1 Introduction

In the Appendix of the 1994 paper by Connes, Sullivan and Teleman, [1], the following is claimed as a theorem,

Let $d > 1$. Let T be a zeroth order pseudodifferential operator which we assume to be dilation invariant, translation invariant and nonzero. Let f be a function on $\mathbb{R}^d$. Then we have that the commutator $[T, M_f]$ is in $\mathcal{L}^{d,\infty}$ if and only if f is in the Sobolev space $W^{1,d}(\mathbb{R}^d)$.

This is false, and the purpose of this document is to explain carefully a counterexample, and to provide a correct restatement of the theorem.

2 A counterexample

We recall some definitions,

Definition 1. *A linear map $T : \mathcal{S}(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$ is called a generalised pseudodifferential operator if there exists a (measurable) function $\sigma : \mathbb{R}^d \to \mathbb{C}$ which increases no more rapidly than a polynomial, such that*

$$Tf(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\langle \xi, x\rangle} \sigma(\xi) \hat{f}(\xi)\, d\xi \tag{1}$$

where $\hat{f}$ is the Fourier transform of f.

The function σ is called the symbol of T, and we write $T = \sigma(\mathcal{D})$.

Furthermore, we say that T is zeroth order if $\sigma \in L^\infty(\mathbb{R}^d)$.

Proposition 1. *Let $T = \sigma(\mathcal{D})$ be a zeroth order generalised pseudodifferential operator, then T has an extention to $L^2(\mathbb{R}^d)$ that is a bounded map $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.*

Definition 2. *A generalised pseudodifferential operator is said to be dilation invariant if it commute with any dilation, and translation invariant if it commutes with any translation.*

Definition 3. *A function $\sigma : \mathbb{R}^d \to \mathbb{R}$ is said to be zeroth degree homogeneous if for any $\lambda > 0$ and all $\xi \in \mathbb{R}^d$, we have $\sigma(\lambda\xi) = \sigma(\xi)$.*

Theorem 1. *Generalised pseudodifferential operators, as we have defined them, are always translation invariant.*

Proof. Let $h \in \mathbb{R}^d$. Then we simply compute,

$$Tf(x+h) = \int_{\mathbb{R}^d} e^{i\langle \xi, h\rangle + i\langle x, h\rangle} \sigma(\xi)\hat{f}(\xi) d\xi. \tag{2}$$

But we observe that $e^{i\langle \xi, h\rangle}\hat{f}(\xi)$ is the Fourier transform of the function $g(x) := f(x+h)$. Hence T commutes with translations. □

Theorem 2. *Let $T = \sigma(\mathcal{D})$ be a generalised pseudodifferential operator. Then if σ is homogeneous of degree 0, then T is dilation invariant.*

Proof. This is a straightforward computation. Let $\lambda > 0$ and $f \in \mathcal{S}(\mathbb{R}^d)$, then we have

$$\begin{aligned} Tf(\lambda x) &= \int_{\mathbb{R}^d} e^{i\langle \xi, \lambda x\rangle} \sigma(\xi)\hat{f}(\xi) d\xi & (3) \\ &= \int_{\mathbb{R}^d} e^{i\langle \xi, x\rangle} \sigma(\xi/\lambda)\hat{f}(\xi/\lambda) d(\xi/\lambda) & (4) \\ &= \int_{\mathbb{R}^d} e^{i\langle \xi, x\rangle} \sigma(\xi)\frac{1}{\lambda}\hat{f}(\xi/\lambda) d\xi. & (5) \end{aligned}$$

However, we see that $\lambda^{-1}\hat{f}(\xi/\lambda)$ is the fourier transform of $g(x) := f(\lambda x)$. Hence T commutes with dilation. □

Putting these results together, we see that if σ is a nonzero bounded function which is homogeneous of order 0, then $\sigma(\mathcal{D})$ is a zeroth order generalised pseudodifferential operator which defines a bounded linear operator on $L^2(\mathbb{R}^d)$ and commutes with dilations and translations.

Proposition 2. *There exists a $\sigma \in L^\infty(\mathbb{R}^3)$ which is homogeneous of order zero, and a function $f \in \mathcal{S}(\mathbb{R}^3)$ such that the commutator $[\sigma(\mathcal{D}), M_f]$ is not compact on $L^2(\mathbb{R}^3)$.*

Proof. Denote the coordinates on $\mathbb{R}^3$ as $x = (x_1, x_2, x_3)$. Consider the function,

$$\sigma(x) := \frac{x_1}{\sqrt{x_1^2 + x_2^2}}. \tag{6}$$

Then σ is homogeneous of order zero and bounded. Hence the associated pseudodifferential operator $\sigma(\mathcal{D})$ is bounded on $L^2(\mathbb{R})$, and commutes with all dilations and translations.

Recall that we have the embedding $L^\infty(\mathbb{R}^2) \otimes L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}^3)$ which takes an elementary tensor $f_1 \otimes f_2$ to the function $f(x_1, x_2, x_3) = f_1(x_1, x_2) f_2(x_3)$.

Similarly, we have an embedding,

$$\mathcal{B}(L^2(\mathbb{R}^2)) \otimes \mathcal{B}(L^2(\mathbb{R})) \to \mathcal{B}(L^2(\mathbb{R}^3)) \tag{7}$$

which maps the elementary tensor $A \otimes B$ to the operator on $L^2(\mathbb{R}^3)$ which acts on functions of the form $f_1(x_1, x_2)f_2(x_3)$ by $Af_1 \otimes Bf_2$, and this extends to an operator on $L^2(\mathbb{R}^3)$ by density.

Now let $f_1 \in \mathcal{S}(\mathbb{R}^2)$ and $f_2 \in \mathcal{S}(\mathbb{R})$. Define $f := f_1 \otimes f_2 \in \mathcal{S}(\mathbb{R}^3)$. As operators on $L^2(\mathbb{R}^3)$, we have $M_f = M_{f_1} \otimes M_{f_2}$.

Now we compute,

$$[\sigma(\mathcal{D}), M_f] = [\sigma(\mathcal{D}), M_{f_1} \otimes M_{f_2}]. \tag{8}$$

However since σ has no dependence on x_3, we see that $\sigma(\mathcal{D})$ acts on $1 \otimes f_2$ as the identity, so we have

$$[\sigma(\mathcal{D}), M_f] = [\sigma(\mathcal{D}), M_{f_1}] \otimes M_{f_2}. \tag{9}$$

However the first term on the left hand side, $[\sigma(\mathcal{D}), M_{f_1}]$ is simply an operator on $L^2(\mathbb{R}^2)$. Hence we have that $[\sigma(\mathcal{D}), M_f]$ is an elementary tensor of the form $A \otimes M_{f_2}$. Hence, since M_{f_2} is not compact, we conclude that $[\sigma(\mathcal{D}), M_f]$ is not compact. □

3 A correct version

The following theorem might be true.

Theorem 3. *Let $\sigma(x_1, x_2) = x_1/\sqrt{x_1^2 + x_2^2}$. Then we have $[\sigma(\mathcal{D}), M_f] \in \mathcal{L}^{2,\infty}$ if and only if $f \in W^{1,2}(\mathbb{R}^2)$.*

References

[1] Alain Connes, Dennis Sullivan, and Nicolas Teleman. Quasiconformal mappings, operators on hilbert space, and local formulae for characteristic classes. *Topology*, 33(4):663–681, 1994.