Introduction

The purpose of these notes is to describe the relationship between function spaces on the circle and on the line by the Cayley transform.

Notation

$\mathbb{H} = \{z \in \mathbb{C} \,:\, \Im(z) > 0\}$ denotes the upper half plane, and $\mathbb{D} = \{z \in \mathbb{C} \,:\, |z| < 1\}$ denotes the open unit ball. $\mathbb{T} = \{z \in \mathbb{C} \,:\, |z| = 1\}$.
We use normalised Haar measure on $\mathbb{T}$, denoted $\boldsymbol{m}$. Lebesgue measure on $\mathbb{R}$ is denoted λ, and two dimensional Lebesgue measure on $\mathbb{C}$ is denoted $\boldsymbol{m}_2$.
Throughout these notes, ω denotes the *Cayley transform*. $\omega : \mathbb{C} \setminus \{1\} \to \mathbb{C}$, and

$$\omega(\zeta) = i\frac{1+\zeta}{1-\zeta},\ \zeta \in \mathbb{D}.$$

For a Banach space E, and a measure space (X, Σ, μ), we define

$$\|f\|_{L^p(X;E)} = \left(\int_X \|f\|_E^p \, d\mu\right)^{1/p}$$

for $p \in (0, \infty)$, and

$$\|f\|_{L^\infty(X;E)} = \sup_{x \in X} \|f(x)\|_E$$

for a weakly measurable $f : X \to E$. We define $L^p(X; E)$ as the set of measurable $f : X \to E$ with $\|f\|_{L^p(X;E)} < \infty$. As usual, we identify together functions on a measure space (X, Σ, μ) which agree μ-almost everywhere.
$L^0(X; E)$ denotes the set of all (μ-almost everywhere equivalence classes of) weakly measurable functions from X to E.
Suppose $\zeta \in \mathbb{T}$. Provided that $\zeta \neq 1$, we see that $\omega(\zeta)$ is defined, and ω maps $\mathbb{T} \setminus \{1\}$ smoothly to $\mathbb{R}$. Thus for $f \in L^0(\mathbb{R}; E)$, we can define $\tilde{f} \in L^0(\mathbb{T}; E)$ by

$$\tilde{f} := f \circ \omega^{-1}.$$

Thus we can define the important operator $\mathcal{U} : L^0(\mathbb{T}; E) \to L^0(\mathbb{R}; E)$,

$$(\mathcal{U}f)(x) = \frac{1}{\sqrt{\pi}} \frac{(f \circ \omega^{-1})(x)}{x + i}.$$

It is the purpose of these notes to collate various results concerning the images of certain subspaces of $L^0(\mathbb{T}; E)$ under $\mathcal{U}$.

Spaces of smooth functions

The spaces $C(\mathbb{T}; E)$ and $C(\mathbb{R}; E)$ denote spaces of continuous E-valued functions on the circle and line respectively.

Since $C(\mathbb{T};E) \subseteq L^2(\mathbb{T};E)$, we may define for $f \in C(\mathbb{T};E)$ and $s \in \mathbb{C}$ with $\Re(s) > 0$.

$$I_s(f) = \sum_{j\in\mathbb{Z}} \frac{1}{(1+|j|)^s}\hat{f}(j)z^j.$$

I_s is the fractional integral operator of order s, also called a Bessel potential. We define

$$C^s(\mathbb{T};E) = I_s C(\mathbb{T};E).$$

Initial results

It is obvious from the definition that $\mathcal{U}$ is linear.
Now define, for $g \in L^0(\mathbb{R};E)$,

$$(\mathcal{F})g(\zeta) = \frac{\sqrt{\pi}}{2i}\frac{(g\circ\omega)(\zeta)}{1-\zeta},\ \zeta \in \mathbb{T}.$$

Lemma 1. *$\mathcal{U}$ and $\mathcal{F}$ are inverse functions, hence $\mathcal{U}$ is a bijection.*

Proof. Let $g \in L^0(\mathbb{R};E)$, and let $t \in \mathbb{R}$. Then we simply compute,

$$\begin{aligned}(\mathcal{U}\circ\mathcal{F})(g)(t) &= \frac{\sqrt{\pi}}{2i}\frac{g(t)}{1-\omega^{-1}(t)}\frac{1}{\sqrt{\pi}}\frac{1}{t+i} \\ &= \frac{1}{2i(1-\omega^{-1}(t))(t+i)}g(t) \\ &= g(t).\end{aligned}$$

So $\mathcal{U}\circ\mathcal{F}$ is the identity function on $L^0(\mathbb{R})$.
Similarly, let $f \in L^0(\mathbb{R};E)$, and $\zeta \in \mathbb{T}$, then

$$\begin{aligned}(\mathcal{F}\circ\mathcal{U})(f)(\zeta) &= \frac{\sqrt{\pi}}{2i}\frac{1}{1-\zeta}\frac{g(\zeta)}{i+\omega(\zeta)} \\ &= g(\zeta).\end{aligned}$$

So $\mathcal{F}\circ\mathcal{U}$ is the identity function on $L^0(\mathbb{T};E)$. $\square$

ω may be regarded as a function from $\mathbb{T}\setminus\{1\} \to \mathbb{R}$. If we define $\omega(1)$ to be some arbitrary value, say $\omega(1) = 0$, we have a measureable function $\omega : \mathbb{T} \to \mathbb{R}$. Thus there is a pushforward of the Haar measure $\boldsymbol{m}$ on $\mathbb{T}$ to $\mathbb{R}$, denoted $\omega_*(\boldsymbol{m})$, defined by

$$\omega_*(\boldsymbol{m})(A) = \boldsymbol{m}(\omega^{-1}(A))$$

for all Lebesgue measurable sets A.
We may describe this with the following result,

Lemma 2. *The pushforward measure, $\omega_*(\boldsymbol{m})$ has Lebesgue Radon-Nikodym derivative*

$$\frac{d\omega_*(\boldsymbol{m})}{d\lambda} = \frac{1}{\pi|i+t|^2}$$

Proof. Let a be the arc length measure on $\mathbb{T}$, so $\boldsymbol{m} = \frac{a}{2\pi}$, now let $A \subseteq \mathbb{R}$ be lebesgue measurable, then

$$\begin{aligned} \omega_*(a)(A) &= \int_A \left| \frac{d(\omega^{-1}(t)}{dt} \right| d\lambda(t) \\ &= \int_A \frac{2}{|i+t|^2} d\lambda(t). \end{aligned}$$

Hence, the required result follows. □

A less obvious result is the following,

Theorem 1. *$\mathcal{U}$ is an isometry from $L^2(\mathbb{T}; E)$ to $L^2(\mathbb{R}; E)$.*

Proof. Let $f \in L^2(\mathbb{T}; E)$. Then

$$\begin{aligned} \|\mathcal{U}f\|^2_{L^2(\mathbb{R};E)} &= \int_{\mathbb{R}} \frac{1}{\pi|i+t|^2} \|(f \circ \omega^{-1})(t)\|^2_E \, d\lambda \\ &= \int_{\mathbb{R}} \|(f \circ \omega^{-1})(t)\|^2_E \, d(\omega_* \boldsymbol{m})(t) \\ &= \int_{\mathbb{T}} \|f\|^2_E d\boldsymbol{m}. \end{aligned}$$

So $\mathcal{U}$ embeds $L^2(\mathbb{T}; E)$ into $L^2(\mathbb{R}; E)$ isometrically. We may similarly prove the opposite embedding. □

We also have,

Theorem 2. *$\mathcal{U}L^\infty(\mathbb{T}; E) \subset L^\infty(\mathbb{R}; E)$. The inclusion here is continuous, and it is not true that $L^\infty(\mathbb{R}; E) = \mathcal{U}L^\infty(\mathbb{T}; E)$.*

Proof. This is evident from the definition of $\mathcal{U}$. Let $f \in L^\infty(\mathbb{T}; E)$. Then,

$$\begin{aligned} \|\mathcal{U}f\|_{L^\infty(\mathbb{R};E)} &= \sup_{t \in \mathbb{R}} \frac{1}{\sqrt{\pi}|i+t|} \|(f \circ \omega^{-1})(t)\|_E \\ &< \sup_{z \in \mathbb{T}} \|f(z)\|_E \\ &= \|f\|_{L^\infty(\mathbb{T};E)}. \end{aligned}$$

However, consider a constant function $c \in L^\infty(\mathbb{R}; E)$. We see that $\mathcal{U}^{-1}c$ is unbounded. □

A natural question is then to determine $\mathcal{U}^{-1}L^\infty(\mathbb{R}; E)$, this will be answered in section **??**.
Similarly to how $L^\infty(\mathbb{T}; E)$ is mapped into $L^\infty(\mathbb{R}; E)$, spaces of continuous and differentiable functions on $\mathbb{T}$ are mapped into continuous and differentiable functions on $\mathbb{R}$ but not vice versa.

Hardy spaces

We now introduce the crucial Hardy spaces.

Definitions

Hardy spaces are initially defined as spaces of functions analytic in the unit ball or the upper half plane. In particular, for a function $f : \mathbb{D} \to E$ that is complex differentiable, and $p \in (0, \infty)$, we define

$$\|f\|_{H^p(\mathbb{D};E)} = \sup_{r \in (0,1)} \left(\int_{\mathbb{T}} \|f(r\zeta)\|_E^p \, d\boldsymbol{m}(\zeta) \right)^{1/p}.$$

and

$$\|f\|_{H^\infty(\mathbb{D};E)} = \sup_{z \in \mathbb{D}} \|f(z)\|_E.$$

There is a very similar definition for Hardy spaces on the upper half plane, if $f : \mathbb{H} \to E$ is complex differentiable, we define

$$\|f\|_{H^p(\mathbb{H};E)} = \sup_{y > 0} \left(\int_{\mathbb{R}} \|f(x + iy)\|_E^p \, d\lambda(x) \right)^{1/p}$$

for $p \in (0, \infty)$ and

$$\|f\|_{H^\infty(\mathbb{H};E)} = \sup_{z \in \mathbb{H}} \|f(z)\|_E.$$

Thus, we consider the Hardy spaces $H^p(\mathbb{D}; E)$ and $H^p(\mathbb{H}; E)$ of complex differentiable functions f on $\mathbb{D}$ and g on $\mathbb{H}$ respectively such that $\|f\|_{H^p(\mathbb{D};E)} < \infty$ and $\|g\|_{H^p(\mathbb{H};E)} < \infty$ respectively for $p \in (0, \infty]$.
Typically, we do not consider Hardy spaces as spaces of functions on the unit ball or upper half plane. This is justified by the following pair of theorems,

Theorem 3. *For almost all $\zeta \in \mathbb{T}$, and $f \in H^p(\mathbb{D}; E)$, the radial limit*

$$\tilde{f}(\zeta) = \lim_{r \to 1^-} f(r\zeta).$$

exists, defines a function $\tilde{f} \in L^p(\mathbb{T}; E)$.

Theorem 4. *For almost all $x \in \mathbb{R}$, and $f \in H^p(\mathbb{H}; E)$, the limit*

$$\tilde{f}(x) = \lim_{y \to 0^+} f(x + iy)$$

exists, defines a function $\tilde{f} \in L^p(\mathbb{R}; E)$.

Bounded and Vanishing Mean Oscillation

References

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