SCHOOL OF MATHEMATICS AND STATISTICS
UNIVERSITY OF NEW SOUTH WALES

MATH5425 Graph Theory Semester 2 2015

Problem Sheet 7, Ramsey Theory

1. Prove that $R(3) = 6$ and that $R(2, s) = s$ for all $s \geq 2$.

2. Suppose that there exists $p \in [0, 1]$ such that

$$\binom{n}{s} p^{\binom{s}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1.$$

   Prove that $R(s, t) > n$.

3. In lectures we proved Erdős' 1947 result (Alon & Spencer Proposition 1.1.1) which shows that for $s \geq 3$, if

$$\binom{n}{s} 2^{1-\binom{s}{2}} < 1 \qquad (*)$$

   then $R(s) > n$. From this we deduced that $R(s) > \lfloor 2^{s/2} \rfloor$ for $s \geq 3$. We now obtain a slightly improved lower bound using sharper analysis.

   (a) Show that if

$$n \leq \frac{s}{\sqrt{2}\, e}\, 2^{s/2}$$

   then

$$n^s < 2^{s(s-1)/2} \sqrt{\frac{\pi s}{2}} \left(\frac{s}{e}\right)^s. \qquad (**)$$

   (b) Using one of Stirling's inequalities, explain why $(**)$ implies that Erdős' condition $(*)$ holds.

   (c) Hence conclude that $R(s) > \dfrac{s}{\sqrt{2}e}\, 2^{s/2}$ for $s \geq 3$.

(... Please turn over for Questions 4 and 5)

4. Prove that $R(C_4, C_4) = 6$, using the following steps. (Recall that C_4 is a 4-cycle.)
   (a) Use Bollobás Chapter 6 Theorem 11 (from lectures) to show that $R(C_4, C_4) \geq 5$.
   (b) Prove that $R(C_4, C_4) > 5$ by displaying a red-blue colouring of the edges of K_5 with no monochromatic 4-cycle.
   (c) Prove that in any red-blue colouring of the edges of K_6 there must be a monochromatic 4-cycle.
   Hint: For a contradiction, suppose that there is a red-blue colouring of the edges of K_6 with no monochromatic C_4. The fact that $R(3) = 6$ may help you to get started. Work out the colour of various edges and end up with a contradiction. You may have to analyse a couple of different cases.

For $k \in \mathbb{Z}^+$ let $[k] = \{1, 2, \ldots, k\}$. Let A, $B_1, B_2, \cdots B_k$ be events in Ω. We say that A is *mutually independent of* $B_1, \ldots, B_k$ if for all $I \subseteq [k]$ we have

$$\Pr(A \cap \bigcap_{i \in I} B_i) = \Pr(A) \times \Pr\left(\bigcap_{i \in I} B_i\right).$$

The Local Lemma (Erdős & Lovász, 1975) *Let $A_1, \ldots A_n$ be events in some probability space. Suppose that each event A_i is mutually independent of a set of all but at most d of the other events, and that $\Pr(A_i) \leq p$ for $1 \leq i \leq n$. If $ep(d+1) \leq 1$ then*

$$\Pr\left(\bigcap_{j=1}^{n} \overline{A_j}\right) > 0.$$

We now obtain another slight improvement on the lower bound for $R(s)$ by applying the Local Lemma, following Spencer (1975).

5. Colour the edges of K_n red or blue with probability $\frac{1}{2}$, independently. For each subset R of s vertices of K_n, let A_R be the event that $K_n[R]$ is monochromatic.
   (a) Let $d = \binom{s}{2}\binom{n}{s-2}$. Prove that A_R is mutually independent from all but a set of at most d events in $\{A_T : \ T \text{ is a subset of } s \text{ vertices of } K_n\}$.
   (b) Apply the Local Lemma to prove that if
   $$e\left(\binom{s}{2}\binom{n}{s-2} + 1\right) 2^{1-\binom{s}{2}} \leq 1$$
   then $R(s) > n$.
   (c) Arguing as in Question 3, deduce that as $s \to \infty$ we have
   $$R(s) > (1 + o(1)) \frac{\sqrt{2}\, s}{e} 2^{s/2}.$$
   (Here $o(1)$ denotes a function of s which tends to 0 as $s \to \infty$.)