UNSW AUSTRALIA

University of New South Wales

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Assignment 1

Algebraic Topology

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Question 1

Let $A = \{a, b, c, d\}$ be an ordered set, ordered alphabetically.

Proposition 1. *Let*

$$K = \{a, b, c, d, ab, bc, bd\}$$

and

$$K' = \{a, b, c, d, ab, bc, abc\}.$$

Then K is a simplicial complex, and K' is not.

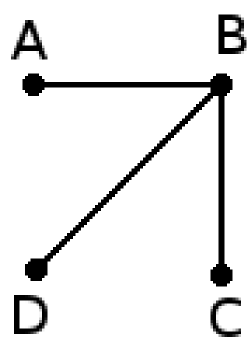
Proof. We need to check that K is closed under taking non-empty subsets. Indeed, the only elements of K which are not singletons are ab, bc, bd, and the only non-empty subsets of these are a, b, c, d, which are contained in K. Hence K is a simplicial complex.

However, we have $abc \in K'$, and $ac \subset abc$ but $ac \notin K'$. Hence K' is not a simplicial complex. □

Proposition 2. *If*

$$K = \{a, b, c, d, ab, bc, bd\}$$

then we can visualise $|K|$ as



Proof. There are only four singletons in K, so therefore four vertices. There are no three element subjects, so the only simplices in K can be 1-simplexes. So there are four vertices, and since there are three two element subsets, we have three lines. They are the lines joining a to b, b to c and b to d. □

Question 2

Proposition 3. *Let $\phi : \mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}$ be given by $\phi(n) = (n, -2n)$ and $\psi : \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}$ be $\psi(n, m) = 2n + m$. Then*

$$0 \to \mathbb{Z} \xrightarrow{\phi} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\psi} \mathbb{Z} \to 0$$

is a chain complex.

The homology groups are all 0, i.e. the trivial group.

Proof. We only need to check that $\operatorname{im}\phi \subseteq \ker\psi$. Let $(n, -2n) \in \operatorname{im}\phi$. Then $\psi(n, -2n) = 2n - 2n = 0$. Hence $(n, -2n) \in \ker\psi$. Thus we have a chain complex.

Starting from the left, the first homology group is just $\ker(\phi)$. But we have $\phi(n) = 0$ if and only if $(n, -2n) = (0, 0)$, so this is true if and only if $n = 0$. Thus $\ker\phi = 0$.

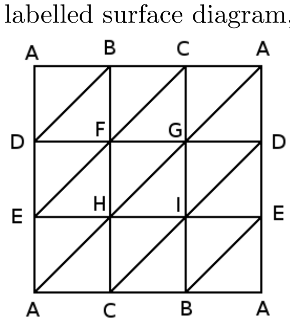
The next homology group is $\ker\psi/\operatorname{im}\phi$.

Let $(n, m) \in \ker\psi$. Then $2n + m = 0$, so $m = -2n$. Thus $(n, m) = \phi(n)$. Therefore, $\ker\psi \subseteq \operatorname{im}\phi$. Hence $\ker\psi/\operatorname{im}\phi$ is the trivial group.

The final homology group is $\ker(\mathbb{Z} \to 0)/\operatorname{im}\psi$. Now $\ker(\mathbb{Z} \to 0) = \mathbb{Z}$, however $\operatorname{im}\psi = \mathbb{Z}$ since if $n \in \mathbb{Z}$ we can write $n = \psi(0, n)$. Hence the final cohomology group is $\mathbb{Z}/\operatorname{im}\psi = \mathbb{Z}/\mathbb{Z}$. So all homology groups are trivial. ☐

Question 3

For this question, we let K be the simplicial complex associated to the following labelled surface diagram,



Proposition 4. *The homology of K is as follows,*

$$H_p(K) \cong 0 \text{ for } p \geq 3$$
$$H_2(K) \cong 0$$
$$H_1(K) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$$
$$H_0(K) \cong \mathbb{Z}$$
$$H_p(K) \cong 0 \text{ for } p < 0.$$

Proof. There are no p-chains in K for $p \geq 3$, so for $p \geq 3$ we have $C_p(K) = 0$. Hence $H_p(K) = 0$. Similarly for $p < 0$ we have $C_p(K) = 0$, so $H_p(K) = 0$.

The only potentially non-trivial cases are $p = 0, 1, 2$.

First we compute $H_0(K)$. Since $C_{-1}(K) = 0$, we have $\ker(\partial_0) = C_0(K)$.

Now if $x, y \in C_0(K)$, since $|K|$ is connected we can create a path $x_0, x_1, x_2, \ldots, x_n$ such that $x = x_0$ and $y = x_n$ so that $x - y = \partial_1([x_0x_1] + [x_1x_2] + \cdots + [x_{n-1}x_n])$. Hence x and y are in the same congruence class modulo $B_0(K)$. Thus there is only one congruence class for all the points of K, thus $H_0(K) \cong \mathbb{Z}$.

Let L be the subcomplex generated by $\{a, b, c, d, e\}$.

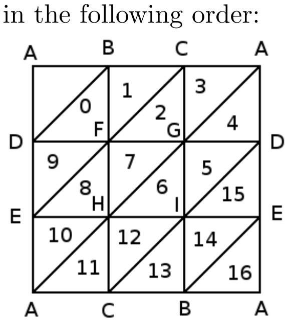
Now we compute $H_1(K) := \ker\partial_1 / \operatorname{im}\partial_2$. We have,

$$[AB] + [BC] + [CA],\ [AD] + [DE] + [EA] \in \ker\partial_1.$$

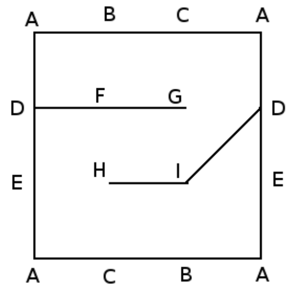
We wish to show that these are generators for $H_1(K)$. To this end, we wish to show that any element of $\ker(\partial_1)$ is homologous to an element carried by $C_1(L)$.

Let $c \in \ker(\partial_1)$. We need to show that we can add elements of $B_1(K)$ to c to obtain a chain carried by L.

Add multiples of elements of $B_1(K)$ to c corresponding to the simplices labelled in the following order:



So that the only edges which are not annihilated are the ones shown below:



Hence c is homologous to a chain carried by the above edges. Now write,

$$c + B_1(K) = \sum_{\sigma \in K, |\sigma|=2} m_\sigma \sigma + B_1(K).$$

where the only coefficients m_σ not equal to zero are the ones shown above. Now since $\partial_1(c) = 0$, we must have that $n_{FG} = n_{DF} = 0$, and $n_{HI} = n_{DI} = 0$.

Hence c is homologous to a chain carried by L.

Let $b \in \ker(\partial_1) \cap C_1(L) = Z_1(L)$. Suppose that

$$b = \sum_{\sigma \in L, |\sigma|=2} k_\sigma \sigma.$$

Then we must have $k_{AB} = k_{BC} = k_{AC}$, since if $\partial_2(B) = 0$, we need $k_{AB} - k_{BC} = k_{AC} - k_{BC} = 0$. Similarly, we must have $k_{AD} = k_{DE} = k_{AE}$.

Hence, b is in the subspace generated by

$$[AB] + [BC] + [CA],\ [AD] + [DE] + [EA] \in \ker \partial_1.$$

Hence $H_1(K)$ is the subgroup generated by these two elements. Since the first one has order 2 and the second has order 0, this means

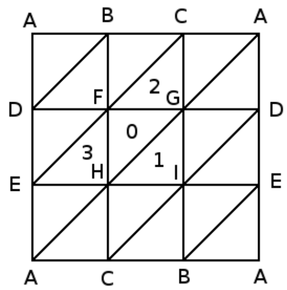
$$H_1(K) \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

To compute $H_2(K)$, note that $C_3(K) = 0$, since there are no 3-simplices in K. Hence $H_2(K) = \ker \partial_2$. Suppose that $c \in C_2(K)$ is a chain such that $\partial_2(C)$ is carried by L. Let

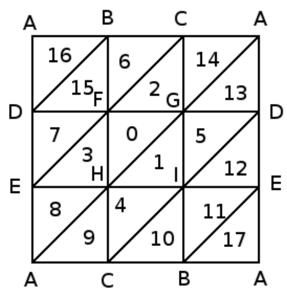
$$c = \sum_{\sigma \in K, |\sigma|=3} n_\sigma \sigma.$$

Now we prove that all the n_σ are equal.

Let $\sigma_0 = K_{FGH}$. Then none of the lines composing the boundary of σ_0 are in L, so if we consider all the simplicies surrounding σ_0, which we call σ_1,σ_2,σ_3. If we label the simplex σ_i with the number i, this is



We see that we must have $n_{\sigma_0} = n_{\sigma_1} = n_{\sigma_2} = n_{\sigma_3}$, since the edges FG, FH and HG are not in L. We now expand outward, labelling the simplices σ_i, for $i > 3$, as follows,



See that since the interior edges must cancel out, we must have that all the n_σ are equal. Hence,

$$c = n \sum_{\sigma \in K, |\sigma|=3} \sigma$$

For some $n \in \mathbb{Z}$.

Now we compute $\partial_2(c)$.

$$\partial_2(c) = n \sum_{\sigma \in K, |\sigma|=3} \partial_2(\sigma) = 2nK_{DE}.$$

Hence $\partial_2(c) = 0$ if and only if $n = 0$, so $c = 0$. Hence $\ker(\partial_2) = 0$, so $H_2(K) = 0$.

□

Question 4

Proposition 5. *Suppose that $f_{1\bullet}, f_{2\bullet} : C_\bullet \to C'_\bullet$ be chain maps, and suppose $s_\bullet$ is a chain homotopy from $f_{1\bullet}$ to $f_{2\bullet}$.*

Suppose $g_\bullet : C'_\bullet \to C''_\bullet$ is another chain map.

Then there is a homotopy from $g_\bullet f_{1\bullet}$ to $g_\bullet f_{1\bullet}$.

Proof. Let $p \in \mathbb{Z}$. Then we have,

$$f_{1p} - f_{2p} = \partial'_{p+1} s_p + s_{p-1}\partial_p.$$

Hence, since g_p is a homomorphism of abelian groups,

$$g_p f_{1p} - g_p f_{2p} = g_p \partial'_{p+1} s_p + g_p s_{p-1}\partial_p.$$

However by assumption, $g_\bullet$ is a chain map. So $g_p\partial'_{p+1} = \partial''_{p+1} g_{p+1}$.

Define $t_p := g_p s_{p-1}$. Then we have

$$g_p f_{1p} - g_p f_{2p} = \partial''_{p+1} t_{p+1} + t_p \partial_p.$$

Hence $t_\bullet$ is a homotopy from $g_\bullet f_{1\bullet}$ to $g_\bullet f_{2\bullet}$. ☐

Question 5

The following is a useful lemma from general topology. I am including it here since I cannot find a good reference for it.

Lemma 1. *Let X and Y be compact Hausdorff spaces. Suppose that $\varphi : X \to Y$ is continuous. The quotient space X/φ is the space of equivalence classes where x is equivalent to y when $\varphi(x) = \varphi(y)$ given the quotient topology. The image space $\operatorname{im}\varphi$ is given the subspace topology from Y. Then we have a homeomorphism,*

$$X/\varphi \cong \operatorname{im}\varphi.$$

Proof. Define $\psi : X/\varphi \to \operatorname{im}\varphi$ as follows.

Let $[x] \in X/\varphi$ be the equivalence class containing x. Define $\psi([x]) = \varphi(x)$. This is well defined, since if $\varphi(x) = \varphi(y)$, then $\psi([x]) = \psi([y])$.

ψ is surjective since if $\varphi(x) \in \operatorname{im}\varphi$, then $\varphi(x) = \psi([x])$.

Let $\pi : X \to X/\varphi$ denote the quotient map.

Now we must show that ψ is continuous. Let $U \subseteq \operatorname{im}\varphi$ be open. Then $U = \operatorname{im}\varphi \cap V$ for some open $V \subseteq \varphi$. Then $\varphi^{-1}(V)$ is open in X. Hence, since $\varphi = \psi \circ \pi$, we have that $\varphi^{-1}(V) = \pi^{-1} \circ \psi^{-1}(V)$. Since X/φ is given the topology, we have $W \subseteq X/\varphi$ is open if and only if $\pi^{-1}(W)$ is open in X. Hence, $\psi^{-1}(V) = \psi^{-1}(U)$ is open in X/φ.

Hence ψ is continuous. We also have that ψ is injective, since if $\psi([x]) = \psi([y])$, then $\varphi(x) = \varphi(y)$, so $[x] = [y]$.

Let $K \subseteq X/\varphi$ be closed. Hence it is compact, hence $\psi(K)$ is compact, hence closed since Y is Hausdorff.

Thus ψ is a closed continuous bijection, hence a homeomorphism. ☐

Proposition 6. *Let K be a simplicial complex, and let X be a topological space such that $\theta : |K| \to X$ is a triangulation. Define*

$$Y := (X \times [0,1])/\sim$$

where $\sim$ is the equivalence relation generated by $(x,1) \sim (x',1)$ for all $x, x' \in X$. Denote the equivalence class containing the point (x,t) by $[(x,t)]_\sim$. Then a triangulation for Y is given by the simplicial complex, $K' = K * w$, where $w \notin K^{(0)}$ is a new vertex, and $K * w$ denotes the cone on K.

Proof. Let $T : X \to |K|$ be the inverse of the triangulation, $T = \theta^{-1}$, and let n be the number of vertices of K.

Let $T' : X \times [0,1] \to |K| \times [0,1]$ be the product map such that $T'(x,t) = (T(x),t)$ for all $x \in X$ and $t \in [0,1]$.

Now we define a continuous surjection $\varphi : |K| \times [0,1] \to |K'|$.

Let $\iota : |K| \to |K'|$ be the embedding, so that $\iota(p) = (p,0)$ for $p \in |K|$. Now if $t \in [0,1]$ and $p \in |K|$, define $\varphi(p,t) = (p(1-t),t)$.

Hence $\varphi(p,t) \in |K'|$, and $\varphi(p,t) = \varphi(q,s)$ if and only if $s = t = 1$. Hence, $\varphi \circ T'$ is a continuous function from $X \times [0,1]$ to $|K'|$. Thus by lemma 1, we have a homeomorphism

$$(X \times [0,1]/\sim) \cong |K'|$$

where $(x,t) \sim (x',t')$ if and only if $t = t' = 1$.

$\square$