University of New South Wales

School of Mathematics and Statistics

Assignment 1

MATH5425 - Graph Theory

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Question 1

For this question, G is a graph with n vertices.

Proposition 1 (Part (a))**.** *Suppose that G has at least n edges. Then G contains a cycle.*

Proof. Suppose that G has k connected components, $G_1, \ldots, G_k$. There must exist a connected component G_j with $|E(G_j)| \geq |V(G_j)|$, since otherwise we would have $|E(G)| < |V(G)|$, but we are assuming that $|E(G)| \geq n$.

Let G_j be a connected component with $|E(G_j)| \geq |V(G_j)|$. We will show that G_j contains a cycle. If G_j contains no cycle, then G_j is a tree by definition. However by Corollary 1.5.3 in the course notes, this can only be true if $|E(G_j)| = |V(G_j)| - 1$. But this contradicts $|E(G_j)| \geq |V(G_j)|$. Hence G_j contains a cycle. $\square$

Proposition 2 (Part (b))**.** *Suppose that G has strictly more than n edges. Then G contains two distinct (not necessarily disjoint) cycles.*

Proof. Let $e \in E(G)$. Then $G - e$ has at least n edges, so by Proposition 1 $G - e$ contains a cycle. Call such a cycle C_1.

Now choose $f \in E(C)$. Again by proposition 1, $G - f$ contains a cycle, call it C_2.

Hence C_1 and C_2 are two cycles in G, and since $f \in C_1$ and $f \notin C_2$, the cycles are distinct. $\square$

Proposition 3 (Part (c))**.** *Suppose that G has minimum degree $\delta(G) \geq 3$. Then G contains a cycle of even length.*

Proof. Observe that G always contains at least one cycle, since by the handshaking lemma,

$$2|E(G)| = \sum_{v \in V(G)} d_G(v) \geq 3|V(G)|. \tag{1}$$

Hence $E(G) \geq V(G)$, so by Proposition 1 G contains a cycle.

We will perform a proof by contradiction. We will assume that G contains only odd cycles, then construct a cycle of even length.

Assume that G contains only odd cycles.

First suppose that G is connected. Let T be a spanning tree for G, and let w be a leaf of T connected to the edge $t \in E(T)$. Hence $d_T(w) = 1$ but $d_G(v) \geq 3$, so there exist two distinct edges, $e, f \in E(G)$ with endpoints at w not contained in T.

Now, since a tree is maximally acyclic, $T + e$ contains a cycle C_1 and $T + f$ contains a cycle C_2. We must have t as an edge of C_1 and C_2, since C_1 must pass through e, and hence through t since $d_{T+e}(w) = 2$, similarly C_2 must pass through t. By assumption C_1 and C_2 have odd length.

Suppose that C_1 and C_2 contain k common edges. We must have $k \geq 1$ since t is common. Consider the cycle C obtained by joining together C_1 and C_2, and removing the common edges this is possible since $k \geq 1$. Hence $|E(C)| = |E(C_1)| + |E(C_2)| - 2k$. But then $|E(C)|$ is even, since by assumption $|E(C_1)|$ and $|E(C_2)|$ are odd.

Hence G contains an even cycle when G is connected.

If G is not connected, then each connected component H satisfies $\delta(H) \geq 3$, so we may run the above argument to show that H has a cycle of even length, so hence G has a cycle of even length.

□

Question 2

We shall say that a graph G on n vertices has degree sequence $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)$ if $\boldsymbol{d}$ is a nondecreasing sequence and the degrees of the vertices of G when arranged in nondecreasing order are the entries of $\boldsymbol{d}$. We consider the following condition:

$$\sum_{i=1}^{n} d_i = 2n - 2. \tag{*}$$

Proposition 4 (Part (a))**.** *Suppose that T on n vertices with degree sequence $\boldsymbol{d}$. Then $\boldsymbol{d}$ satisfies (*).*

Proof. By Corollary 1.5.3 in the course notes, T has $n - 1$ edges. Hence by the handshaking lemma,

$$\sum_{i=1}^{n} d_i = 2|E(T)| = 2(n-1) = 2n - 2. \tag{2}$$

So (*) holds. □

Proposition 5 (Part (b))**.** *Now suppose that $n = 2$, and $\boldsymbol{d}$ is a sequence such that (*) holds. Then there exists a tree T with degree sequence $\boldsymbol{d}$.*

Proof. If (*) holds, then $d_1 + d_2 = 2(2) - 2 = 2$. Hence we can choose T to be the unique tree on two vertices. Then we have $d_1 = d_2 = 1$. □

Proposition 6 (Part (c))**.** *Now let $n \geq 3$, and let $\boldsymbol{d}$ be a sequence such that (*) holds. Let $\delta = d_1$ and $\Delta = d_n$ be the minimum and maximum entries of $\boldsymbol{d}$ respectively. Then,*

1. *$\delta = 1$ and $\Delta \geq 2$.*
2. *There exists a tree T with degree sequence $\boldsymbol{d}$.*

Proof. First we prove (1). Suppose that $\delta \geq 2$. This means that

$$\sum_{i=1}^{n} d_i \geq 2n. \tag{3}$$

But then $2n - 2 \geq 2n$, which is impossible. Now if we assume that $\Delta < 2$, then we must have $\Delta = 1$, so $\boldsymbol{d} = (1, 1, \ldots, 1)$. Thus,

$$n = \sum_{i=1}^{n} d_i = 2n - 2. \tag{4}$$

But then $n = 2$, which is impossible since $n \geq 3$ by assumption.

Now we prove (2) by induction. Suppose that for all $k < n$, we know that for any degree sequence $(d_1, d_2, \ldots, d_k)$ with $\sum_{i=1}^{k} d_i = 2k - 2$ there exists a tree on k vertices with degree sequence k.

Now let $\boldsymbol{d} = (d_1, \ldots, d_n)$ be a sequence satisfying (*).

Let

$$q = \min\{j\ d_j > 1\}. \tag{5}$$

Consider the sequence of $n - 1$ numbers,

$$\tilde{\boldsymbol{d}} = (d_2 \ldots, d_q - 1, \ldots, d_n). \tag{6}$$

if $q < n$, and $\tilde{\boldsymbol{d}} = (d_2, \ldots, d_n - 1)$ if $q = n$. Then we can see that $\boldsymbol{d}$ is a sequence satisfying (*), since for $q < n$:

$$d_2 + \cdots + (d_q - 1) + \cdots + d_n = 2n - 2 - 1 - d_n = 2n - 4 = 2(n-1) - 2. \tag{7}$$

If $q = n$, then we must have $\boldsymbol{d} = (1, 1, \ldots, 1, d_n)$. Hence $n - 1 + d_n = 2n - 2$, so $d_n = n - 1$, and

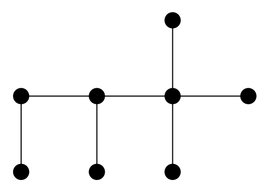
$$d_2 + \cdots + d_n - 1 = \underbrace{1 + \cdots + 1}_{n-2 \text{ times}} + d_n - 1 = 2(n-2) = 2(n-1) - 2. \tag{8}$$

Hence, there exists a tree with degree sequence $\tilde{\boldsymbol{d}}$ by our inductive hypothesis. Let T be such a tree, and let v be a vertex with degree $d_q - 1$.

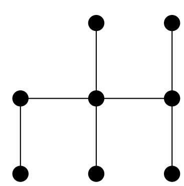
Consider the tree F obtained by joining a new leaf to v. Then F has n vertices, and has degree sequence $(1, d_2, \ldots, d_n) = \boldsymbol{d}$. Thus the assertion is proved by induction since by Proposition 5 it is true for $n = 2$. ☐

Proposition 7 (Part (d)). *There exist two non-isomorphic trees with identical degree sequence.*

Proof. Consider the following two trees. T_1:



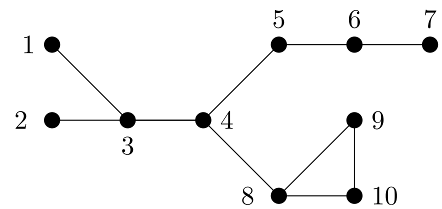
and T_2:



We can see that T_1 and T_2 both have degree sequence $(1, 1, 1, 1, 1, 2, 3, 4)$, but they are not isomorphic, since T_1 has a vertex of degree 4 attached to 3 leaves, but T_2 only has a vertex of degree 4 attached to 2 leaves. ☐

Question 3

For this question, we let G be the following graph on 10 vertices:



Proposition 8 (Part (a))**.** *There is no perfect matching of G.*

Proof. By Tutte's theorem (Theorem 2.1.1 in the course notes), it suffices to find $S \subseteq V(G)$ such that the number of connected components of $G - S$ with an odd number of vertices, $q(G - S)$ exceeds $|S|$.

Take $S = \{4\}$. Then the connected components of $G - S$ have vertices $\{1, 2, 3\}$, $\{5, 6, 7\}$ and $\{8, 9, 10\}$. Hence $q(G - S) = 3$. But $|S| = 1$, so $q(G - S) > |S|$. Thus there is no perfect matching of G. □

Proposition 9 (Part (b))**.** *A maximum matching of G is:*

$$M = \{23, 45, 67, 89\}. \tag{9}$$

Proof. It is clear from the picture that M is a matching. Since M covers 8 vertices, and there is no matching of G with 10 vertices by Proposition 8, and a matching must cover an even number of vertices, we conclude that M is maximum. □

Question 4

Proposition 10 (Part (a))**.** *Let G be a graph on $2r$ vertices, with minimum degree $\delta(G) \geq r$ with $r \geq 1$. Then G has a perfect matching.*

Proof. By Dirac's theorem (Theorem 10.1.1 in the course notes), G has a Hamilton cycle. Label the vertices of G as $v_1, v_2, \ldots, v_{2r}$ so that the cycle passes through $v_1 v_2 \cdots v_{2r}$. Now match up vertices according to the cyclic order, $\{v_1, v_2\}, \{v_3, v_4\}, \ldots, \{v_{2r-1}, v_{2r}\}$. This is possible since $2r$ is even. Hence G has a perfect matching. □

For the remainder of this question, G is a graph on n vertices with minimum degree $\delta(G) \geq 1$, and $\Delta := \Delta(G)$ is the maximum degree. Let F be a maximum matching of G, and let $\nu := \nu(G) = |F|$ be the size of a maximum matching in G. Recall that we say that a vertex x of G is covered by F if x is an endpoint of an edge in F.

Proposition 11 (Part (b) i)**.** *Let x be a vertex not covered by F. Then every neighbour of x is covered by F.*

Proof. If x has a neighbour y not covered by F, then $F \cup \{xy\}$ is a matching strictly larger than F, which contradicts the assumption that F is maximum. Hence every neighbour of x is covered by F. □

Proposition 12 (Part (b) ii)**.** *Let x, y be two distinct vertices not covered by F, and $a, b \in V(G)$. If $xa, yb \in E(G)$, then $ab \notin F$.*

Proof. If a and b are not distinct, then $ab \notin F$. So assume that $a \neq b$. Suppose that $xa, yb \in E(G)$, and $ab \in F$. Then consider the set $F' = (F \setminus \{ab\}) \cup \{xa, yb\}$. Then F' is a matching strictly larger than F, contradicting the assumption that F is maximum. □

Proposition 13 (Part (b) iii)**.** *The number of vertices of G not covered by F is at most $(\Delta - 1)\nu$.*

Proof. Let $v_1, v_2, \ldots, v_\nu$ be a set of representative vertices of each edge in F. That is, for each $e \in F$ there is exactly one k so that v_k is incident on e. Now, for each $1 \leq k \leq \nu$, v_k has at most $\Delta - 1$ unmatched neighbours since it has at most Δ neighbours in total and it connected to at least one matched vertex. This means that there cannot be more than $\nu(\Delta - 1)$ unmatched vertices. □

Proposition 14 (Part (b) iv)**.** *We have $\nu \geq n/(\Delta + 1)$.*

Proof. Let k be the number of vertices of G not covered by F. However each edge of F connects to two unique vertices, so the number of edges covered by F is 2ν. Thus $n = k + 2\nu$. But by Proposition 13, $k \leq (\Delta - 1)\nu$. Hence,

$$\begin{aligned} n &= k + 2\nu && (10) \\ &\leq (\Delta - 1)\nu + 2\nu && (11) \\ &= (\Delta + 1)\nu. && (12) \end{aligned}$$

So $\nu \geq n/(\Delta + 1)$. □

Remark 1. *The above estimate is sharp. Consider a triangle graph. In this case, $n = 3$, $\Delta = 2$ and $\nu = 1$.*

Question 5

For this question, G is a 2-edge connected graph. That is, G is connected and for any $e \in E(G)$, $G-e$ is connected. We define a relation $\sim$ on $E(G)$ as follows: $e \sim f$ if and only if $e = f$ or $G - \{e, f\}$ is disconnected.

Proposition 15 (Part (a))**.** *Let $e, f \in E(G)$ be such that every cycle containing e contains f and vice versa. Then $e \sim f$.*

Proof. We prove the contrapositive: we show that if $e \nsim f$, then there is a cycle containing e but not f.

Assume that $e \nsim f$. Then e and f are distinct, and $G-\{e, f\}$ is connected. Let $e = xy$. Then there is a path P joining x and y in $G - \{e, f\}$. Hence $P + e$ is a cycle in G containing e but not f.

Hence if every cycle containing e contains f, then $e \sim f$. Thus *a forteriori*, if every cycle containing e contains f and vice versa, $e \sim f$. □

Proposition 16 (Part (b))**.** *Suppose we have $e, f \in E(G)$ with $e \sim f$. Then every cycle which contains e also contains f.*

Proof. If $e = f$ the result is trivial, so we consider $e \neq f$.

Hence, $G-\{e, f\}$ is disconnected, but since G is 2-edge connected, we have that $G - e$ and $G - f$ are connected.

Assume, to find a contradiction, that C be a cycle in G which contains e but not f. Let $a, b \in V(G)$ be two vertices which are disconnected in $G - \{e, f\}$. Let P be a path joining a and b in $G - f$. Hence P must pass through e.

But if P passes through e, then this is a contradiction since we can adjoin $C-e$ to the path $P - e$ to get a path joining a and b in $G - \{e, f\}$.

Thus, every cycle containing e must contain f. □

Proposition 17 (Part (c))**.** *$\sim$ is an equivalence relation on $E(G)$, and the equivalence classes of $\sim$ are subsets of cycles of G.*

Proof. We must prove that $\sim$ is reflexive, symmetric and transitive.

We immediately get that for any edge e, $e \sim e$ since $e = e$.

Now if $e \sim f$, then $e = f$ or $G - \{e, f\}$ is disconnected. But this holds true if and only if $f = e$ and $G - \{f, e\}$ is connected. Hence $f \sim e$ so $\sim$ is symmetric.

Now we prove that $\sim$ is transitive. Let e, f, g be edges with $e \sim f$ and $f \sim g$. Then by Proposition 16, every cycle which contains e also contains f, and every cycle which contains f contains g. Hence every cycle containing e also contains g. Hence, by Proposition 15 we conclude that $e \sim f$.

Let e is an edge, and $[e]_\sim$ is the equivalence class of $\sim$ containing e. e must be contained in a cycle, since otherwise $G - e$ is disconnected, which contradicts 2-edge-connectivity.

Let C be a cycle containing e. Then every element of $[e]_\sim$ is contained in $\sim$ by Proposition 16. Thus $[e]_\sim \subseteq E(C)$. $\square$

Proposition 18 (Part (d))**.** *Let $P \subseteq E(G)$ be an equivalence class of $\sim$. Then every connected component of $G - P$ with at least two vertices is 2-edge-connected.*

Proof. Let H be a connected component of $G - P$ with at least two vertices, and let $x, y \in V(H)$. It is required to prove that $H - xy$ is connected. In other words, we must prove that H contains a cycle containing xy. Note that there exists a cycle containing xy in G since G is 2-edge-connected.

We now show that there exists a cycle containing xy in G which is disjoint from every element of P.

Suppose that C is a cycle in G containing xy which also contains $e \in P$. But then since C is a cycle containing e, it must also contain all f such that $f \sim e$. Hence if C is a cycle containing some element of P, it must contain all elements of P. So it suffices to show that there is a cycle containing xy which does not contain a particular element of P.

Let $e \in P$, then since $e \nsim xy$, by Proposition 15, there is a cycle containing xy but not e.

Hence, there is a cycle C containing xy disjoint from P, so C is a cycle in $G - P$ containing xy. Hence, H contains a cycle containing xy. Therefore, H is 2-edge-connected. $\square$