SCHOOL OF MATHEMATICS AND STATISTICS
UNIVERSITY OF NEW SOUTH WALES

MATH5425 Graph Theory Semester 2 2015

Assignment 1 (25%): due Week 5
Wednesday 26 August (in the lecture)

You may discuss this assignment with other students, or look at textbooks if you wish (though the assignment is designed to be attempted based on the material covered in lectures and in Diestel). But it is very important that you write up your solution *on your own*. Do not copy work from other students or from textbooks.

If a textbook or fellow student gave you an idea which helped you, please acknowledge this help in your solutions. (For example, write something like "Discussions with XXX were helpful" or "The main idea for part (b) came from XXX, page number XXX".) This is how professional mathematicians acknowledge assistance from their colleagues or from prior work.

In addition to the marks for each question shown below, up to **3 marks** will be awarded for the overall clarity of exposition and logic, elegance of solutions, neatness (lack of typos), etc. Hence the assignment will **marked out of 50**.

Use of LaTeX is optional and will not affect the mark awarded. The LaTeX file for the assignment will be made available on Moodle.

Please hand in your assignment in **hardcopy** (i.e., a printout, or your handwritten assignment) by the end of the Wednesday lecture in Week 5. If you think you will be unable to do this, please contact me in advance (say by the end of Monday of week 5) to make alternative arrangements.

Please also hand in a signed cover sheet, available from

https://www.maths.unsw.edu.au/sites/default/files/assignment.pdf

1. **(8 marks)** Let G be graph with n vertices.

   (a) Suppose that G has at least n edges. Prove that G contains a cycle.

   (b) Now suppose that G has more than n edges. Prove that G contains at least two cycles. (Note: the cycles need not be disjoint).

   (c) Suppose that G has minimum degree $\delta(G) \geq 3$. Show that G contains a cycle of even length.

2. **(13 marks)**
   Fix $n \geq 2$ and let $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)$ be a sequence of positive integers which satisfy

$$d_1 \geq d_2 \geq \cdots \geq d_n.$$

   We say that the graph G has *degree sequence* $\boldsymbol{d}$ if G has n vertices and the ith vertex has degree d_i, for $i = 1, \ldots, n$ (after relabelling the vertices, if necessary). In this question we prove that a tree T exists with degree sequence $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)$ if and only if

$$\sum_{i=1}^{n} d_i = 2n-2. \tag{$*$}$$

   (a) Prove that the condition $(*)$ is necessary: that is, if a tree T exists with degree sequence $\boldsymbol{d}$ then $(*)$ holds.

   (b) Suppose that $(*)$ holds and that $n = 2$. Prove that a tree T exists with degree sequence $\boldsymbol{d}$.

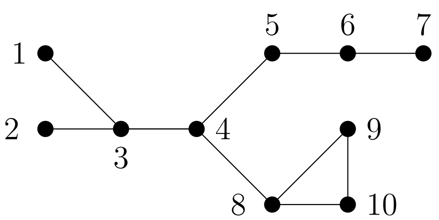
   (c) Now assume that $(*)$ holds and that $n \geq 3$. Let $\delta = d_n$ and $\Delta = d_1$ denote the minimum and maximum entry of $\boldsymbol{d}$, respectively.

      (i) Prove that $\delta = 1$ and $\Delta \geq 2$.

      (ii) Using induction on n, or otherwise, show that a tree T exists with degree sequence $\boldsymbol{d}$.

   (d) Draw two non-isomorphic trees with the same degree sequence. Also provide a brief proof that the two trees are not isomorphic.

3. **(5 marks)** Let G be the following graph with 10 vertices.



   (a) Use Tutte's Theorem to prove that G has no perfect matching.

   (b) Find a maximum matching M of G. Include a brief proof.

4. **(9 marks)**

(a) Let G be a graph with $2r$ vertices and minimum degree $\delta(G) \geq r$, where $r \geq 1$. Prove that G has a perfect matching.

(b) Let G be a graph with minimum degree $\delta(G) \geq 1$ and maximum degree $\Delta = \Delta(G)$. Let F be a maximum matching in G and let $\nu = \nu(G) = |F|$ be the size of a maximum matching in G. Say that a vertex is *covered by* F if it is the endvertex of an edge in F.

(i) Let x be a vertex not covered by F. Show that every neighbour of x is covered by F.

(ii) Let x, y be two distinct vertices not covered by F. Show that if $xa, yb \in E$ then $ab \notin F$.

(iii) Hence show that the number of vertices not covered by F is at most $(\Delta - 1)\,\nu$.

(iv) Prove that $\nu \geq n/(\Delta + 1)$.

5. **(12 marks)**

Let G be a 2-edge-connected graph. Define a binary relation on the edges of G, denoted $\sim$, as follows: for $e, f \in E(G)$ we have $e \sim f$ if and only if $e = f$ or the graph $G - \{e, f\}$ is disconnected.

(a) Let $e, f \in E(G)$. Suppose that every cycle in G which contains e also contains f, and vice-versa. Prove that $e \sim f$.

(b) Suppose that $e, f \in E(G)$ and that $e \sim f$. Prove that every cycle which contains e also contains f.

(c) Show that $\sim$ is an equivalence relation (just give a brief explanation). Further, prove that each equivalence class is a subset of the edges of some cycle of G.

(d) Let $P \subseteq E(G)$ be an equivalence class of $\sim$. Prove that each connected component of $G - P$ with at least two vertices is 2-edge-connected.