SCHOOL OF MATHEMATICS AND STATISTICS
UNIVERSITY OF NEW SOUTH WALES

MATH5425 Graph Theory Semester 2 2015

Assignment 2 (25%)

Due Wednesday 7 October, Week 10

You may discuss this assignment with other students, or look at textbooks if you wish (though the assignment is designed to be attempted based on the material covered in lectures and in Diestel). But it is very important that you write up your solution *on your own*. Do not copy work from other students or from textbooks.

If a textbook or fellow student gave you an idea which helped you, please acknowledge this help in your solutions. (For example, write something like "Discussions with XXX were helpful" or "The main idea for part (b) came from XXX, page number XXX".) This is how professional mathematicians acknowledge assistance from their colleagues or from prior work.

In addition to the marks for each question shown below, up to **3 marks** will be awarded for the overall clarity of exposition and logic, elegance of solutions, neatness (lack of typos), etc. Hence the assignment will **marked out of 50**.

Use of LaTeX is optional and will not affect the mark awarded. The LaTeX file for the assignment will be made available on Moodle.

Please hand in your assignment in **hardcopy** (i.e., a printout, or your handwritten assignment) by the end of the Wednesday lecture in Week 10. If you think you will be unable to do this, please contact me in advance (say by the end of Monday of Week 10) to make alternative arrangements. Marks may be deducted for assignments that are late without warning.

Please also hand in a signed cover sheet, available from
https://www.maths.unsw.edu.au/sites/default/files/assignment.pdf

(Please turn over...)

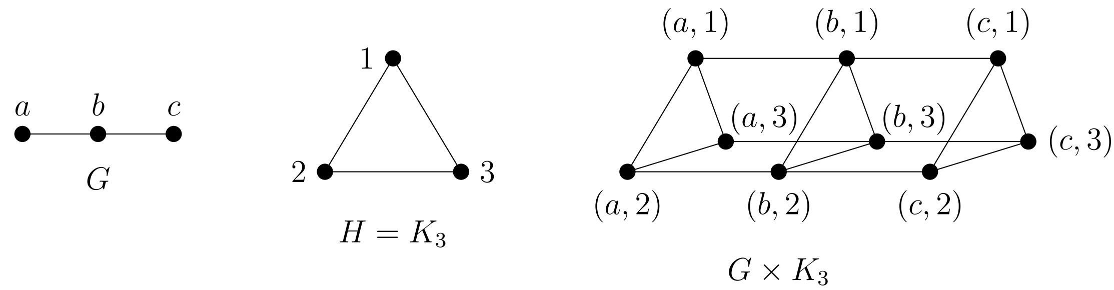
1. **(13 marks)**
Let $G = (V, E)$ and $H = (W, F)$ be two graphs. Define the *product* of G and H to be the graph $G \times H$ with vertex set

$$V \times W = \{(v, w) \mid v \in V, \ \ w \in W\},$$

where vertices (v_1, w_1) and (v_2, w_2) of $G \times H$ are adjacent if and only if

$$(v_1 = v_2 \text{ and } w_1w_2 \in F) \quad \text{ or } \ (v_1v_2 \in E \text{ and } w_1 = w_2)\,.$$

An example is shown below, where G is a path of length 2 and $H = K_3$.



Suppose that G is a graph with $n \geq 2$ vertices. We consider the product of G with K_r, the complete graph on r vertices.

Recall that $\alpha(H)$ denotes the *independence number* of H, which is the number of vertices in the largest independent set of H.

(a) Prove that $\alpha(G \times K_r) \leq n$.

(b) Prove that equality holds in (a) when $r = \chi(G)$.

(c) Hence, or otherwise, prove that

$$\chi(G) = \min\{\, r \in \mathbb{Z}^+ \ \mid \ \alpha(G \times K_r) = n\,\}.$$

2. **(7 marks)** Let G be a 3-connected graph and let $xy \in E(G)$. Recall that G/xy denotes the graph obtained from G by contracting the edge xy to give the new vertex v_{xy}.

(a) If G/xy is 3-connected, prove that $G - \{x, y\}$ is 2-connected.

(b) If G/xy is not 3-connected, prove that $G - \{x, y\}$ is not 2-connected.

3. **(12 marks)** For each nonnegative integer k, let $P_G(k)$ denote the number of k-colourings of G. Recall from Problem Sheet 4, Question 3 that P_G is a polynomial in k, called the *chromatic polynomial* of G. (The results proved in Problem Sheet 4, Question 3 may help in answering this question.)

   (a) Prove that there is no graph G with chromatic polynomial $P_G(k)$ given by

   $$P_G(k) = k^4 - 4k^3 + 8k^2 - 5k = k(k-1)(k^2 - 3k + 5).$$

   (b) Let v be a vertex of the graph G and let k be a positive integer. Calculate the number of k-colourings of G which assign colour 1 to vertex v, with a brief explanation. (Your answer should involve $P_G(k)$.)

   (c) Suppose that G has connected components $G_1, \ldots, G_r$. How can $P_G(k)$ be expressed in terms of $P_{G_1}(k), \ldots, P_{G_r}(k)$? Give a brief proof of your answer.

   (d) Suppose that G has r connected components. What does that tell you about the *coefficients* of $P_G(k)$? Make a hypothesis and prove it.

   (e) Draw a graph which has chromatic polynomial $k^2(k-1)^3(k-2)$. (You do not need to provide a proof, just the graph.)

4. **(7 marks)** Let $G = (V, E)$ be a k-edge-connected graph, and suppose that $F_1, \ldots, F_m \subseteq E$ are distinct sets of k edges which separate G. That is, for each $i = 1, \ldots, m$,

   $$|F_i| = k \quad \text{and} \quad G - F_i \text{ is disconnected.}$$

   Let $\mathcal{F} = F_1 \cup F_2 \cup \cdots \cup F_m$, and suppose that $G - \mathcal{F}$ has t connected components, denoted $C_1, \ldots, C_t$.

   (a) Prove that for each $i = 1, \ldots, t$,

   $$|\{e \in \mathcal{F} : e \cap C_i \neq \emptyset\}| \geq k.$$

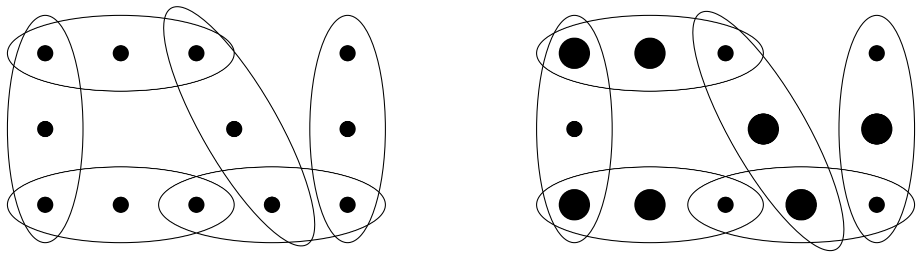
   (That is, at least k edges $\mathcal{F}$ are incident with at least one vertex of C_i in G.)

   (b) Hence find a lower bound on $|\mathcal{F}|$, with proof.

   (c) By finding an upper bound on $|\mathcal{F}|$, or otherwise, prove that $t \leq 2m$.

5. **(8 marks)**
A *3-uniform hypergraph* $H = (V, E)$ consists of a finite set V of vertices and a set E of *hyperedges*, where each hyperedge $e \subseteq V$ contains 3 vertices.

An *independent set* in a hypergraph H is a set $U \subseteq V$ of vertices such that $e \not\subseteq U$ for all hyperedges $e \in E$. A 3-uniform hypergraph with 12 vertices and 6 hyperedges is shown on the left, and an independent set in this hypergraph is shown on the right (with larger vertices belonging to the independent set):



Fix a 3-uniform hypergraph H with n vertices and $m \geq n/3$ hyperedges. Let U be a random subset of V such that $v \in U$ with probability p, independently for each $v \in V$.

(a) Write down the expected value of $|U|$.

(b) Let Y be the number of hyperedges of H which are contained in U. Calculate the expected value of Y, with explanation.

(c) Hence prove that H has an independent set which contains at least

$$\frac{2\,n^{3/2}}{3\sqrt{3}\,\sqrt{m}}$$

vertices.