UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Assignment 2

Graph Theory

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Question 1

For graphs G and H, we let consider the graph product $G \times H$ as having the vertex set $V(G) \times V(H)$, and we have an edge $(v,w)(v',w') \in E(G \times H)$ if and only if $v = v'$ and $ww' \in E(H)$ or $vv' \in E(G)$ and $w = w'$.

For a graph G, $\alpha(G)$ denotes the size of the largest independent set in G.

Lemma 1 (Part (a)). *Let G be a graph on $n \geq 2$ vertices, and let $r \geq 1$. Then*

$$\alpha(G \times K_r) \leq n.$$

Proof. It is sufficient to show that any set of $n+1$ vertices in $G \times K_r$ has an adjacent pair. Let $(v_1, w_1), (v_2, w_2), \ldots, (v_{n+1}, w_{n+1}) \in V(G \times K_r)$, where each $v_k \in G$ and each $w_k \in K_r$.

By the pigeonhole principle, not all the v_k can be distinct, so we must have $v_k = v_j$ for some $j \neq k$. Now since K_r is complete, $w_k w_j \in E(K_r)$. Hence, $(v_k, w_k)(v_j, w_j) \in E(G \times K_r)$, so any set of $n+1$ vertices must have an adjacent pair.

Hence, $\alpha(G \times K_r) \leq n$. $\square$

Lemma 2 (Part (b)). *For a graph G on $n \geq 2$ vertices, and $r \geq 2$, we have $\alpha(G \times K_r) = n$ if and only if $r = \chi(G)$.*

Proof. Suppose first that $r = \chi(G)$. We must show that $\alpha(G \times K_r)$ has an independent set of size n. Choose an r-colouring for G, $c : G \to \{1, 2, \ldots, r\}$. Let $V_j = c^{-1}(\{j\})$ be the set of vertices of G with colour j.

Label the vertices of K_r as $w_1, w_2, \ldots, w_r$.

Consider the set of vertices $S = \{(v, w_j) \ v \in V_j, 0 \leq j \leq r\}$. We shall show that S is an independent set. Let $(v, w_j), (v', w_k) \in S$. Then the vertices $(v, w_j), (v', w_k)$ are adjacent if and only if one of two possibilities holds:

Firstly, if $w_j = w_k$, then $(v, w_j), (v', w_k)$ are adjacent if and only if $vv' \in E(G)$. But this is impossible since $v, v' \in V_j$. So v and v' both have colour j so cannot be adjacent.

The second case is that $j \neq k$, but we have $v = v'$. But then v has colour j, and v' has colour k. But this is impossible since $v = v'$.

Thus, the vertices $(v, w_j), (v', w_k)$ are never adjacent.

Now, the size of S is just $|S| \sum_{j=1}^{r} |\{(v, w_j) \ : \ v \in V_j\}| = \sum_{j=1}^{r} |V_j| = |V(G)|$, since the colour clases form a partition of G.

Hence, $|S| = n$, so we have an independent set of size n. $\square$

Corollary 1 (Part (c))**.** *Hence for a graph G on $n \geq 2$ vertices, we have:*

$$\chi(G) = \min\{r > 0 \ : \ \alpha(G \times K_r) = n\}.$$

Proof. By lemma 1b, we have that $\alpha(G \times K_r) = n$ if and only if $r = \chi(G)$.

Hence, if $r < \chi(G)$, we must have $\alpha(G \times K_r) < n$.

Thus the minimum value of r such that $\alpha(G \times K_r) = n$ must be $\chi(G)$, and so we are done. ☐

Question 2

For this question, G is a 3-connected graph and $xy \in E(G)$. We use the notation G/xy to denote the graph obtained from G by contracting xy.

Proposition 1 (Part (a))**.** *If G/xy is 3-connected, then $G-\{x,y\}$ is 2-connected.*

Proof. To show that $G - \{x, y\}$ is 2-connected, at the very least it must have at least three vertices. To show this, we note that by assumption G/xy is 3-connected, and $|V(G/xy)| = |V(G)| - 1$. Since G/xy is itself 3-connected, we have $|V(G/xy)| > 3$. Hence, $|V(G)| > 4$. Thus, $|V(G - \{x, y\})| > 2$.

Now we need to show that $G - \{x, y\}$ is connected, and for any vertex $v \in V(G - \{x, y\})$, $G - \{x, y, v\}$ is connected.

Now since G is by assumption 3-connected, automatically we have that $G-\{x, y\}$ is connected. Hence it is only required to show that $G - \{x, y, v\}$ is connected for all vertices $v \in V(G - \{x, y\})$.

So let $v \in V(G - \{x, y\})$. Hence $v \in V(G/xy)$. Let w be the vertex in G/xy formed from merging x and y.

Thus, $G/xy - \{v, w\}$ is connected since G/xy is 3-connected by assumption.

Let $p, q \in V(G - \{x, y, v\})$. Since $G/xy - \{v, w\}$ is connected, there is a path P joining p and q in G/xy which avoids v and w.

Hence every edge of P consists of edges of $G/xy - \{v, w\}$. Since every edge of $G/xy - \{v, w\}$ is an edge of G, P can be considered as a path in G. Since it avoids v and w, it must avoid x, y and v in G.

Thus $G - \{x, y, v\}$ is connected, and so $G - \{x, y\}$ is connected. ☐

Proposition 2 (Part (b))**.** *Now if G/xy is* not *3-connected, then $G - \{x, y\}$ is not 2-connected.*

Proof. We shall prove the contrapositive. Suppose that $G - \{x, y\}$ is 2-connected. In particular, this means that $G - \{x, y\}$ has at least three vertices, so G has at least five vertices. Hence, G/xy has at least four vertices.

Now to complete the proof we need to show that removing any 2 element subset of G/xy does not disconnect G/xy.

Now by assumption, $G - \{x, y\}$ is 2-connected. In particular this means that it is connected, and for any $v \in V(G) - \{x, y\}$ we have that $G - \{x, y, v\}$ is connected.

Now, let $u, w \in V(G/xy)$. Then there are two possibilities:

1. One of u or v is formed by contracting x and y.
2. Neither of u and v is formed by contracting x and y.

In the first case, we may assume without loss of generality that u is formed from contracting x and y. We now consider $G/xy - \{u, w\}$. This is simply $G - \{x, y, w\}$. But by assumption this is connected, hence $G/xy - \{u, w\}$ is connected.

In the second case, we have $u, v \in V(G - \{x, y\})$. Now, let $H = G/xy - \{u, v\}$.

Now, let $p, q \in V(H)$. We must show that there is a path joining p and q in H. Since neither u nor v is formed from contracting x and y, it is sufficient to show that there is a path joining p and q in G which avoids u and v. Hence, we search for a path in $G - \{u, v\}$. However, by assumption G is 3-connected, so such a path exists. Hence, H is connected.

Thus, G/xy is 3-connected, as required. □

Question 3

For this question, G is a graph and $P_G(k)$ denotes the number of k-colourings of G. It is known that $P_G(k)$ is polynomial in k, and is hence called the chromatic polynomial of G.

From the problem sheets, we know that if G has n vertices then $P_G(k)$ is a monic polynomial of degree n, and if G has m edges then the coefficient of k^{n-1} is $-m$.

Since this is not proved in the notes, we give a short proof here. Let $x, y \in V(G)$, and let $G' = G - xy$ and G'' be the graph obtained by contracting x and y. Now a k-colouring of G is exactly a k-colouring of G' which assigns different colours to x and y. The k-colourings of G which assign the same colours to x and y are exactly the k-colourings of G''. Hence, $P_G(k) = P_{G'}(k) - P_{G''}(k)$. Hence if we do induction on the number of edges of G, we can prove the required properties of $P_G(k)$.

Lemma 3 (Part (a))**.** *There is no graph G with chromatic polynomial*

$$P_G(k) = k^4 - 4k^3 + 8k^2 - 5k.$$

Proof. Let G be such a graph. Then G must have 4 vertices and 4 edges. Hence, by the handshaking lemma the average degree of G is 2. Hence G is 2-regular or has a vertex of degree 3.

If G is 2-regular, then $G = C_4$ is a cyclic graph on 4 vertices. Hence, in this case, the number of 2-colourings of G is 2.

However $P_G(2) = 16 - 32 + 32 - 10 = 6$, so $G \neq C_4$.

The only other case is that G has a vertex of degree 3. Since $|V(G)| = 4$, this means that G has a vertex v, connected to every other vertex. Hence if we have a 2-colouring of G, $c : V(G) \to \{1, 2\}$ we either have $c(v) = 1$ and all other vertices have colour 2, or $c(v) = 2$ and all other vertices have colour 1. Hence there are at most two 2-colourings of G. So we cannot have $P_G(k)$ as the given polynomial. □

Lemma 4 (Part (b))**.** *Let G be a graph, with $v \in V(G)$. Then the number of k-colourings of G which assign colour 1 to v is given by $P_G(k)/k$.*

Proof. Let C_i be the number of colourings which assign colour i to v. Then,

$$P_G(k) = C_1 + C_2 + \cdots + C_k.$$

Now by symmetry, we have $C_1 = C_i$ for all i, hence,

$$P_G(k) = kC_1$$

as required. □

Lemma 5 (Part (c)). *Let G be a graph with connected components $G_1, G_2, \ldots, G_r$. Then*

$$P_G(k) = \prod_{j=1}^{r} P_{G_j}(k).$$

Proof. Let $k \geq 1$. Then for each $1 \leq j \leq r$, a k-colouring of G induces a k-colouring of G_j. Thus each k-colouring of G is produced by an individual k colouring of each G_j. Hence, the number of k-colourings of G is given by a k-colouring of G_1, times the number of k-colourings of G_2, times the number of k-colourings of G_2, and etc., until we have a k-colouring for G. □

Lemma 6 (Part (d)). *Let G have r connected components. If:*

$$P_G(k) = c_0 + c_1 k + c_2 k^2 + \cdots + c_n k^n$$

we have $c_0 = c_1 = \ldots = c_{r-1} = 0$ and $c_r \neq 0$.

In other words, if G has r connected components, then r is the minimal positive integer such that the coefficient of k^r in $P_G(k)$ is nonzero.

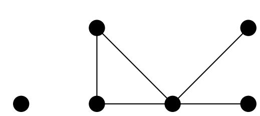
Alternatively, r is the number of times k divides $P_G(k)$.

Proof. Let the connected components of G be $\{G_i\}_{i=1}^r$. Then by lemma 5, we have:

$$P_G(k) = \prod_{i=1}^{r} P_{G_i}(k).$$

By lemma 4, k divides $P_{G_i}(k)$. Hence, k divides $P_G(k)$ at least r times. □

Now for Part (e), the following graph has chromatic polynomial $P_G(k) = k^2(k-1)^3(k-2)$:



Question 4

For this question, $k \geq 1$ and $G = (V, E)$ is a k-edge connected graph. We let $F_1, \ldots, F_m \subseteq E$ be distinct sets of k edges which disconnected G. We let $\mathcal{F} = F_1 \cup F_2 \ldots \cup F_m$, and let $C_1, \ldots, C_t$ be the connected components of $G - \mathcal{F}$.

Lemma 7 (Part (a))**.** *For each $i = 1, \ldots, m$, we have:*

$$|\{e \in \mathcal{F} \ : \ e \cap C_i \neq \emptyset\}| \geq k.$$

Proof. Since G is k-connected, it suffices to show that $S_i = \{e \in \mathcal{F} \ : \ e \cap C_i \neq \emptyset\}$ disconnects G.

Consider $G - S_i$. Suppose that e is an edge joining C_i to $G - C_i$. Then since C_i is a connected component of $G - \mathcal{F}$, we must have $e \in \mathcal{F}$, and in particular $e \in S_i$. Hence $e \notin G - S_i$, so $G - S_i$ contains no edges joining C_i to $G - C_i$. Thus $G - S_i$ is disconnected, and hence $|S_i| \geq k$. □

Corollary 2 (Part (b))**.** *We have $|\mathcal{F}| \geq kt/2$.*

Proof. Now, each $e \in \mathcal{F}$ is incident with at least one C_i, and at most two C_i. Let S_i as in lemma 7, and we have:

$$\mathcal{F} = S_1 \cup S_2 \cup \ldots \cup S_t.$$

However, each element of $\mathcal{F}$ occurs in at most two of the S_i. Thus the sum

$$\sum_{i=1}^{t} |S_i|$$

counts each element of $\mathcal{F}$ at most twice. Hence,

$$\sum_{i=1}^{t} |S_i| \leq 2|\mathcal{F}|.$$

Hence, since for each i we have $|S_i| \geq k$ from lemma 7,

$$kt \leq 2|\mathcal{F}|.$$

So the result follows. □

Corollary 3 (Part (c))**.** *We have $|\mathcal{F}| \leq mk$, and hence $t \leq 2m$.*

Proof. Since $\mathcal{F} = F_1 \cup F_2 \cup \cdots \cup F_m$, hence $|\mathcal{F}| \leq |F_1| + |F_2| + \cdots + |F_m|$. Since each $|F_i| = k$, we then have $|\mathcal{F}| \leq mk$. Thus,

$$kt/2 \leq |\mathcal{F}| \leq mk.$$

and hence $t \leq 2m$. □

Question 5

For this question, H denotes a 3-uniform hypergraph on n vertices with n element vertex set V and m element hyperedge set E, and $m \geq n/3$.

Lemma 8 (Part (a))**.** *Let $U \subset V$ be a random vertex set, with $v \in U$ with probability p. Then $\mathbb{E}(|U|) = np$.*

Proof. Let $X : G \to \{0,1\}$ be the indicator function of the set U. Then,

$$\mathbb{E}(|U|) = \sum_{v \in V} \mathbb{E}(X(v)).$$

Now, $X(v) = 1$ with probability p and 0 with probability $1-p$. Hence $\mathbb{E}(X(v)) = p$, so $\mathbb{E}(|U|) = np$. □

Lemma 9 (Part (b))**.** *Define Y to be the number of hyperedges contained in U, randomly chosen as in lemma 8, that is,*

$$Y = |\{e \in E \ : \ e \subseteq U\}|. \tag{1}$$

Then $\mathbb{E}(Y) = mp^3$.

Proof. Let $X : E \to \{0,1\}$ be the indicator function of the set of hyperedges in U. Then,

$$\mathbb{E}(Y) = \sum_{e \in E} \mathbb{E}(X(e)).$$

So we need to compute $\mathbb{E}(X(e))$ for a given hyperedge e.

Since $X(e) = 1$ if $e \subseteq U$, and 0 otherwise, we have that $\mathbb{E}(X(e))$ equal the probability that $e \subseteq U$. In otherwords, $\mathbb{E}(X(e))$ is the probability that U contains e.

Suppose that $e = \{u, v, w\}$. Then $\mathbb{E}(X(e)) = \mathbb{P}(u, v, w \in U)$. Since each vertex is chosen to be in U with probability p independently, we have that $\mathbb{E}(X(e)) = p^3$. Thus,

$$\begin{aligned}\mathbb{E}(Y) &= \sum_{e \in E} p^3 \\ &= mp^3.\end{aligned}$$

□

Corollary 4 (Part (c))**.** *Hence, H contains an independent set with at least $\frac{2n^{3/2}}{3\sqrt{3m}}$ vertices.*

Proof. We will use the probabilistic method: we will define a random independent subset of H and show that the expected value of its size is at least $\frac{2n^{3/2}}{3\sqrt{3m}}$.

Let $U \subseteq H$ be chosen as in lemma 8, and let Y be the number of edges in U as in lemma 9.

If we remove one vertex from every edge in U, then we create an independent set. Hence we can remove Y vertices from H to form an independent set of size $|U| - Y$. By the linearity of expectation, and lemmas 8 and 9:

$$\begin{aligned}\mathbb{E}(|U| - Y) &= \mathbb{E}(|U|) - \mathbb{E}(Y) \\ &= np - mp^3.\end{aligned}$$

Hence there exists an independent set of size at least $np - mp^3$.

Now we choose p: let $p = \sqrt{n/3m}$. This is a true probability since by assumption $m \geq n/3$. Then,

$$\begin{aligned}np - mp^3 &= \sqrt{\frac{n}{3m}}(n - m\frac{n}{3m}) \\ &= \frac{2n^{3/2}}{3\sqrt{3m}}.\end{aligned}$$

Hence there is an independent set of at least this size. □