University of New South Wales

School of Mathematics and Statistics

Assignment 2

Algebraic Topology

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Question 1

Proposition 1. *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories, and $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{E}$ are functors. Then the composite $\mathcal{G}\mathcal{F} : \mathcal{C} \to \mathcal{E}$, defined by $(\mathcal{G}\mathcal{F})(x) = \mathcal{G}(\mathcal{F}(x))$ for $x \in \mathrm{Obj}(\mathcal{C})$ and $(\mathcal{G}\mathcal{F})(f) = \mathcal{G}(\mathcal{F}(f))$ for a morphism f, is a functor.*

Proof. It is necessary to prove,

1. If $x \in \mathrm{Obj}(\mathcal{C})$, then $(\mathcal{G}\mathcal{F})(\mathrm{id}_x) = \mathrm{id}_{\mathcal{G}\mathcal{F}(x)}$.
2. If f and g are morphisms in $\mathcal{C}$ such that gf is defined, then $(\mathcal{G}\mathcal{F})(gf) = (\mathcal{G}\mathcal{F})(g)(\mathcal{G}\mathcal{F})(f)$.

To prove 1, we simply compute,

$$\begin{aligned} (\mathcal{G}\mathcal{F})(\mathrm{id}_x) &= \mathcal{G}(\mathrm{id}_{\mathcal{F}(x)}) \\ &= \mathrm{id}_{\mathcal{G}\mathcal{F}(x)}. \end{aligned}$$

Similarly, we prove 2,

$$\begin{aligned} (\mathcal{G}\mathcal{F})(gf) &= \mathcal{G}(\mathcal{F}(g)\mathcal{F}(f)) \\ &= (\mathcal{G}\mathcal{F})(g)(\mathcal{G}\mathcal{F})(f). \end{aligned}$$

□

Question 2

Lemma 1. *Let $f : A \to B$ be a morphism in* **Ab** *that has a left inverse $g : B \to A$. Then f is injective and B is the internal direct sum $B = f(A) \oplus \ker g$.*

Proof. Let $x, y \in A$ with $f(x) = f(y)$. Then $x = g(f(x)) = g(f(y)) = y$, so f is injective.

Let $b \in B$. Then

$$b = f(g(b)) + (b - f(g(b)))$$

See that $g(b - f(g(b))) = g(b) - g(b) = 0$, so $b - f(g(b)) \in \ker(g)$, and $f(g(b)) \in f(A)$, so we have the sum $B = f(A) + \ker(g)$.

To show that this sum is direct, we need to prove that $f(A) \cap \ker(g) = \{0\}$.

To this end, $x \in f(A) \cap \ker(g)$. Then $x = f(y)$ for some $y \in A$. Then $0 = g(x) = g(f(y)) = y$. Hence $x = f(0) = 0$. □

Lemma 2. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a covariant functor. If $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is left invertible then so is $F(f)$.*

Proof. Let $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ be a left inverse for f. Then,

$$F(g)F(f) = F(gf) = F(\mathrm{id}_X) = \mathrm{id}_{F(X)}.$$

Hence $F(g)$ is a left inverse for $F(f)$. □

Recall that a retract of a topological space X is a subspace A such that the inclusion map $\iota : A \to X$ is left invertible.

Proposition 2. *There is no retract A of the Klein bottle K with $H_1(A) \cong \mathbb{Z}^2$.*

Proof. If there were a left invertible map $A \to K$, then there would be a left invertible map $\mathbb{Z}^2 \cong H_1(A) \to H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Hence there would be an injective map $\varphi : \mathbb{Z}^2 \to \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Hence there are elements $x, y \in \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ such that there are no non-zero integers a, b such that $xa + yb = 0$. However if $2x = (c, 0)$ and $2y = (d, 0)$ then

$2cy - 2dx = 0$.

Thus φ cannot exist, hence A cannot exist. □

Proposition 3. *There is no topological space X with $H_1(X) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/4\mathbb{Z}$ such that the Klein bottle K is a retract of X.*

Proof. If K is a retract of X. then there is an injection $H_1(K) \to H_1(X)$ such that the image of $H_1(K)$ is a direct summand of $H_1(X)$.

Hence $\mathbb{Z}^2 \oplus \mathbb{Z}/4\mathbb{Z}$ has a subgroup $A \cong \mathbb{Z} \oplus \mathbb{Z}/2$ and a subgroup B such that we have the (internal) direct sum $\mathbb{Z}^2 \oplus \mathbb{Z}/4\mathbb{Z} = A \oplus B$.

See that $\mathbb{Z}^2 \oplus \mathbb{Z}/4\mathbb{Z}$ has a unique element of order 2, $(0, 0, 2)$. Since A must have an element of order 2, we conclude that $(0, 0, 2) \in A$.

Note that A has no element of order 4, hence $(0, 0, 1) \notin A$. Hence there is some nonzero $b \in B$ such that $b - (0, 0, 1) \in A$.

Thus $2b - (0, 0, 2) \in A$, hence $2b \in A$. But $b \in B$, so $2b \in B \cap A$. Hence $b = 0$, so $(0, 0, 1) \in A$, which is a contradiction.

□

Question 3

Proposition 4. *Let $f : S^n \to S^n$ be a continuous function with non-zero degree. Then f is surjective.*

Proof. Let $f : S^n \to S^n$ be a continuous function that is not surjective. Let $y \in S^n$ be not in the image of f. Let $p : S^n \to \mathbb{R}^n$ be stereographic projection from the point y. Consider the functions $r : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ that contracts by t, $r(t,x) = (1-t)x$. r is a polynomial so r is continuous.

Then $H(t,x) = p^{-1}(r(t,p(f(x))))$ continuous, and $H(0,x) = p^{-1}(p(f(x)) = f(x)$, and $H(1,x) = p^{-1}(0)$. Thus H is a homotopy of f to a constant map.

Hence f is homotopic to a constant map, so $\deg(f) = 0$. □

Question 4

We consider S^{n-1} as a subset of S^n by embedding it into the equator.

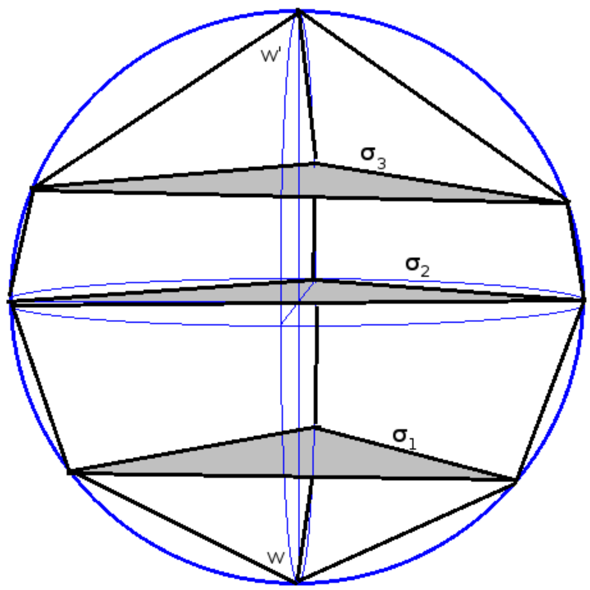
Proposition 5. *The relative homology $H_p(S^n, S^{n-1})$ is as follows:*

$$H_p(S^n, S^{n-1}) \cong \begin{cases} \mathbb{Z}^2, \ p = n \\ 0, \ \textit{otherwise.} \end{cases}$$

Proof. We shall compute the homology with a specific triangulation of S^n. Let $\sigma_1, \sigma_2, \sigma_3$ be three distinct n-simplices, and let w, w' be two vertices not in σ_i,$i = 1,2,3$.

We construct a triangulation K as follows:
Place σ_2 at the equator of S^n, and put σ_1 and σ_3 below and above σ_2 respectively.

Place the vertex w on the surface of the sphere, beneath σ_1, and w' is the antipodal point to w above σ_3. Now we connect $K_{\sigma_1}^{(n-1)} * w$ to σ_2 as illustrated below in the case for $n = 2$,



similarly $K^{(n-1)}_{\sigma_3} * w'$ is connected to σ_2.

Thus we have a triangulation K of S^n, and $K^{(n-1)}_{\sigma_2}$ corresponds to S^{n-1}.

Now let $p > 0$. We need to compute $C_p(K)/C_p(K^{(n-1)}_{\sigma_2})$.

Let L be the simplicial complex that is the image of K where all of $K^{n-1}_{\sigma_2}$ is identified to a point $x \in L$.

We claim that there is an isomorphism $C_p(L) \cong C_p(K)/C_p(K^{(n-1)}_{\sigma_2})$.

Let $\varphi : C_p(K) \to C_p(L)$ be the map that sends simplices in $K^{(n-1)}_{\sigma_2}$ to 0. Hence $\ker \varphi = C_p(K^{(n-1)}_{\sigma_2})$, so we have the required isomorphism.

It is then easy to see that the boundary map ∂_p is the same on $C_p(L)$ as it is on $C_p(K)/C_p(K^{(n-1)}_{\sigma_2}$. Hence the relative homology $H_p(S^n, S^{n-1})$ is the same as the homology $H_p(L)$ for $p > 0$.

Now L is a union of two n-spheres, so we have $H_p(L) = 0$ for $0 < p < n$ and $H_n(L) = \mathbb{Z}^2$.

For the case $p = 0$, we consider $C_0(K)/C_0(K^{(n-1)}_{\sigma_2})$. Now $Z_0(K, K^{(n-1)}_{\sigma_2}) = C_0(K)/C_0(K^{(n-1)}_{\sigma_2})$.

However, for every vertex $x \in K \setminus K^{(n-1)}_{\sigma_2}$, there is a one chain joining a vertex in $K^{(n-1)}_{\sigma_2}$ to x. Hence every point $x \in K \setminus K^{(n-1)}_{\sigma_2}$ is the 0-boundary of a 1-chain, modulo points of $K^{(n-1)}_{\sigma_2}$. Hence $H_0(S^n, S^{n-1}) \cong 0$. □

Question 5

Let $X = \mathbb{R}^2 \setminus \{p, q\}$ for distinct points $p, q \in \mathbb{R}^2$. Let A be the union of the circles centred at p and q with radius half the distance between p and q.

Proposition 6. *A is a weak deformation retract of X.*

Proof. By a change of coordinates, we can set $p = (1, 0)$ and $q = (-1, 0)$. Let C_p be the disc centred at p with radius 1, and C_q is similarly the disc centred at q with radius 1.

First we note that the relation of being a weak deformation retract is transitive, in the sense that if $A \subseteq B \subseteq X$ is a nested triple of topological spaces, and B is a weak deformation retract of X, and A is a weak deformation retract of B, then A is a weak deformation retract of X.

Let D be the closed disc centred at $(0, 0)$ of radius 2.

It is easy to see that $D \setminus \{p, q\}$ is a weak deformation retract of X, simply project the exterior of D onto D.

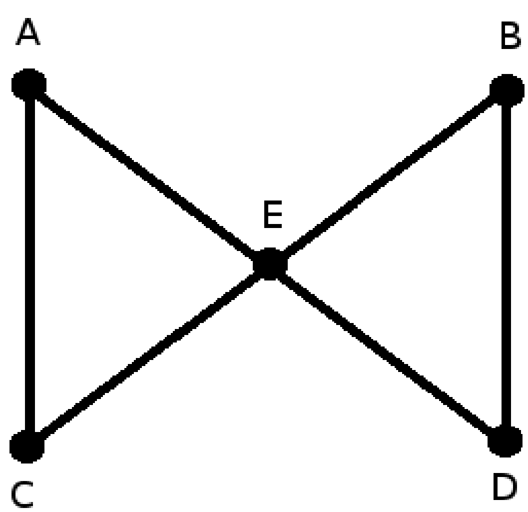
Now we show that A is a weak deformation retract of D.

We now define a function $H : [0, 1] \times D \to D$. Let H move points outside of $C_p \cup C_q$ onto the boundary of $C_p \cup C_q$ by moving them towards the x-axis.

For a point $a \in C_p \setminus \{p\}$, let H radially move a towards the boundary of C_p. Similarly H moves points in C_q towards the boundary of C_q.

$\square$

Proposition 7. *We can triangulate A as follows:*



Let K be the associated simplicial complex.

Proof. Simply perform a projection from $|K|$ to A. □

Proposition 8. *The homology of A is*

$$H_p(A) \cong \begin{cases} \mathbb{Z}^2, \ p = 1 \\ \mathbb{Z}, \ p = 0 \\ 0 \ \textit{otherwise.} \end{cases}$$

Proof. By the main theorem of the course, $H_p(A) = H_p(K)$. Since K has no p-simplices for $p \neq 0, 1$, the only potentially non-trivial homology groups are $H_1(K)$ and $H_0(K)$.

Since K is connected, we have $H_0(K) \cong \mathbb{Z}$.

So we now need only compute $H_1(K)$. Since $C_2(K) = 0$, this is exactly $Z_1(K)$. Let $a \in Z_1(K)$. Write,

$$a = \sum_{\sigma} a_\sigma \sigma$$

where the sum runs over all 1-simplices in K. Since $\partial a = 0$, we have that $a_{AC} = a_{CE} = a_{AE}$ and $a_{EB} = a_{BD} = a_{DE}$. Hence $Z_1(K)$ is generated by the 1-chains $[AC] + [CE] + [AE]$ and $[EB] + [BD] + [DE]$. No multiple of these chains can be zero, so we have two order zero generators of $Z_1(K)$. Since they are independent over $\mathbb{Z}$, we have $H_1(K) \cong \mathbb{Z}^2$. □

Proposition 9. $\mathbb{R}^2 \setminus \{p, q\}$ *is not homeomorphic to* $\mathbb{R}^2 \setminus \{p\}$.

Proof. It was proved in class that $\mathbb{R}^2 \setminus \{p\}$ is a weak deformation retract of S^1. Hence $H_1(\mathbb{R}^2 \setminus \{p\}) = \mathbb{Z}$. Thus $\mathbb{R}^2 \setminus \{p\}$ and $\mathbb{R}^2 \setminus \{p, q\}$ have different homology. □