SCHOOL OF MATHEMATICS AND STATISTICS
UNIVERSITY OF NEW SOUTH WALES
MATH5425 Graph Theory Semester 2 2015
Problem Sheet 6, Planar Graphs

1. Prove that every graph can be embedded in $\mathbb{R}^3$ with all edges straight and no edges crossing.

2. Recall that the girth of a graph is the size of its smallest cycle. Let $g \geq 3$ be a fixed integer and let $\mathcal{H}$ be the set of all plane graphs with n vertices, minimum degree at least 1 and girth g. Say that a graph $G \in \mathcal{H}$ is *maximally $\mathcal{H}$-plane* if we cannot add an edge to G to give a new plane graph $G' \in \mathcal{H}$.

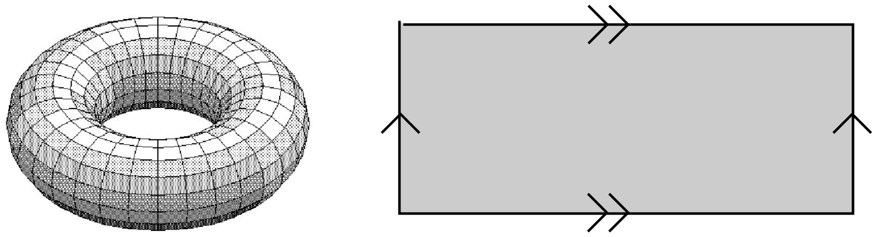
   (a) Suppose that G is a plane graph in $\mathcal{H}$ such that every face is bounded by a g-cycle. Prove that G is maximally $\mathcal{H}$-plane.

   (b) Suppose that G is a plane graph such that every face is bounded by a g-cycle. Further suppose that G has n vertices and m edges. Prove that
   $$(g-2)m = g(n-2).$$

3. Without using Kuratowski's Theorem, prove that a planar graph does not contain any subdivision of $K_{3,3}$ or K_5 as a subgraph.

4. (a) Show that $K_{3,3}$ with one edge deleted is a planar graph, and similarly for K_5.

   (b) Show that you can draw $K_{3,3}$ and K_5 on the torus with no edges crossing. (A torus is shown on the left below. It may help to consider the torus as a rectangle with pairs of opposite sides identified, with no twists, as in the figure on the right.)



   (c) Hence, or otherwise, conjecture which constant c appears on the right hand side in the analogue of Euler's formula for the torus:
   $$n - m + \ell = c$$
   for all connected non-planar graphs with n vertices and m edges which can be embedded in the torus with no edges crossing and with ℓ faces.

5. Find an embedding of $K_{3,3}$ on the torus such that every face is bounded by a Hamilton cycle. (This is just for fun: don't waste too much time on it.)