Introduction

The purpose of these notes is to describe the relationship between function spaces on the circle and on the line by the Cayley transform.

Notation

$\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ is the upper half plane, and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc. $\boldsymbol{m}$ denotes the normalised Haar measure on $\mathbb{T}$, and λ is the Lebesgue measure on $\mathbb{R}$. $\boldsymbol{m}_2$ denotes the area measure on $\mathbb{C}$.
E denotes a Banach space. By $L^p(X;E)$, we mean

$$\|f\|_{L^p(X;E)}^p = \int_X \|f\|_E^p \, d\mu$$

for $0 < p < \infty$, and we say $f \in L^\infty(X;E)$ when $\|f\|_E \in L^\infty(X)$.

The Cayley Transform

The Cayley transform is a conformal mapping of the complex plane to itself, given by

$$\omega : z \mapsto \frac{z-i}{z+i}.$$

Defined for $z \neq -i$. In particular, ω maps the upper half plane to the unit disc, and the real line to the unit circle with 1 removed. Note that

$$\omega^{-1} : z \mapsto -i\frac{z+1}{z-1}.$$

Since this is not a bijection, we must make careful note of the following technical point:

Remark 1. *Throughout these notes, as is typical in analysis, functions on $\mathbb{R}$ or $\mathbb{T}$ are defined only up to almost everywhere equivalence. The spaces of almost-everywhere equivalence classes of measurable E-valued functions on a measure space X is denoted $L^0(X;E)$.*
If X is a topological space, we consider $C(X;E) \subseteq L^0(X;E)$ by identifying a continuous function with an equivalence class of functions that agree with it almost everywhere.

Given $f \in L^0(\mathbb{R};E)$, we define

$$\mathcal{U}f := f \circ \omega^{-1} \in L^0(\mathbb{T};E).$$

and for $g \in L^0(\mathbb{T};E)$

$$\mathcal{U}^{-1}g = g \circ \omega.$$

It is evident that $\mathcal{U}$ is a well defined linear isomorphism of $L^0(\mathbb{R};E)$ to $L^0(\mathbb{T};E)$.

Integrability and the Cayley transform

The first result of use is,

Lemma 1. *For $0 < p \leq \infty$, U maps continuously from $L^p(\mathbb{R})$ to $L^p(\mathbb{T})$.*

Proof. Let $f \in L^p(\mathbb{R})$. Then we simply compute,

$$\begin{aligned} \|\mathcal{U}f\|_p^p &= \int_{\mathbb{T}} \left| f\left(-i\frac{z+1}{z-1}\right)\right|^p d\boldsymbol{m}(z) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \end{aligned}$$

blah blah. □

A natural question is to ask for which p is $\mathcal{U}L^p(\mathbb{R}) = L^p(\mathbb{T})$? It is obvious that this is true for $p = \infty$, less obvious is that this works for $p = 2$.

Hardy spaces

Hardy spaces on $\mathbb{D}$ as defined as spaces of holomorphic functions as follows,

Definition 1. *Let $0 < p < \infty$. Given a holomorphic $f : \mathbb{D} \to E$, we define*

$$\|f\|_{H^p(\mathbb{D};E)}^p = \sup_{0<r<1}\left(\int_{\mathbb{T}} \|f(rz)\|_E^p \, d\boldsymbol{m}(z)\right).$$

Similarly, define

$$\|f\|_{H^\infty(\mathbb{D};E)} = \sup_{z\in\mathbb{D}} \|f(z)\|_E.$$

We say that $f \in H^p(\mathbb{D}; E)$ if $\|f\|_{H^p(\mathbb{D};E)} < \infty$ where $0 < p \leq \infty$.

There are analogous Hardy spaces on the upper half plane,

Definition 2. *Let $0 < p < \infty$. Given a holomorphic $f : \mathbb{H} \to E$, we define*

$$\|f\|_{H^p(\mathbb{H};E)}^p = \sup_{y>0}\left(\int_{\mathbb{R}} \|f(x+iy)\|_E^p \, d\lambda(x)\right).$$

Similarly, define

$$\|f\|_{H^p(\mathbb{H};E)}^p = \sup_{z\in\mathbb{H}} \|f(z)\|_E$$

We say that $f \in H^p(\mathbb{R}; E)$ if and only if $\|f\|_{H^p(\mathbb{D};E)} < \infty$ for $0 < p \leq \infty$

References

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