$D[\tilde{\theta}_3^2] = \frac{13}{15}\theta^2 \geq \frac{\theta^2}{3} \rightarrow$ эф. по Крамеру-Рао

№4

$\xi \sim R[\theta, 2\theta]$

а) $\tilde{\theta}_3 = \frac{1}{5}(X_{min} + 2X_{max})$

ОММ: $\alpha_1(\theta) = M[\xi] = \int_\theta^{2\theta} x \cdot \frac{1}{\theta} dx = \frac{x^2}{2\theta} \Big|_\theta^{2\theta} = \frac{3}{2}\theta$

$\Rightarrow \tilde{\theta}_1 = 2/3\, \bar{X}$

ОМП: $L(\theta) = \prod^n p(x_i, \theta) = \frac{1}{\theta^n} (\theta \leq X_i \leq 2\theta) \rightarrow max$

$\Rightarrow \tilde{\theta}_3 = x_{max}/2$

б) • исследуем первую оценку

$\tilde{\theta}_1 = \frac{2}{3}\bar{X}$

$M[\tilde{\theta}_1] = M[\frac{2}{3}\bar{X}] = \frac{2}{3} M\xi = \frac{2}{3} \cdot \frac{3}{2}\theta = \theta \rightarrow$ несмещ.

$D[\tilde{\theta}_1] = D[\frac{2}{3}\bar{X}] = \frac{4}{9} \cdot D[\frac{1}{n}\sum x_i] = \frac{4}{9n} \cdot D\xi \xrightarrow[n\to\infty]{} 0$

$\Rightarrow$ состоятельна

• исследуем вторую оценку $\tilde{\theta}_2 = \frac{X_{max}}{2}$

$M[\tilde{\theta}] = M[\frac{X_{max}}{2}] = \frac{1}{2} M[X_{max}] =$

$= \frac{1}{2} \int_\theta^{2\theta} x n \cdot F^{n-1} \cdot f(x)\, dx = \frac{n}{2} \int_\theta^{2\theta} x \left(\frac{x-\theta}{\theta}\right)^{n-1} \cdot \frac{1}{\theta} dx = \theta - \frac{\theta}{2(n+1)} =$

$= \frac{2(n+1)\theta - \theta}{2(n+1)} = \theta \cdot \left(\frac{2n+2-1}{2(n+1)}\right) \neq \theta$ - смещ.

Исправляем: $\tilde{\theta}_2' = \frac{2(n+1)}{2n+1} \tilde{\theta}_2 = \frac{2(n+1)}{2n+1} \cdot \frac{X_{max}}{2}$ - несмещ.

$$D[\tilde{\theta}_2'] = D\left[\frac{(n+1)X_{max}}{2n+1}\right] = \frac{(n+1)^2}{(2n+1)^2}\cdot D[X_{max}]$$

$$M[X_{max}^2] = \int_\theta^{2\theta} x^2\cdot n\left(\frac{x-\theta}{\theta}\right)^{n-1}\cdot\frac{1}{\theta}dx = \int_\theta^{2\theta} x^2 d\left(\frac{x-\theta}{\theta}\right)^n =$$

$$= x^2\cdot\left(\frac{x-\theta}{\theta}\right)^n\Big|_\theta^{2\theta} - \int_\theta^{2\theta}\left(\frac{x-\theta}{\theta}\right)^n\cdot 2x\,dx = 4\theta^2\cdot\frac{n}{n+1} + \frac{2\theta^2}{(n+1)(n+2)}$$

$\Rightarrow$ ~~$D[\theta_2$~~

$$D[\tilde{\theta}_2'] = \frac{(n+1)^2}{(2n+1)^2}\cdot\left(4\theta^2\cdot\frac{n}{n+1} + \frac{2\theta^2}{(n+1)(n+2)} - \frac{(2n+1)^2\theta^2}{(n+1)^2}\right) =$$

$$= \frac{4\theta^2 n(n+2) + 2\theta^2}{(n+1)(n+2)}\cdot\frac{(n+1)^2}{(2n+1)^2} - \theta^2 = \frac{n\theta^2}{(n+2)(2n+1)} \xrightarrow[n\to\infty]{} 0 \Rightarrow \text{сост}$$

. исследуем третью оценку $\tilde{\theta}_3 = \frac{1}{5}(X_{min} + 2X_{max})$

$$M[\tilde{\theta}_3] = \frac{1}{5}M[X_{min} + 2X_{max}] = \frac{1}{5}M[X_{min}] + \frac{2}{5}M[X_{max}]$$

$$M[X_{min}] = \int_\theta^{2\theta} xn\left(1-\left(\frac{x-\theta}{\theta}\right)\right)^{n-1}\cdot\frac{1}{\theta}dx = \frac{n+2}{n+1}\theta$$

тогда $M[\tilde{\theta}_3] = \frac{1}{5}\cdot\frac{n+2}{n+1}\theta + \frac{2}{5}\cdot\frac{(2n+1)}{n+1}\cdot\theta =$

$= \theta\cdot\frac{5n+4}{5(n+1)} \to$ смещ.

исправляем: $\tilde{\theta}_3' = \frac{5(n+1)}{5n+4}\cdot\frac{1}{5}\cdot(X_{min} + 2X_{max})$ – несмещ.

$$D[\tilde{\theta}_3'] = D\left[\frac{n+1}{5n+4}\cdot(X_{min} + 2X_{max})\right] = \frac{(n+1)^2}{(5n+4)^2}\big(D[X_{min}] +$$

$$+ 4D[X_{max}] + 4\operatorname{cov}(X_{min}, X_{max})\big)$$

$$\operatorname{cov}(X_{min}, X_{max}) = M[X_{min}\cdot X_{max}] - M[X_{min}]\cdot M[X_{max}]$$

$$\delta(y,z) = \frac{\partial^2 K}{\partial y\partial z};\quad K = \begin{cases} F^n(z), & z<y \\ F^n(z) - (F(z)-F(y))^n, & z\ge y\end{cases}$$

отсюда $f(y,z) = n(n-1)\left(F(z)-F(y)\right)^{n-2} \cdot F'_y \cdot F'_z =$

$= n(n-1) \cdot \left(\frac{z-y}{\theta}\right)^{n-2} \cdot \frac{1}{\theta^2}$

$M[X_{min} \cdot X_{max}] = \iint\limits_{-\infty}^{+\infty} yz f\, dy\, dz = \int\limits_{\theta}^{2\theta} dy \int\limits_{\theta}^{y} zy\, n(n-1) \cdot \left(\frac{z-y}{\theta}\right)^{n-2} \frac{1}{\theta^2}$

$= \frac{(2n+5)}{n+2} \theta^2$

$cov(X_{min}, X_{max}) = \frac{\theta^2}{(n+2)(n+1)^2}$

$D[X_{min}] = M[X_{min}^2] - M^2[X_{min}]$

$M[X_{min}^2] = \int\limits_{\theta}^{2\theta} x^2 n \cdot \left(1 - \left(\frac{x-\theta}{\theta}\right)\right)^{n-1} \cdot \frac{1}{\theta} dx = \theta^2 \cdot \frac{(n^2+5n+8)}{(n+1)(n+2)}$

$\Rightarrow D[X_{min}] = \theta^2 \left(\frac{n^2+5n+8}{(n+1)(n+2)} - \frac{(n+2)^2}{(n+1)^2}\right) = \theta^2 \cdot \frac{n}{(n+1)^2(n+2)}$

$\Longrightarrow D[\tilde{\theta}_3'] = \frac{(n+1)^2}{(5n+4)^2}\left(\theta^2 \cdot \frac{n}{(n+1)^2(n+2)} + 4\frac{n\theta^2}{(n+1)^2(n+2)} +\right.$

$\left. + \frac{4\theta^2}{(n+2)(n+1)^2}\right) = \frac{\theta^2}{(5n+4)(n+2)} \xrightarrow[n\to\infty]{} 0 \Rightarrow$ сост.

c) $D[\tilde{\theta}_1] = \frac{4}{9n}\theta^2$

$D[\tilde{\theta}_2'] = \frac{n\theta^2}{(n+2)(2n+1)}$

$D[\tilde{\theta}_3'] = \frac{\theta^2}{(5n+4)(n+2)}$

$\tilde{\theta}_3'$ - самая эффективная

~~Ср. 9n~~

$\frac{1}{5n+4} < \frac{n}{2n+1}$

$2n+1 < 5n^2+4n$

d) $F(x) = \begin{cases} 0, & x < \theta \\ (x-\theta)/\theta, & x \in [\theta, 2\theta] \\ 1, & x > 2\theta \end{cases}$

$F_\theta(y) = P(\max(\overline{x}_n) \le \theta y) = (P(x_1 \le \theta y))^n = (F(\theta y))^n$

$F_\theta(y) = \begin{cases} 0, & y < 1 \\ (y-1)^n, & 1 \le y \le 2 \\ 1, & y > 2 \end{cases}$ $\qquad p_\theta(y) = \begin{cases} 0, & y < 1 \text{ или } y > 2 \\ n(y-1)^{n-1}, & 1 \le y \le 2 \end{cases}$

$\int\limits_1^{f_1} n(y-1)^{n-1}\,dy = \frac{1-0,95}{2} \to (f_1-1)^n = 0,025 \to f_1 = \sqrt[n]{0,025}+1$

$\int\limits_{f_2}^{2} n(y-1)^{n-1}\,dy = \frac{1-0,95}{2} \to f_2 = \sqrt[n]{0,975}+1$

$I = (f_1, f_2) = (\sqrt[n]{0,025}+1, \sqrt[n]{0,975}+1)$ – для $\frac{\max \overline{x}_n}{\theta}$

$\to \theta \in \left(\frac{\max(\overline{x}_n)}{1+\sqrt[n]{0,025}} ; \frac{\max(\overline{x}_n)}{1+\sqrt[n]{0,975}} \right)$ – точный довериt. интервал

e) ассимптотический довериt. интервал

ОМН: $\tilde{\theta}_1 = \frac{2}{3}\overline{x} = \frac{2}{3}\hat{\alpha}_1$

$f(\alpha_1) = \frac{2}{3}\alpha_1$

$k = k_{11} = \alpha_2 - \alpha_1^2 \quad \Rightarrow \sigma = \sqrt{\nabla^T f(\alpha) \cdot k \cdot \nabla f(\alpha)}$

$\nabla f = \frac{2}{3}$

$\sigma = \sqrt{\frac{4}{9}(\alpha_2 - \alpha_1^2)}$

$\tilde{\sigma} \xrightarrow{P} \sigma \to \frac{f(\hat{\alpha}) - f(\alpha)}{\tilde{\sigma}}\sqrt{n} \sim N(0,1)$

$\frac{\tilde{\theta} - \theta}{\sqrt{\frac{4}{9}(\alpha_2 - \alpha_1^2)}} \cdot \sqrt{n} \sim N(0,1) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$

$$t_1 < \frac{\tilde{\theta}-\theta}{\sqrt{\frac{4}{9}(\tilde{\alpha}_2-\tilde{\alpha}_1^2)}}\sqrt{n} < t_2$$

$$\Downarrow$$

$$\frac{\frac{2}{3}\sqrt{\tilde{\alpha}_2-\tilde{\alpha}_1^2}}{\sqrt{n}}\cdot t_1 < \tilde{\theta}-\theta < \frac{\frac{2}{3}\sqrt{\tilde{\alpha}_2-\tilde{\alpha}_1^2}}{\sqrt{n}}\cdot t_2$$

$$\tilde{\theta} - t_2\cdot\frac{2/3\sqrt{\tilde{\alpha}_2-\tilde{\alpha}_1^2}}{\sqrt{n}} < \theta < \tilde{\theta} - t_1\cdot\frac{2/3\sqrt{\tilde{\alpha}_2-\tilde{\alpha}_1^2}}{\sqrt{n}}$$

№5

распределение Парето: $p(x) = \begin{cases} \frac{\theta-1}{x^\theta}; & x \geq 1 \\ 0, & x < 1 \end{cases}$ $\theta > 1$

а) X_n – выборка, объём n

ОМП: $L(\theta) = \prod^n p(x_i, \theta) = \frac{(\theta-1)^n}{(\prod^n x_i)^\theta}$

$$\ln L(\theta) = n\ln(\theta-1) - \theta\ln(\prod x_i) = n\cdot\ln(\theta-1) - \theta\cdot\sum\ln x_i$$

$$\frac{\partial \ln L(\theta)}{\partial\theta} = n\cdot\frac{1}{\theta-1} - \sum\ln x_i$$

$$\frac{\partial \ln L(\theta)}{\partial\theta} = 0 \to n\cdot\frac{1}{\theta-1} = \sum\ln x_i$$

Отсюда $\tilde{\theta} = \frac{n}{\sum\ln x_i} + 1$

$$\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} = \frac{-n}{(\theta-1)^2} < 0 \to \max$$

в) доверительный интервал для медианы

$$\int_1^{\infty} \frac{(\theta-1)}{x^\theta} dx = \frac{1}{2} \text{ - медиана}$$

$$\int_1^{x} \frac{(\theta-1)}{x^\theta} dx = 1 - x^{1-\theta} = \frac{1}{2} \to \quad x^{1-\theta} = \frac{1}{2}$$

$$x = \left(\frac{1}{2}\right)^{1/1-\theta} = 2^{\frac{1}{\theta-1}}$$

$$\frac{f(\tilde{\theta}) - f(\theta)}{\sigma} \sqrt{n} \to N(0,1)$$

$$\sigma = \sqrt{\nabla^T f \cdot I^{-1} \cdot \nabla f}$$

$$f(\theta) = 2^{\frac{1}{\theta-1}}$$

$$\frac{\partial f}{\partial \theta} = \ln 2 \cdot 2^{\frac{1}{\theta-1}} \left(-\frac{1}{(\theta-1)^2}\right)$$

$$I(\theta) = -M\left[\frac{\partial^2 \ln p}{\partial \theta^2}\right]$$

$$\ln p = \ln(\theta-1) - \theta \ln x$$

$$\frac{\partial \ln p}{\partial \theta} = \frac{1}{\theta-1} - \ln x \to \quad \frac{\partial^2 \ln p}{\partial \theta^2} = \frac{-1}{(\theta-1)^2}$$

$$\Rightarrow I(\theta) = \frac{1}{(\theta-1)^2}$$

тогда

$$\frac{2^{\frac{1}{\tilde{\theta}-1}} - 2^{1/\theta-1}}{\ln 2 \cdot 2^{\frac{1}{\tilde{\theta}-1}}} \cdot (\tilde{\theta}-1)\sqrt{n} \to N(0,1)$$

$$\beta = 0{,}95 \Rightarrow \quad -1{,}96 < \frac{2^{\frac{1}{\tilde{\theta}-1}} - med}{\ln 2 \cdot 2^{\frac{1}{\tilde{\theta}-1}}} (\tilde{\theta}-1)\sqrt{n} < 1{,}96$$

$$-2^{\frac{1}{\tilde{\theta}-1}} - \frac{1{,}96 \ln 2 \cdot 2^{\frac{1}{\tilde{\theta}-1}}}{\sqrt{n}(\tilde{\theta}-1)} < -med < \frac{1{,}96 \ln 2 \cdot 2^{\frac{1}{\tilde{\theta}-1}}}{\sqrt{n}(\tilde{\theta}-1)} - 2^{\frac{1}{\tilde{\theta}-1}}$$

$$\Rightarrow med \in \left(2^{\frac{1}{\theta-1}} - \frac{1{,}96\cdot \ln 2 \cdot 2^{\frac{1}{\theta-1}}}{\sqrt{n}(\theta-1)} ;\ 2^{\frac{1}{\theta+1}} + \frac{\ln 2 \cdot 1{,}96 \cdot 2^{\frac{1}{\theta+1}}}{\sqrt{n}(\theta-1)}\right)$$

c) байесовская оценка параметра для

$$p(y) = \begin{cases} e^{1-y}, & y \geq 1 \\ 0, & y < 1 \end{cases}$$

$$\cdot\ p(\theta, \bar{x}_n) = \frac{p(\bar{x}_n / \theta) \cdot p(\theta)}{p(x_n)} = \begin{cases} \frac{e^{1-\theta} \prod p(x_i, \theta)}{p(x_n)} \cdot C, & \theta \geq 1 \\ 0, & \theta < 1 \end{cases}$$

$$p(\theta / x_n) = C \cdot L(\theta) \cdot P(\theta)$$

$$\ln p(\theta, x_n) = \ln C + \ln L + \ln P = \ln C + n \cdot \ln(\theta - 1) -$$

$$- \theta \sum \ln x_i + (1 - \theta)$$

$$\partial \ln p / \partial \theta = \frac{n}{\theta - 1} - 1 - \sum \ln x_i$$

$$\partial \ln p / \partial \theta = 0 \rightarrow \tilde{\theta} = \frac{n}{\sum \ln x_i + 1} + 1$$

$$\int_1^{t_1} p(\theta / \bar{x}_n) d\theta = 0{,}025$$

$$\int_{t_2}^{\infty} p(\theta / x_n) d\theta = 0{,}025 \quad \rightarrow \quad I = (t_1, t_2)$$

доверит. интервал

d) асимптотич. дов. интервал

$$I(\theta) = \frac{1}{(\theta - 1)^2} \rightarrow \frac{\tilde{\theta} - \theta}{(\tilde{\theta} - 1)} \sqrt{n} \sim N(0, 1)$$

$$-1{,}96 < \frac{\tilde{\theta} - \theta}{\tilde{\theta} - 1} \sqrt{n} < 1{,}96$$

$$\Rightarrow \tilde{\theta} - \frac{1,96\,(\tilde{\theta}-1)}{\sqrt{n}} < \theta < \tilde{\theta} + \frac{1,96\,(\tilde{\theta}-1)}{\sqrt{n}}$$

$$\tilde{\theta} = \frac{n}{\sum \ln x_i} + 1$$