j-i-1} = 600\cdot C^{j-i-1}_{24-i}\cdot e^{-x-25y}$$

$$\cdot(1-e^{-x})^{i-1}\cdot(e^{-x}-e^{-y})^{j-i-1}$$

$x>0$ $\quad i<j$

$y>0$

№3

$$p(x) = \begin{cases} e^{-x/\theta}/\theta, & x \geq 0 \\ 0, & x < 0 \end{cases} \quad \theta > 0$$

$\tilde{\theta}_1 = \bar{x}$; $\tilde{\theta}_2 = \frac{x_{min} + x_{max}}{2}$; $\tilde{\theta}_3 = x_{(2)}$

$$M\xi = \int_0^\infty \frac{x}{\theta} \cdot e^{-x/\theta} dx = \frac{1}{\theta} \int_0^\infty x e^{-x/\theta} dx = \frac{1}{\theta} \cdot \theta^2 = \theta$$

а) $\tilde{\theta}_1 = x$

$\cdot\ M[\tilde{\theta}_1] = M[\frac{1}{n} \sum x_i] = \frac{1}{n} \sum M x_i = \theta \to$ несмещ.

$$\cdot\ M\xi^2 = \int_0^\infty \frac{x^2}{\theta} e^{-x/\theta} dx = \frac{1}{\theta} \int_0^\infty x^2 e^{-x/\theta} dx =$$

$$= \frac{1}{\theta} \cdot 2\theta^3 = 2\theta^2$$

$$\to D\xi = 2\theta^2 - \theta^2 = \theta^2$$

$$D[\tilde{\theta}_1] = D[\frac{1}{n} \sum x_i] = \frac{1}{n^2} \sum D[x_i] = \frac{\theta^2}{n} \xrightarrow[n \to \infty]{} 0$$

$\Rightarrow$ состоят.

б) $M[\tilde{\theta}_2] = M[\frac{x_{min} + x_{max}}{2}] = \frac{1}{2}(M x_{min} + M x_{max})$

Рассмотрим отдельно: $\cdot\ \tilde{\theta} = x_{min}$

$$\Phi = 1 - (1 - F(y))^n$$

$$\varphi = \Phi' = n(1 - F(y))^{n-1} \cdot p(y)$$

тогда $M\tilde{\theta} = \int_0^\infty y n (1 - F(y))^{n-1} \cdot p(y) dy =$

$$= \int_0^\infty y n (1 - (1 - e^{-y/\theta}))^{n-1} \cdot \frac{e^{-y/\theta}}{\theta} dy =$$

$$= \int_0^\infty \frac{yn}{\theta} e^{-\frac{ny}{\theta}} dy = \frac{n}{\theta} \cdot \frac{\theta^2}{n^2} = \frac{\theta}{n} = \frac{\theta}{3}$$

$$M\tilde{\theta}^2 = \frac{n}{\theta}\int_0^\infty y^2 e^{-ny/\theta} dy = 2\theta^2/n^2 \to D\tilde{\theta} = \frac{\theta^2}{n^2} = \frac{\theta^2}{9}$$

$\cdot\ \tilde{\theta} = x_{max}$

$$\varphi = (F(y))^n \to f = \varphi' = n(F(y))^{n-1}\cdot f(y)$$

тогда $$M\tilde{\theta} = \int_0^\infty yn(1-e^{-y/\theta})^2 \cdot \frac{e^{-y/\theta}}{\theta} dy =$$

$$= \frac{n}{\theta}\int_0^\infty y(e^{-y/\theta} - 2e^{-2y/\theta} + e^{-3y/\theta})dy = \frac{n}{\theta}\cdot\theta^2 - \frac{n}{\theta}\cdot\frac{\theta^2}{2} +$$

$$+ \frac{n}{\theta}\cdot\frac{\theta^2}{9} = n\theta - \frac{n\theta}{2} + \frac{n\theta}{9} = \frac{11}{18}n\theta = \frac{11}{6}\theta$$

$$M\tilde{\theta}^2 = \int_0^\infty y^2 n/\theta \cdot e^{-y/\theta} dy - \int_0^\infty \frac{2y^2 n}{\theta} e^{-2y/\theta} dy +$$

$$+ \frac{n}{\theta}\int_0^\infty y^2 e^{-3y/\theta} dy = \frac{n}{\theta}\cdot 2\theta^3 - \frac{n}{\theta}\cdot\frac{\theta^3}{2} + \frac{n}{\theta}\cdot\frac{2\theta^3}{27} =$$

$$= 2n\theta^2 - \frac{1}{2}n\theta^2 + \frac{2}{27}n\theta^2 = 85/54\, n\theta^2 = 85/18\,\theta^2$$

$$D[\tilde{\theta}] = \frac{85}{18}\theta^2 - \frac{121}{36}\theta^2 = \frac{49}{36}\theta^2$$

вернёмся к исходной оценке: $M[\tilde{\theta}_2] =$

$$= \frac{1}{2}\left(\frac{\theta}{3} + \frac{11}{6}\theta\right) = \frac{11}{12}\theta + \frac{\theta}{6} = \frac{13}{12}\theta \to \text{смещ.}$$

исправление: $\tilde{\theta}_2' = \frac{12}{13}\tilde{\theta}_2$

$$D[\tilde{\theta}_2] = Dx_{min} + Dx_{max} + 2\,cov(x_{min}, x_{max})$$

$$cov(x_{min}, x_{max}) = M(x_{min}\cdot x_{max}) - M(x_{min})\cdot M(x_{max})$$

$$K(y,z)=\begin{cases} F^n(z)-(F(z)-F(y))^n, & y\le z \\ F^n(z), & y>z \end{cases}$$

$$\mathcal{X}=\frac{\partial^2 K}{\partial y\partial z}=\begin{cases} n(n-1)(F(z)-F(y))^{n-2}\cdot F'_y\cdot F'_z \\ 0 \end{cases}$$

тогда $M(x_{min}\cdot x_{max})=\int\limits_{-\infty}^{+\infty} yz\,\mathcal{X}\,dy\,dz=$

$$=\theta\int\limits_0^{+\infty}dz\int\limits_0^{z} yz\left(e^{-y/\theta}-e^{-z/\theta}\right)\cdot\frac{1}{\theta^2}\cdot e^{-\frac{z+y}{\theta}}dy=$$

$$\frac{1}{\theta}\int\limits_0^{+\infty}dz\int\limits_0^{z} yz\left(e^{-2z+y/\theta}-e^{\frac{+2y-z}{\theta}}\right)dy=$$

$$=\frac{13}{18}\theta^2\to \operatorname{cov}(x_{min},x_{max})=\theta^2\left(\frac{13}{18}-\frac{11}{6}\cdot\frac{1}{3}\right)=$$

$$=\frac{1}{9}\theta^2\Rightarrow D[\tilde\theta_2]=\frac{61}{144}\theta^2$$

$$D[\tilde\theta_2']=D\left[\frac{12}{13}\tilde\theta_2\right]=\frac{144}{169}\cdot\frac{61}{144}\theta^2=\frac{61}{169}\theta^2<1\to$$

состоятельная

в) $\tilde\theta_3=x_{(2)}$

$$F_{(k)}=n(n-1)\cdot C^{l-k-1}_{n-k-1}\cdot F(y)^{k-1}(1-F(y))^{n-k}$$

$k=2;\ l=n\to \mathcal{X}=6\cdot F(y)(1-F(y))\cdot \rho(y)$

тогда $M[\tilde\theta_3]=\int\limits_0^{+\infty}y\cdot\mathcal{X}(y)dy=$

$$=\int\limits_0^{\infty}\theta e^{-y/\theta}\cdot y\left(1-e^{-y/\theta}\right)\frac{e^{-y/\theta}}{\theta}dy=$$

$=\frac{5}{6}\theta$ - не смещ.

$\tilde{\theta}_3' = \frac{6}{5}\theta_3$

• $M[\tilde{\theta}_3^2] = \frac{19}{18}\theta^2 \to D[\tilde{\theta}_3] = \frac{19}{18}\theta^2 - \frac{25}{36}\theta^2 = \frac{13\theta^2}{36}$

$D[\tilde{\theta}_3'] = D[\frac{6}{5}\tilde{\theta}_3] = \frac{36}{25} \cdot \frac{13}{36}\theta^2 = \frac{13}{25}\theta^2$

Выберем самую эффективную:

$D[\tilde{\theta}_1] = \frac{\theta^2}{n} = \frac{\theta^2}{3}$

$D[\tilde{\theta}_2'] = \frac{61}{169}\theta^2 \longrightarrow$ первая эффективнее

$D[\tilde{\theta}_3'] = \frac{13}{25}\theta^2 \qquad \frac{1}{3} < \frac{61}{169} < \frac{13}{25}$

• Эффективность по Крамеру-Рао.

$D[\tilde{\theta}] \geq \frac{1}{nI(\theta)}$

Посчитаем информацию Фишера: $I(\theta) = +M[(\frac{\partial \ln p}{\partial \theta})^2] = -M[\frac{\partial^2 \ln p}{\partial \theta^2}]$

$\frac{\partial \ln p}{\partial \theta} = \frac{\partial \ln e^{-x/\theta}/\theta}{\partial \theta} = \frac{\partial(-x/\theta - \ln\theta)}{\partial\theta} = \frac{x}{\theta^2} - \frac{1}{\theta}$

тогда $\frac{\partial}{\partial\theta}(\frac{x}{\theta^2} - \frac{1}{\theta}) = \frac{1}{\theta^2} - \frac{2x}{\theta^3}$

$I(\theta) = -M[\frac{1}{\theta^2} - \frac{2x}{\theta^3}] = -(-\frac{2}{\theta^3}M[x] + \frac{1}{\theta^2}) = \frac{1}{\theta^2}$

тогда $D[\tilde{\theta}] \geq \frac{\theta^2}{3} \qquad g(\theta) = g(\tilde{\theta}_1) \Rightarrow g(\tilde{\theta}_1') = 1$

$D[\tilde{\theta}_1] = \frac{1}{3}\theta^2$ нер-во выполн. $\Rightarrow$ эф. по Крамеру-Рао

$D[\tilde{\theta}_2] = \frac{61}{169}\theta^2$; $g(\theta) = g(\tilde{\theta}_2') = 2$

$\to$ для второй оценки $\frac{61}{169}\theta^2 > \frac{2}{3/\theta^2}$ - не выполн

$\Rightarrow$ не эфф. по Крамеру-Рао

$\mathcal{D}[\tilde{\theta}_3^{2}] = \frac{13}{15}\theta^2 \geq \frac{\theta^2}{3}$ → эф. по Крамеру-Рао