

# MATH5665: Algebraic Topology- Course notes

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## Abstract

These are the lecture notes for an Honours course in algebraic topology. They are based on standard texts, primarily Munkres's "Elements of algebraic topology" and to a lesser extent, Spanier's "Algebraic topology".

## 1 What's algebraic topology about?

**Aim lecture:** We preview this course motivating it historically.

Recall that a continuous map  $f : X \rightarrow Y$  of topological spaces is a *homeomorphism* if it is bijective and  $f^{-1}$  is also continuous. In this case we say  $X$  and  $Y$  are *homeomorphic* and write  $X \simeq Y$ .

**Major Q** in topology is how to determine if two spaces  $X, Y$  are homeomorphic or not.

**A** depends on whether they are homeomorphic or not. If they are, one usually guesses the homeomorphism. If not, one needs to be a bit more sophisticated. One usually uses *invariants* to distinguish them. Historically, the first example is below.

### Euler Characteristic

**Notation 1.1** We denote the  $n$ -dim unit ball by  $B^n := \{\vec{v} \in \mathbb{R}^n \mid |\vec{v}| \leq 1\}$  and the  $n$ -dim unit sphere by  $S^n := \{\vec{v} \in \mathbb{R}^{n+1} \mid |\vec{v}| = 1\}$

Consider a regular polyhedron  $K \subset \mathbb{R}^3$  or more generally, any polyhedron homeomorphic to the unit sphere  $S^2$ . Let  $V$  = no. vertices,  $E$  = no. edges and  $F$  = no. faces.

**Theorem 1.2 (Euler)**  $V - E + F = 2$ .

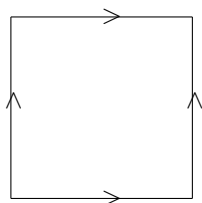
**E.g.** Check the cube, tetrahedron etc.

The quantity  $e(K) := V - E + F$  is called the *Euler number* of  $K$ .

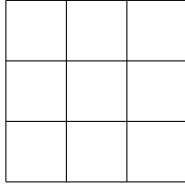
### Quotient spaces

Let  $X$  = topological space and  $\sim$  be an equivalence relation on the underlying set so there is a set map  $\pi : X \rightarrow X/\sim : x \mapsto [x]$  where  $X/\sim$  is the set of equivalence classes  $[x]$ . We put the strongest topology on  $X/\sim$  so that  $\pi$  is continuous, viz,  $U \subset X/\sim$  is open iff  $\pi^{-1}(U)$  is open. The resulting topological space is called the *quotient space*.

**E.g.** Let  $I = [0, 1] \subset \mathbb{R}$  and  $X = I \times I$ . We put the weakest equivalence relation on  $X$  s.t.  $(0, x) \sim (1, x)$ ,  $(x, 0) \sim (x, 1)$  for  $x \in I$ . We sometimes sum up this info in the following picture:



Then  $X/\sim$  is homeomorphic to the 2-torus  $\mathbb{T}^2 = S^1 \times S^1$ . Consider the following polyhedron  $K$  which is homeomorphic to  $\mathbb{T}^2$ :



Note  $e(K) = 9 - 18 + 9 = 0 \neq 2!$

**Upshot**  $\mathbb{T}^2$  is not homeomorphic to  $S^2$ .

### Topological invariants

The Euler number is an example of a *topological invariant*, that is, an object associated to a top space that doesn't change when you pass to an homeomorphic space. Such invariants are often algebraic in nature, and algebraic topology is about the use of such invariants to show spaces are non-homeomorphic and deduce other interesting topological facts.

In this case, we will closely examine this Euler number, refining it to certain invariants called *Betti numbers* and more conceptually, to *homology groups*.

### Phenomenon of independence of choice

The Euler number (as defined above) of a topological space requires choosing a polyhedron homeomorphic to it. There are many possibilities and the surprise is that it is independent of choice. This phenomenon is typical in the theory of homology and is something we will examine and study in depth.

### Proof of Euler's thm

Actually, Euler's thm is not so hard to prove. Use induction. Maybe do next time as plenty of time. ex.

## 2 Simplicies

**Aim Lecture** We introduce simplices which are the building blocks for many topological spaces.

### Barycentric co-ordinates

**Notation 2.1** Given a list  $a_0, \dots, a_n$ , we let  $a_0, \dots, \widehat{a_i}, \dots, a_n$  to denote the same list except with  $a_i$  omitted.

**Proposition-Definition 2.2** Let  $a_0, \dots, a_n \in \mathbb{R}^N$  and  $H \subset \mathbb{R}^{n+1}$  be the hyperplane  $H := \{\vec{t} = (t_0, \dots, t_n) : \sum t_i = 1\}$ . TFAE:

- i.  $a_0 - a_i, \dots, \widehat{a_i - a_i}, \dots, a_n - a_i$  is linearly independent.
- ii. The map  $\beta : H \rightarrow \mathbb{R}^N : \vec{t} \mapsto \sum_j t_j a_j$  is injective.

In this case, we say that  $a_0, \dots, a_n$  are geometrically independent and  $\vec{t}$  are the barycentric co-ordinates of  $\sum t_j a_j \in \text{im } \beta$ .

**Proof.** Note that  $\vec{s} \in H \implies s_i = 1 - \sum_{j \neq i} s_j$ . Hence  $\beta(\vec{s}) = \beta(\vec{t})$  iff

$$0 = \sum_{j \neq i} (s_j - t_j) a_j + (-\sum_{j \neq i} s_j + \sum_{j \neq i} t_j) a_i = \sum_{j \neq i} (s_j - t_j) (a_j - a_i)$$

□

**E.g.** 3 collinear points are geom dependent.

E.g. Points with barycentric co-ords  $(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ . Regions defined by  $t_2 = 0, < 0$ .

## Simplices

**Definition 2.3** Let  $a_0, \dots, a_n \in \mathbb{R}^N$ . The span of  $a_0, \dots, a_n \in \mathbb{R}^N$  is

$$\sigma = \langle a_0 \dots a_n \rangle := \{ \sum t_j a_j \mid \sum_j t_j = 1, t_j \geq 0 \}.$$

The span is thus the convex hull. If further  $a_0, \dots, a_n$  are geom independent, then we say  $\langle a_0 \dots a_n \rangle$  is a (geometric)  $n$ -simplex since then  $\dim \beta$  is  $n$ -dimensional.

A (resp proper) *face* of the simplex  $\sigma = \langle a_0 \dots a_n \rangle$  is any simplex spanned by a (resp proper) subset of  $\{a_0, \dots, a_n\}$ .

**E.g.**

The  $i$ -th face of  $\langle a_0 \dots a_n \rangle$  is  $\langle a_0 \dots \widehat{a_i} \dots a_n \rangle$ .

**Proposition-Definition 2.4** Let  $\langle a_0 \dots a_n \rangle$  be a simplex and  $b_0, \dots, b_n \in \mathbb{R}^M$ .

- i. We have a well-defined map  $f : \langle a_0 \dots a_n \rangle \longrightarrow \langle b_0 \dots b_n \rangle : \sum_j t_j a_j \mapsto \sum_j t_j b_j$ . We call it affine linear and denote it  $l(\langle a_0 \dots a_n \rangle, \langle b_0 \dots b_n \rangle)$ .
- ii. In particular, any two  $n$ -simplices are homeomorphic.

**Proof.** Note  $f$  is well-defined being the composite of  $\beta^{-1}$  with  $\vec{t} \mapsto \sum t_j b_j$ . Also (ii) holds since  $l(\langle a_0 \dots a_n \rangle, \langle b_0 \dots b_n \rangle), l(\langle b_0 \dots b_n \rangle, \langle a_0 \dots a_n \rangle)$  are inverse continuous maps. □

**Proposition 2.5** An  $n$ -simplex  $\sigma$  is homeomorphic to  $B^n$  and its boundary (- union of its proper faces) is homeomorphic to  $S^{n-1}$ .

**Proof.** This is an ex. Pictorially the argument is □

### Topology of polytopes

A *polytope*  $X$  is a finite union of simplices  $\sigma_1, \dots, \sigma_r \in \mathbb{R}^N$  such that  $\sigma_i \cap \sigma_j$  is a face of both  $\sigma_i, \sigma_j$  for all  $i, j$ . Note  $X$  is compact since each  $\sigma_i$  is.

**E.g.**

The next “gluing lemma” allows one to glue locally defined continuous maps to obtain a global one.

**Lemma 2.6** *Let top space  $X$  be a finite union  $X_1 \cup \dots \cup X_r$  of closed subsets  $X_i$ .*

- i.  $Y \subset X$  is closed (resp open) in  $X$  iff every  $Y \cap X_i$  is closed (resp open) in  $X_i$ .*
- ii. If  $f_i : X_i \rightarrow Z$  are continuous maps for all  $i$  such that*
  - (\*)  $f_i, f_j$  agree on  $X_i \cap X_j$  for all  $i, j$ .**Then there is a global continuous map  $f : X \rightarrow Z$  which restricts to each  $f_i$ .*

**Proof.** (i) If  $Y \cap X_i$  is closed in  $X_i$ , then it is closed in  $X$ . Hence  $Y = \cup_i (Y \cap X_i)$  is closed too.

(ii) Condition (\*) ensures that the  $f_i$  glue to a set map  $f : X \rightarrow Z$ , whilst (i) ensures that it's continuous. □

## 3 Simplicial complexes and triangulation

**Aim lecture** Introduce simplicial complexes to describe how to glue simplices together to form interesting topological spaces.

### Simplicial complex

Below, we let  $\mathcal{P}'(S)$  to denote the power set of  $S$  less the element  $\emptyset$ .

**Definition 3.1** *A simplicial complex  $K$  consists of a finite ordered set  $A$  of vertices and a subset  $K$  of  $\mathcal{P}'(A)$  such that*

*(\*) if  $\sigma \in K$  and  $\emptyset \neq \tau \subset \sigma$  then  $\tau \in K$ .*

*We write the elements  $\sigma$  in the form  $a_0 \dots a_d$  where  $a_0 < \dots < a_d$  and call such an element an (abstract)  $d$ -simplex of  $K$ . A face of  $\sigma$  is any non-empty subset of  $\sigma$ .*

*A subcomplex  $K'$  of  $K$  is any subset satisfying (\*).*

e.g.

**Proposition-Definition 3.2** *i. For  $p \in \mathbb{N}$ , the  $p$ -skeleton of  $K$  is the subset  $K^{(p)}$  consisting of all  $d$ -simplices with  $d \leq p$ . It is a subcomplex of  $K$ .*

*ii. For  $\sigma \in K$ , the subset  $K_\sigma := \mathcal{P}'(\sigma)$  is a subcomplex of  $K$  called the complex associated to  $\sigma$ .*

**Proof.** Easy. □

**N.B.**  $K(0)$  is just the set of vertices of  $K$ .

### Polytope of a simplicial complex

Let  $K$  be a complex with  $K^{(0)} = A$ . Let  $\mathbb{R}^A$  = set of functions  $A \rightarrow \mathbb{R}$  which is naturally a top space on identifying  $\mathbb{R}^A$  with  $\mathbb{R}^{|A|}$ . Let  $\varepsilon_a \in \mathbb{R}^A$  be the standard basis vector with  $\varepsilon_a(a) = 1, \varepsilon_a(a') = 0$  if  $a' \neq a$ . Note that the  $\varepsilon_a$  are geometrically independent.

The following shows how a simplicial complex determines the combinatorial data to glue simplices together and form a topological space.

**Proposition-Definition 3.3** *The polytope  $|K|$  of  $K$  is the union in  $\mathbb{R}^A$  of the following simplices: for each abstract simplex  $\sigma = a_0 \dots a_d \in K$ , the geometric simplex  $\langle \sigma \rangle := \langle \varepsilon_{a_0} \dots \varepsilon_{a_d} \rangle$ .*

*i.  $|K|$  is a polytope.*

*ii.  $\tau \subseteq \sigma \implies \langle \tau \rangle$  is a face of  $\langle \sigma \rangle$  so  $|K_\sigma| = \langle \sigma \rangle$ .*

*iii. In fact, if we write  $K = K_{\sigma_1} \cup \dots \cup K_{\sigma_r}$  for some  $\sigma_i \in K$ , then  $|K| = |K_{\sigma_1}| \cup \dots \cup |K_{\sigma_r}|$  and  $|K_{\sigma_i}| \cap |K_{\sigma_j}| = |K_{\sigma_i \cap \sigma_j}|$  (if we define  $|K_\emptyset| = \emptyset$ ).*

**Proof.** (ii) is clear whilst (iii)  $\implies$  (i). Finally, (iii) is easy (look at which barycentric co-ords are 0) and clear from any example. □

**E.g.**  $K = K_{abc} \cup K_{bcd} = \{a, b, c, ab, bc, ac, abc, d, bd, cd, bcd\}$ .

**Example 3.4** *Sphere.*

Let  $K = K_\sigma$  where  $\sigma$  is an  $n$ -simplex so  $|K_\sigma| \simeq B^n$ . Then  $|K_\sigma^{(n-1)}|$  = boundary of  $\langle \sigma \rangle$  so  $|K_\sigma^{(n-1)}| \simeq S^{n-1}$ .

**Example 3.5**  $K = K_{abc}$ .

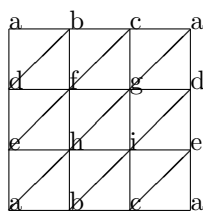
As we've seen,  $|K_{abc}^{(2)}|$  is homeomorphic to the 2-sphere. The 1-skeleton  $|K_{abc}^{(1)}|$

## Triangulation

**Definition 3.6** A triangulation of a top space  $X$  is a simplicial complex  $K$  and homeomorphism  $f : |K| \rightarrow X$ .

Below, we use *labelled surface diagrams* to describe triangulations of well-known surfaces. Here  $K$  consists of 3-simplices given by the triangles and all other simplices are their subsets.

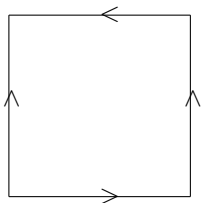
**Example 3.7** 2-torus  $\mathbb{T}^2$ .



The gluing lemma 2.6 shows there is indeed a triangulation of  $\mathbb{T}^2$ . Indeed, we identify  $\mathbb{T}^2$  with  $I \times I / \sim$  as in lecture 1 so the geometric simplices like  $\langle abd \rangle$  above can be thought as subsets of  $I \times I$  and hence in fact also of  $\mathbb{T}^2$ . Then the affine linear maps such as  $l(\langle \varepsilon_a \varepsilon_b \varepsilon_d \rangle, \langle abd \rangle)$  glue together to an homeomorphism  $|K| \rightarrow \mathbb{T}^2$ .

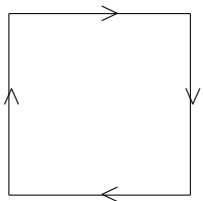
Note you can't omit a row!!

**Example 3.8** Klein bottle



**Example 3.9** (real) Projective plane  $\mathbb{R}P^2$ .

Be careful!



## 4 Homology groups

**Aim lecture** Intro homology groups which we'll see later give topological invariants.  
 $K$  = simplicial complex.

**$p$ -chains**

**Definition 4.1** The group of  $p$ -chains  $C_p(K)$  of  $K$ , is the free abelian group on the set of  $p$ -simplices in  $K$ , i.e.

$$C_p(K) = \left\{ \sum_{\sigma} n_{\sigma} \sigma \mid n_{\sigma} \in \mathbb{Z}, \sigma \in K \text{ is a } p\text{-simplex} \right\}$$

$$=: \bigoplus_{\sigma \in K \text{ is a } p\text{-simplex}} \mathbb{Z} \sigma$$

Its elements are called  $p$ -chains. By default  $C_p(K) = 0$  if  $p < 0$  or there are no  $p$ -chains.

Given a  $p$ -simplex  $a_0 \dots a_p$  and a permutation  $\pi \in \text{Perm}\{0, \dots, p\}$ , we let

$$[a_{\pi(0)} \dots a_{\pi(p)}] = \text{sgn}(\pi) a_0 \dots a_p.$$

These are called *oriented  $p$ -simplices*. The idea is that the ordering on  $K^{(0)}$  gives an *orientation* on all the simplices.

e.g. If  $p = 1$  then  $[ab] \neq [ba]$  can be represented by

$$a \longrightarrow b \text{ and } a \longleftarrow b$$

.

Similarly, for 2-simplices

### Boundary operator

We define a group hom  $\partial_p : C_p(K) \longrightarrow C_{p-1}(K)$  by

$$\partial_p(a_0 \dots a_p) = \sum_{i=0}^p (-1)^i a_0 \dots \hat{a}_i \dots a_p$$

and extending linearly. We call it the  $p$ -th *boundary operator*.

### Proposition 4.2

$$(*) \quad \partial_p[a_0 \dots a_p] = \sum_{i=0}^p (-1)^i [a_0 \dots \hat{a}_i \dots a_p].$$

**Proof.** We can use a permutation  $\pi$  to put the  $a_i$  in order. Now  $\pi$  is a product of, say  $\ell$  transpositions of form  $(jj+1)$  so it suffices by induction on  $\ell$  to show

$$\partial_p[a_0 \dots a_j a_{j+1} \dots a_p] = -\partial_p[a_0 \dots a_{j+1} a_j \dots a_p]$$

if we compute both sides using (\*). We leave this as an exercise. □

The choice of orientation in  $\partial_p$  is best seen in an

**e.g.**  $\partial_2(abc) = bc - ac + ab = [bc] + [ca] + [ab]$ . Pictorially,

$$\text{Note } \partial_1 \partial_2(abc) = (b - c) + (c - a) + (a - b) = 0!!$$

## Homology groups

**Lemma 4.3**  $\partial_{p-1} \partial_p = 0$ .

**Proof.**

$$\begin{aligned} \partial_{p-1} \partial_p(a_0 \dots a_p) &= \partial_{p-1} \sum_i (-1)^i a_0 \dots \hat{a}_i \dots a_p \\ &= \sum_{j < i} (-1)^{i+j} a_0 \dots \hat{a}_j \dots \hat{a}_i \dots a_p + \sum_{j > i} (-1)^{i+j-1} a_0 \dots \hat{a}_i \dots \hat{a}_j \dots a_p \end{aligned}$$

Note that the  $(i, j)$  term cancels with the  $(j, i)$ -term. □

We define *p-cycles* and *p-boundaries* to be resp, elements of the groups

$$Z_p(K) = \ker(\partial_p : C_p(K) \longrightarrow C_{p-1}(K)) \leq C_p(K), \quad B_p(K) = \text{im}(\partial_{p+1} : C_{p+1}(K) \longrightarrow C_p(K)) \leq C_p(K).$$

Lemma  $\implies \partial_p(B_p) = 0 \implies B_p(K) \leq Z_p(K)$ . We define the *p-th homology group* to be

$$H_p(K) := Z_p(K)/B_p(K).$$

## 5 Examples

**Aim lecture** We compute some examples illustrating how homology detects holes.

### Circle

**Example 5.1**  $K = K_{abc}^{(1)}$



The  $C^p(K)$  and boundary maps are:

$$0 = C_2(K) \xrightarrow{\partial_2} C_1(K) = \mathbb{Z}ab \oplus \mathbb{Z}bc \oplus \mathbb{Z}ac \xrightarrow{\partial_1} C_0(K) = \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c \xrightarrow{\partial_0} C_{-1}(K) = 0$$

where

$$\partial_1 : [ab] = ab \mapsto b - a, \quad [bc] \mapsto c - b, \quad [ca] = -ac \mapsto a - c$$

Note if  $p \neq 0$  or  $1$ , then  $Z_p(K) \leq C_p(K) = 0$  so  $H_p(K) = 0$ .

$p = 0$   $Z_0(K) = C_0(K) = \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c \simeq \mathbb{Z}^3$ ,  $B_0(K) = \mathbb{Z}(b - a) + \mathbb{Z}(c - b) + \mathbb{Z}(a - c)$ . Note  $\overline{Z_0(K)} = \mathbb{Z}a + B_0(K)$  while (ex)  $\mathbb{Z}a \cap B_0(K) = 0$  so isomorphism thm  $\implies$

$$H_0(K) = Z_0(K)/B_0(K) \simeq \mathbb{Z}a/(\mathbb{Z}a \cap B_0(K)) = \mathbb{Z}a/0 \simeq \mathbb{Z}.$$

$p = 1$   $B_1(K) = 0$ . Let  $\gamma = \gamma_1[ab] + \gamma_2[bc] + \gamma_3[ca]$ . Then  $\gamma \in Z_1(K)$  iff

$$0 = \partial(\gamma) = (\gamma_3 - \gamma_1)a + (\gamma_1 - \gamma_2)b + (\gamma_2 - \gamma_3)c$$

iff  $\gamma_1 = \gamma_2 = \gamma_3$ . Hence  $Z_1(K) = \mathbb{Z}\gamma$  where  $\gamma = [ab] + [bc] + [ca]$ . We think of  $\gamma$  as surrounding the “hole” of the circle which lives in the homology group  $H_1(K)$ .

The argument above suggests the following

**Proposition 5.2** For a simplicial complex  $K$ ,  $H_0(K) \simeq \mathbb{Z}^n$  where  $n$  is the no. connected components of  $|K|$ .

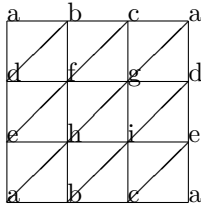
**Proof.** See problem sheets. □

**Definition 5.3** Let  $K =$  simplicial complex. We say  $\gamma, \gamma' \in Z_p(K)$  are homologous if  $\gamma - \gamma' \in B_p(K)$ , i.e.  $\gamma \equiv \gamma' \pmod{B_p(K)}$  so define the same element in  $H_p(K)$ .

Let  $L =$  subcomplex of  $K$  so  $C_p(L) \leq C_p(K)$ . We say  $\gamma \in C_p(K)$  is carried by  $L$  if  $\gamma \in C_p(L)$ .

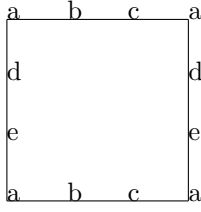
## 2-torus

**Example 5.4** Let  $T$  be the triangulation of the torus below



$p = 0$   $H_0(T) \simeq \mathbb{Z}$ .

Let  $L$  be the subcomplex  $K_{abc} \cup K_{ade}$  so  $|L|$  is a “wedge” of 2 circles.



$p = 2$

Let  $\{\sigma_r\}$  be the set of all 2-simplices in  $T$  oriented anti-clockwise and  $\Sigma$  be their sum.

**Proposition 5.5** *i. If  $\gamma = \sum_r n_r \sigma_r \in C_2(T)$  is s.t.  $\partial(\gamma)$  is carried by  $L$ , then all  $n_r$  are equal so  $\gamma = n\Sigma$  for some  $n \in \mathbb{Z}$ .*

*ii.  $Z_2(T) = \mathbb{Z}\Sigma \simeq \mathbb{Z}$ .*

*iii.  $H_2(T) = Z_2(T)/B_2(T) = \mathbb{Z}\Sigma/0 \simeq \mathbb{Z}$ .*

**Proof.** (i) If  $\sigma_r, \sigma_s$  are neighbouring simplices with common internal edge  $\varepsilon$ , then the co-efficient of  $\varepsilon$  in  $\partial(\gamma)$  is  $\pm(n_r - n_s) = 0$ . Hence all  $n_r$  are equal.

(ii) Note  $\partial(\Sigma) = 0$  so  $\mathbb{Z}\Sigma \subseteq Z_2(T)$ . Also (i)  $\implies Z_2(T) \subseteq \mathbb{Z}\Sigma$ .

(iii) Follows as  $B_2(T) = 0$ . □

$p = 1$

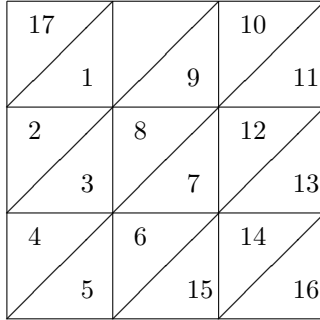
Let  $\alpha = [ab] + [bc] + [ca]$ ,  $\beta = [ad] + [de] + [ea]$ .

**Proposition 5.6** *i.  $Z_1(L) = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$ .*

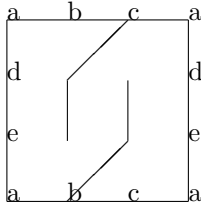
*ii. Any  $\gamma \in Z_1(T)$  is homologous to one carried by  $L$ , i.e.  $Z_1(T) = Z_1(L) + B_1(T)$ .*

*iii. By 5.5(i),  $Z_1(L) \cap B_1(T) = 0$  so  $H_1(T) = (Z_1(L) + B_1(T))/B_1(T) \simeq Z_1(L)/(Z_1(L) \cap B_1(T)) \simeq \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$ .*

**Proof.** (i) follows as in e.g. 5.1 so suffice prove (ii). Adding appropriate multiple of  $[bdf]$  to  $\gamma$ , we may replace  $\gamma$  with a homologous cycle with co-efficient of  $[bdf] = 0$ . Continue inductively in order below



Can assume the  $\gamma$  is homologous to one carried by



But  $\gamma$  a cycle  $\implies$  its carried by  $L$ .

□

## 6 Chain complexes

**Aim lecture** Intro general algebraic framework for homology.

### Chain complexes

**Definition 6.1** A (chain) complex of abelian groups (resp vector spaces), is a sequence

$$C_\bullet : \dots \longrightarrow C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \longrightarrow \dots$$

of abelian groups  $C_p$  (resp vector spaces) and group homomorphisms  $\partial_p$  (resp linear maps) such that  $\partial^2 := \partial_{p-1}\partial_p = 0$ .

**Example 6.2** If  $K$  = simplicial complex then we have the chain complex

$$C_\bullet(K) : \dots \longrightarrow C_{p+1}(K) \xrightarrow{\partial_{p+1}} C_p(K) \xrightarrow{\partial_p} C_{p-1}(K) \longrightarrow \dots$$

where the  $\partial$  are boundary operators.

**Proposition-Definition 6.3** Let  $C_\bullet$  be a chain complex. The group of  $p$ -cycles is  $Z_p(C_\bullet) = Z_p = \ker \partial_p$  and the group of  $p$ -boundaries is  $B_p(C_\bullet) = B_p = \text{im } \partial_{p+1}$ . Then  $B_p \leq Z_p$  so we may define the  $p$ -th homology group to be

$$H_p(C_\bullet) = Z_p/B_p.$$

**E.g.** 6.2 again.  $H_p(C_\bullet(K)) = H_p(K)$ .

### Reduced homology

**Example 6.4** The augmented chain complex of a simplicial complex  $K$ .

Let

$$\tilde{C}_p(K) = \begin{cases} C_p(K) & \text{if } p \neq -1, \\ \mathbb{Z} & \text{if } p = -1 \end{cases}$$

The boundary maps  $\partial$  will be the same as for  $C_\bullet(K)$  except now

$$\partial_0 : \tilde{C}_0(K) \longrightarrow \tilde{C}_{-1}(K) = \mathbb{Z} : a \mapsto 1$$

for any vertex  $a \in K^{(0)}$ .

Note  $\partial_{p-1}\partial_p = 0$  holds for  $p \neq 1$  since it holds in  $C_\bullet(K)$ , and it holds for  $p = 1$  since  $\partial^2(ab) = \partial(b - a) = 1 - 1 = 0$ .

**Definition 6.5** We define the  $p$ -th reduced homology of  $K$  to be  $\tilde{H}_p(K) = H_p(\tilde{C}_\bullet(K))$ .

**Ex**  $\tilde{H}_p(K) = H_p(K)$  if  $p \neq 0$  and  $\tilde{H}_0(K) = \mathbb{Z}^{n-1}$  where  $|K|$  has  $n$  connected components.

### Chain maps

**Definition 6.6** A subchain complex of a chain complex  $C_\bullet$ , is a collection of subgroups  $D_p \leq C_p$  such that  $\partial_p(D_p) \subseteq D_{p-1}$ . In this case,  $D_\bullet$  is a chain complex too. We write  $D_\bullet \leq C_\bullet$ .

**E.g.** If  $L$  is a subcomplex of  $K$  then  $C_\bullet(L)$  is a chain subcomplex of  $C_\bullet(K)$ .

Let  $C_\bullet, C'_\bullet$  be chain complexes. A chain map  $f_\bullet : C_\bullet \rightarrow C'_\bullet$  or morphism of chain complexes is

$$\begin{array}{ccccccc} \dots & \longrightarrow & C_{p+1} & \xrightarrow{\partial_{p+1}} & C_p & \xrightarrow{\partial_p} & C_{p-1} \longrightarrow \dots \\ & & \downarrow f_{p+1} & & \downarrow f_p & & \downarrow f_{p-1} \\ \dots & \longrightarrow & C'_{p+1} & \xrightarrow{\partial'_{p+1}} & C'_p & \xrightarrow{\partial'_p} & C'_{p-1} \longrightarrow \dots \end{array}$$

a sequence of group homs  $f_p, p \in \mathbb{Z}$  s.t. the diagram commutes in the sense that  $\partial'_p \circ f_p = f_{p-1} \circ \partial_p$  for all  $p$ .

**E.g.** If  $L$  is a subcomplex of  $K$ , then the inclusion maps  $C_p(L) \hookrightarrow C_p(K)$  define a chain map  $\iota : C_\bullet(L) \rightarrow C_\bullet(K)$ .

**Proposition 6.7** The identity  $id_\bullet : C_\bullet \rightarrow C_\bullet$  is a chain map. Given chain maps  $f_\bullet : C_\bullet \rightarrow D_\bullet, g_\bullet : D_\bullet \rightarrow E_\bullet$ , the composite  $h_\bullet := g_\bullet f_\bullet : C_\bullet \rightarrow E_\bullet$  defined by  $h_p = g_p f_p$  is a chain map.

**Proof.** Good easy ex. □

**N.B.** All the above definitions and results hold if the chain complexes are of vector spaces and we replace group homs with linear maps.

### Functoriality of $H_p$

**Proposition-Definition 6.8** Let  $f_\bullet : C_\bullet \rightarrow D_\bullet$  be a chain map.

i.  $f_p(Z_p(C_\bullet)) \subseteq Z_p(D_\bullet), \quad f_p(B_p(C_\bullet)) \subseteq B_p(D_\bullet).$

ii. Hence  $f_\bullet$  induces a well-defined group hom

$$f_* := H_p(f_\bullet) : H_p(C_\bullet) \rightarrow H_p(D_\bullet) : \gamma + B_p(C_\bullet) \mapsto f_p(\gamma) + B_p(D_\bullet).$$

**Proof.** (i) For  $\gamma \in Z_p(C_\bullet)$ ,

$$\partial f(\gamma) = f \partial(\gamma) = f(0) = 0 \implies f(\gamma) \in Z_p(D_\bullet).$$

and

$$f(B_p(C_\bullet)) = f \partial(C_{p+1}) = \partial f(C_{p+1}) \subseteq B_p(D_\bullet).$$

Finally, (i) gives (ii). □

**Proposition 6.9** i.  $H_p(id_\bullet) = id_{H_p}$ .

ii. Given chain maps  $f_\bullet : C_\bullet \rightarrow D_\bullet, g_\bullet : D_\bullet \rightarrow E_\bullet$ , the following diagram commutes

$$\begin{array}{ccc} H_p(C_\bullet) & \xrightarrow{(gf)_*} & H_p(E_\bullet) \\ \downarrow f_* & \nearrow g_* & \\ H_p(D_\bullet) & & \end{array}$$

i.e.  $(gf)_* = g_* f_*$ .

## 7 Homotopy

**Aim lecture** We recall the notion of homotopy = deformations of continuous maps from MATH3701. Most notions in algebraic topology are invariant under homotopy.

### Homotopy of pairs

A *topological pair*  $(X, A)$  consists of a topological space  $X$  and a subspace  $A$ . We will identify  $(X, \emptyset)$  with  $X$ . We formulate our definitions and results for pairs, and setting  $A = \emptyset$  gives the definitions for top spaces.

A *continuous map of pairs*  $f : (X, A) \rightarrow (X', A')$  is a continuous map  $f : X \rightarrow X'$  such that  $f(A) \subseteq A'$ . Let  $I = [0, 1]$ . Given two continuous maps  $f_0, f_1 : (X, A) \rightarrow (X', A')$ , a *homotopy*  $F : f_0 \approx f_1$  is a continuous map  $F : (X \times I, A \times I) \rightarrow (X', A')$  such that  $f_0(x) = F(x, 0)$ ,  $f_1(x) = F(x, 1)$  for all  $x \in X$ . We say  $f_0, f_1$  are *homotopic* in this case.

If we further write  $f_t = F(-, t) : X \rightarrow X'$ , then we can visualise the family  $\{f_t | t \in [0, 1]\}$  as a continuous deformation from  $f_0$  to  $f_1$

DRAW PICTURE

**Proposition 7.1** *The homotopic relation  $\approx$  is an equivalence relation on the set of continuous maps  $(X, A) \rightarrow (X', A')$ .*

**Proof.** Ex. Pictorially obvious. □

**Proposition 7.2** *Let  $f_0, f_1 : (X, A) \rightarrow (X', A')$ ,  $g_0, g_1 : (X', A') \rightarrow (X'', A'')$  be continuous. If  $F : f_0 \approx f_1$ ,  $G : g_0 \approx g_1$  are homotopies, then  $g_0 f_0 \approx g_1 f_1$  i.e. composites of homotopic maps are homotopic.*

**Proof.** We have homotopies

$$G \circ f_0 \times \text{id}_I : g_0 f_0 \approx g_1 f_0 : (X \times I, A \times I) \xrightarrow{f_0 \times \text{id}_I} (X' \times I, A' \times I) \xrightarrow{G} (X'', A'').$$

$$g_1 \circ F : g_1 f_0 \approx g_1 f_1 : (X \times I, A \times I) \xrightarrow{F} (X', A') \xrightarrow{g_1} (X'', A'').$$

Transitivity  $\implies g_0 f_0 \approx g_1 f_1$ . □

### Homotopy equivalence

**Definition 7.3** *Let  $f : (X, A) \rightarrow (Y, B)$  be continuous. A homotopy inverse to  $f$  is a continuous map  $g : (Y, B) \rightarrow (X, A)$  such that  $fg$  is homotopic to  $\text{id}_Y$  and  $gf$  is homotopic to  $\text{id}_X$ . In this case we say  $f$  is a homotopy equivalence and  $(X, A), (Y, B)$  are homotopy equivalent.*

Let  $A$  be a subspace of  $X$  &  $\iota : A \hookrightarrow X$  be the inclusion. We say  $A$  is a *weak deformation retract* of  $X$  if  $\iota$  is a homotopy equivalence.

### Star convex

**Definition 7.4** *We say that  $X \subseteq \mathbb{R}^n$  is star convex with respect to  $x \in X$  if for every point  $y \in X$ , the line segment  $xy$  lies in  $X$ . Note every convex set is star convex (wrt some, in fact any, point).*

E.g.

**Proposition 7.5** *Let  $X$  be star convex. With notation as above, the identity map  $id : X \rightarrow X$  is homotopic to the constant map  $c : X \rightarrow \{x\} \hookrightarrow X$ . In particular,  $\{x\}$  is a weak deformation retract of  $X$ .*

**Proof.** Note that  $c$  is a left inverse to inclusion  $\{x\} \rightarrow X$  so it suffices to prove the first assertion.

Translating by  $-x$ , we will assume that  $x = \vec{0}$ . The homotopy is given by  $H : X \times I \rightarrow X : (y, t) \mapsto ty$ .  $\square$

### Chain homotopy

Consider two chain maps  $f_\bullet, g_\bullet : C_\bullet \rightarrow C'_\bullet$ . A *chain homotopy*  $s_\bullet$  between  $f_\bullet$  and  $g_\bullet$  is a sequence of group homs  $s_p : C_p \rightarrow C'_{p+1}$  such that

$$f_p - g_p = \partial'_{p+1}s_p + s_{p-1}\partial_p.$$

DRAW DIAGRAM

The relation with topological homotopy is suggested by the following picture:

**Proposition 7.6** *Let  $s_\bullet$  be a chain homotopy between  $f_\bullet, g_\bullet : C_\bullet \rightarrow C'_\bullet$ .*

- i. Then  $H_p(f_\bullet) = H_p(g_\bullet) : H_p(C_\bullet) \rightarrow H_p(C'_\bullet)$  for all  $p$ .*
- ii. In particular, if  $id : C_\bullet \rightarrow C_\bullet$  (defined by  $id_p = id_{C_p}$ ) is homotopic to the zero map  $0_p = 0 : C_\bullet \rightarrow C_\bullet$ , then  $H_p(C_\bullet) = 0$ .*

**Proof.** Note (i)  $\implies$  (ii), for in this case, the identity map on  $H_p(C_\bullet)$  is the same as the zero map. We now prove (i).

Let  $\gamma \in Z_p(C_\bullet)$ . Then

$$f_p(\gamma) = g_p(\gamma) + \partial'_{p+1}s_p(\gamma) + s_{p-1}\partial_p(\gamma) = g_p(\gamma) + \partial'_{p+1}s_p(\gamma) \equiv g_p(\gamma) \pmod{B_p(C'_\bullet)}.$$

Hence  $H_p(f_\bullet)(\gamma + B_p(C_\bullet)) = H_p(g_\bullet)(\gamma + B_p(C_\bullet))$ .  $\square$

## 8 Homology of cones and spheres

**Aim lecture** We compute the homology of two important simplicial complexes whose polytopes are homeomorphic to the ball and sphere.

### Abstract cones

Given a simplicial complex  $K$ , we pick a new vertex say  $w \notin K^{(0)}$  and for definiteness, let  $w > a$  for all  $a \in K^{(0)}$ . We define the *cone on  $K$*  to be the simplicial complex whose simplices consists of those of  $K$  as well as those of the form  $a_0 \dots a_n w$  whenever  $a_0 \dots a_n \in K$ . (CHECK it's a simplicial complex). We denote it  $K * w$ .

**E.g.** Draw  $K_{abc}^{(1)} * d$ .

The following explains the terminology.

**Proposition 8.1**  $|K * w|$  is star convex.

**Proof.** We let  $a \in K^{(0)}$  also denote the corresponding point of  $|K * w|$  and show that for any  $x \in |K * w|$ , the line segment joining  $x$  to  $w$  lies in  $|K * w|$  too. Indeed,  $x$  must lie in a geometric simplex of  $|K * w|$  of form  $\langle a_0 \dots a_n w \rangle$  and this is convex so closed under line segments. It contains  $x, w$  so we're done.  $\square$

### Homology of cones

**Proposition 8.2** *i.*  $\tilde{H}_p(K * w) = 0$ .

$$\textit{ii. } H_p(K * w) = \begin{cases} 0 & \text{if } p \neq 0, \\ \mathbb{Z} & \text{if } p = 0 \end{cases}$$

**Proof.** Note  $|K * w|$  is connected being, star convex, so (i)  $\implies$  (ii). We prove (i) by constructing a chain homotopy  $s_\bullet$  between  $\text{id}, 0 : \tilde{C}_\bullet(K * w) \longrightarrow \tilde{C}_\bullet(K * w)$ .

We define  $s_p : \tilde{C}_p(K * w) \longrightarrow \tilde{C}_{p+1}(K * w)$  by

$$s_p(\sigma) = (-1)^{p+1} \sigma w, \quad s_{p+1}(\sigma w) = 0$$

for  $\sigma$  a  $p$ -simplex in  $K$ .

Note  $\partial(\sigma w) = (\partial\sigma)w + (-1)^{p+1} \sigma$ . Hence

$$(\partial s + s\partial)\sigma = (-1)^{p+1}((\partial\sigma)w + (-1)^{p+1} \sigma) + (-1)^p(\partial\sigma)w = \sigma.$$

Similarly,

$$(\partial s + s\partial)\sigma w = s((\partial\sigma)w + (-1)^{p+1} \sigma) = \sigma w.$$

Hence  $\partial s + s\partial = \text{id}$  and the proposition is proved.  $\square$

### Homology of spheres

**Definition 8.3** A simplicial complex  $K$  is said to be acyclic if  $\tilde{H}_p(K) = 0$  for all  $p$ .

**Theorem 8.4** Consider the  $n$ -simplex  $\sigma = a_0 \dots a_n$ .

*i.*  $K_\sigma$  is acyclic.

ii. Let  $\Sigma^{n-1} = K_\sigma^{(n-1)}$  where  $n > 1$ . Then

$$H_p(\Sigma^{n-1}) = \begin{cases} 0 & \text{if } p \neq 0 \text{ or } n-1 \\ \mathbb{Z} & \text{if } p = 0 \text{ or } n-1 \end{cases}.$$

Furthermore,  $Z_{n-1}(\Sigma^{n-1})$  is generated by  $\partial \sigma$ .

**Proof.** (i) Note that  $K_\sigma$  is the cone  $K_{a_0 \dots a_{n-1}} * a_n$  so it has zero reduced homology by 8.2.

(ii) Note that  $H_0(\Sigma^{n-1}) = \mathbb{Z}$  as  $|\Sigma^{n-1}| \simeq S^{n-1}$  which is connected when  $n > 1$ . We compute  $H_p(\Sigma^{n-1})$  for  $p > 0$ . Now  $\tilde{C}_p(K_\sigma) = \tilde{C}_p(\Sigma^{n-1})$  for  $p \neq n$  so  $0 = \tilde{H}_p(K_\sigma) = \tilde{H}_p(\Sigma^{n-1})$  except possibly when  $p = n-1, n, n+1$ .

Now  $\Sigma^{n-1}$  is a subcomplex of  $K_\sigma$  so there is a chain map

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & C_{n-1}(\Sigma^{n-1}) & \longrightarrow & \dots \\ & & \downarrow & & \parallel & & \parallel \\ 0 & \longrightarrow & C_n(K_\sigma) = \mathbb{Z}\sigma & \longrightarrow & C_{n-1}(K_\sigma) & \longrightarrow & \dots \end{array}$$

Note  $0 = H_n(\Sigma^{n-1}) = H_{n+1}(\Sigma^{n-1})$ . Also,

$$H_{n-1}(\Sigma^{n-1}) = Z_{n-1}(\Sigma^{n-1}) = Z_{n-1}(K_\sigma) \stackrel{(i)}{=} B_{n-1}(K_\sigma) = \mathbb{Z}(\partial \sigma).$$

□

## Suspension

Let  $K$  be a simplicial complex. We consider two new vertices  $w, w' \notin K^{(0)}$  and form the new simplicial complex  $S(K) = K * w \cup K * w'$  called the *suspension of  $K$* . This is an extremely important operation in algebraic topology although we won't get much opportunity to look at it.

**E.g.**  $|S(K_{abc}^{(1)})|$ .

Given any  $\sigma \in K$  a  $p$ -simplex, we have a homomorphism

$$\nu : \tilde{C}_p(K) \longrightarrow \tilde{C}_{p+1}(S(K)) : \sigma \mapsto \sigma w - \sigma w'.$$

Needless to say, this cannot give a chain map  $\tilde{C}_\bullet(K) \longrightarrow \tilde{C}_\bullet(S(K))$ , but if we re-index  $\tilde{C}_\bullet(S(K))$  by letting  $\tilde{C}_\bullet(S(K))[1]$  be the same sequence of abelian groups and boundary maps but indexed so  $\tilde{C}_p(S(K))[1] = \tilde{C}_{p+1}(S(K))$ , then  $\nu$  is a chain map. Indeed,

$$\partial \nu \sigma = \partial(\sigma w - \sigma w') = \partial \sigma w + (-1)^{p+1} \sigma - \partial \sigma w' + (-1)^{p+1} \sigma = \nu \partial \sigma.$$

Thus by functoriality proposition 6.8, we obtain a homomorphism  $\tilde{H}_p(K) \longrightarrow \tilde{H}_{p+1}(S(K))$ .

It turns out this is actually an isomorphism, a fact we won't prove as it is usually proved using the Mayer-Vietoris sequence. We will however, see a similar result for singular homology. You might like to check the isomorphism on a toy example, see problem sheets.

## 9 Homology with co-efficients. Betti numbers

**Aim lecture** Introduce numerical invariants called Betti numbers derived from homology with co-efficients.



Today,  $\mathbb{F}$  denotes a field and  $K$  = simplicial complex.

### Homology with Co-efficients

We define a chain complex  $C_\bullet(K, \mathbb{F})$  of vector spaces over  $\mathbb{F}$  by:

$$C_p(K, \mathbb{F}) = \bigoplus_{\sigma \in K \text{ is a } p \text{ simplex}} \mathbb{F} \sigma$$

that is, the vector space with basis the  $p$ -simplices of  $K$ . The boundary map  $\partial$  is the unique  $\mathbb{F}$ -linear map such that

$$\partial(a_0 \dots a_p) = \sum_{i=0}^p (-1)^i a_0 \dots \hat{a}_i \dots a_p.$$

Our old proof also gives

**Proposition-Definition 9.1**  $C_\bullet(K, \mathbb{F})$  is a chain complex of  $\mathbb{F}$ -spaces. Its homology, denoted  $H_p(K, \mathbb{F})$  is called the homology of  $K$  with co-efficients in  $\mathbb{F}$ . These are  $\mathbb{F}$ -spaces.

The following examples can be proved using the same techniques as for usual homology.

**E.g.**

- Given a cone  $K * w$  we have  $H_p(K * w, \mathbb{F}) = 0$  for  $p > 0$  and  $H_0(K * w, \mathbb{F}) = \mathbb{F}$ .
- Consider the “boundary” of an  $n$ -simplex  $\sigma$ , denote it  $\Sigma^{n-1} = K_\sigma^{(n-1)}$ . Then

$$H_p(\Sigma^{n-1}, \mathbb{F}) = \begin{cases} 0 & \text{if } p \neq 0 \text{ or } n-1 \\ \mathbb{F} & \text{if } p = 0 \text{ or } n-1 \end{cases}.$$

- Let  $T$  be the triangulation of the 2-torus  $\mathbb{T}^2$  given in e.g. 3.7. Then  $H_0(T, \mathbb{F}) = H_2(T, \mathbb{F}) = \mathbb{F}$ ,  $H_1(T, \mathbb{F}) = \mathbb{F}^2$  and all other homology is 0.

Unfortunately, we will not prove the following result whose proof depends on the universal co-efficient theorem.

**Theorem 9.2** If  $H_p(K) \simeq \mathbb{Z}^r \oplus T$  for some finite abelian group  $T$ , then  $H_p(K, \mathbb{F}) \simeq \mathbb{F}^r$  for any field  $\mathbb{F}$  of characteristic 0.

**Definition 9.3** The  $p$ -th Betti number of  $K$  is the vector space dimension  $\beta_p(K) := \dim_{\mathbb{Q}} H_p(K, \mathbb{Q})$ .

### Euler number

**Definition 9.4** The Euler number of  $K$  is the alternating sum of the Betti numbers:

$$e(K) := \sum_{p=0}^{\infty} (-1)^p \dim_{\mathbb{Q}} H_p(K, \mathbb{Q}).$$

N.B. The sum is finite.

Remarkably, we will show that the Euler number can be computed without calculating homology!

**Lemma 9.5** Let  $W < V$  be finite dim  $\mathbb{F}$ -spaces. Then  $\dim V/W = \dim V - \dim W$ .

**Proof.** Let  $W'$  be a vector space complement to  $W$  in  $V$ . Then  $W + W' = V, W \cap W' = 0$  so by the isomorphism theorem (for vector spaces),

$$V/W = (W + W')/W \simeq W'.$$

Hence  $\dim V/W = \dim W' = \dim V - \dim W$ . □

**Theorem 9.6**

$$e(K) = \sum_{p=0}^{\infty} (-1)^p \dim_{\mathbb{Q}} C_p(K, \mathbb{Q}).$$

**Proof.** The first isomorphism thm shows that  $B_p(K, \mathbb{Q}) \simeq C_{p+1}(K, \mathbb{Q})/Z_{p+1}(K, \mathbb{Q})$  so from lemma 9.5 we have

$$\dim B_p(K, \mathbb{Q}) = \dim C_{p+1}(K, \mathbb{Q}) - \dim Z_{p+1}(K, \mathbb{Q}).$$

Lemma 9.5 also shows that

$$\begin{aligned} e(K) &= \sum_p (-1)^p \dim_{\mathbb{Q}} H_p(K, \mathbb{Q}) \\ &= \sum_p (-1)^p \dim_{\mathbb{Q}} Z_p(K, \mathbb{Q}) - \sum_p (-1)^p \dim_{\mathbb{Q}} B_p(K, \mathbb{Q}) \\ &= \sum_p (-1)^p \dim_{\mathbb{Q}} Z_p(K, \mathbb{Q}) - \sum_p (-1)^p \dim_{\mathbb{Q}} C_{p+1}(K, \mathbb{Q}) + \sum_p (-1)^p \dim_{\mathbb{Q}} Z_{p+1}(K, \mathbb{Q}) \\ &= \sum_p (-1)^{p+1} \dim_{\mathbb{Q}} C_{p+1}(K, \mathbb{Q}) \end{aligned}$$

□

**Examples**

**Example 9.7** *Triangulated surfaces*

If  $K$  is a simplicial complex, all of whose simplices have  $\dim \leq 2$ , then

$$\dim C_0(K, \mathbb{F}) = \text{no. vertices of } K$$

$$\dim C_1(K, \mathbb{F}) = \text{no. edges of } K$$

$$\dim C_2(K, \mathbb{F}) = \text{no. faces of } K$$

so the thm says our new definition agrees with the old.

Let  $T$  be the triangulation of the 2-torus in the previous e.g. By definition,  $e(T) = 1 - 2 + 1 = 0$ .

**Example 9.8** *Triangulations of  $S^1$ .*

Let  $K_n$  be the  $n$ -gon  $K_n = K_{a_1 a_2} \cup K_{a_2 a_3} \cup \dots \cup K_{a_{n-1} a_n} \cup K_{a_n a_1}$ . Hence  $|K_n| \simeq S^1$ .

PICTURE

Note  $H_0(K_n, \mathbb{Q}) = \mathbb{Z}$  whilst  $H_1(K_n, \mathbb{Q}) = \mathbb{Q}([a_1 a_2] + \dots [a_n a_1])$ . Hence  $e(K_n) = 1 - 1 = 0$ . Alternatively, note

$$\dim C_0(K, \mathbb{Q}) - \dim C_1(K, \mathbb{Q}) = n - n = 0.$$

**Example 9.9** Let  $\Sigma^{n-1}$  be the boundary of the  $n$ -simplex as in the first example.

$$\text{Here } e(\Sigma^{n-1}) = 1 + (-1)^{n-1}.$$

## 10 Categories and functors

**Aim lecture** Intro higher order abstraction notions of categories and functors. This is the language which allows us to use algebra to study topology.

### Categories

**Definition** A *category*  $\mathcal{C}$  consists of the following data:

- i. a class of *objects* denoted  $\text{Obj } \mathcal{C}$ ,
- ii. disjoint sets of *morphisms*  $\text{Hom}_{\mathcal{C}}(X, Y)$  for any  $X, Y \in \text{Obj } \mathcal{C}$ ,
- iii. for any  $X, Y, Z \in \text{Obj } \mathcal{C}$ , a *composition map*

$$\text{Hom}_{\mathcal{C}}(Y, Z) \times \text{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \text{Hom}_{\mathcal{C}}(X, Z) : (f, g) \mapsto fg$$

satisfying the following axioms

- i. composition is associative whenever it's defined
- ii. for each  $X \in \text{Obj } \mathcal{C}$ , there exists an element  $\text{id}_X \in \text{Hom}_{\mathcal{C}}(X, X)$  such that

$$\text{id}_X g = g, \quad f \text{id}_X = f$$

whenever the LHS is defined. We call  $\text{id}_X$  the *identity on X*.

When lazy we write  $X \in \mathcal{C}$  instead of  $X \in \text{Obj } \mathcal{C}$ . For  $f \in \text{Hom}_{\mathcal{C}}(X, Y)$  we also write  $f : X \longrightarrow Y$ .

**Example 10.1** Below are some categories. Composition is the usual composition of functions unless otherwise stated. The same goes for the identity.

- $\mathcal{C} = \underline{\text{Grp}}$ , the category of groups.  $\text{Obj } \mathcal{C} =$  class of groups,  $\text{Hom}_{\underline{\text{Grp}}}(X, Y) =$  the set of group homomorphisms of form  $\phi : X \longrightarrow Y$ .
- $\mathcal{C} = \underline{\text{Ab}}$ .  $\text{Obj } \mathcal{C} =$  class of abelian groups,  $\text{Hom}_{\underline{\text{Ab}}}(X, Y) =$  the set of group homomorphisms of form  $\phi : X \longrightarrow Y$ .
- $\mathcal{C} = \underline{\text{Top}}$ .  $\text{Obj } \mathcal{C} =$  class of topological spaces,  $\text{Hom}_{\underline{\text{Top}}}(X, Y) =$  the set of continuous maps of form  $\phi : X \longrightarrow Y$ .
- $\mathcal{C} = \underline{\text{TopPair}}$ .  $\text{Obj } \mathcal{C} =$  class of topological pairs  $(X, A)$ ,  $\text{Hom}_{\underline{\text{TopPair}}}((X, A), (Y, B)) =$  the set of continuous maps of pairs.
- $\mathcal{C} = \underline{\text{Ch}}$ .  $\text{Obj } \mathcal{C} =$  class of chain complexes,  $\text{Hom}_{\underline{\text{Ch}}} =$  chain maps. Composition is the composite of chain maps (OK by 6.7).

- $\mathcal{C} = \underline{\text{HTop}}$ .  $\text{Obj } \mathcal{C}$  = class of topological spaces,  $\text{Hom}_{\underline{\text{HTop}}}(X, Y)$  = the set of homotopy equivalence classes  $[f]$  of continuous maps  $f : X \longrightarrow Y$ . Composite  $[f][g] := [fg]$  well-defined by proposition 7.2. We call this the *homotopy category*.

## Isomorphisms

**Definition 10.2** Let  $\mathcal{C}$  be a category and  $f \in \text{Hom}_{\mathcal{C}}(X, Y)$ . We say  $f$  is an isomorphism in  $\mathcal{C}$  if there is some morphism  $g \in \text{Hom}_{\mathcal{C}}(Y, X)$  such that  $fg = \text{id}_Y$  and  $gf = \text{id}_X$ . In this case we say  $g$  is the (unique) inverse of  $f$  and  $X, Y$  are isomorphic in  $\mathcal{C}$ .

**Example 10.3** We give the notion of isomorphism for various categories  $\mathcal{C}$ .

- $\mathcal{C} = \underline{\text{Grp}}$ . Isomorphisms are group isomorphisms.
- $\mathcal{C} = \underline{\text{Top}}$ . Isomorphisms are homeomorphisms.
- $\mathcal{C} = \underline{\text{HTop}}$ . Isomorphisms are called *homotopy equivalences* by definition.

**Proposition 10.4** The composite of isomorphisms is an isomorphism.

**Proof.** Ex. □

## Functors

Let  $\mathcal{C}, \mathcal{D}$  be categories.

**Definition 10.5** A (covariant) functor  $F : \mathcal{C} \longrightarrow \mathcal{D}$  consists of the data:

- i. A function  $\text{Obj } \mathcal{C} \longrightarrow \text{Obj } \mathcal{D} : X \mapsto F(X)$  and,
- ii. for each  $X, Y \in \mathcal{C}$  a function

$$F : \text{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y))$$

such that

- i.  $F(\text{id}_X) = \text{id}_{F(X)}$  and,
- ii.  $F(fg) = F(f)F(g) : F(X) \xrightarrow{F(g)} F(Y) \xrightarrow{F(f)} F(Z)$  given  $X \xrightarrow{g} Y \xrightarrow{f} Z$  in  $\mathcal{C}$ .

We also say in this case,  $F(X)$  is functorial in  $X$ .

Proposition 6.9 can be restated as

**Proposition 10.6**  $H_p : \underline{\text{Ch}} \longrightarrow \underline{\text{Ab}}$  is a covariant functor.

**Example 10.7** We have a functor  $Q : \underline{\text{Top}} \longrightarrow \underline{\text{HTop}}$  defined by  $Q(X) = X$  and  $Q(f) = [f]$ .

**Proposition 10.8** Let  $F : \mathcal{C} \longrightarrow \mathcal{D}$  be a functor and  $f \in \text{Hom}_{\mathcal{C}}$  be an isomorphism in  $\mathcal{C}$ . Then  $F(f)$  is also an isomorphism in  $\mathcal{D}$ .

**Proof.** Let  $g \in \text{Hom}_{\mathcal{C}}$  be the inverse to  $f$  in  $\mathcal{C}$ . Then  $F(g)$  is the inverse to  $F(f)$  for

$$F(f)F(g) = F(fg) = F(\text{id}) = \text{id}$$

and similarly  $F(g)F(f) = \text{id}$ . □

We mostly use this in the contrapositive form

**Corollary 10.9** *Let  $F : \underline{HTop} \rightarrow \underline{Ab}$  be a functor and  $X, Y \in \underline{Top}$ . If  $F(X) \not\cong F(Y)$  then  $X, Y$  are not homotopically equivalent and in particular, not homeomorphic.*

## 11 Simplicial maps. Homotopy groups

**Aim lecture** We introduce 3 important functors arising in topology.

### Category of simplicial complexes

$K, L = \text{simplicial complexes}$

Below, we let  $a \in K^{(0)}$  also denote the corresponding point of  $|K|$ .

**Proposition-Definition 11.1** *A simplicial map  $f : K \rightarrow L$  is a map on vertices  $f : K^{(0)} \rightarrow L^{(0)}$  such that if  $a_0 \dots a_p \in K$  then  $f(a_0) \dots f(a_p)$  are the (not necessarily distinct) vertices of a simplex in  $L$ .*

*Given such a simplicial map, the affine linear maps  $l(\langle a_0 \dots a_p \rangle, \langle f(a_0) \dots f(a_p) \rangle)$  glue together to form a continuous map denoted  $|f| : |K| \rightarrow |L|$ .*

**Proof.** This follows immediately from the gluing lemma 2.6. □

**E.g.**

**Proposition 11.2** *There is a category  $\mathcal{C} = \underline{Simp}$  of simplicial complexes with  $\text{Obj } \mathcal{C} = \text{class of simplicial complexes}$ ,  $\text{Hom}_{\mathcal{C}} = \text{simplicial maps}$  and composition the usual composition of (vertex) maps.*

**Proof.** It suffices to show that the composite of simplicial maps  $K \xrightarrow{f} L \xrightarrow{g} M$  is simplicial for composition is associative the identity is clearly simplicial. This is clear. □

### Polytope functor

**Proposition-Definition 11.3** *We have a polytope functor  $|\cdot| : \underline{Simp} \rightarrow \underline{Top}$  defined on objects by  $K \mapsto |K|$  and morphisms by  $f \mapsto |f|$ .*

**Proof.** Note  $l(\langle \sigma \rangle, \langle \sigma \rangle) = \text{id}$  so  $|\text{id}_K| = \text{id}_{|K|}$ . Consider simplicial maps  $K \xrightarrow{f} L \xrightarrow{g} M$ . Note  $|gf| = |g||f|$  since they both send

$$\sum_i t_i a_i \mapsto \sum_i t_i f(a_i) \mapsto \sum_i t_i (gf)(a_i).$$

□

**Proposition 11.4** *Given functors  $F : \mathcal{C} \rightarrow \mathcal{D}, G : \mathcal{D} \rightarrow \mathcal{E}$ , there is a composite functor  $GF : \mathcal{C} \rightarrow \mathcal{E}$  defined on objects by  $GF(X) = G(F(X))$  and morphisms by  $GF(f) = G(F(f))$ .*

**Proof.** Good ex. □

We often call the composed functor  $Q|\cdot| : \underline{\text{Simp}} \rightarrow \underline{\text{Top}} \rightarrow \underline{\text{HTop}}$  the polytope functor as well and, abusing notation, denote it also by  $|\cdot|$ .

### Homology functor

We have a functor  $C_\bullet : \underline{\text{Simp}} \rightarrow \underline{\text{Ch}}$  defined as follows. On objects it maps  $K \mapsto C_\bullet(K)$ . On a simplicial map  $f : K \rightarrow L$  the chain map  $c_\bullet := C_\bullet(f) : C_\bullet(K) \rightarrow C_\bullet(L)$  is defined by

$$c_p : C_p(K) \rightarrow C_p(L) : a_0 \dots a_p \mapsto \begin{cases} [f(a_0) \dots f(a_p)] & \text{if } f(a_0), \dots, f(a_p) \text{ are distinct,} \\ 0 & \text{otherwise} \end{cases}.$$

Note  $c_\bullet$  is a chain map since if  $f(a_0), \dots, f(a_p)$  are distinct,

$$\partial c_p(a_0 \dots a_p) = \sum_i (-1)^i [f(a_0) \dots \widehat{f(a_i)} \dots f(a_p)] = c_p(\partial(a_0 \dots a_p)).$$

If they are not distinct, one can show  $c_p(\partial(a_0 \dots a_p)) = 0$  with a little care taking cases ex.

It's easy to see  $C_\bullet$  is functorial ex. We thus obtain a  $p$ -th homology functor  $H_p : \underline{\text{Simp}} \rightarrow \underline{\text{Ab}}$  as the composite

$$\underline{\text{Simp}} \xrightarrow{C_\bullet} \underline{\text{Ch}} \xrightarrow{H_p} \underline{\text{Ab}}.$$

Similarly, we can define *homology with co-efficients* in a field  $\mathbb{F}$  functor  $H_p(-, \mathbb{F}) : \underline{\text{Simp}} \rightarrow \underline{\text{Vect}}_{\mathbb{F}}$  where  $\underline{\text{Vect}}_{\mathbb{F}}$  is the category of  $\mathbb{F}$ -spaces and  $\mathbb{F}$ -linear maps.

### Homotopy groups

Consider the *category of sets*  $\underline{\text{Set}}$  whose objects are sets, and morphisms are functions.

**Proposition 11.5** *Let  $\mathcal{C}$  be a category and  $C \in \mathcal{C}$ . We have a functor  $\text{Hom}_{\mathcal{C}}(C, -) : \mathcal{C} \rightarrow \underline{\text{Set}}$  defined on an object  $X \in \mathcal{C}$  as the set of morphisms  $\text{Hom}_{\mathcal{C}}(C, X)$  and given a morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$ ,*

$$\text{Hom}_{\mathcal{C}}(C, f) : \text{Hom}_{\mathcal{C}}(C, X) \rightarrow \text{Hom}_{\mathcal{C}}(C, Y) : g \mapsto fg.$$

**Proof.** Note  $\text{Hom}_{\mathcal{C}}(C, \text{id}_X) : g \mapsto \text{id}_X g = g$  is the identity on  $\text{Hom}_{\mathcal{C}}(C, X)$ . Also given  $X \xrightarrow{f} Y \xrightarrow{f'} Z$  in  $\mathcal{C}$  we have

$$\text{Hom}_{\mathcal{C}}(C, f'f)(g) = f'fg = \text{Hom}_{\mathcal{C}}(C, f')(fg) = \text{Hom}_{\mathcal{C}}(C, f')\text{Hom}_{\mathcal{C}}(C, f)(g)$$

and  $\text{Hom}_{\mathcal{C}}(C, -)$  is a functor. □

Let  $\underline{\text{HPtTop}}$  be the *homotopy category of pointed topological spaces* whose objects are topological pairs of the form  $(X, x)$  for some  $x \in X$ , and morphisms are homotopy equivalence classes of continuous maps of pairs. We define the functor  $\pi_n : \underline{\text{HPtTop}} \rightarrow \underline{\text{Set}}$  to be  $\pi_n = \text{Hom}_{\underline{\text{HPtTop}}}((S^n, \text{pt}), -)$ .

**Theorem 11.6** *For each  $(X, x) \in \underline{\text{HPtTop}}$ , there is a group structure on  $\pi_n(X, x)$  which makes  $\pi_n$  a functor  $\underline{\text{HPtTop}} \rightarrow \underline{\text{Grp}}$  (i.e.  $\pi_n(f)$  is also a group homomorphism). We call  $\pi_n(X, x)$  the  $n$ -th homotopy group of  $(X, x)$ .*

**Proof.** We will not prove this, but you should have seen  $\pi_1$  in MATH3701. □

## 12 Homology: the topological invariant

**Aim lecture** We state the Main Theorem of this course, the existence of a homology functor  $H_* : \underline{HTop} \rightarrow \underline{Ab}$  and its computability via homology of simplicial complexes.

Let  $X =$  topological space

### Natural isomorphisms

It's rare that two functors found in different situations are equal, usually they are only *naturally isomorphic* in the sense below.

**Definition 12.1** Let  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  be two functors. A natural transformation  $\eta : F \xrightarrow{\sim} G$  consists of morphisms (in  $\mathcal{D}$ )  $\eta_X : F(X) \rightarrow G(X)$  for each  $X \in \mathcal{C}$  such that for every morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$  the square below is commutative

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ \downarrow F(f) & & \downarrow G(f) \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array}$$

If all  $\eta_X$ 's are isomorphisms, we say  $\eta$  is a natural isomorphism, and  $F, G$  are naturally isomorphic. We sometimes write  $F \simeq G$ .

### Main Theorem

**Theorem 12.2** For each  $p \in \mathbb{N}$ , there is a functor  $H_p : \underline{HTop} \rightarrow \underline{Ab}$  (resp  $\tilde{H}_p, H_p(-, \mathbb{F}) : \underline{HTop} \rightarrow \underline{Vect}_{\mathbb{F}}$ ) such that the simplicial homology functor  $H_p : \underline{Simp} \rightarrow \underline{Ab}$  (resp reduced homology, homology with co-efficients) is naturally isomorphic to the composite functor

$$\underline{Simp} \xrightarrow{|\cdot|} \underline{HTop} \xrightarrow{H_p} \underline{Ab}$$

(resp ...).

In particular, given any  $K \in \underline{Simp}$ , we have  $H_p(|K|) \simeq H_p(K)$ .

**Proof.** much later in section 24

□

We call  $H_p(X)$  the  $p$ -th homology group of  $X$ , and similarly for  $\tilde{H}_p(X), H_p(X, \mathbb{F})$ .

### Homology of some topological spaces

Since functors preserve isomorphisms, homotopically equivalent spaces have isomorphic homology groups.

- i. Suppose  $X$  is homotopy equivalent to a point, e.g. this occurs if  $X$  is star convex by proposition 7.5. Then  $\tilde{H}_p(X) \simeq \tilde{H}_p(\text{pt})$  is isomorphic to the reduced homology of the 0-simplex  $K_a$  i.e. 0.
- ii. If  $\sigma$  is an  $n$ -simplex then  $H_p(S^{n-1}) \simeq H_p(K_\sigma^{(n-1)}) = \mathbb{Z}$  if  $p = 0$  or  $n - 1$  and 0 otherwise.
- iii. From e.g. 5.4 we know

$$H_p(\mathbb{T}^2) \simeq \begin{cases} \mathbb{Z} & \text{if } p = 0, 2 \\ \mathbb{Z}^2 & \text{if } p = 1 \\ 0 & \text{otherwise} \end{cases}$$

## Topological applications

**Proposition 12.3** *If  $m \neq n$ , then  $S^n, S^m$  are not homeomorphic.*

**Proof.** Since  $H_m(S^m) \simeq \mathbb{Z} \neq 0 = H_m(S^n)$ . □

Similarly, we also see that the 2-torus and 2-sphere are not homeomorphic.

**Lemma 12.4** *The space  $U = \mathbb{R}^n - 0$  is homotopy equivalent to  $S^{n-1}$ .*

**Proof.** We show the retraction  $f : U \rightarrow S^{n-1} : v \mapsto v/|v|$  is inverse to the inclusion  $\iota : S^n \hookrightarrow U$  in  $\text{HTop}$ . Since  $f\iota = \text{id}_{S^{n-1}}$ , it suffices to show that  $\iota f$  is homotopy equivalent to  $\text{id}_U$ .

The desired homotopy is given by  $H : U \times I \rightarrow U : (v, t) \mapsto (1-t)v + \frac{tv}{|v|}$ . □

**Theorem 12.5** *If  $m \neq n$ , then  $\mathbb{R}^m, \mathbb{R}^n$  are not homeomorphic.*

**Proof.** Suppose by contradiction that  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is an homeomorphism. Then  $\mathbb{R}^m - 0$  is homeomorphic to  $\mathbb{R}^n - f(0)$ . The lemma 12.4 shows then that  $S^{m-1}$  and  $S^{n-1}$  are homotopically equivalent (since homotopy equivalence is an equivalence relation ex). This is false as they have different homology groups in degree  $m-1$  and  $n-1$ . □

## Betti numbers of topological spaces

We define the  $p$ -th Betti number of  $X$  to be  $\dim_{\mathbb{Q}} H_p(X, \mathbb{Q})$ . If all these numbers are finite and  $H_p(X) = 0$  for  $p \gg 0$ , then we can define the Euler number of  $X$  to be

$$e(X) = \sum_p (-1)^p \dim_{\mathbb{Q}} H_p(X, \mathbb{Q}).$$

If  $X$  and  $Y$  are homotopically equivalent then the vector spaces  $H_p(X, \mathbb{Q}) \simeq H_p(Y, \mathbb{Q})$  so they have the same Betti numbers, and Euler number (if defined).

**Proposition 12.6** *Homotopically equivalent topological spaces have the same Betti numbers, and hence the same Euler number, if it's defined.*

**Remark** Recall that if  $X$  has a triangulation  $h : |K| \rightarrow X$ , then it has infinitely many triangulations. From our main theorem and theorem 9.6,

$$e(X) = \sum_p (-1)^p \dim_{\mathbb{Q}} C_p(K, \mathbb{Q}).$$

The remarkable fact is that though the individual dimensions  $\dim_{\mathbb{Q}} C_p(K, \mathbb{Q})$  vary as you vary  $K$ , the alternating sum does not.



## 13 Topological applications of the degree map

**Aim lecture** Intro the degree of a self map of spheres, and give applications to Brouwer's fixed point theorem and the fundamental thm of algebra.

### Degree

Need

**Lemma 13.1** Any group hom  $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$  is multiplication by  $d$  where  $d = \phi(1)$ .

**Proof.**  $\phi(n) = n\phi(1) = dn$ . □

Let  $f : S^n \rightarrow S^n$  be cont. The lemma shows that  $H_p(f) : H_n(S^n) \simeq \mathbb{Z} \rightarrow H_n(S^n)$  must be multiplication by an integer  $d =: \deg f$  which we call the *degree* of  $f$ .

Here are some basic properties.

**Proposition 13.2** *i. If  $f \approx g$  (i.e homotopy equiv), then  $\deg f = \deg g$ .*

*ii.  $\deg id = 1$ .*

*iii. If  $f$  extends to a continuous map  $h : B^{n+1} \rightarrow S^n$ , then  $\deg f = 0$ .*

*iv.  $\deg fg = (\deg f)(\deg g)$ .*

**Proof.** (i)  $[f] = [g] \implies H_n([f]) = H_n([g])$ .

(ii)  $H_n$  a functor  $\implies H_n(id) = id$  i.e. multn by 1.

(iii)  $H_n(f)$  factors as

$$H_n(S^n) \xrightarrow{H_n(\text{inc})} H^n(B^{n+1}) = 0 \xrightarrow{H_n(h)} H^n(S^n)$$

so is 0.

(iv)  $H_n(fg) = H_n(f)H_n(g)$  is the composite of multiplications

$$H_n(S_n) \xrightarrow{\deg f} H_n(S_n) \xrightarrow{\deg g} H_n(S_n)$$

which is multiplication by  $(\deg f)(\deg g)$ . □

### Degree and winding

Below, we view  $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ . Recall, that  $\mathbb{Z}$  has 2 generators only,  $\pm 1$ .

**Proposition 13.3** Let  $f : S^1 \rightarrow S^1 : z \mapsto z^d$  (DRAW PICTURE). Then  $\deg f = d$ .

**Proof.** Best understood pictorially (see lecture). We'll only prove  $\deg f = \pm d$  which is all we need.

We use 2 different triangulations of  $S^1$ . Consider vertex sets  $K_1^{(0)} = \mathbb{Z}/3d\mathbb{Z}$ ,  $K_2^{(0)} = \mathbb{Z}/3\mathbb{Z}$  and simplicial complexes

$$K_1 = \cup_{a \in \mathbb{Z}/3d\mathbb{Z}} K_{a,a+1}, \quad K_2 = \cup_{a \in \mathbb{Z}/3\mathbb{Z}} K_{a,a+1}.$$

Hence  $|K_1|$  is a regular  $3d$ -gon and  $|K_2|$  is an equilateral  $\triangle$ . Radial projections  $h_1 : |K_1| \rightarrow S^1, h_2 : |K_2| \rightarrow S^1$  give triangulations of the circle.

Consider the simplicial map

$$\phi : K_1^{(0)} = \mathbb{Z}/3d\mathbb{Z} \rightarrow K_2^{(0)} = \mathbb{Z}/3\mathbb{Z} : a + 3d\mathbb{Z} \mapsto a + 3\mathbb{Z}.$$

Observe that  $f \approx h_2 \circ |\phi| \circ h_1^{-1}$  by problem set 2, Q6 and that  $\phi_*$  sends the generator  $\sum_{a \in \mathbb{Z}/3d\mathbb{Z}} [a \ a+1] \mapsto d \sum_{a \in \mathbb{Z}/3\mathbb{Z}} [a \ a+1]$ ,  $d$  times the generator  $\sum_{a \in \mathbb{Z}/3\mathbb{Z}} [aa+1]$  for  $H_1(K_2)$ . Now  $H_1(f)$  is the composite

$$H_1(S^1) \xrightarrow{H_1(h_1^{-1})} H_1(K_1) \xrightarrow{H_1(\phi)} H_1(K_2) \xrightarrow{H_1(h_2)} H_1(S^1).$$

This must be multn by  $\pm d$ . □

### Brouwer fixed point theorem

**Theorem 13.4** *Any cont map  $h : B^{n+1} \longrightarrow B^{n+1}$  has a fixed point.*

**Proof.** If  $h$  has no fixed points, we have a cont map  $h' : B^{n+1} \longrightarrow S^n$  defined by

$$h'(v) = \frac{v - h(v)}{|v - h(v)|}.$$

We consider the restriction  $f := h'|_{S^n} : S^n \longrightarrow S^n$  which has degree 0 by lemma 13.2(iii). However,  $f$  is homotopic to id by the homotopy

$$H(v, t) = \frac{v - th(v)}{|v - th(v)|}$$

(note denominator non-zero), so  $\deg f = 1$  too, a contradiction. □

### Fundamental theorem of algebra

**Theorem 13.5** *let  $p(z) = z^d + a_{d-1}z^{d-1} + \dots + a_0$  be a complex polynomial of degree  $d > 0$ . Then  $p$  has a complex root in the disc  $B_c := \{z \in \mathbb{C} \mid |z| \leq c\}$  for any  $c > d \max\{1, |a_{d-1}|, \dots, |a_0|\}$ .*

**Proof.** We view  $p$  as a continuous map  $\mathbb{C} \longrightarrow \mathbb{C}$ . Suppose the theorem is false so  $p$  restricts to a map  $p|_{B_c} : B_c \longrightarrow \mathbb{C} - 0 =: U$ . Let  $S_c =$  boundary of  $B_c$  and  $r : U \longrightarrow S_c : z \mapsto \frac{cz}{|z|}$  be a retraction. We consider the composite  $f = rp|_{S_c} : S_c \longrightarrow S_c$ . First note that  $f$  extends to  $B_c$  so  $\deg f = 0$  by proposition 13.2.

Let  $q : S_c \longrightarrow U : z \mapsto z^d$ . We wish to show that  $f$  is homotopy equivalent to  $f' : z \mapsto c^{-(d-1)}z^d = rq(z)$ . This will give the desired contradiction as proposition 13.3 shows  $\deg f' = d$ .

It suffices by Proposition 7.2 to show that  $p|_{S_c}, q : S_c \longrightarrow U$  are homotopy equivalent. We show that  $H : S_c \times I \longrightarrow U$  defined below is a homotopy.

$$H(z, t) = z^d + t(a_{d-1}z^{d-1} + \dots + a_0).$$

The only thing that needs to be checked is whether  $\text{im } H$  lies in  $U$  or not. However, the triangle inequality and our assumption on  $c$  ensure

$$|a_{d-1}z^{d-1} + \dots + a_0| < |z^d|$$

so  $H(z, t)$  is never zero are we are done. □

## 14 Hairy coconut theorem

**Aim lecture** We compute the degree of the antipodal map to study vector fields on spheres.

### Degree of the antipode

Below we consider the following triangulation of  $S^n$ . First let  $\sigma = a_0 \dots a_n$  be an  $n$ -simplex and  $K = K_\sigma^{(n-1)}$  so radial projection from the barycentre  $\hat{\sigma}$  of  $\sigma$  gives a triangulation  $\bar{\theta} : |K| \xrightarrow{\sim} S^{n-1}$ .

Consider the suspension  $S(K) = K * w_+ \cup K * w_-$ . We construct a triangulation  $\theta : |S(K)| \rightarrow S^n \subset \mathbb{R}_{x_0, \dots, x_n}^{n+1}$  which extends the triangulation  $\bar{\theta}$ . Let

$$E_\pm := \{\vec{x} \in S^n \mid \pm x_n \geq 0\}$$

be the upper and lower hemispheres of  $S^n$  so the equator  $E := E_+ \cap E_- \simeq S^{n-1}$ . Place the simplex  $\langle a_0 \dots a_n w_+ \rangle$  in  $\mathbb{R}^{n+1}$  with the base  $K$  inscribed in the equator  $E$ , and  $w_+$  above the equator. DRAW PICTURE.

Radial projection from  $\hat{\sigma}$  gives a homeomorphism  $\theta_+ : |K * w_+| \rightarrow E_+$  which we may assume restricts to  $\bar{\theta}$  on  $K$ . Similary, there is a homeomorphism  $\theta_- : |K * w_-| \rightarrow E_-$ . They glue together to give the desired triangulation  $\theta : |S(K)| \rightarrow S^n$ .

**Lemma 14.1** *The radial projection map  $r : \mathbb{R}^{n+1} - 0 \rightarrow S^n : v \mapsto \frac{v}{|v|}$  commutes with reflection  $\tau : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1} : (x_0, \dots, x_n) \mapsto (x_0, \dots, x_{n-1}, -x_n)$  about the  $x_n = 0$  hyperplane.*

**Proof.**  $r\tau(\vec{x}) = (x_0, \dots, x_{n-1}, -x_n) / \pm (x_0, \dots, x_{n-1}, -x_n) = \tau r(\vec{x})$ . □

Below we consider the group hom  $[-w_\pm] : C_p(K) \rightarrow C_p(K * w_\pm)$  defined on the free basis by  $[a_{i_0} \dots a_{i_p}] \mapsto [a_{i_0} \dots a_{i_p} w_\pm]$ .

**Proposition 14.2** *Let  $\tau : S^n \rightarrow S^n$  be reflection about a co-ordinate hyperplane. Then  $\deg \tau = -1$ .*

**Remark** Since  $\tau^2 = \text{id}$  and  $\deg$  is multiplicative, we know  $(\deg \tau)^2 = 1$ .

**Proof.** We may assume that the hyperplane is  $x_n = 0$ . We first consider the vertex map  $f : S(K)^{(0)} \rightarrow S(K)^{(0)}$  which switches  $w_\pm$  and leaves all other vertices fixed. Clearly this is a simplicial map. Moreover, from the construction of our triangulation  $\theta : |S(K)| \rightarrow S^n$  and lemma 14.1,  $|\phi|$  corresponds to  $\tau$  in the sense that

$$\tau = \theta |\phi| \theta^{-1} : S^n \xrightarrow{\theta^{-1}} |S(K)| \xrightarrow{|\phi|} |S(K)| \xrightarrow{\theta} S^n.$$

It thus suffices to show that  $H_n(\phi)$  is multn by -1 (ex make sure you know why). To this end, let  $z \in Z_n(S(K))$  be an  $n$ -cycle which must have the form

$$z = [z_+ w_+] + [z_- w_-]$$

for some  $n-1$ -chains  $z_+, z_- \in C_{n-1}(K)$ . Being cycle means

$$0 = \partial z = [\partial z_+ w_+] + (-1)^{n+1} z_+ [\partial z_- w_-] + (-1)^{n+1} z_-.$$

We must thus have  $\partial z_{\pm} = 0, z_+ + z_- = 0$ . Hence  $z = [z_+ w_+] - [z_+ w_-]$  and

$$\phi_* z = [z_+ w_-] - [z_+ w_+] = -z$$

and  $H_n(\phi)$  must be multn by -1 as desired.  $\square$

**Definition 14.3** The antipode map is  $a : S^n \longrightarrow S^n : v \mapsto -v$ .

**Corollary 14.4**  $\deg a = (-1)^{n+1}$ .

**Proof.** Let  $\tau_0, \dots, \tau_n : S^n \longrightarrow S^n$  be reflection about the co-ordinate hyperplanes  $x_0 = 0, \dots, x_n = 0$ . Then  $a = \tau_0 \circ \dots \circ \tau_n$ . Hence multiplicativity of the degree map and proposition 14.2 gives  $\deg a = (-1)^{n+1}$ .  $\square$

### Fixed point theorems for spheres

**Theorem 14.5** Let  $f : S^n \longrightarrow S^n$  be continuous. If  $\deg f \neq (-1)^{n+1}$ , then  $f$  has a fixed point.

**Proof.** Suppose that  $f$  has no fixed point. By corollary 14.4, it suffices to show  $f$  is homotopic to the antipode  $a$ . In fact, the homotopy is  $H : S^n \times I \longrightarrow S^n$  defined by

$$H(v, t) = \frac{(1-t)f(v) - tv}{|(1-t)f(v) - tv|}.$$

We need only check the denominator is never 0. Suppose instead that  $(1-t)f(v) = tv$ . Since  $f(v), v$  both have length 1, we must have  $t = \frac{1}{2}$ . Then  $f(v) = v$  giving the fixed point  $v$ , a contradiction.  $\square$

### Vector fields on spheres

A continuous vector field  $v$  on the sphere  $S^n$  is a continuous map of the form  $x \in S^n \mapsto v_x T_x S^n = x^\perp$  where  $T_x S^n$  is the tangent space to  $S^n$  at  $x$ , which we identify with the subspace  $x^\perp \subset \mathbb{R}^{n+1}$ . We say  $v$  is *nowhere vanishing* if  $v_x \neq 0$  for every  $x$ .

**Theorem 14.6** If  $S^n$  has a nowhere vanishing continuous vector field, then  $n$  is odd.

**Remark** Explain name hairy coconut thm.

**Proof.** Suppose  $v$  is a nowhere vanishing continuous vector field. We can then define the continuous map  $f : S^n \longrightarrow S^n$  by  $x \mapsto \frac{v_x}{|v_x|}$ . Since  $x \perp v_x, f(x)$ , we know that  $f$  has no fixed point. Hence  $\deg f = (-1)^{n+1}$  by theorem 14.5. Similarly,  $fa$  has no fixed point so

$$(\deg f)(\deg a) = (-1)^{n+1} \implies \deg f = 1.$$

This gives the equality  $(-1)^{n+1} = 1$  so  $n$  must be odd.  $\square$

**Addendum** When  $n$  is odd, it is easy to construct continuous nowhere vanishing vector fields.

## 15 Singular homology

**Aim lecture** We introduce the homology functor on topological spaces. We will begin also our proof of the Main Theorem, and so in particular, we will not assume it until we've finally proved it.

Let  $X$  = topological space

### Group of singular $p$ -chains

Consider the Euclidean space  $\mathbb{R}^{p+1}_{x_0, \dots, x_p}$  with co-ordinates  $x_0, \dots, x_p$ . Let the standard basis vectors be  $\varepsilon_0, \dots, \varepsilon_p$ . The *standard (geometric)  $p$ -simplex* is

$$\Delta_p := \langle \varepsilon_0 \dots \varepsilon_p \rangle.$$

A *singular  $p$ -simplex* of  $X$  is a continuous map  $T : \Delta_p \rightarrow X$ .

**E.g.**

- i. If  $x \in X$ , then the constant map  $T_x : \Delta_0 \rightarrow \{x\} \hookrightarrow X$  is a singular 0-simplex.
- ii. The identity map  $T_{\Delta_p} = \text{id}_{\Delta_p} : \Delta_p \rightarrow \Delta_p$  is a singular  $p$ -simplex.
- iii. The affine linear map  $l_i := l(\langle \varepsilon_0 \dots \widehat{\varepsilon}_i \dots \varepsilon_p \rangle, \Delta_p) : \Delta_{p-1} \rightarrow \Delta_p$  is a singular  $p-1$ -simplex of  $\Delta_p$  representing the  $i$ -th face.
- iv. The affine linear map  $l_{ij} := l(\langle \varepsilon_0 \dots \widehat{\varepsilon}_i \dots \widehat{\varepsilon}_j \dots \varepsilon_p \rangle, \Delta_p) : \Delta_{p-2} \rightarrow \Delta_p$  is a singular  $p-2$ -simplex of  $\Delta_p$ .

Let  $S_p(X)$  be the free abelian group generated by the singular  $p$ -simplices. The elements are called *singular  $p$ -chains*. Given a continuous map  $f : X \rightarrow Y$ , consider the group homomorphism  $S_p(f) : S_p(X) \rightarrow S_p(Y)$  defined on generators (i.e.  $p$ -simplices) by

$$S_p(f) : T \mapsto f \circ T =: f_{\#}(T).$$

**Lemma 15.1**  $S_p : \underline{\text{Top}} \rightarrow \underline{\text{Ab}}$  is a covariant functor.

**Proof.**  $S_p(\text{id}) = \text{id}$  since  $S_p(\text{id})$  maps  $T \mapsto \text{id} \circ T = T$  so is the identity on generators. Also given continuous  $X \xrightarrow{f} Y \xrightarrow{g} Z$  note  $S_p(gf) = S_p(g)S_p(f)$  since on a  $p$ -simplex  $T$ , they both map  $T \mapsto (gf)T = g(fT)$ .  $\square$

### Singular chain complex

We wish to assemble the  $S_p(X)$  for  $p \in \mathbb{Z}$  so that we obtain a chain complex and  $S_{\bullet}$  becomes a functor  $\underline{\text{Top}} \rightarrow \underline{\text{Ch}}$ . For a  $p$ -simplex  $T$ , its  $i$ -th face is the  $(p-1)$ -simplex

$$Tl_i : \Delta_{p-1} \xrightarrow{l_i} \Delta_p \xrightarrow{T} X.$$

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The boundary operator is defined by

$$\partial_p : S_p(X) \rightarrow S_{p-1}(X) : T \mapsto \sum_{i=1}^p (-1)^i Tl_i.$$

**Proposition 15.2**  $\partial^2 = 0$  so we have a complex

$$S_\bullet(X) : \dots \longrightarrow S_2(X) \xrightarrow{\partial} S_1(X) \xrightarrow{\partial} S_0(X) \longrightarrow 0$$

called the singular chain complex of  $X$ .

**Proof.** Note that affine linear maps are determined by their values on vertices, and that the composite of two is again affine linear. Hence  $l_i l_j = l_{ij}$  if  $i < j$  and  $l_i l_j = l_{j,i+1}$  if  $i \geq j$ . We compute  $\partial^2$  on a  $p$ -simplex  $T$ .

$$\partial^2 T = \partial\left(\sum_i (-1)^i T l_i\right) = \sum_{i,j} (-1)^{i+j} T l_i l_j = \sum_{i < j} (-1)^{i+j} T l_{ij} + \sum_{i' > j} (-1)^{i'+j-1} T l_{ji'} = 0$$

where we split the sum and performed a change of variable  $i' = i + 1$ .  $\square$

### Singular homology functor

**Proposition 15.3** Given a continuous map  $f : X \longrightarrow Y$ , the collection of maps  $S_\bullet(f) = \{S_p(f)\} : S_\bullet(X) \longrightarrow S_\bullet(Y)$  is a chain map. This makes  $S_\bullet : \underline{Top} \longrightarrow \underline{Ch}$  a covariant functor.

**Notn** We also write  $f_*$  for  $S_p(f)$ .

**Proof.** The second assertion follows easily from the first by lemma 15.1. It thus suffices to check  $\partial S_p(f) = S_{p-1}(f)\partial$ . For a singular  $p$ -simplex  $T$  we have

$$\partial S_p(f)T = \partial(fT) = \sum_i (-1)^i (fT) l_i = S_{p-1}(f)\left(\sum_i (-1)^i T l_i\right) = S_{p-1}(f)\partial T.$$

$\square$

**Corollary 15.4** We have a composite functor  $H_p : \underline{Top} \xrightarrow{S_\bullet} \underline{Ch} \xrightarrow{H_p} \underline{Ab}$ . It is called the singular  $p$ -th homology functor and  $H_p(X)$  is the  $p$ -th homology group of  $X$ .

We will show later that  $H_p(f)$  only depends on the homotopy equivalence class of  $f$  so we will actually have a functor  $H_p : \underline{HTop} \longrightarrow \underline{Ab}$ .

Unlike simplicial homology, one usually cannot compute them from definitions because when  $X$  is infinite, all the  $S_p(X)$  are infinitely generated.

### Variants

Just as for simplicial homology, we can define reduced homology and homology with co-efficients in the setting of singular homology. Analogues of all the above results hold and we will not spell this out in future.

For reduced singular homology, we replace  $S_\bullet(X)$  with the complex  $\tilde{S}_\bullet(X)$ , which is the same as  $S_\bullet(X)$  except in degree -1 we have  $\tilde{S}_{-1}(X) = \mathbb{Z}$  and the boundary map

$$\partial_0 : \tilde{S}_0(X) \longrightarrow \tilde{S}_{-1}(X) : T \mapsto 1$$

for any 0-simplex  $T$ . We then get the  $p$ -th reduced homology of  $X$  as  $\tilde{H}_p(X) := H_p(\tilde{S}_\bullet(X))$ .

Given a field  $\mathbb{F}$ , we define homology with co-efficients in  $\mathbb{F}$ , by replacing all the  $S_p(X)$  with

$$S_p(X, \mathbb{F}) = \bigoplus_{p\text{-simplex } T} \mathbb{F}T$$

i.e. the vector space over  $\mathbb{F}$  with basis the  $p$ -simplices of  $X$ . This gives functors  $H_p(-, \mathbb{F}) : \underline{Top} \longrightarrow \underline{Vect}_{\mathbb{F}}$ .

## 16 Acyclicity of simplices

**Aim lecture** We show that star convex top spaces have trivial reduced homology.

$$X = \text{topological space}$$

### Bracket operation

For a simplicial cone  $K * w$ , we used the map  $C_p(K) \longrightarrow C_{p+1}(K * w) : \sigma \mapsto \sigma w$ . We need the topological analogue called the *bracket operation*.

We consider the  $p+1$ -st face  $l_{p+1} : \Delta_p \longrightarrow \Delta_{p+1} : \varepsilon_i \mapsto \varepsilon_i$  which we view as the natural embedding. We extend this to a surjective continuous map  $\phi : \Delta_p \times I \longrightarrow \Delta_{p+1} : (x, t) \mapsto (1-t)l_{p+1}x + t\varepsilon_{p+1}$ . This identifies  $\Delta_{p+1}$  as quotient space of  $\Delta_p \times I$ , more precisely,  $\phi$  induces a homeomorphism  $(\Delta_p \times I) / \sim \simeq \Delta_{p+1} : [(v, t)] \mapsto \phi(v, t)$  where the equivalence relation is generated by  $(v, 1) \sim (v', 1)$  for all  $v, v' \in \Delta_p$ .

Suppose  $X$  is star convex relative to  $w$ . Given a  $p$ -simplex  $T : \Delta_p \longrightarrow X$  we can construct a  $p+1$ -simplex  $[T, w]$  as follows. Consider the continuous map  $T_I : \Delta_p \times I \longrightarrow X : (v, t) \mapsto (1-t)T(v) + tw$ . Because  $T_I(v, 1) = T_I(v', 1)$ , it induces a continuous map  $[T, w] : \Delta_{p+1} \longrightarrow X$ .

DRAW commutative diagram in class instead + picture.

Extending linearly, gives a group hom  $[-, w] : \tilde{S}_p(X) \longrightarrow \tilde{S}_{p+1}(X)$ . By default we set  $\tilde{S}_{-1}(X) \longrightarrow \tilde{S}_0(X) : n \mapsto nT_w$ .

### Acyclicity

**Lemma 16.1** *Let  $X$  be star convex rel to  $w$ .*

*i. For a  $p$ -simplex  $T$ ,  $[T, w]l_i = [Tl_i, w]$  for  $i \leq p$  (beware the  $l_i$  on the LHS is different from the  $l_i$  on the RHS).*

*ii. For  $c \in \tilde{S}_p(X)$ , we have*

$$\partial[c, w] = [\partial c, w] + (-1)^{p+1}c.$$

**Proof.** (i) is an ex but is best understood from a picture.

(ii) By linearity of LHS & RHS, it suffices to note

$$\partial[T, w] = \sum_{i=0}^{p+1} (-1)^i [T, w]l_i = \sum_{i=0}^p (-1)^p [Tl_i, w] + (-1)^{p+1} [T, w]l_{p+1} = [\partial T, w] + (-1)^{p+1}T.$$

□

**Definition 16.2** *We say  $X$  is acyclic if  $\tilde{H}_p(X) = 0$  for all  $p$ .*

**Theorem 16.3** *Let  $X$  be star convex rel to  $w$ . Then  $X$  is acyclic. In particular, any geometric simplex is acyclic.*

**Proof.** We show that any  $c \in \tilde{Z}_p(X)$  lies in  $\tilde{B}_p(X)$ . Indeed

$$\partial[c, w] = [\partial c, w] + (-1)^{p+1}c = (-1)^{p+1}c$$

so  $\partial(-1)^{p+1}[c, w] = c$ . □

### Compact support axiom

**Proposition 16.4** *i. Let  $\alpha \in H_p(X)$ . Then there is a compact subspace  $X_0 \subseteq X$ , and  $\alpha_0 \in H_p(X_0)$  such that  $\iota_*(\alpha_0) = \alpha$  where  $\iota : X_0 \hookrightarrow X$  is the inclusion.*

*ii. Let  $Y \subseteq X$  be a compact subspace and  $\beta \in H_p(Y)$  be such that  $\iota_*(\beta) = 0$  where  $\iota : Y \hookrightarrow X$  is inclusion. Then there is a compact subspace  $X_1 \subseteq X$  containing  $Y$  such that  $\iota'_*\beta = 0$  where  $\iota' : Y \hookrightarrow X_1$  is inclusion.*

**Proof.** Both are similar, so we just prove (ii) (the harder one and the one we need). Let  $b \in Z_p(Y)$  be a singular  $p$ -cycle representing  $\beta$ . Since  $\iota_*\beta = 0$ , we know there is a singular  $p+1$ -chain  $d \in S_{p+1}(X)$  such that  $b = \partial d$ . Write  $d = \sum_i n_i T_i$  for some singular  $p$ -simplices  $T_i$  and  $n_i \in \mathbb{Z}$ . Note that  $\text{im}(T_i : \Delta_{p+1} \rightarrow X)$  is compact so  $X_1 := Y \cup \bigcup_i \text{im } T_i$  is too. This is the desired subspace as clearly  $\iota'_*\beta = 0$  too. □

### $H_0(X)$

Just for simplicial homology,  $H_0$  measures the path connected components.

**Proposition 16.5** *Let  $I$  be the set of path components of  $X$  and  $\{x_i\}_{i \in I}$  be a set of representative points, i.e.  $x_i$  is a point in the path component  $i$ . Then*

$$H_0(X) \simeq \oplus_{i \in I} \mathbb{Z} T_{x_i}.$$

**Proof.** Let  $A := \oplus_i \mathbb{Z} T_{x_i} \subseteq S_0(X) = Z_0(X)$ . Note  $A + B_0(X) = Z_0(X)$  since DRAW PICTURE.

Also  $A \cap B_0(X) = 0$  since if  $\sum_i n_i T_{x_i} \in B_0(X)$  then DRAW PICTURE

Then  $H_0(X) \simeq A/(A \cap B_0(X)) = A$ . □

## 17 Relative homology

**Aim lecture** We introduce a version of singular homology for topological pairs.

$X$  = topological space,  $(X, A)$  = topological pair

### Quotient complexes



Let  $C_\bullet$  be a complex and  $C'_\bullet$  be a subcomplex. We define the *quotient complex*  $C_\bullet/C'_\bullet$  to be the complex  $C''_\bullet$  with  $C''_p = C_p/C'_p$  and boundary maps  $\partial''$  induced by the boundary map  $\partial$  of  $C_\bullet$ , i.e.

$$\partial''_p : C''_p = C_p/C'_p \longrightarrow C''_{p-1} = C_{p-1}/C'_{p-1} : c + C'_p \mapsto \partial c + C'_{p-1}.$$

Note it is indeed a complex as  $\partial^2 = 0 \implies \partial''^2 = 0$ .

Suppose now we have a chain map  $f_\bullet : C_\bullet \longrightarrow D_\bullet$ , and that  $D'_\bullet$  is a subcomplex of  $D_\bullet$  such that  $f_p(C'_p) \subseteq D'_p$ . Then there is an induced chain map  $\bar{f}_\bullet : C_\bullet/C'_\bullet \longrightarrow D_\bullet/D'_\bullet$  defined by

$$\bar{f}_p : C_p/C'_p \longrightarrow D_p/D'_p : c + C'_p \mapsto f(c) + D'_p.$$

It is readily seen to be well-defined and is a chain map because  $f_\bullet$  is.

### Relative homology

Note  $S_\bullet(A)$  is a subcomplex of  $S_\bullet(X)$  so we may define  $S_p(X, A)$  to be the quotient complex  $S_\bullet(X)/S_\bullet(A)$  and the (relative) *homology of*  $(X, A)$  to be

$$H_p(X, A) := H_p(S_\bullet(X, A)).$$

Suppose  $f : (X, A) \longrightarrow (Y, B)$  is a continuous map of pairs. Then  $f_\#(S_p(A)) \subseteq S_p(B)$  so there is an induced chain map  $S_\bullet(X, A) \longrightarrow S_\bullet(Y, B)$  which induces group homomorphisms  $f_* = H_p(f) : H_p(X, A) \longrightarrow H_p(Y, B)$ . We easily obtain

**Proposition 17.1**  $H_p : \underline{TopPair} \longrightarrow \underline{Ab}$  is a covariant functor.

### Exact sequences

Let  $A_\bullet$  be a complex. We say that  $A_\bullet$  is *exact* at  $A_p$  if  $H_p(A_\bullet) = 0$ . If it is exact at every  $A_p$ , then we say  $A_\bullet$  is *exact* or *acyclic*.

**E.g.** Given a subgroup  $B < A$  with  $A$  abelian, we get an exact sequence

$$0 \longrightarrow B \xrightarrow{\iota} A \xrightarrow{\pi} A/B \longrightarrow 0$$

where  $\iota$  is inclusion and  $\pi : a \mapsto a + B$ . It's easy to check this.

The converse is the following:

**Proposition 17.2** Consider a complex of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

(which just means  $gf = 0$ ).

- i. Exactness at  $A$  means  $f$  is injective.
- ii. Exactness at  $C$  means  $g$  is surjective.
- iii. Exactness of the sequence thus means that  $g$  induces an isomorphism  $C \simeq B/f(A)$  via the first isomorphism theorem. In this case we say the complex is a short exact sequence (SES for short).

**Proof.** i) Exact at  $A$  iff  $\ker f = 0$  iff  $f$  is inj.

ii) Exact at  $C$  iff  $\operatorname{im} g = C$ .

iii) Exact at  $B$  iff  $\ker g = f(A)$ , so first isomorphism thm completes the proof.  $\square$

The following is the famous 5-lemma.

**Lemma 17.3** Consider the commutative diagram of abelian groups and group homomorphisms below.

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{d_1} & A_2 & \xrightarrow{d_2} & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\ B_1 & \xrightarrow{e_1} & B_2 & \xrightarrow{e_2} & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

Suppose that the two rows are exact and that  $f_1, f_2, f_4, f_5$  are isomorphisms. Then  $f_3$  is an isomorphism too.

**Proof.** This is done by “diagram chasing” and best shown on a board (see lectures). We’ll briefly sketch the proof here, but it reads rather terse.

We’ll show  $f_3$  is injective, surjectivity being similar. Let  $a_3 \in \ker f_3$ . Let its image in  $A_4, B_3, B_4$  be  $a_4, b_3, b_4$  so  $b_3 = b_4 = 0$ . Now  $f_4$  is bijective so  $a_4 = 0$ . Also exactness at  $A_3$  means that there’s some  $a_2 \in A_2$  with  $d_2(a_2) = a_3$ . Now setting  $b_2 := f_2(a_2)$  then  $e_2(b_2) = f_3 d_2(a_2) = b_3 = 0$  so exactness at  $B_2$  means there’s some  $b_1$  with  $e_1(b_1) = b_2$ . Bijectivity of  $f_1$  means that we can find  $a_1 \in A_1$  with  $f_1(a_1) = b_1$  and  $f_2$  bijective ensures  $d_1(a_1) = a_2$ . Hence  $a_3 = d_2 d_1(a_1) = 0$ .  $\square$

## Long exact sequence in homology

**Proposition 17.4** Let  $(X, A)$  be a top pair. Let  $i : A \hookrightarrow X$  &  $\pi : X = (X, \emptyset) \rightarrow (X, A)$  be inclusion maps. Then there is a long exact sequence associated to  $(X, A)$

$$\begin{aligned} \dots \rightarrow H_p(A) &\xrightarrow{i_*} H_p(X) \xrightarrow{\pi_*} H_p(X, A) \\ &\rightarrow H_{p-1}(A) \xrightarrow{i_*} H_{p-1}(X) \xrightarrow{\pi_*} H_{p-1}(X, A) \rightarrow \dots \end{aligned}$$

If  $f : (X, A) \rightarrow (Y, B)$  is a continuous map of pairs, then the maps  $H_p(f)$  assemble to give a chain map between the two LES = (long exact sequences).

**Proof.** Follows from the LES proved in 18.1.  $\square$

**E.g.** Let  $(X, A)$  be a topological pair with  $A$  acyclic. Then  $\text{LES} \implies H_p(X, A) \simeq H_p(X)$  for  $p > 1$ . If  $A$  and  $X$  are connected, then  $H_1(X, A) \simeq H_1(X)$ .

## 18 Long exact sequence

**Aim lecture** We relate the homologies of a complex with that of a subcomplex and the corresponding quotient.

$C_\bullet, D_\bullet, E_\bullet$  are chain complexes with boundary maps  $\partial_C, \partial_D, \partial_E$

### Short exact sequence of complexes

We say

$$0 \longrightarrow C_\bullet \xrightarrow{f_{bu}} D_\bullet \xrightarrow{g_\bullet} E_\bullet \longrightarrow 0$$

is a *short exact sequence* of chain complexes if for every  $p$  we have

$$0 \longrightarrow C_p \xrightarrow{f_p} D_p \xrightarrow{g_p} E_p \longrightarrow 0$$

is a SES of groups.

**E.g.** If  $C'_\bullet$  is a subcomplex of  $C_\bullet$  then the canonical inclusion and quotient maps give a SES of chain complexes

$$0 \longrightarrow C'_\bullet \longrightarrow C_\bullet \longrightarrow C_\bullet/C'_\bullet \longrightarrow 0.$$

A *morphism* of SES of chain complexes is a commutative diagram in Ch of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C'_\bullet & \xrightarrow{f'_\bullet} & D'_\bullet & \xrightarrow{g'_\bullet} & E'_\bullet & \longrightarrow & 0 \\ & & \downarrow \gamma & & \downarrow & & \downarrow \varepsilon & & \\ 0 & \longrightarrow & C_\bullet & \xrightarrow{f_\bullet} & D_\bullet & \xrightarrow{g_\bullet} & E_\bullet & \longrightarrow & 0 \end{array} \quad (1)$$

where the rows are SES of chain complexes.

Note that for any zero chain map  $0_\bullet : C_\bullet \rightarrow D_\bullet$  we have  $H_p(0_\bullet) = 0$  so given a SES  $0 \rightarrow C_\bullet \rightarrow D_\bullet \rightarrow E_\bullet \rightarrow 0$ , we obtain a complex

$$0 \rightarrow H_p(C_\bullet) \rightarrow H_p(D_\bullet) \rightarrow H_p(E_\bullet) \rightarrow 0.$$

One might suspect or hope that this sequence is also exact. The truth (below) is both more complicated and infinitely more interesting.

### Statement of Main Theorem

**Theorem 18.1** *Let  $0 \rightarrow C_\bullet \xrightarrow{f_\bullet} D_\bullet \xrightarrow{g_\bullet} E_\bullet \rightarrow 0$  be a SES of chain complexes.*

- i. There is a group hom  $\partial_p : H_p(E_\bullet) \rightarrow H_{p-1}(C_\bullet)$  which is natural wrt morphisms of SESs of chain complexes. i.e. given a morphism as in (1), there is a commutative diagram.*

$$\begin{array}{ccc} H_p(E'_\bullet) & \xrightarrow{\partial'_p} & H_{p-1}(C'_\bullet) \\ \downarrow & & \downarrow \\ H_p(E_\bullet) & \xrightarrow{\partial_p} & H_{p-1}(C_\bullet) \end{array}$$

$\partial_p$  is called the connecting homomorphism and is induced by  $\partial_D$ .

- ii. There is a long exact homology sequence*

$$\begin{aligned} \dots \rightarrow H_p(C_\bullet) &\xrightarrow{f_*} H_p(D_\bullet) \xrightarrow{g_*} H_p(E_\bullet) \\ &\xrightarrow{\partial_p} H_{p-1}(C_\bullet) \xrightarrow{f_*} H_{p-1}(D_\bullet) \xrightarrow{g_*} H_{p-1}(E_\bullet) \rightarrow \dots \end{aligned}$$

*Note that part i) and functoriality of  $H_p$  mean that the whole long exact sequence is natural wrt morphisms of SESs of chain complexes.*

**Remark** This proves the LES for a top pair  $(X, A)$ , proposition 17.4 on applying the theorem to the SES

$$0 \rightarrow S_\bullet(A) \rightarrow S_\bullet(X) \rightarrow S_\bullet(X, A) \rightarrow 0.$$

**Proof.** We spend the rest of this lecture proving this. Our first goal is to define the

### Connecting homomorphism

This is best defined on a board using the diagram

$$\begin{array}{ccccccc}
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & C_p & \xrightarrow{f_p} & D_p & \xrightarrow{g_p} & E_p \longrightarrow 0 \\
 & & \partial_{C,p} \downarrow & & \partial_{D,p} \downarrow & & \partial_{E,p} \downarrow \\
 0 & \longrightarrow & C_{p-1} & \xrightarrow{f_{p-1}} & D_{p-1} & \xrightarrow{g_{p-1}} & E_{p-1} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow
 \end{array} \quad . \quad (2)$$

Our definition initially depends on many choices. Let  $e_p \in E_p$  be a  $p$ -cycle representing the homology class  $[e_p] \in H_p(E_\bullet)$ . We wish to define  $\partial_p[e_p] \in H_{p-1}(C_\bullet)$ . Exactness of the row at  $E_p$  means we can find  $d_p \in D_p$  such that  $g_p(d_p) = e_p$ . Let  $d_{p-1} = \partial_D(d_p)$ . Since the RH square commutes,  $g(d_{p-1}) = \partial_E(e_p) = 0$ . Exactness of the row at  $D_{p-1}$  means that there's some  $c_{p-1} \in C_{p-1}$  with  $f(c_{p-1}) = d_{p-1}$ . Exactness of the row at  $C_{p-1}$  ensures that  $c_{p-1}$  is uniquely determined by  $d_{p-1}$ . We define

$$\partial_p[e_p] = [c_{p-1}].$$

We check independence of the choice of  $d_p$  (here we leave  $e_p$  fixed). We could have changed it by  $f(c_p)$  for some  $c_p \in C_p$ , that is, replaced  $d_p$  with  $d_p + f(c_p)$ . Since the LH square commutes, this just changes  $c_{p-1}$  by  $\partial_C(c_p)$  which is a  $(p-1)$ -boundary. Hence the homology class  $[c_{p-1}]$  is independent of the choice of  $d_p$ .

We need also check independence of the choice of  $e_p$ . We could have changed it by a  $p$ -boundary  $\partial_E(e_{p+1})$  where  $e_{p+1} \in E_{p+1}$ . Picking  $d_{p+1}$  st  $g(d_{p+1}) = e_{p+1}$  means  $d_p$  changes by  $\partial_D(d_{p+1})$ . This changes  $d_{p-1}$  by  $\partial^2(d_{p+1}) = 0$ .

This shows that  $\partial_p$  is well-defined, and by construction, it is additive i.e. a group hom.

To see naturality, we need to consider the 3-dimensional lattice of morphisms obtained by placing a primed version of (2) on top of (2). All squares of the cube commute and chasing through the defns shows naturality.

### Proof of exactness

At  $H_p(D_\bullet)$  We have already noted that  $g_*f_* = 0$  so we need only check if  $[d_p] \in \ker H_p(g_\bullet) = g_*$ , then  $[d_p] \in \text{im } H_p(f_\bullet) = f_*$ . Now  $[d_p] \in \ker H_p(g_\bullet) \implies$  we can find  $e_{p+1}$  st  $\partial(e_{p+1}) = g(d_p)$ . Pick  $d_{p+1} \in D_{p+1}$  st  $g(d_{p+1}) = e_{p+1}$ . Now commutativity of RH square means that  $g(d_p - \partial(d_{p+1})) = e_p - e_p = 0$  so there's  $c_p \in C_p$  with  $f(c_p) = d_p - \partial(d_{p+1})$ . We conclude

$$f_*[c_p] = [d_p - \partial(d_{p+1})] = [d_p].$$

At  $H_p(C_\bullet)$  By construcion of the connecting hom,  $f_*\partial_p = 0$  so it suffices to show given  $[c_{p-1}] \in \ker f_*$ , we can find  $[e_p] \in H_p(E_\bullet)$  with  $\partial_p[e_p] = [c_{p-1}]$ . Now  $[c_{p-1}] \in \ker f_* \implies$  there's  $d_p \in D_p$  with  $\partial_D(d_p) = f(c_{p-1})$  and if  $e_p := g(d_p)$  is a  $p$ -cycle, then we are done as then  $\partial_p[e_p] = [c_{p-1}]$ . But  $\partial_E(e_p) = g(\partial_D d_p) = gf(c_{p-1}) = 0$  so we're done.

At  $H_p(C_\bullet)$  Similar proof left as ex.

□

## 19 Homotopy invariance

**Aim lecture** We show that homotopic maps induce the same map on homology so the homology functor gives functor  $\underline{H}\text{Top} \rightarrow \underline{Ab}$ .

$$f_0, f_1 : X \rightarrow Y \text{ continuous, } F : f_0 \approx f_1 : X \times I \rightarrow Y \text{ homotopy}$$

DRAW PICTURE

We define  $h_0, h_1 : X \rightarrow X \times I$  by  $h_j(x) = (x, j)$  and note  $H := \text{id}_{X \times I} : X \times I \rightarrow X \times I$  is a homotopy from  $i_0$  to  $i_1$ . In fact, it is the universal homotopy out of  $X$  in the sense that  $F = FH$  is the homotopy from  $f_0 = Fh_0$  to  $f_1 = Fh_1$ . When  $X = \Delta_p$  we use the more specialised notation  $H^\Delta, h_0^\Delta, h_1^\Delta$ .

### Main theorem

**Theorem 19.1** *With above notation,  $f_{0*} = f_{1*} : H_p(X) \rightarrow H_p(Y)$ . Hence the homology functor can be viewed as a functor  $H_p : \underline{H}\text{Top} \rightarrow \underline{Ab}$ .*

**Remark:** There is a similar thm (with similar proof) for reduced homology and homology of a top pair.

**Proof.** By 7.6, it suffices to produce a chain homotopy between  $f_{0\#} = S_\bullet(f_0)$  and  $f_{1\#} = S_\bullet(f_1)$ . We do this in the special case of the universal homotopy  $H$  above.

**Lemma 19.2** *There are maps  $D_X = D_{X,p} : S_p(X) \rightarrow S_{p+1}(X \times I)$  such that*

i.  $D_X$  is a chain homotopy between  $h_{0\#}, h_{1\#}$  i.e. For singular  $p$ -simplex  $T : \Delta_p \rightarrow X$ ,

$$\partial D_X T + D_X \partial T = h_{1\#} T - h_{0\#} T. \quad (3)$$

ii.  $D_X$  is natural in  $X$ , i.e. given cont  $g : X \rightarrow Y$ , there is a commutative square

$$\begin{array}{ccc} S_p(X) & \xrightarrow{D_X} & S_{p+1}(X \times I) \\ g_\# \downarrow & & (g \times I)_\# \downarrow \\ S_p(Y) & \xrightarrow{D_Y} & S_{p+1}(Y \times I) \end{array}$$

**Proof thm assuming lemma:** The required chain homotopy is  $D = F_\# D_X$  since the lemma gives

$$\partial D = \partial F_\# D_X = F_\# \partial D_X = F_\# (h_{1\#} - h_{0\#} - D_X \partial) = (Fh_1)_\# - (Fh_0)_\# - F_\# D_X \partial = f_{1\#} - f_{0\#} - D \partial.$$

□

### Proof lemma

**Proof.** This is proved by induction on  $p$ , using the *method of acyclic models*. For  $p = 0$  we define

$$D_X T_x : \Delta_1 \simeq I \longrightarrow X \times I : t \mapsto (x, t).$$

DRAW PICTURE

Inductive step Key is to define  $D_X T$  first for  $X = \Delta_p$  &  $T = T_{\Delta_p} = T_\Delta = \text{id}_\Delta$ . Consider  $s_p \in S_p(\Delta_p \times I)$  given by

$$s_p = h_{1\#}^\Delta T_\Delta - h_{0\#}^\Delta T_\Delta - D_{\Delta_p}(\partial T_\Delta)$$

which is well-defined by induction. By induction

$$\partial D_{\Delta_p}(\partial T_\Delta) = h_{1\#}^\Delta(\partial T_\Delta) - h_{0\#}^\Delta(\partial T_\Delta) - D_{\Delta_p}(\partial^2 T_\Delta).$$

Now  $s_p$  is a cycle because

$$\partial s_p = \partial h_{1\#}^\Delta T_\Delta - \partial h_{0\#}^\Delta T_\Delta - (h_{1\#}^\Delta(\partial T_\Delta) - h_{0\#}^\Delta(\partial T_\Delta)) = 0.$$

Now  $\Delta_p \times I$  is convex, hence star convex and thus acyclic by theorem 16.3. Thus  $H_p(\Delta_p \times I) = 0$  and the  $p$ -cycle  $s_p$  must be a boundary say  $s_p = \partial r_p$ . We define

$$D_{\Delta_p}(T_\Delta) = r_p$$

so eqn (3) holds for  $X = \Delta_p, T = T_\Delta$ .

Now for general  $X$  and  $T = T \text{id}_{\Delta_p} = T_\#(T_\Delta)$ , we let

$$D_X T = (T \times \text{id}_I)_\# D_{\Delta_p}(T_\Delta).$$

Proving parts i) & ii) are easy. For i) note that  $(T \times \text{id}_I)h_j^\Delta = h_j T$  so by the above we have

$$\begin{aligned} \partial D_X T &= \partial(T \times \text{id}_I)_\# D_{\Delta_p}(T_\Delta) = (T \times \text{id}_I)_\# \partial D_{\Delta_p}(T_\Delta) \\ &= (T \times \text{id}_I)_\# (h_{1\#}^\Delta - h_{0\#}^\Delta - D_{\Delta_p} \partial)(T_\Delta) = (h_{1\#} T_\# - h_{0\#} T_\#)(T_\Delta) - D_X \partial(T_\Delta) \end{aligned}$$

and the last equality follows by naturality in dimension  $p - 1$ .

For ii), let  $g : X \longrightarrow W$  be continuous and note

$$D_W g_\#(T) = D_W(gT) = (gT \times \text{id}_I)_\# D_{\Delta_p}(T_\Delta) = (g \times \text{id}_I)_\# (T \times \text{id}_I)_\# D_{\Delta_p}(T_\Delta) = (g \times \text{id}_I)_\# D_X(T).$$

□

## 20 Barycentric subdivision operator

**Aim lecture** We introduce the useful tool of barycentric subdivision, which allows us to replace chains with arbitrarily fine chains.

$$X = \text{topological space}$$

**Basic idea**

Let  $a, b, c \in \mathbb{R}^N$ . Then  $T = l(\Delta_2, \langle abc \rangle)$  has boundary

$$\partial T = l(\Delta_2, \langle bc \rangle) - l(\Delta_2, \langle ac \rangle) + l(\Delta_2, \langle ab \rangle).$$

If  $a, b, c$  are collinear this can be pictured as

We can repeat this process as we like.

We wish to generalise this to higher dimensions.

### Subdivision operator

We wish to define a chain map  $\text{sd}_X \bullet : \tilde{S}_\bullet(X) \rightarrow \tilde{S}_\bullet(X)$ . We do so by induction on degree with  $\text{sd}_{Xp} = 0$  for  $p = -1, 0$ .

Let  $p > 0$ . We first look at the case  $X = \Delta_p, T = T_\Delta \in S_p(X)$  and let  $w_p$  be the barycentre of  $\Delta_p$ , i.e. the point with barycentric co-ords  $\frac{1}{p+1}(1, 1, \dots, 1)$ . We define  $\text{sd}_p(T_\Delta) = (-1)^p[\text{sd}_{p-1}(\partial T_\Delta), w_p]$  where the bracket operation here was defined in lecture 16.

For general  $X$ , we define  $\text{sd}_X : S_p(X) = \tilde{S}_p(X) \rightarrow S_p(X)$  on singular  $p$ -chains by

$$\text{sd}_X(T) = T_\#(\text{sd}_{\Delta_p}(T_\Delta)).$$

**Proposition 20.1**  $\text{sd}_X$  is natural in  $X$ , i.e. given continuous  $f : X \rightarrow Y$ , the following diagram commutes

$$\begin{array}{ccc} \tilde{S}_p(X) & \xrightarrow{\text{sd}_X} & \tilde{S}_p(X) \\ f_\# \downarrow & & f_\# \downarrow \\ \tilde{S}_p(Y) & \xrightarrow{\text{sd}_Y} & \tilde{S}_p(Y) \end{array}$$

**Proof.** Let  $T$  be a singular  $p$ -simplex. Then

$$f_\# \text{sd}_X(T) = f_\# T_\# \text{sd}_{\Delta_p}(T_\Delta) = (fT)_\# \text{sd}_{\Delta_p}(T_\Delta) = \text{sd}_Y(fT) = \text{sd}_Y f_\# T.$$

□

**Proposition 20.2**  $\text{sd}_X$  is a chain map.

**Proof.** We prove  $\text{sd}_{p-1} \partial_p = \partial_{p-1} \text{sd}_p$  by induction on  $p$ . For  $p \leq 0$  this holds as both sides are 0. Given  $T$  a  $p$ -simplex, we compute using the inductive hypothesis and naturality

$$\begin{aligned} \partial \text{sd}(T) &= \partial(-1)^p T_\# [\text{sd} \partial T_\Delta, w_p] = (-1)^p T_\# \partial [\text{sd} \partial T_\Delta, w_p] \\ &= (-1)^p T_\# ([\partial \text{sd} \partial T_\Delta, w_p] + (-1)^p \text{sd} \partial T_\Delta) \\ &= (-1)^p T_\# ([\text{sd} \partial^2 T_\Delta, w_p] + (-1)^p \text{sd} \partial T_\Delta) \\ &= \text{sd}_X T_\# \partial T_\Delta = \text{sd}_X \partial T \end{aligned}$$

□

### Diameter of subdivisions

Let  $\langle \sigma \rangle = \langle a_0 \dots a_p \rangle$  be a geometric  $p$ -simplex and  $T_\sigma = l(\Delta_p, \langle \sigma \rangle)$  be the corresponding singular  $p$ -simplex. Then  $\text{sd}^m T_\sigma$  will be a sum of  $p$ -simplices  $T$  and we call  $\text{im } T$  a geometric simplex of  $\text{sd}^m \sigma$  (note we haven't defined the latter, but we won't need to).

**Proposition 20.3** *The diameters (wrt say the  $|\cdot|_\infty$  norm) of the geometric simplices of  $\text{sd}^m \sigma$  tend to 0 as  $m \rightarrow \infty$ .*

**Proof.** We omit the tedious but elementary proof of this fact which is readily seen from any picture.

The key fact is that the simplices of  $\text{sd } \sigma$  have diameter at most  $\frac{p}{p+1}$  that of  $\sigma$ . □

### Admissible covers and subdivision

Let  $X$  be a top space and  $\mathcal{A}$  be a collection of subsets. We say  $\mathcal{A}$  is an *admissible cover* if the interiors of  $A \in \mathcal{A}$  form an open cover of  $X$ . We say a singular  $p$ -chain is  $\mathcal{A}$ -small if it is a linear combination of  $p$ -simplices  $T$  such that  $\text{im } T \subseteq A_T$  for some  $A_T \in \mathcal{A}$ .

Hopefully, you know the following from MATH3611.

**Lemma 20.4** *Let  $X$  be a compact metric space and  $\mathcal{U}$  be an open cover. There is a positive real number  $\lambda$  (called the Lebesgue number for covers) such that any subset  $Z \subseteq X$  with diameter  $< \lambda$  is contained in some  $U \in \mathcal{U}$ .*

**Proof.** If no such  $\lambda$  exists, then for each  $n \in \mathbb{N}$ , we can find a subset  $Z_n$  with diameter  $< \frac{1}{n}$  which does not lie in any  $U \in \mathcal{U}$ . Pick  $z_n \in Z_n$ . By compactness, the sequence  $z_n$  has convergent subsequence, whose limit is say  $z \in X$ . Pick  $U \in \mathcal{U}$  containing  $z$  so there is some  $\varepsilon$ -ball  $B_z$  centred at  $z$  which lies in  $U$ . We obtain a contradiction as for  $n \gg 0$  we must have  $Z_n \subseteq B_z$ . □

Proposition 20.3 and this lemma give

**Proposition 20.5** *Let  $\mathcal{A}$  be an admissible cover of  $X$ . For any  $p$ -simplex  $T$  of  $X$ ,  $\text{sd}^m T$  is a linear combination of  $\mathcal{A}$ -small simplices for  $m$  large enough.*

**Proof.** We may assume  $\mathcal{A}$  is an open cover by replacing each  $A \in \mathcal{A}$  with its interior. Then  $\{T^{-1}(U) | U \in \mathcal{A}\}$  is an open cover of  $\Delta_p$  with say, Lebesgue number  $\lambda$ . Any  $m$  such that the diameters of the geometric simplices of  $\text{sd}^m \Delta_p$  are  $< \lambda$  will do. □

## 21 Excision

**Aim lecture** We prove the excision theorem which allows cutting & pasting type arguments.

$X$  = topological space,  $\mathcal{A}$  = admissible cover

**Subdivision is homotopy equivalent to the identity**



**Theorem 21.1** Fix  $m$ . For each  $p$ , there is a natural transformation  $D : \tilde{S}_p \longrightarrow \tilde{S}_{p+1}$  which induces a chain homotopy between  $\text{sd}^m$  and  $\text{id}$ , i.e. for any  $p$ -simplex  $T$  of  $X$ , we have

$$\partial D_X T + D_X \partial T = \text{sd}_X^m T - T. \quad (4)$$

**Proof.** (Sketch) We use the method of acyclic models as in Lemma 19.2. We use induction on  $p$  with  $D_X = 0$  if  $p \leq 0$ .

Assume  $D_X$  defined satisfying (4) and naturality for  $d$ -simplices,  $d < p$ . We next define  $D_X T$  for  $X = \Delta_p, T = T_\Delta$  so (4) holds:

$$\partial D_{\Delta_p} T_\Delta + D_{\Delta_p} \partial T_\Delta = \text{sd}^m T_\Delta - T_\Delta. \quad (5)$$

Now  $\Delta_p$  is acyclic, so this is possible so long as

$$z_p := -D_{\Delta_p} \partial T_\Delta + \text{sd}^m T_\Delta - T_\Delta$$

is a cycle. This follows by a simple computation using induction.

We can now define for general  $X, T$ ,  $D_X T = T_\# D_{\Delta_p} T_\Delta$ . Naturality follows readily and applying  $T_\#$  to (5) using proposition 20.1 gives (4).  $\square$

#### $\mathcal{A}$ -small chain complex

We let  $\tilde{S}_p^{\mathcal{A}}(X)$  be the subgroup of  $\tilde{S}_p(X)$  generated by  $\mathcal{A}$ -small simplices. Note that these define a subcomplex  $\tilde{S}_\bullet^{\mathcal{A}}(X)$ . For a topological pair  $(X, B)$  we define  $S_\bullet^{\mathcal{A}}(X, B) = \tilde{S}_\bullet^{\mathcal{A}}(X) / \tilde{S}_\bullet^{\mathcal{A}}(B)$ .

**Lemma 21.2** The quotient chain complex  $\bar{S}_\bullet(X) := \tilde{S}_\bullet(X) / \tilde{S}_\bullet^{\mathcal{A}}(X)$  is acyclic.

**Proof.** Let  $c_p + \tilde{S}_p^{\mathcal{A}}(X)$  be a  $p$ -cycle in  $\bar{S}_p(X)$ . This means that  $\partial c_p \in \tilde{S}_{p-1}^{\mathcal{A}}$ . Pick  $m$  large enough so  $\text{sd}_X^m c_p \in \tilde{S}_p^{\mathcal{A}}(X)$ .

We use the chain homotopy  $D_X$  of theorem 21.1 between  $\text{sd}_X^m$  and  $\text{id}$ . Then

$$\partial D_X c_p = -D_X \partial c_p + \text{sd}_X^m c_p - c_p.$$

Now naturality of  $D_X$  ensures that  $D_X \partial c_p \in \tilde{S}_p^{\mathcal{A}}(X)$  too so

$$\partial(D_X c_p + \tilde{S}_p^{\mathcal{A}}(X)) = -c_p + \tilde{S}_p^{\mathcal{A}}(X).$$

Hence  $c_p + \tilde{S}_p^{\mathcal{A}}(X)$  is also a boundary.  $\square$

**Theorem 21.3** The inclusion  $\iota_\bullet : \tilde{S}_\bullet^{\mathcal{A}}(X) \longrightarrow \tilde{S}_\bullet(X)$  chain map induces an isomorphism on homology.

**Proof.** We use the long exact homology sequence on

$$0 \longrightarrow \tilde{S}_\bullet^{\mathcal{A}}(X) \longrightarrow \tilde{S}_\bullet(X) \longrightarrow \tilde{S}_\bullet(X) / \tilde{S}_\bullet^{\mathcal{A}}(X) \longrightarrow 0$$

which, by acyclicity proven in the lemma, is

$$0 \longrightarrow H_p(\tilde{S}_\bullet^{\mathcal{A}}(X)) \xrightarrow{\iota_*} H_p(\tilde{S}_\bullet(X)) \longrightarrow 0.$$

The theorem follows as then  $\iota_*$  is both injective and surjective.  $\square$

**Corollary 21.4** For a topological pair  $(X, B)$ , the inclusion  $\iota_\bullet : S_\bullet^{\mathcal{A}}(X, B) \longrightarrow S_\bullet(X, B)$  induces an isomorphism on homology.

**Proof.** Consider the map of SES of chain complexes

$$\begin{array}{ccccccc}
0 & \longrightarrow & \tilde{S}_{\bullet}^{\mathcal{A}}(B) & \longrightarrow & \tilde{S}_{\bullet}^{\mathcal{A}}(X) & \longrightarrow & \tilde{S}_{\bullet}^{\mathcal{A}}(X, B) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \tilde{S}_{\bullet}(B) & \longrightarrow & \tilde{S}_{\bullet}(X) & \longrightarrow & \tilde{S}_{\bullet}(X, B) \longrightarrow 0
\end{array}$$

There is a corresponding map on LES in homology which by the previous theorem are isomorphisms on the homologies of  $B$  and  $X$ . The 5-lemma gives the corollary.  $\square$

### Excision theorem

**Theorem 21.5** *Let  $(X, A)$  be a top pair, and  $U \subset A$  be such that  $\bar{U}$  is in the interior  $\text{Int}A$  of  $A$ . Then the inclusion  $(X - U, A - U) \rightarrow (X, A)$  induces an isomorphism in homology.*

**Proof.** We consider the admissible cover  $\mathcal{A} = \{X - U, A\}$ . By corollary 21.4, it suffices to prove that the natural maps

$$\frac{S_p(X - U)}{S_p(A - U)} \longrightarrow \frac{S_p^{\mathcal{A}}(X)}{S_p^{\mathcal{A}}(A)}$$

are isomorphisms. This follows from the subgroup isomorphism theorem since  $S_p(X - U) + S_p^{\mathcal{A}}(A) = S_p^{\mathcal{A}}(X)$  and  $S_p(X - U) \cap S_p^{\mathcal{A}}(A) = S_p(A - U)$ .  $\square$

## 22 Mayer-Vietoris sequence

## 23 Ordered homology

## 24 Compatibility of simplicial and singular homology

## 25 Jordan curve theorem

## 26 Cohomology

## 27 Cup products

## 28 Applications