School of Mathematics and Statistics University of New South Wales

MATH5665: Algebraic Topology (2015,S1)
Assignment 2 [1]

This assignment is due during the week 10 wednesday lecture.

1. Prove that the composite of functors is a functor.

2. We say that a morphism $f : X \longrightarrow Y$ in a category $\mathcal{C}$ is *left invertible* if there is a morphism $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ with $gf = \mathrm{id}_X$. If an inclusion of topological spaces $\iota : A \longrightarrow X$ is left invertible (in $\underline{\mathrm{Top}}$) then we say A is a *retract* of X.

   (a) Show that if $f : A \longrightarrow B$ is a left invertible morphism in $\underline{\mathrm{Ab}}$, then i) f is injective and ii) B is the (internal) direct sum of $f(A)$ and $\ker g$ where g is any left inverse to f.

   (b) Prove that if $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a covariant functor and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is left invertible, then so is $F(f)$.

   (c) Is there a retract A of the Klein bottle K with $H_1(A) \simeq \mathbb{Z}^2$? (You may assume your calculations in assignment 1 viz. $H_1(K) \simeq \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, H_2(K) = 0$.) Justify your answer fully!

   (d) Can K be the retract of a topological space X with $H_1(X) \simeq \mathbb{Z}^2 \oplus \mathbb{Z}/4\mathbb{Z}$? Justify your answer fully!

3. Let $f : S^n \longrightarrow S^n$ be a continuous map with non-zero degree. Prove that f is surjective.

4. We view S^{n-1} as the equator of S^n. Compute the relative homology of (S^n, S^{n-1}). (You will probably need to prove that $H_p(S^n, S^{n-1})$ is finitely generated in the process).

5. Let $p, q \in \mathbb{R}^2$ be distinct points and $X = \mathbb{R}^2 - \{p, q\}$. Let $A \subset \mathbb{R}^2$ be the union of two circles centred at p, q respectively and both with radii half the distance between p and q (so they intersect tangentially).

   (a) Show that A is a weak deformation retract of X.

   (b) Find a triangulation of A.

   (c) Using the main theorem of the course (lecture 12) or otherwise, compute the homology of X.

   (d) Prove that X is not homeomorphic to $\mathbb{R}^2 - \{p\}$.

[1] by Daniel Chan