SCHOOL OF MATHEMATICS AND STATISTICS
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MATH5425 Graph Theory   Semester 2 2015

Revision of discrete probability

Let Ω be a finite set. A *probability distribution* on Ω is a function $\pi : \Omega \to [0,1]$ such that

$$\sum_{z\in\Omega} \pi(z) = 1.$$

We say that (Ω, π) forms a *probability space* with underlying set Ω. Often our probability distributions will be *uniform*, meaning that every element of Ω has equal probability: that is, $\pi(z) = 1/|\Omega|$ for all $z \in \Omega$. In this case (Ω, π) is called the *uniform probability space* on Ω. If z is chosen from Ω according to the uniform distribution, then we say that z is chosen *uniformly at random* from Ω.

An *event* is a subset of Ω. For any event $A \subseteq \Omega$ we define

$$\pi(A) = \sum_{z\in A} \pi(z).$$

(Since we only consider finite sets Ω, we do not have to worry about many of the intricacies of probability theory. In particular, all subsets of Ω are measurable.)

Once π is defined we often just write $\Pr(z)$ or $\Pr(A)$ instead of $\pi(z)$ or $\pi(A)$, where $z \in \Omega$, $A \subseteq \Omega$.

Exercises: Let $A, B \subseteq \Omega$ be events.

(i) Suppose that π is the uniform distribution on Ω and let A be an event in Ω. Calculate $\Pr(A)$.

(ii) If A and B are events in Ω, calculate $\Pr(A \cup B)$ in terms of $\Pr(A)$ and $\Pr(B)$.

(iii) Write down an expression for $\Pr(A \cup B)$ when A and B are disjoint events.

Let A and B be events in Ω with $\Pr(B) \neq 0$. The *conditional probability* of event A, given B, is denoted by $\Pr(A|B)$ and defined by

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

Two events A, B are said to be *independent* if

$$\Pr(A \cap B) = \Pr(A)\Pr(B).$$

Exercise:

(iv) Suppose that A and B are independent events and that $\Pr(B) \neq 0$. Explain why $\Pr(A|B) = \Pr(A)$.

A *random variable* on Ω is a function $X : \Omega \to \mathbb{R}$. Given any $T \subseteq \mathbb{R}$, the predicate "$X \in T$" defines an event $A = \{z \in \Omega \mid X(z) \in T\}$ and hence

$$\Pr(X \in T) = \Pr(A).$$

The *expectation* or *expected value* of a random variable X, denoted by $\mathbb{E}[X]$ (or sometimes just $\mathbb{E}X$), is given by

$$\mathbb{E}[X] = \sum_{z \in \Omega} X(z)\Pr(z).$$

Exercises:

(v) Let X be a random variable with $\mathbb{E}[X] = \mu$. Prove that there exists $z, w \in \Omega$ such that $X(z) \leq \mu$ and $X(w) \geq \mu$.

(vi) For X as above, when is it not true that there exists $z, w \in \Omega$ such that $X(z) < \mu$ and $X(w) > \mu$?

(vii) What does $\mathbb{E}[X]$ equal for a uniform probability space?

Let $X_1, \ldots, X_k$ be random variables defined on the same probability space and let $c_1, \ldots c_k \in \mathbb{R}$. Define the random variable $X = c_1 X_1 + \cdots + c_k X_k$. Then

$$\mathbb{E}[X] = c_1\,\mathbb{E}[X_1] + c_2\,\mathbb{E}[X_2] + \cdots + c_k\,\mathbb{E}[X_k].$$

This relationship is known as *linearity of expectation.* It is surprisingly powerful since is does not require any special conditions on $X_1, \ldots, X_k$.

Exercise: (viii) Prove linearity of expectation.

Given an event $A \subseteq \Omega$, the *indicator variable* for the event A is the random variable I_A such that

$$I_A(z) = \begin{cases} 1 & \text{if } z \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise: (ix) Calculate the expectation of I_A.

The following lemma, Markov's lemma or Markov's inequality, is extremely useful. It applies to *nonnegative* random variables (i.e., those which only take nonnegative values).

Markov's inequality. *Suppose that* $X : \Omega \to [0, \infty)$ *is a nonnegative random variable on* Ω *and let* $k > 0$*. Then*

$$\Pr(X \geq k) \leq \frac{\mathbb{E}[X]}{k}.$$

Exercise:

(x) Let A denote the event "$X(z) \geq k$". Prove that $k\, I_A(z) \leq X(z)$ for all $z \in \Omega$.

(xi) Hence, or otherwise, prove Markov's inequality.

Let k be an integer, $k \geq 2$. Events $A_1, \ldots, A_k$ are said to be *mutually independent* if for all j, $\ell_1, \ldots, \ell_j$ with $2 \leq j \leq k$ and $1 \leq \ell_1 < \ell_2 < \cdots < \ell_j \leq k$, we have

$$\Pr\left(\bigcap_{i=1}^{j} A_{\ell_i}\right) = \prod_{i=1}^{j} \Pr(A_{\ell_i}).$$

Sometimes we just say "independent" instead of "mutually independent". We now explore what this definition means in the case $k = 3$.

Exercises:

First, consider the set $\Omega = \{1, 2, 3, 4, 5, 6\}$ with the uniform distribution, considered as the outcome of the roll of a die, say. Let $E_1, E_2, E_3 \subseteq \Omega$ be the events defined by the predicates "is greater than 3", "is even" and "is not a multiple of 3", respectively.

(xii) Calculate $\Pr(E_1 \cap E_2)$ and compare with $\Pr(E_1) \Pr(E_2)$.

(xiii) Calculate $\Pr(E_1 \cap E_2 \cap E_3)$ and compare with $\Pr(E_1)\Pr(E_2)\Pr(E_3)$.

(xiv) What conclusion can you draw about testing mutual independence of three events?

Now consider the uniform probability space over $\Omega = \{4, 5, 6, 7\}$. Consider the events E_1, E_2 and E_3 defined by the predicates "is even", "is a factor of 20" and "is a factor of 30", respectively.

(xv) Calculate $\Pr(E_i \cap E_j)$ and compare with $\Pr(E_i)\Pr(E_j)$, for $1 \leq i < j \leq 3$.

(xvi) Calculate $\Pr(E_1 \cap E_2 \cap E_3)$ and compare with $\Pr(E_1)\Pr(E_2)\Pr(E_3)$.

(xvii) What can you conclude about testing three events for mutual independence?