MATH5725: Galois Theory (2014, S2) Assignment 1

Due Date
This assignment is due at 5pm on Wednesday 3 September.

Marks
The assignment is worth 10% of your total mark. It is marked out of 10.

Submission guidelines
You can submit the assignment by email to s.meagher@unsw.edu.au, or by giving me a hard copy, or by uploading to moodle.

Typed submissions are preferred, but if you do not have access to mathematical typesetting software (e.g. latex), or if you are not familiar with the use of such software, handwritten submissions are also acceptable.

Notes on references
If you use a theorem or a formula, you do not need to give a full reference if it is something mentioned in the lectures, but just a short reference will suffice.

Question
Let $n > 1$ and k be a field such that k contains a primitive nth root of unity ζ_n (e.g. $k = \mathbf{Q}(e^{2\pi i/n})$) and such that the characteristic of k does not divide n.
For each $a \in k$ such that there is no $b \in k$ satisfying $b^n = a$, let $P_a(X) = X^n - a \in k[X]$.
Let $K_{a,n} = k(\sqrt[n]{a})$ be a field extension of k so that

$$K_{a,n} \to k[X]/(P_a(X)) : \sqrt[n]{a} \mapsto X + P_a(X)$$

is an isomorphism.
In particular, it follows that (and you may assume that):
(a) if $m \mid n$ then $K_{a,m} \subset K_{a,n}$ is a subfield generated by $\sqrt[m]{a} = (\sqrt[n]{a})^{n/m}$ over k.
(b) each $\alpha \in K_{a,n}$ can be written uniquely in the form

$$\alpha = u_0 + u_1(\sqrt[n]{a}) + u_2(\sqrt[n]{a})^2 + \cdots + u_{n-1}(\sqrt[n]{a})^{n-1}$$

with $u_i \in k$.

1. *(1 Mark)* Write P_a as a product of n distinct linear factors in $K_{a,n}[X]$.

2. *(4 Marks)* Let $\lambda \in (\mathbf{Z}/n)$ and consider the map given by the rule

$$\sigma_\lambda : K_{a,n} \longrightarrow K_{a,n} : f(\sqrt[n]{a}) \mapsto f(\sqrt[n]{a}\zeta_n^\lambda),$$

for $f \in k[X]$.
(i) Show that σ_λ respects addition and multiplication.
(ii) Show that σ_λ is well-defined, i.e. if $f, g \in k[X]$ satisfy $f(\sqrt[n]{a}) = g(\sqrt[n]{a})$ then $\sigma_\lambda(f(\sqrt[n]{a})) = \sigma_\lambda(g(\sqrt[n]{a}))$.
(iii) Show that σ_λ is injective.
(iv) Show that $\sigma_\lambda(u) = u$ if $u \in k$ and that σ_λ is surjective.
Conclude that σ_λ is a field automorphism.

3. *(2 Marks)* Show that if $\sigma : K_{a,n} \longrightarrow K_{a,n}$ is a field automorphism such that $\sigma(u) = u$ for $u \in k$ then $P(\sigma(\sqrt[n]{a})) = 0$. Show therefore that $\sigma = \sigma_\lambda$ for some $\lambda \in (\mathbf{Z}/n)$. Write down an isomorphism between the group

$$\mathrm{Aut}(K_{a,n}/k) = \{\sigma : K_{a,n} \longrightarrow K_{a,n} \mid \sigma \text{ is a field automorphism and } \sigma(u) = u \text{ for } u \in k\}$$

and the group $(\mathbf{Z}/n)$, and prove that it is an isomorphism of groups.

4. *(3 Marks)* Let $m \mid n$ be an integer and let $m(\mathbf{Z}/n)$ be the subgroup of $(\mathbf{Z}/n)$ generated by the multiples of m. Show that $\sqrt[m]{a}$ is fixed by each σ_λ with $\lambda \in m(\mathbf{Z}/n)$. Show that $\sigma_\lambda(u) = u$ for $\lambda \in m(\mathbf{Z}/n)$ if and only if $u \in K_{a,m}$.