UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Assignment 2

Graph Theory

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Question 1

For graphs G and H, we let consider the graph product $G \times H$ as having the vertex set $V(G) \times V(H)$, and we have an edge $(v, w)(v', w') \in E(G \times H)$ if and only if $v = v'$ and $ww' \in E(H)$ or $vv' \in E(G)$ and $w = w'$.

For a graph G, $\alpha(G)$ denotes the size of the largest independent set in G.

Lemma 1 (Part (a)). *Let G be a graph on $n \geq 2$ vertices, and let $r \geq 1$. Then*

$$\alpha(G \times K_r) \leq n.$$

Proof. It is sufficient to show that any set of $n + 1$ vertices in $G \times K_r$ has an adjacent pair. Let $(v_1, w_1), (v_2, w_2), \ldots, (v_{n+1}, w_{n+1}) \in V(G \times K_r)$, where each $v_k \in G$ and each $w_k \in K_r$.

By the pigeonhole principle, not all the v_k can be distinct, so we must have $v_k = v_j$ for some $j \neq k$. Now since K_r is complete, $w_k w_j \in E(K_r)$. Hence, $(v_k, w_k)(v_j, w_j) \in E(G \times K_r)$, so any set of $n+1$ vertices must have an adjacent pair.

Hence, $\alpha(G \times K_r) \leq n$. □

Lemma 2 (Part (b)). *For a graph G on $n \geq 2$ vertices, and $r \geq 2$, we have $\alpha(G \times K_r) = n$ if and only if $r = \chi(G)$.*

Proof. Suppose first that $r = \chi(G)$. We must show that $\alpha(G \times K_r)$ has an independent set of size n. Choose an r-colouring for G, $c : G \to \{1, 2, \ldots, r\}$.

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□

Corollary 1 (Part (c)). *Hence for a graph G on $n \geq 2$ vertices, we have:*

$$\chi(G) = \min\{r > 0 \ : \ \alpha(G \times K_r) = n\}.$$

Proof. By lemma 1b, we have that $\alpha(G \times K_r) = n$ if and only if $r = \chi(G)$.

Hence, if $r < \chi(G)$, we must have $\alpha(G \times K_r) < n$.

Thus the minimum value of r such that $\alpha(G \times K_r) = n$ must be $\chi(G)$, and so we are done. □

Question 2

For this question, G is a 3-connected graph and $xy \in E(G)$. We use the notation G/xy to denote the graph obtained from G by contracting xy.

Proposition 1 (Part (a))**.** *If G/xy is 3-connected, then $G - \{x, y\}$ is 2-connected.*

Proof. To show that $G - \{x, y\}$ is 2-connected, at the very least it must have at least three vertices. To show this, we note that by assumption G/xy is 3-connected, and $|V(G/xy)| = |V(G)| - 1$. Since G/xy is itself 3-connected, we have $|V(G/xy)| > 3$. Hence, $|V(G)| > 4$. Thus, $|V(G - \{x, y\})| > 2$.

Now we need to show that $G - \{x, y\}$ is connected, and for any vertex $v \in V(G - \{x, y\})$, $G - \{x, y, v\}$ is connected.

Now since G is by assumption 3-connected, automatically we have that $G - \{x, y\}$ is connected. Hence it is only required to show that $G - \{x, y, v\}$ is connected for all vertices $v \in V(G - \{x, y\})$.

So let $v \in V(G - \{x, y\})$. Hence $v \in V(G/xy)$. Let w be the vertex in G/xy formed from merging x and y.

Thus, $G/xy - \{v, w\}$ is connected since G/xy is 3-connected by assumption.

Let $p, q \in V(G - \{x, y, v\})$. Since $G/xy - \{v, w\}$ is connected, there is a path P joining p and q in G/xy which avoids v and w.

Hence every edge of P consists of edges of $G/xy - \{v, w\}$. Since every edge of $G/xy - \{v, w\}$ is an edge of G, P can be considered as a path in G. Since it avoids v and w, it must avoid x, y and v in G.

Thus $G - \{x, y, v\}$ is connected, and so $G - \{x, y\}$ is connected. $\square$