SCHOOL OF MATHEMATICS AND STATISTICS
UNIVERSITY OF NEW SOUTH WALES

MATH5425 Graph Theory Semester 2 2015

Problem Sheet 4, Colourings

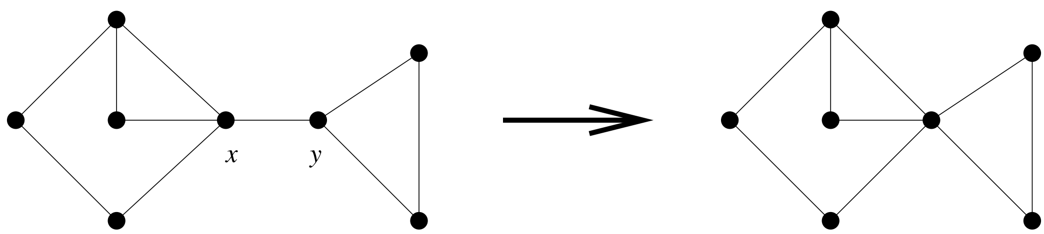
1. Check that $\chi(G) = \Delta(G) + 1$ if G is a complete graph or an odd cycle.

2. Let G be a graph with n vertices. Prove that

$$n/\alpha(G) \leq \chi(G) \leq n + 1 - \alpha(G),$$

where $\alpha(G)$ is the independence number of G.

3. Let $G = (V, E)$ be a graph with n vertices. Given $xy \in E$, let $G' = G - xy$ and form G'' from G by *contracting* the edge xy to a point, as shown below:



Let $P_G(k)$ be the number of distinct k-colourings $c : V \to \{1, \ldots, k\}$ of G. (This means distinct as functions from $V \to \{1, \ldots, k\}$, taking the vertex labels into account.)

   (a) Find a formula relating $P_G(k)$ to $P_{G'}(k)$ and $P_{G''}(k)$.

   (b) By induction on $m = |E(G)|$, prove that $P_G(k)$ is a polynomial of degree n in k, where the coefficient of k^n is 1 and the coefficient of k^{n-1} is $-m$. (We call $P_G(k)$ the *chromatic polynomial* of G.)

4. In the proof of Brooks' theorem, the key step is to show that the graph G is Δ-colourable, under the inductive hypothesis (namely, that all graphs H with fewer vertices than G and which are not regular or an odd cycle are $\Delta(H)$-colourable). Show that this argument is a lot shorter if the graph G is not regular.

(... Please turn over for Questions 5, 6 and 7)

5. A *latin square* is an $n \times n$ array such that each row and each column contain the integers $\{1, \ldots, n\}$ once each. Model the problem of constructing an $n \times n$ latin square into a graph colouring problem. How many vertices do you need? (Consider edge colourings as well as vertex colourings.)

6. Let G be a cubic (3-regular) graph with a 3-edge-colouring

$$c : E(G) \to \{1, 2, 3\}.$$

   (a) Prove that each colour class of c forms a perfect matching.

   (b) Now suppose that up to permutation of the colours $1, 2, 3$, the 3-edge-colouring c is unique. (That is, any other 3-edge-colouring gives the same partition of the edge set of G into colour classes.) Show that G is Hamiltonian.

7. Without using Proposition 5.3.1 (König 1916), show that $\chi'(G) = k$ for all k-regular bipartite graphs G.