# MATH5725: Galois Theory (2014, S2) Problem Set 3

The purpose of this problem set is to consider examples of

$$\mathrm{Iso}_\psi(K/k, K'/k') = \{\Psi : K \longrightarrow K' \mid \Psi \text{ is a field isomorphism and } \Psi(a) = \psi(a) \text{ for } a \in k\}$$

for $\psi : k \longrightarrow k'$ and $f \in k[x]$.

In the questions below, each step should seem trivial or just a result of unpacking a definition. The point is to show that the theorem showing that $\mathrm{Iso}_\psi(K/k, K'/k')$ is non-empty and equal to the degree of $[K : k]$ when all the roots of the irreducible factors of $f$ are distinct is the result of simple steps.

**Note:** The rest of this course will not be so heavy in notation, the goal of this exercise is to let you see that behind the notation used to prove the theorem mentioned in the previous paragraph lies something fairly straightforward.

Let $k = \mathbf{F}_3(u)$ and let $k' = k$ and let $\psi : k \longrightarrow k'$ be given by sending $u$ to $1 + u$. Let $f \in k[X]$ be given by

$$(X^3 - u)(X^2 - u).$$

A splitting field for $f$ over $k$ is therefore

$$K = k(\sqrt{u}, \sqrt[3]{u}).$$

Put

$$E = k(\sqrt[3]{u}) \text{ and } E' = k(\sqrt[3]{u+1}),$$

and put $g = X^3 - u$ and $h = X^2 - u$ so that $f = gh$.

1. Calculate the size of

$$\mathrm{Iso}_\psi(E/k, E'/k)$$

by doing the following:
(a) Show that $\psi(g) = X^3 - (u + 1)$ and write down a factorisation of $\psi(g)$ in $E'[X]$ and $g$ in $E[X]$.
(b) Let $\alpha \in E$ be a root of $g$. Show that if $\Psi \in \mathrm{Iso}_\psi(E/k, E'/k)$ that $\Psi(\alpha)$ is a root of $\psi(g)$.
(c) Conclude that $\Psi(\alpha)$ is uniquely determined and therefore $\mathrm{Iso}_\psi(E/k, E'/k)$ has one element.

2. Let $\psi'$ be the unique element of $\mathrm{Iso}_\psi(E/k, E'/k)$. Calculate

$$\mathrm{Iso}_{\psi'}(E(\sqrt{u})/E, E'(\sqrt{u+1})/E')$$

by doing the following:
(a) Show that $\psi'(h) = X^2 - (u + 1)$ and write down a factorisation of $\psi'(h)$ in $K' = E'(\sqrt{u+1})$.
(b) Let $\alpha \in E(\sqrt{u})$ be a root of $h$. Show that if $\Psi \in \mathrm{Iso}_{\psi'}(E(\sqrt{u})/E, E'(\sqrt{u+1})/E')$ that $\Psi(\alpha)$ is a root of $\psi'(h)$.
(c) Conclude that $\Psi(\alpha)$ has two possible values and therefore $\mathrm{Iso}_{\psi'}(E(\sqrt{u})/E, E(\sqrt{u})'/E')$ has two elements.

3. Calculate the size of (where $K'$ is as in 2(a))

$$\mathrm{Iso}_\psi(K/k, K'/k)$$

by doing the following:
(a) Write down a map of sets

$$\mathrm{Iso}_\psi(K/k, K'/k) \to \bigcup_{\psi' \in \mathrm{Iso}_\psi(E/k, E'/k)} \mathrm{Iso}_{\psi'}(E(\sqrt{u})/E, E'(\sqrt{u+1})/E')$$

by considering how an element $\Psi \in \mathrm{Iso}_\psi(K/k, K'/k)$ defines an element of $\mathrm{Iso}_\psi(E/k, E'/k)$ and $\mathrm{Iso}_{\psi'}(E(\sqrt{u})/E, E'(\sqrt{u+1})/E')$.
(Hint: an element of

$$\mathrm{Iso}_\psi(K/k, K'/k)$$

is an isomorphism $\Psi : K \longrightarrow K'$ and fits into the following commutative diagram

$$\begin{tikzcd} K \arrow[r, "\Psi"] & K' \\ E \arrow[u] \arrow[r, "?"] & E' \arrow[u] \\ k \arrow[u] \arrow[r, "\psi"] & k \arrow[u] \end{tikzcd}$$

where the vertical arrows are inclusions of fields. You need to think about how the field isomorphism labelled "?" is defined and what it means.)

(b) Show that the map in (a) is injective and surjective.

(c) Conclude that

$$\mathrm{Iso}_{\psi}(K/k, K'/k)$$

has exactly two elements.